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Ranking Income Distributions Using the Geometric Mean and a Related General Measure*

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I. Introduction

The recent application of stochastic dominance ranking rules to the evaluation of income distributions has stimulated new interest in welfare orderings. While the power of stochastic dominance rules is quite remarkable [19; 6], there are still important cases where incomplete orderings emerge. Bishop, Formby, and Thistle, (hereafter referred to as BFT) [6] use rank dominance (first degree stochastic dominance) and generalized Lorenz dominance (second degree stochastic dominance) to order the income distributions of two international data sets. While the generalized Lorenz (rank) dominance orders approximately 82–84% (75–78%) of the pairwise comparisons of the two data sets, some of the most interesting cases are among those left unordered. For example, Japan's income distribution dominates 16 others, but is unordered compared to 9 distributions.

This paper proposes a ranking index that provides a complete ordering of income distributions. This ordering is consistent with the partial ordering from generalized Lorenz dominance. The ranking index includes a subjective parameter (ε) that allows the observer to set the desired degree of equity preference relative to efficiency preference and meets the minimum requirements of Schur-concavity and the weak Pareto principle for all admissible values of ε, while satisfying strict Schur-concavity and the strong Pareto principle for all interior values of ε. It gives a measure that is decomposable into the arithmetic mean and a measure of dispersion. Furthermore this generalized index encompasses as special cases, the arithmetic mean ($\varepsilon = 0$), the Rawlsian criteria ($\varepsilon \to \infty$), and the geometric mean ($\varepsilon = 1$).

Rank dominance [18; 20; 21] is based on the strong Pareto principle and is shown by Saposnik [20; 21] to be equivalent to first degree stochastic dominance (FDSD). BFT [6] note that a possible objection to rank dominance is that it does not take account of the degree of income inequality. Generalized Lorenz dominance [19; 11; 12] is based on the strong Pareto principle and the principle of transfers and is shown by Thistle [24] to be equivalent to second degree stochastic dominance (SDSD). While FDSD may be objected to for not taking account of the degree of income inequality, SDSD may be objected to for very weak equity preference. To illustrate

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this, consider a lexicographic ranking obtained by using the arithmetic mean and an inequality measure such as the Gini coefficient only as a tie-breaker when the means of two distributions are the same. This lexicographic ranking incorporates equity preference via the inequality measure in essentially a second order manner. Significantly, SDSD produces a partial ordering that does not conflict with the complete lexicographic ordering. Apparently, then, equity preference is an order of magnitude lower than efficiency preference in SDSD rankings as well. BFT [6, 1409] come to largely the same conclusion and report that: ". . . much of the power of generalized Lorenz dominance to order distributions derives from efficiency preference rather than equity preference."

The incomplete ordering provided by both FDSD and SDSD leaves some ambiguity in ranking distributions. Some economists have argued that this ambiguity is appropriate and that the subjective weighting should be left to the policy analysts and decision makers [17]. However, with a rigorously specified criterion that provides a complete ordering, the effect of the subjective weights on the final rankings can be determined. Hence, there remains a role for a ranking index with a subjective parameter that allows the observer to determine the desired degree of equity preference relative to efficiency preference and which can provide complete orderings of income distributions.

II. Measuring Welfare

There are broadly acceptable requirements that may serve as minimum desirable properties for a welfare index that can be summarized in two basic axioms for a welfare function, W, and an income distribution, $\mathbf{y}$.

AXIOM 1. *Weak Pareto Principle: W is non-decreasing in* $\mathbf{y}$.

This axiom follows from having utility functions that are non-decreasing in y_i. It captures the minimum broadly acceptable efficiency preference of W. The strong Pareto principle would require that W is monotonically increasing in $\mathbf{y}$. This is required for FDSD.

AXIOM 2. *Schur-Concavity: Let W be a welfare function and* $\mathbf{y}$ *be the income vector, then for all bistochastic matrices* $\mathbf{B}$, $W(\mathbf{By}) \geq W(\mathbf{y})$.[1]

This property insures that mean preserving regressive transfers do not increase W and also incorporates the symmetrical treatment of individuals (or anonymity) since any permutation matrix is a bistochastic matrix. This property captures the minimum broadly acceptable equity preference of W. Requiring strict Schur concavity [i.e., $W(\mathbf{By}) > W(\mathbf{y})$ for bistochastic matrices $\mathbf{B}$, excluding permutation matrices] serves to incorporate the welfare equivalent of the Pigou-Dalton condition of inequality measures; that is, that rank preserving progressive transfers causes W to increase. This transfer principle along with the strong Pareto principle are required for SDSD.

A welfare index that satisfies these two axioms and gives a complete ordering is constructed by borrowing the structure of the Svejnar [22; 23] generalization of the Nash-Zeuthen-Harsanyi bargaining problem. Let $W = \Pi_i U_i^{\gamma_i}$, for $i = 1$ to n, where Π_i signifies the Cartesian product over i, U_i is the individual's utility function, and γ_i serves as a monotonic transformer of the

1. A bistochastic matrix is one in which all the elements are non-negative and the elements of any row or column will sum to unity.

utility function. The simplest implementation of W would have $U_i = y_i$ and $\gamma_i = (1/n)$ for all i. This implies that W is the geometric mean of individual incomes y_i. Specifically, $W = \Pi_i y_i^{(1/n)}$.

While the arithmetic mean is widely used in ranking income distributions the geometric mean is largely ignored.[2] Nygard and Sandstrom [14] mention it in passing yet they fail to elucidate the very desirable properties that it possesses. Jean [10] demonstrates that a geometric mean ranking of a distribution is a necessary (but not sufficient) condition for any degree stochastic dominance. The range of the geometric mean for all distributions is from 0 to μ (where μ is the arithmetic mean). It is straightforward to show that the geometric mean decomposes into the arithmetic mean and a term that is well suited to be used as an equality measure (so phrased because it increases as inequality decreases). Let q_i represent the fraction of the sum of all individuals' incomes that is received by the ith agent, and μ represent the arithmetic mean of the income distribution. Now define:

$$E = \{n^*[\Pi_i(q_i)]^{(1/n)}\}, \quad \text{so} \tag{1}$$

$$W = \mu^* E. \tag{2}$$

The equality measure, E, conveniently falls in the range 0 to 1, where 0 is the value attained for the concentrated distribution (i.e., when a single agent receives all of the income and other agents in the society receive nothing) and 1 is the value attained for the egalitarian distribution (i.e., when each agent in a society receives μ). It gives a complete ordering that is consistent with the Lorenz criterion's partial ordering. This can be verified by recognizing that the equality measure, E, is identifiable as the variable element of the Atkinson [2] inequality index when $U_i = \ln y_i$ (the specific case in which the Atkinson inequality aversion parameter is set to unity).

The axiomatic attributes and other characteristics of the Atkinson index are well known and are rigorously and extensively examined in Fields and Fei [8] and Champernowne [7]. Fields and Fei establish that the Atkinson index is one of the few that will satisfy the three basic axioms of Scale Irrelevance, Symmetry (or anonymity), and Rank-Preserving equalizations (or the Pigou-Dalton condition).

The Atkinson Inequality index is:

$$A_\varepsilon = 1 - y_{ede}/\mu, \tag{3}$$

where y_{ede} is the equally distributed equivalent income and can be found from:

$$y_{ede} = U^{-1}\left\{\sum\nolimits_i U(y_i)/n\right\}. \tag{4}$$

The family of utility functions specified by Atkinson [2] reflects the use of the inequality aversion parameter, ε.[3] Atkinson [2] specifies U, for all i, as:

$$U(y_i) = B(y_i^{1-\varepsilon}/1-\varepsilon), \quad \text{for } \varepsilon \geq 0 \text{ and } \varepsilon \;/\!= 1, \tag{5a}$$

$$U(y_i) = \ln(y_i), \quad \text{for } \varepsilon = 1. \tag{5b}$$

2. Basmann and Slottje [4; 5] incorporate the geometric mean into a measure of income inequality.

3. Atkinson's inequality aversion parameter is mathematically analogous to and is taken from the Arrow [1] and Pratt [15] results on risk aversion.

Substituting (5) into (4) yields:

$$y_{ede} = \left[\sum_i (y_i^{1-\varepsilon})/n\right]^{1/(1-\varepsilon)}, \quad \text{for } \varepsilon \geq 0 \text{ and } \varepsilon \neq 1, \tag{6a}$$

$$y_{ede} = e^{[\Sigma_i \ln(y_i)/n]}, \quad \text{for } \varepsilon = 1. \tag{6b}$$

The right hand side of equation (6b) is simply an expression of the geometric mean of $\mathbf{y}$. Therefore, E [from (2)] is equivalent to the variable component of the Atkinson index (for the case $\varepsilon = 1$). Specifically:

$$E = y_{ede}/\mu = \left\{e^{[\Sigma_i \ln(y_i)/n]}\right\}/\mu. \tag{7}$$

III. The Generalized Welfare Measure

The equivalency of E to the variable component of the Atkinson index for one of its cases suggests that the equally distributed equivalent income can be used as a generalized welfare measure. Thus, substituting from (6) above:

$$W = \left[\sum_i (y_i^{1-\varepsilon})/n\right]^{1/(1-\varepsilon)}, \quad \text{for } \varepsilon \geq 0 \text{ and } \varepsilon \neq 1, \tag{8a}$$

$$W = e^{[\Sigma_i \ln(y_i)/n]} = \Pi_i y_i^{(1/n)}, \quad \text{for } \varepsilon = 1.\text{[4]} \tag{8b}$$

The equivalency of W to y_{ede} means that this welfare measure has cardinal as well as ordinal significance. Also, W easily decomposes multiplicatively into μ and the variable component of the Atkinson inequality index.

The parameter, ε, is used to set the relative degree of equity preference to efficiency preference allowing this generalized welfare measure to incorporate other standard welfare criteria as special cases. For example, when $\varepsilon = 0$, $W = \mu$ for all income distributions. Thus a ranking using the arithmetic mean represents an extreme case of efficiency preference and, while it is Schur-concave (satisfying the bare minimum criterion for equity preference), it will not exhibit strict Schur-concavity.[5] The limit of W as $\varepsilon \to \infty$ is the minimum income in the distribution. This represents the extreme case of equity preference, but remains within the minimum bounds set for efficiency preference (and thus satisfies the weak Pareto principle). The case for the use of the minimum income as a welfare measure is made by Rawls [16]. Note that for all interior values of ε this generalized welfare measure satisfies strict Schur-concavity and the Pareto principle.

The geometric mean is the specific case of the generalized welfare measure (when $\varepsilon = 1$) in which W responds equally to equal *proportionate* increases of y_i for all i. That is, for any given distribution, a 10% increase in one agent's income causes the same increase in W as would a 10% increase in any other agent's income, whether higher or lower. Therefore, for all values of $\varepsilon > 1$, W is more sensitive to equal *proportionate* increases of lower incomes. For all values of $\varepsilon < 1$, W is more sensitive to equal *proportionate* increases of higher incomes.

4. This is akin to the iso-elastic form of the welfare function in Atkinson and Stiglitz [3, 340].

5. In this case the welfare index reduces to the Benthamite welfare function divided by n, the population size, for the case of $U_i = y_i$.

Table I. Welfare Rankings for 26 Countries

COUNTRY	Rank	*GDPc*	Rank	$W_{\varepsilon=0.5}$	Rank	$W_{\varepsilon=1.0}$	Rank	$W_{\varepsilon=2.0}$
U.S.	1	4790	1	4163	1	3557	1	2494
Sweden	2	4148	2	3655	2	3167	2	2275
W. Germany	3	3569	4	2962	4	2523	5	1967
Denmark	4	3516	3	3137	3	2753	4	2028
Norway	5	3276	5	2893	5	2427	8	1341
New Zealand	6	3094	6	2739	8	2348	7	1526
Finland	7	3022	10	2462	10	1877	11	852
U.K.	8	2994	7	2686	6	2398	6	1892
Netherlands	9	2984	9	2507	9	2055	9	1279
Japan	10	2764	8	2561	7	2374	3	2054
Spain	11	1902	11	1685	11	1496	10	1206
Uruguay	12	1705	12	1371	12	1066	13	620
Panama	13	1336	13	1125	13	928	14	597
Gabon	14	1245	15	870	15	640	15	432
Barbados	15	1145	14	1026	14	921	12	753
Brazil	16	1102	16	767	16	557	16	360
Columbia	17	838	18	634	18	470	21	245
Malaysia	18	829	17	657	17	504	20	260
Peru	19	781	21	474	22	313	22	197
Ivory Coast	20	742	19	582	19	461	17	324
Tunisia	21	656	20	529	20	427	18	295
Sri Lanka	22	456	22	398	21	345	19	263
Kenya	23	351	23	248	24	186	25	131
India	24	290	24	242	23	205	23	152
Tanzania	25	252	26	194	26	155	26	111
Indonesia	26	245	25	205	25	176	24	139

An increase in a lower income will have a greater effect on W than an equal *absolute* increase in any higher income whenever $\varepsilon > 0$. That is, a \$100 increase in income of one agent increases W more than a \$100 increase in any richer agent's income. For the arithmetic mean ($\varepsilon = 0$), equal *absolute* increases affect W equally.

IV. Comparison to the Stochastic Dominance Rankings

A data set with income distributions for 26 countries is used to demonstrate the complete rankings that can be provided by the generalized welfare index and the sensitivity of the rankings to the subjective parameter (ε). This is one of two data sets used in BFT [6].[6] Using the same data allows for direct comparison to the partial orderings using FDSD and SDSD found in BFT [6].

Table I lists the countries in descending order of *GDP* per capita (*GDPc*). The generalized welfare index for four values of ε is computed and listed and preceded by the rank of each country associated with the computed welfare index values. (The *GNPc* is equivalent to the generalized

6. The data are from Jain [9] and Kravis, Heston, and Summers [13]. One needs to bear in mind the usual caveats about the problems associated with such data concerning the treatment of casual workers, income recipients below tax or sample thresholds, income from home production, and the host of exchange rate difficulties that affect comparability across countries.

welfare measure with $\varepsilon = 0$.) The values for the welfare measure are in terms of 1970 US Dollars and are per capita figures.

Comparison of Table I with the stochastic dominance rankings in BFT [6] confirms that all FDSD and SDSD ranked pairs are consistent with the ranking provided by the generalized welfare measure. Furthermore, for a wide range of values of ε, the generalized welfare measure yields a consistent pair-wise ordering when the generalized Lorenz criterion is not able to. Consider, for example, the comparison of the United Kingdom and Denmark. While generalized Lorenz dominance will not provide an ordering of the pair, the rankings in Table I indicate that the income distribution in Denmark welfare dominates that of the United Kingdom except for cases of extreme equity preference ($\varepsilon > 2$, not reported in the table).

The magnitude of the decrease of the welfare measure for a country, going across the rows from left to right roughly indicates the degree of inequality present in the country's income distribution. For example, Finland's welfare index value decreases by 72% as ε goes from 0 to 2, whereas the United Kingdom's only decreases by 37% for the same change. Countries that have relatively less inequality (such as Japan) tend to move up in the rankings as ε increases and countries that have relatively more inequality (such as W. Germany) tend to move down.

V. Conclusion

This paper illustrates the usefulness of a generalized welfare measure for ranking income distributions that are left unordered by stochastic dominance and illuminates the properties of the measure. Other welfare criteria such as the arithmetic mean and Rawls criteria are special cases of this generalized measure. This generalized welfare measure illustrates that use of these special cases involves a subjective judgment by the observer of the relative importance of equity preference to efficiency preference.

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