Allais-anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth I: Consensual optimality

Mabrouk, Mohamed

ESSAI-Tunis

7 April 2006

Allais-anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth
I: Consensual optimality

Mohamed Mabrouk
Ecole Supérieure de Statistique et d'Analyse de l'Information de Tunisie (Université 7 November à Carthage), 6 rue des métiers, Charguia II, Tunis, Tunisia
Faculté des Sciences Economiques et de Gestion (Université Tunis El Manar), Campus Universitaire, Tunis 1060, *Tunisia*
Personal address : 7 rue des Lys, El Menzah 5, Tunis 1004, Tunisia
m_b_r_mabrouk@yahoo.fr
tel: 21621141575

version: April 4, 2006
A previous version of this paper circulated under the title "Intergenerational anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth: I Consensual optimality"

Acknowledgement: I would like to thank G.B. Asheim and D. Donaldson for helpful comments. The calculations as well as the opinions presented are exclusively assumed by the author.

Abstract

The objective of this work is to try to define and calculate the optimal growth path, in the presence of exogenous technical change, without resorting to the discounted-sum criterion. The solution suggested is to consider an optimality criterion expressing an Allais-anonymous intergenerational consensus. The partial characterization of consensual optimality was made possible thanks to the decomposition of the dual of the space of sub-geometric sequences of reason p. The main finding is a relation between the marginal rate of substitution between bequest and heritage, and the growth rate, relation which is a necessary condition for consensual optimality. The necessary study of the Pareto-optimality of the consensual optimum is the subject of a forthcoming paper "Allais-anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth II: Pareto optimality and some economic interpretations".

JEL classification: D90; C61; D71;D63; O41; O30.

Keywords: Intergenerational anonymity; Intergenerational equity; Optimal growth; Technical change; Time-preference; Discounted-sum criterion; Consensual criterion; OG economy.

1 Introduction

The question of the modeling of the intertemporal choice, or more particularly that of the intergenerational choice, is significant for the normative problem of the definition of optimal growth. The obvious reason is that any theory on economic dynamics depends on this modeling. Since [Koopmans 1960, Koopmans-Diamond-Williamson 1964, Diamond 1965], the social choice theory tries to propose an axiomatization of the intergenerational choice which makes it possible indeed to model it. There is a rather rich series of work which studies compatibility between the various desirable properties of the intergenerational choice ([Fleurbaey-Michel 2003, Lauwers 2000] offer useful syntheses starting from different approaches). Anonymity, which is, roughly speaking, insensitivity to permutations on the generations, is one of these properties.

Although certain questions related to this property, such as for example the question of the degree of anonymity to require ([Lauwers 1998, Fleurbaey-Michel 2003, Mitra-Basu 2005]), or that of the incompatibility with the sensitivity to short-run changes (which can be regarded as the main topic of the line of research initiated by [Koopmans 1960] and continued, among others, by [Diamond 1965], [Svensson 1980], [Chichilnisky 1996], [Lauwers 1998], [Mihara 1999], [Basu-Mitra 2003], [Fleurbaey-Michel 2003], [Sakai 2003b], [Bossert-Sprumont-Suzumura 2004] [Blackorby-Bossert-Donaldson 2005], [Hara-Shinotsuka-Suzumura-Xu 2005] and [Asheim-Mitra-Tungodden 2006]), are still open questions, the objective of this work is to try to define and calculate the optimal growth path on the basis of an anonymous intergenerational consensus.

The reason of this orientation is the conviction that the traditional criterion of the exponential discounted-sum of utility levels, based on time-preference assumptions, presents even more disadvantages than anonymity and thus, it should be avoided. Section 2 presents a summary discussion of the problems related to anonymity and the discounted-sum, although a thorough analysis of these problems, which should include a comparison of intergenerational choice axioms, exceeds the range of this work, restricted to presenting an optimality criterion then to calculating and commenting on the optimal path thus obtained.

In growth theory, one can distinguish three major approaches to model intergenerational choice.

The traditional approach is that of the exponential discounted-sum.

[Frederick-Loewenstein-O'Donoghue 2002] offers an overall picture of research on this approach and close methods. A question arises then: which discount rate to choose?[Bayer-Cansier 1998, Guesnerie 2004, Gollier 2002, Gollier 2004].

The second approach is the recursive approach. Generally, this approach is used to be freed from the assumption of time-additively separable preferences. That makes it possible to consider a discount rate depending on the level of instantaneous consumption (or utility) and to thus solve the Ramsey problem which is that the assumption of time-additively separable preferences leads to the conclusion that in the long run, the most patient agent will own all the wealth ([Boyd 1990, Farmer-Lahiri 2005, Le Van-Vailakis 2005]).

The recent contribution [Asheim-Mitra-Tungodden 2006] shows that this approach extends beyond the discounted-sum to include representable intergenerational preferences checking conditions of intergenerational equity.

The third approach, and the one of interest here, consists in maximizing the criterion

$$\Psi_0 = \lim_n U_n$$

where U_n is the utility level of the n-th generation. Since [Samuelson 1958], this approach is often used by macroeconomists when working in an OG context. It is also called: the maximization of the utility of the representative agent (for example: [Groth 2003, Heidjra-van der Poeg 2002, Mankiw 2001]) or the green golden rule [Beltratti-Chichilnisky-Heal 1994, Heal 2001].

Why being interested more particularly in the third approach? The reason is that it seems to me that it is the most intuitive criterion satisfying anonymity. Indeed, it is obvious that the discounted-sum approach does not satisfy anonymity. As for the recursive approach as extended by

[Asheim-Mitra-Tungodden 2006], it can give place to various criteria according to the axioms used. It is probable that one could also characterize Ψ_0 in this manner. That would be certainly advantageous within the framework of a comparative analysis of optimality criteria. But although such a work is essential, it clearly exceeds the object of this paper.

The definition of anonymity that I use here (see section 7), is different from other versions met in the literature, although they both imply that the criterion is insensitive to finite permutations. I chose label "Allais-anonymity", since the concept was suggested by a passage from [Allais 1947] (quoted in section 2). In fact, I chose the definition which appeared to me easiest to handle in the context of the assumptions of the present paper and sufficiently expressing the idea of an equal importance of the generations to the eyes of the criterion, without seeking to compare between the various definitions which one finds in the literature. As said above, that is not in the range of this paper.

I show that criterion Ψ_0 is closely related to any Allais-anonymous criterion. The property of Allais-anonymity presupposes that the criterion is differentiable using the supmetric (continuity and differentiability are considered in the supmetric throughout the paper). This is in conformity with [Lauwers 1998] who proves that criterion Ψ_0, or rather its linear extension to nonsteady states , which corresponds to medial limits, are the only continuous linear criteria satisfying anonymity (in the sense of [Lauwers 1998]).

The followed method is to consider a social welfare function Ψ checking Allais-anonymity and to show that the differential of Ψ is a generalization of criterion Ψ_0 to sequences not necessarily bounded and convergent. Then, one quite simply characterizes the optimum by the nullity of the differential of Ψ with respect to state variables k_1, k_2...(see section 3).

This formalization can be regarded as complementary to the various approaches of intertemporal modeling quoted above, to the extent that these approaches are applied to the intra-life intertemporal choice and the present approach to the intergenerational choice. It is an extension of [Mabrouk 2005] to economies with exogenous technical change. Let's announce that, compared to [Mabrouk 2005], I replaced the word "egalitarianism" by "Allais-anonymity".

Admittedly, for certain simple models, it is possible to calculate the optimum (with respect to criterion Ψ_0) in the case of a stationary state or stationary growth by maximizing by a direct calculation the utility of the representative agent, like in [Samuelson 1958]. However, on the one hand that shows that the stationary state dominates the other stationary states but there remains possible that it is dominated by a nonstationary state. In addition, as soon as the model is more elaborate, direct calculation is not possible any more. It is the case for example if one adopts the continuous-time neoclassical OG model with exogenous neutral technical change (see example 1).

In this case, it is necessary to consider a numerical resolution. The theoretical framework that I propose here makes it possible to determine a first order condition on the sequence (U_n) called bequest-rule, condition which adapts well to numerical calculation.

On the technical level, the first order condition characterizing the maximizer of the intergenerational optimality criterion is obtained thanks to the decomposition of the dual of l_∞^p, the space of sub-geometric sequences of reason p (see section 4). [Le Van-Saglam 2004] have already applied a similar decomposition (concerning the dual of l_∞) for the determination of the Lagrange multipliers associated with optimization of models similar to that of the present work : OG economy with infinite horizon. [Chichilnisky 1996] and [Lauwers 1998] used a similar theorem of decomposition, applicable to measures on $\mathbb{N}$ (Yosida and Hewitt theorem).

It remains to be announced that, in accordance with the well-known incompatibility between continuous anonymity and the weak Pareto property[1] ([Fleurbaey-Michel 2003]), the welfare relation induced by Ψ is not paretian. Hence, it is not sure that any solution of the first order condition is Pareto optimal (section 2). In this case (non Pareto-optimality), it would be difficult to qualify this solution as an optimal growth path. It is thus necessary to study Pareto optimality separately. This is why the calculation of the optimum will be done in two stages: the present paper, limited to optimality according to the criterion Ψ, called consensual optimality (for reasons explained in section 2), and then the Pareto-optimality which, to respect a correct size for the present paper, will be treated in a forthcoming article : "Allais-anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth II: Pareto optimality and some economic interpretations" (indicated from now on by the abbreviation: Allais-anonymity II).

The paper is organized as follows. Section 2 discusses anonymity. Section 3 presents the model and defines the optimal growth path. Section 4 presents the formal properties of mathematical spaces as well as the decomposition lemma. Section 5 presents the assumptions on the sequence of utility functions (U_i).

[1] A relation $\trianglelefteq$ defined on the set of real sequences satisfies the weak Pareto property iff $(u_j \prec v_j \forall j) \Longrightarrow (u_j) \lhd (v_j)$ where $\lhd$ is the asymetric part of $\trianglelefteq$.
The relation satisfies the strong Pareto property, or is paretian, iff $(u_j \leq v_j \forall j$ and $\exists i : u_j \prec v_j) \Longrightarrow (u_j) \lhd (v_j)$

Section 6 establishes the differentiability of the sequence (U_i) under the assumptions of section 5. Sections 4,5 and 6 relate to the techniques of calculation. The reader who wishes to avoid these questions can thus pass directly to the section 7 which lays down the first-order condition characterizing the consensual optimum (the bequest-rule). Lastly, section 8 gives all the formal proofs.

2 Anonymity and the consensual criterion

2.1 The shortcomings of the discounted-sum criterion

The time-preference assumption consists in always preferring to consume the same good today rather than tomorrow, all other things being equal. This assumption may be acceptable on an individual level. But it is much less when one transposes it on an intergenerational level. To support this claim, I give the following quotations:

[Sidgwick 1907] p.414 , quoted by [Bossert-Sprumont-Suzumura 2004] p.1:

> "the time at which a man exists cannot affect the value of his happiness from a universal point of view"

[Arrow 1995] p.12, quoting the adversaries of "pure time preference" or "social time preference", what includes intergenerational time preference:

> "But the presence of pure time-preference, denoted by ρ, has been very controversial. The English economists, in particular, have tended to be very scornful of pure time preference. Pigou (1932, p.25) stated rather politely that pure time preference "implies...our telescopic faculty is defective". Ramsey and Harrod were more morally assertive. Ramsey (1928, p261): "[I]t is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination". Harrod (1948, p40): "[P]ure time preference [is] a polite expression of rapacity and the conquest of reason by passion". Koopmans, who has in fact given the basic argument *for* discounting, nevertheless holds "an ethical preference for neutrality as between the welfare of different generations" (1965, p239). Robert Solow (1974, p40): "In solemn conclave assembled, so to speak, we ought to act as if the social rate of time preference were zero."

Other authors like [Cass1965, Ferejohn-Page 1978, Michel 1990, Cowen-Parfit 1992, Schelling 1995, Rasmussen 2001] also adopted a skeptic attitude against intergenerational time preference which is the base of intergenerational discounting, i.e. the discounted-sum criterion.

At the origin of this rejection of intergenerational time preference, there is doubtless a requirement to impartially treat the different generations, requirement accentuated today by the problems of exhaustion of certain natural resources and the problems of pollution like that of climate change. [Arrow 1995] informs us on the nature and the source of this requirement: it is a moral requirement whose source is the principle of *universalizability*:

[Arrow 1995] p 7:

> "Abatement and other very long-lived investment are thus definitely matters of moral obligations, not of self-interest"

and p 12:

> "A general principle which guides the formation of moral judgments is what many philosophers call *universalizability*...One way of characterizing universalizability is that it is the view that would be taken by a disinterested spectator."

[Solow 1973], in a paper where he reveals the significant shortcomings of the max-min criterion, ascribes to [Rawls 1971] the call to take account of a social contract between generations:

> "[It is part of Rawl's argument for the max-min criterion that] we should regard earlier and later generations as facing each other contemporaneously when the social contract is being drawn."

Such a contract can only refer, I suppose, to impartiality between generations.

In addition to being partial, the discounted-sum criterion is seen as a short-run criterion. In fact, it is a consequence of partiality. The following example, found in [Lauwers 1998], highlights this fact.

> "...set $\beta = 0.95$ and the value function $F_\beta(u) = \sum_{k=1}^{\infty} \beta^{k-1} u_k$ ranks the stream $v_1 = (1, ...1, 0, 0...)$ [the value 1 appears 13 times] strictly above $v_2 = (0, ...0, 1, 1...)$ [the value 0 appears 13 times] . That the utility level 1 occurs infinitely many times in the future is sacrificed for the sake of obtaining this utility level in the first thirteen periods. Since the example can be constructed for each value of β, it follows that positive discounting does not comply with our notion of intergenerational justice."

2.2 Anonymity, long-term well-being and short-term sacrifices

The merit of Arrow's paper is that it clarifies the ethical and economic bases of intergenerational choice underlying the mathematical developments on this question. However, contrary to the position adopted here, [Arrow 1995] supports intergenerational discounting mainly on the ground that anonymity would impose too large sacrifices on present generations. Yet, it is well-known that impartial treatment of generations, or intergenerational anonymity, leads to the maximum long-term well-being, characterized by growth theory's golden-rule[2].

I would like to quote a passage from [Allais 1947] where, on the one hand one can find the explanation of the mechanism connecting intergenerational anonymity to the golden rule, explanation which the present paper takes as a starting point, but which, on the other hand, panders, in appearance, to the objection of [Arrow 1995] against intergenerational anonymity. That will thus make it possible to introduce the logic underlying this objection. [Allais 1947] p.225:

> "Let us indicate that if the State takes into account the present generation and the future generations, i.e. if it estimates that all the generations are identical to the point of view of the estimate of the increases and the reductions in satisfaction, it will be necessarily led to seek the maximization of the social productivity, whatever sacrifice it can cost for the present generation. We saw, indeed, that the additional saving necessary once created allows an indefinite increase in the real collective income of all the future generations and that thus, of any policy of lowering of the interest rate which keeps an equal account of the generations present and future, the possible sorrow of the present generation to constitute the additional saving necessary is indefinitely compensated by the supplement of satisfaction from which will profit the future generations"[3]

It is the property of estimating "that all the generations are identical to the point of view of the estimate of the increases and the reductions in satisfaction" that is labelled Allais-anonymity in the present paper. Allais-anonymity is discussed and characterized in section 7.

Let's specify that "maximation de la productivité sociale" indicates the golden rule, and that the lowering of the interest rate is seen like a manner of pushing the economy to produce at the golden rule level.

Although this passage indicates that an anonymous treatment of the generations is likely to lead the economy to produce at the highest possible level, one

[2] See [Mankiw 2001] p. 116 for an analysis of this issue in the context of the Solow model, or [Heal 2001] for an application to ecological problems.

[3] Original text : "Indiquons que si l'Etat prend en considération la génération présente et les générations futures, c'est-à-dire s'il estime que toute les générations sont identiques au point de vue de l'estimation des accroissements et des diminutions de satisfaction, il sera nécessairement conduit à rechercher la maximation de la productivité sociale, quelque sacrifice qu'il puisse en coûter pour la génération présente. Nous avons vu, en effet, que l'épargne supplémentaire nécessaire une fois créée permet une augmentation indéfinie du revenu collectif réel de toutes les générations futures et qu'ainsi, de toute politique d'abaissement du taux d'intérêt qui tient un égal compte des générations présente et future, la peine éventuelle de la génération présente pour constituer l'épargne supplémentaire nécessaire est indéfiniment compensée par le supplément de satisfaction dont bénéficieront les générations futures"

can be tempted to deduce from it that *any* sacrifice of present generations is then justified. That is what led [Arrow 1995] to see as unacceptable the principle of intergenerational anonymity since the idea of unlimited sacrifice clearly conflicts with a more fundamental principle: the principle of self-regard.

This charge, of imposing unlimited sacrifice on present generations, is in fact the essential argument against anonymity. It has been criticized in [Asheim-Buchholz 2003]. Their argument is to say that the criterion used in [Arrow 1995] is in fact too malleable to allow precise conclusions. In [Blackorby-Bossert-Donaldson 2005], one finds a proposal, which remains to be explored, to avoid would-be unbearable sacrifices of present generations without having recourse to the discounted-sum.

> "Our view is that, for purpose of social evaluation, the well-being of future generations should not be discounted. If maximization of ethically appropriate objective function requires the present generation to sacrifice most of its consumption for the benefit of others, then such an action can be considered supererogatory: desirable but beyond the call of duty. If these sacrifices are considered to be too demanding, we do not think it is a suitable response to give to future generations a smaller weight in the social ordering."

In what follows, it will be argued that in the social choice process suggested here, present generations will not have to suffer exaggerated sacrifices.

2.3 A two-stage social choice process: consensus then efficiency...

As seen in the introduction, we are interested in a representable criterion Ψ. Moreover, Ψ is supposed to be differentiable using the supmetric. It is well known since [Basu-Mitra 2003] that a social welfare relation representable by a real valued function cannot be at the same time anonymous and strong Pareto. This result generalizes the theorem of Yaari ([Diamond 1965] [4]). That means the impossibility for an anonymous social welfare function of being sensitive to individual changes, what consolidates the idea of sacrifice of the individual vis-a-vis the community. That seems paradoxical because it contradicts the fact that infinity has some importance since it is made up from individuals of null weight. Perhaps it is the reason why the assumption of the representability of the intergenerational choice by a real valued function was generally dropped after [Basu-Mitra 2003] (except in [Asheim-Mitra-Tungodden 2006]).

But contradiction is only seemingly. In fact the individual counts, but through the collectivity. The negligibility of the individual in front of infinity is a usual assumption in physics or in the theory of probability. Nobody is astonished, for example, that an individual outcome has a null probability if

[4] This theorem is sometimes referred as the second impossibility theorem of Diamond, as in [Sakai 2003b], although Diamond ascribes it to Yaari in [Diamond 1965].

there is an infinity of equiprobable outcomes[5]. Even in economics, the assumption of price-taker behavior in the theory of competitive equilibrium, rests on the idea of the negligibility of the individual ([Lauwers 1998]).

What is more awkward is that Ψ is not even weak Pareto, a condition often perceived as essential (for example in [Fleurbaey-Michel 2003]). But, on the other hand, theorem 1 and theorem 2 in [Fleurbaey-Michel 2003] suggest also that in the presence of continuity, weak Pareto is a condition rather close to strong Pareto. In other words, condition weak Pareto is not so weak. Ψ being continuous, one should not thus too much reproach it for not respecting weak Pareto! I will try to illustrate through a simple example why this is so. Suppose that, taking into account the possibilities of production, the optimal asymptotic level of the utility is 100. We have seen above that what is important for the anonymous (and continuous) criterion is only what happens to the infinity. But the infinity of generations which will be as close as one wants to 100, is the same one, whether it starts from the 20th or the 10000th generation. So, be it after 20 generations or 10000 generations that the economy reaches a level close enough to 100, that is indifferent to the anonymous criterion. But, in the first alternative, the utility level is strictly larger than in the second alternative for every generation. Thus, the anonymous criterion violates weak Pareto and the reason is the weight of infinity.

To escape this inevitable shortcoming of Ψ, I propose not to regard Ψ as representing all the social welfare relation, but simply as the voice of the intergenerational consensus, or if you prefer, that of the principle of universalizability, or the voice of the disinterested spectator. Thus Ψ , the *consensual criterion*, would be the first stage of the total social welfare relation, to which one can add considerations of efficiency (i.e. Pareto-optimality) as second stage. This way, the total social welfare relation will be necessarily strong Pareto. However, it will not be continuous.

This diagram of social choice process makes it possible to (partly) have the advantages of continuity while keeping the strong Pareto property, because to impose continuity on the first stage of the social welfare relation does not oblige to give up the strong Pareto property for the total social welfare relation.

We will see through the discret-time example in "Allais-anonymity II" that such a two-stage optimization gives completely realistic results. In fact this assertion is hardly surprising since Ψ is only the generalization of a criterion already well-known and largely used. This underlines the intuitive character of Ψ.

The idea of two-stage optimization has been suggested in [Ferejohn-Page 1978] and was first specified and applied in [Asheim 1991] then in [Lauwers 1998], [Asheim-Buchholz 2003] and [Asheim-Buchholz-Tungodden 2001]. The example in [Lauwers 1998] is closer to the present one since it is also based on medial limits. In the two examples [Asheim 1991, Asheim-Buchholz-Tungodden 2001],

[5] There nevertheless exists a theory of lexicographical probabilities that gives weight to zero probability events. See L. Blume, A. Brandenburger and E. Dekel, Lexicographic probabilities and choice under uncertainty, Econometrica vol59 n°1(1991)61-79. I thank G.B. Asheim for having made me aware of that.

the first stage consists in ruling out ethically undesirable streams by means of Asheim's J-rule in the first example and Suppes-Sen rule in the second one. But an additional assumption is imposed there: the assumption of "immediate productivity". This assumption excludes the economies with possibility of over-accumulation. In [Asheim-Buchholz 2003] both the first and the second stage are based on undiscounted utilitarianism with different utility functions (see [Asheim-Buchholz 2003] page 15). These utility functions are calibrated of kind to make optimal a given increasing consumption stream.

There is in the literature other candidates for a social welfare relation, that have the advantage of being at the same time strong Pareto, anonymous and complete, of course at the expense of supmetric continuity. But then it seems that it is at the cost of their noncomputability ([Svensson 1980, Mihara 1999, Fleurbaey-Michel 2003]), what harms their practical aspect clearly. Let's also notice that [Asheim-Mitra-Tungodden 2006] propose a social welfare relation which is representable and, at the same time, checks an intergenerational equity condition : Hammond equity for the future. This result is obtained by slightly weakening continuity and efficiency.

2.4 ...without exaggerated sacrifices

I come now to the fear that such a criterion would impose to present generations some unbearable sacrifice. Admittedly, taking into account long-run interest would possibly impose some sacrifice on present generations and as long as such a sacrifice significantly improves long-run interest, the anonymous criterion will demand it. But this does not mean, as in [Arrow 1995], that any sacrifice of present generations is desirable. The reason is that, precisely, Ψ violates weak Pareto whereas [Arrow 1995] implicitly uses Ramsey criterion which is strong Pareto. So to speak, it is the good side of the insensitivity of Ψ: Ψ does not demand sacrifices that do not significantly improve long-run interest.

Here are two examples illustrating the link between this characteristic of Ψ and the superfluity of disproportionate sacrifices.

Firstly, the insensitivity of Ψ to short-run changes allows the distribution of the effort requested to the present generations on a great number of generations, as great as one wants. Take again the previous example. Suppose that, to reach the level of utility 100, an additional effort of saving, say Δs, is needed. This effort can thus be indifferently distributed on 10000 generations, that is to say $\frac{\Delta s}{10000}$ per generation. That would make it almost imperceptible for a generation. In other words, the anonymous consensual criterion imposes the destination to which you must go (i.e. the utility level: 100), not the speed that you must employ.

The second example is based on the discrete-time case analysed in "Allais-anonymity II". It can be summarized as follows. When the economy is under-capitalized, the sacrifice of a generation, say g_1, necessarily benefits the following generations. However, in the presence of technical progress and starting from a certain level of sacrifice, only the immediately posterior generations would have a substantial profit from it. For the more distant generations the profit would become negligible, to such a degree that the anonymous consensual criterion would not improve anymore. That is possible because Ψ is not weak Pareto. If the generations $g_2, g_3...$ were tempted to make the same sacrifice as g_1 to improve more the situation of their immediate descendants, they would deteriorate their own situations. So that ultimately, the situation of all the generations would worsen and, consequently, the value of the anonymous consensual criterion would not increase, Ψ being nevertheless nondecreasing. This is the reason why, in the presence of technical progress, it is not optimal to push the capital to the level where net marginal productivity is zero.

In fact, in this example, one does nothing but find the well-known golden-rule: net marginal productivity of capital = rate of technical change (see the proof in "Allais-anonymity II"). By using a Cobb-Douglas production function as well as the data provided in [Mankiw 2001] p.135 on the American economy: marginal productivity of capital $mpk = 0.12$, share of capital in the production $\alpha = 0.3$, stationary rate of savings at the current level of capital $s = 10\%$, optimal marginal productivity of capital $mpk^* = 0.07$, we deduce the optimal savings rate thanks to the relations:

$$\frac{s^*}{s} = \left(\frac{k^*}{k}\right)^{1-0.3} = \frac{mpk^*}{mpk} \simeq 1.7$$

from where $s^* = 17\%$. This savings rate is far from expressing exaggerated sacrifices.

However one has to expect that anonymity indeed increases the rate of savings. But the reason is not as well the importance of the sacrifices as the need for maintaining a high stock of capital making possible a higher consumption level, higher than the one reached with a discounted-sum criterion. Anyway, even if one established that anonymity imposes too high a rate of savings, that would not prove that anonymity is a concept null and void, but simply that the current economic situation deviates from it.

To finish this section, here is now an argument,

due to [Blackorby-Bossert-Donaldson 2005], that reverses the objection of non respect of the Pareto property against the discounted-sum criterion. The idea is to regard the date of birth of an individual as a variable and not a characteristic of this individual. By advancing the date of birth of an individual i, one increases the value of the discounted-sum criterion. It is then possible to decrease the utility of individual i of a quantity Δu_i while keeping the discounted-sum criterion above its initial level, so that the utility stream $(u_1, u_2, ..., u_i - \Delta u_i, ...)$ with premature birth of individual i, is strictly preferred to $(u_1, u_2, ..., u_i, ...)$. Hence, the ordering defined by the discounted-sum criterion "approves of changes in birth dates even when, in term of well-being, no one gains and someone loses" ([Blackorby-Bossert-Donaldson 2005] p.3).

3 Preliminary

3.1 Model

Since the aim is to examine intertemporal choice, or more precisely intergenerational choice, the model suggested avoids what one could call the "spatial dimension" of the economy, i.e. intra-generational exchanges, differentiation of goods, agents and firms, distribution of wealth ...and focuses on the "temporal dimension". Hence, our economy is constituted by a succession of generations, each generation being made up of only one individual who is at the same time consumer and producer.

Another significant characteristic of the model is that capital accumulation is achieved through bequests. This rises from the observation that bequest constitutes ultimately the current shape of transmission of capital from generation to generation.

Moreover, the model avoids monetary questions. The reason is that on the long-run (centuries), it is seems quite clear to me that whatever be monetary transfers, what is important for the economy is the accumulation of nonmonetary capital, the word "nonmonetary capital" being used here in its smithian, physical meaning (in opposition to monetary), of durable goods incorporating a share of human work, including knowledge. In this view, accumulating capital amounts to some extent to storing work.

It is necessary to recognize that there is inevitably here a standpoint in favour of the monetarist idea of long-term neutrality of money. Consequently, the present approach deviates from certain works in the line of [Samuelson 1958] or [Balasko-Shell 1980] utilizing money to allow for example for son-father transfers. On the other hand, it is closer to the demonetized models of [Solow 1956] or [Cass 1972].

From this point of view, although it is possible to observe negative bequests (son-father transfers), along the optimal growth path the nonmonetary capital should probably reach a sufficiently high value so that the most extravagant father nevertheless transmits a positive bequest to his son. Notice that, by excluding other behaviors clearly nonoptimal like the deliberated destruction of capital, and considering that spatial dimension is removed, this extravagant father can waste his capital only via his own consumption. Hence, it would not be restrictive for the determination of this optimal growth path, to suppose that the bequests are all positive. I will thus make this assumption.

For simplicity, there is only one good that is used for consumption, investment, savings and bequests. At the beginning of its active life, a given generation g_i inherits a quantity b_{i-1} of that good. Its only acts during its life are: to consume, produce, invest in order to increase its future consumption and, at the end of the active lifetime, to bequeath b_i to the descent. In doing so, generation g_i achieves a level of life-utility $U_i(b_{i-1}, b_i)$. Also, for simplicity, it is supposed that a generation is economically non-existent apart from its active life. Thus, generations do not really overlap.

The following example shows how one can define U_i starting from the instantaneous utility function u and the production function F. However, to preserve the generality of this work, functions U_i need not be specified, provided that they observe the conditions of section 5.

Example 1 *The continuous-time neoclassical OG model with exogenous neutral technical change:*

$$\begin{aligned}
U_i(h,l) &= \\
&\max_{c(t)} \int_0^T u(c(t))dt \\
\textit{subject to} &: \dot{k} + c(t) = F(N^{i-1}, N^{i-1}k(t)) - a \cdot k(t); \\
k(0) &= h; k(T) = l \textit{ and } c \in \left[0, F(N^{i-1}, N^{i-1}k)\right]
\end{aligned}$$

where N is the factor of neutral technical change, k the capital, a the capital depreciation rate and c the consumption.

3.2 Formalization of the optimal growth path

Thus, consider a sequence of functions $(U_i)_{i\geq 1}$, each function U_i being defined from $D_i \subset \mathbb{R}^2_+$ to $\mathbb{R}$. Conditions to impose on $(U_i)_{i\geq 1}$are discussed further in section 5.

Let k_0 be a real positive number such that the set$\{l \in \mathbb{R}/(k_0, l) \in \text{interior of } D_1\}$is not empty.

Denote:

$$D = \{K = (k_1, k_2, \cdots) \ / \text{ for all } i \geq 1 : (k_{i-1}, k_i) \in D_i\}$$

Consider a consensual criterion represented by a real valued and Frechet-differentiable functional Ψ (on a space to be defined in section 7) such that the consensual value of the sequence $(U_n)_{n\geq 1}$ is given by $\Psi\left((U_n)_{n\geq 1}\right)$. As a function representing preferences, Ψ can be supposed to be concave and nondecreasing, but these assumptions don't play here any mathematical role.

For $K \in D$ and i an integer ≥ 1, denote respectively $P_i(K)$ and $S(\Psi)$ the following programs:

$P_i(K) =$

$$\begin{aligned}
&\max_{B\in D} U_i(b_{i-1}, b_i) \\
&\qquad \textit{subject to} \ : \ U_j(b_{j-1}, b_j) \geq U_j(k_{j-1}, k_j) \ \forall j \geq 1, \ j \neq i
\end{aligned}$$

where b_{i-1}, b_i are the i-1th and the ith components of B and k_{i-1}, k_i are the i-1th and the ith components of K.

$S(\Psi) =$

$$\max_{B\in D}\Psi\left[(U_i(b_{i-1},b_i))_{i\geq 1}\right]$$

Clearly, a bequests plan K is a Pareto-optimal bequests plan if and only if it is solution to $P_i(K)$ for all $i\geq 1$ and K is a consensual optimum for the criterion Ψ if and only if it is solution to $S(\Psi)$. If K is simultaneously Pareto-optimal and consensus-optimal, it is described as optimal growth path or social optimum.

We focus in this paper on program $S(\Psi)$, while programs $P_i(K)$ are the object of "Allais-anonymity II".

4 Properties of the work spaces

4.1 Decomposition lemma and calculus of the differential of a function on l_∞^r

For $r\geq 0$, define $l_\infty^r=\left\{B=(b_1,b_2,...)/b_i\in\mathbb{R}\text{ and}\sup_{i\geq 1}|b_i|\,e^{-ri}\prec +\infty\right\}$,

$l_1^r=\left\{B=(b_1,b_2,...)/\ b_i\in\mathbb{R}\text{ and }\sum_{i=1}^{+\infty}|b_i|\,e^{ri}\prec +\infty\right\}$,

$c_r=\left\{B=(b_1,b_2,...)/\ b_i\in\mathbb{R}\text{ and }b_ie^{-ri}\text{ converges}\right\}$and

$c_0^r=\left\{B=(b_1,b_2,...)/\ b_i\in\mathbb{R}\text{ and }|b_i|\,e^{-ri}\text{ converges to }0\right\}$.

Let δ_∞^r be the functional defined on c_r by: $\delta_\infty^r(x)=\lim_{i\longrightarrow+\infty}x_i.e^{-r.i}$ and δ_∞ the functional defined on the set of real converging sequences c by: $\delta_\infty(x)=\lim_{i\longrightarrow+\infty}x_i$. Denote l_∞^{r*} the dual of l_∞^r.

It is known that l_∞^r and l_1^r are complete normed vector spaces respectively for the norms $\|B\|_r=\sup_{i\geq 1}|b_i|\,e^{-ri}$ and $\|B\|_{1,r}=\sum_{i=1}^{+\infty}|b_i|\,e^{ri}$. Henceforth, I will use the norm $\|.\|_r$ and simply write it $\|.\|$when possible.

It is standard to consider that the set of feasible growth rates is up-bounded, which means that there is $p\geq 0$ such that $D\subset l_\infty^p$.

Lemma 2 *: Let $y\in l_\infty^{r*}$. Then we can write in a unique manner:*

$$y=y_1+y_2$$

where $y_1$verifies:

$$\sum_{i=1}^{+\infty}|y_{1i}|\,e^{r.i}\prec +\infty$$

and y_2 is such that its restriction to c_r is proportional to δ_∞^r.

Let f be a function from l_∞^r to $\mathbb{R}$, Frechet-differentiable at $x_0\in l_\infty^r$. Denote $\delta f(x_0)$ the Frechet-differential of f at x_0. By definition, $\delta f(x_0)\in l_\infty^{r*}$. We then apply the decomposition lemma:

$$\delta f(x_0)=\delta f_1(x_0)+\delta f_2(x_0)$$

where $\delta f_1(x_0) \in l_1^r$ and $\delta f_2(x_0)$ is such that its restriction to c_r is proportional to δ_∞^r.

Denote the restriction of $\delta f_2(x_0)$ to c_r by $\delta f_2(x_0)\lfloor c_r$. Then, there is a real $\alpha(x_0)$ such that:

$$\delta f_2(x_0)\lfloor c = \alpha(x_0)\,\delta_\infty^r$$

We can consider $\delta f_1(x_0)$ as the finite part of the differential, and $\delta f_2(x_0)$ as the infinite part. Denote $\alpha(x_0)$ by $\frac{\partial f}{\partial r\infty}(x_0)$. A formula to calculate $\frac{\partial f}{\partial r\infty}(x_0)$ is given in 8.1.

4.2 The interior of D

Definition 3 *Let $B \in l_\infty^p$.*

$\overline{\pi}(B) = \inf\{\alpha \geq 0 / \limsup |b_n| e^{-n\alpha} \prec +\infty\}$ *is the reason superior of B and* $\underline{\pi}(B) = \sup\{\alpha \geq 0 / \liminf |b_n| e^{-n\alpha} \succ 0\}$ *is the reason inferior of B .*

$\overline{\omega}(B) = \limsup |b_n| e^{-n\overline{\pi}(B)}$ *and* $\underline{\omega}(B) = \liminf |b_n| e^{-n\underline{\pi}(B)}$ *are respectively the amplitude superior and the amplitude inferior of B.*

If $\underline{\pi}(B) = \overline{\pi}(B) = \pi$, B is said to be homogeneous of reason π. If, in addition, $\underline{\omega}(B) \succ 0$ and $\overline{\omega}(B) \prec +\infty$, *$B$ is said strictly of reason π (or π is the strict reason of B). If $\overline{\omega}(B) = \underline{\omega}(B)$, B is said to be convergent of reason π.*

Remark 4 *(a)The reason of a homogeneous bequests plan corresponds to the capital growth rate. (b)K convergent of reason π is equivalent to $K \in c_\pi$.*

Let s_∞^p be the subset of l_∞^p of sequences strictly of reason p. Let $l_{\infty+}^p$ and $s_{\infty+}^p$ be respectively the subset of l_∞^p and s_∞^p of positive sequences. Denote $\overset{\circ}{l}{}_{\infty+}^p$ the interior of $l_{\infty+}^p$ and $s_{\infty++}^p$ the subset of $s_{\infty+}^p$ of strictly positive sequences.

Denote $\overset{\circ}{D}$ the interior of D.

Proposition 5 $\overset{\circ}{D} \subset s_{\infty++}^p$

Remark 6 *If we had $D \subset l_{\infty+}^p$ but not $p = \inf\{\alpha / D \subset l_\infty^\alpha\}$, this would imply that $\overset{\circ}{D} = \varnothing$. Indeed, there would be $x \prec p$ such that $D \subset l_\infty^x$. For all B in D we would have* $\limsup |b_n| e^{-xn} \prec +\infty$ *which implies* $\liminf |b_n| e^{-pn} = 0$. *Then, since $\overset{\circ}{D} \subset s_{\infty++}^p$, we must have$\overset{\circ}{D} = \emptyset$. This would be catastrophic since all optimization theorems require non empty interiors.*

5 Assumptions on the sequence $(U_i)_{i\geq 1}$

5.1 Regularity, variation and concavity

Assume that:

- For all $i \geq 1$, D_i is strictly included in $\mathbb{R}_+^2$, closed and with a non-empty interior
- U_i is of class C_2 on D_i
- $U'_{ih} \succ 0$ (U'_{ih} and U'_{il} are respectively the derivatives of U_i with respect to its first and second variable)
- D_i convex, U_i concave. One can then show easily that D is also convex.

These conditions are standard, except the second one which is needed to set the differentiability of the sequence $(U_i)_{i\geq 1}$.

5.2 Geometricity

5.2.1 Definition

Denote G the mapping that associates to $K \in D$, $G(K) = (U_i(k_{i-1}, k_i))_{i\geq 1}$.

Definition 7 *If $\{\alpha/D \subset l_\infty^\alpha\} \neq \varnothing$ and $p = \inf\{\alpha/D \subset l_\infty^\alpha\}$, D is geometric of reason p. If on top of that $D \subset l_\infty^p$, D is strictly geometric of reason p.*

5.2.2 Geometricity assumptions

Henceforth, it will be assumed that:

- D is strictly geometric of reason p and $\overset{\circ}{D} \neq \varnothing$.
- $G(D)$ is strictly geometric of reason p_1.

The strict geometricity of D is crucial as far as we need non empty interiors to use optimization theorems. The strict geometricity of $G(D)$ doesn't play any role in the formal validity of the results. However, these results are stripped of any economic meaning if $G(D)$ is not strictly geometric, or, more precisely, if Ψ is defined on a space l_∞^q with $q \succ p_1$. Indeed, in this case and for a criterion Ψ fulfilling the assumptions *(3)* and *(4)*, $\delta(\Psi G)(K) = 0$ for all steady-state K in D (see definition of steady-state forward). Thus, all steady-states in D are consensual optima. Obviously, such a consensual criterion Ψ is not very interesting!

5.3 Linearity at infinity of $(U_i)_{i\geq 1}$

5.3.1 Introduction

The assumptions maid above on $(U_i)_{i\geq 1}$ (subsection 5.1) are not enough to guarantee the differentiability of G, which is needed to obtain the first order condition of optimality. That is why I introduce the condition of linearity at infinity of $(U_i)_{i\geq 1}$. This condition is sufficient to have the differentiability of G at interior points. Moreover, one can check that it is fulfilled for discret-time models with standard instantaneous utility functions and production functions and for models with an homothetic sequence $(U_i)_{i\geq 1}$, with rather general conditions on the utility functions, the production functions and $(U_i)_{i\geq 1}$.[6]

5.3.2 The condition of linearity at infinity

Define Θ, the operator from $D \times l_\infty^p$ to $\mathbb{R}^\infty$ (the set of real sequences) such that $\Theta(B, X)$ is the sequence $(\theta_n)_{n\geq 1}$where

$$\theta_n(B,X) = \begin{pmatrix} x_{n-1} & x_n \end{pmatrix} \begin{pmatrix} U''_{nh^2}(b_{n-1}, b_n) & U''_{nlh}(b_{n-1}, b_n) \\ U''_{nhl}(b_{n-1}, b_n) & U''_{nl^2}(b_{n-1}, b_n) \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$$

for $n \geq 1$ (with $x_0 = 0$ and $X = (x_n)_{n\geq 1}$).

Definition 8 *Let $K \in \overset{\circ}{D}$. The condition of linearity at infinity of $(U_i)_{i\geq 1}$at the point K for the reasons (p, p_1) is that there is $\alpha \succ 0$ and $M \geq 0$ such that for all $X \in l_\infty^p$, $\|X\|_p \leq 1$ and for all $B \in S(K, \alpha) \cap D$ we have:*

$$\|\Theta(B, X)\|_{p_1} \leq M.$$

This condition implies that the Hessian of U_n at (b_{n-1}, b_n) tends to 0 when n tends to infinity, what means the linearity of U_n at (b_{n-1}, b_n) when n tends to infinity.

6 Differentiability of G

Let $K \in \overset{\circ}{D}$ and suppose that G fulfills the condition of linearity at infinity at K for (p, p_1). When there is no risk of confusion, denote $u'_{hn} = U'_{nh}(k_{n-1}, k_n)$, $u'_{\ln} = U'_{nl}(k_{n-1}, k_n)$, $U'_h(K)$ the sequence (u'_{hn}) and $U'_l(K)$ the sequence $(u'_{\ln})$.

[6] A discrete-time example is provided in "Allais-anonymity II"

Lemma 9 *Under the above assumptions, the linear transformation $\delta G(K)$ defined for $\Delta K \in l_\infty^p$ (with $\Delta k_0 = 0$) by*

$$\delta G(K) \cdot \Delta K = \left[U_{nh}^{'} \left(k_{n-1}, k_n\right) \Delta k_{n-1} + U_{nl}^{'} \left(k_{n-1}, k_n\right) \Delta k_n \right]_{n \geq 1}$$

has range in $l_\infty^{p_1}$ and checks:

$$\lim_{\|\Delta K\|_p \longrightarrow 0} \frac{\|G(K + \Delta K) - G(K) - \delta G(K) \cdot \Delta K\|_{p_1}}{\|\Delta K\|_p} = 0$$

Moreover, G is Gateaux-differentiable at K in every direction $\Delta K \in l_\infty^p$ and its Gateaux-differential at K with the increment ΔK is $\delta G(K) \cdot \Delta K$. If G is continuous at K, G is Frechet-differentiable at K and $\delta G(K)$ is its Frechet-differential at K.

Proposition 10 *If the condition of linearity at infinity holds at a point $K \in \overset{\circ}{D}$ then the sequences $\left(U_{nh}'(k_{n-1}, k_n)\right)_{n \geq 1}$ and $\left(U_{nl}'(k_{n-1}, k_n)\right)_{n \geq 1}$ are in $l_\infty^{p_1 - p}$.*

Proposition 11 *If the condition of linearity at infinity holds at a point $K \in \overset{\circ}{D}$, G is continuous and Frechet-differentiable at K and $\delta G(K)$ is its Frechet-differential at K.*

7 Consensual optimality

Except in 7.4, suppose that $K \in \overset{\circ}{D}$ and that G (see 5.2 for the definition of G) fulfills the condition of linearity at infinity at K for (p, p_1) (see 5 for the definition of linearity at infinity).

7.1 The differential of $\Psi \circ G$

Definition 12 *$K \in D$ is a steady state bequests plan if and only if $\lim u_{hn}' e^{-(p_1 - p)n}$ and $\lim u_{\ln}' e^{-(p_1 - p)n}$ both exist (u_{hn}' and $u_{\ln}'$ are respectively the derivatives of U_n at (k_{n-1}, k_n) with respect to its first and second variable).*

Remark 13 *(a)It would have been natural to qualify convergent plans as steady state plans but this definition proved difficult to handle. (b)It can be easily shown that if the sequence (U_i) is homothetic, the two definitions are quite near. (c)This definition of steady states is also quite natural because on the one hand, $U_h'(K)$ and $U_l'(K)$ are in $l_\infty^{p_1 - p}$, on the other hand it is natural that in a steady state the marginal rate of substitution between bequest and heritage $\frac{u_{hn}'}{-u_{\ln}'}$ tends to a limit.*

As defined in section 3.2, a consensual optimum is a bequests plan maximizing an inter-generations criterion $\Psi(G(B))$, where $\Psi(x)$ is a Frechet-differentiable function from $l_{\infty}^{p_1}$ to $\mathbb{R}$.

Denote the vector $\delta\Psi_1(G(K)) = (\frac{\partial\Psi}{\partial x_1}(G(K)), \frac{\partial\Psi}{\partial x_2}(G(K)), \cdots)$ by $(\Psi_1', \Psi_2', \cdots)$ and $\frac{\partial\Psi}{\partial p_1\infty}(G(K))$ by $\Psi_{p_1\infty}'$ (the symbol δ indicates the Frechet-differential; for $\frac{\partial\Psi}{\partial p_1\infty}$, see 4).

Thanks to the differentiability of G, we have:

Proposition 14 *If K is a steady state and $\Delta B \in c_p$, then:*

$$\begin{aligned} & \delta\left[\Psi G\right](K) \cdot \Delta B \\ = & \sum_{i=1}^{+\infty} (\Psi_i' u_{li}' + \Psi_{i+1}' u_{hi+1}')\Delta b_i + \Psi_{p_1\infty}'(u_h' e^{-p} + u_l')\Delta b \end{aligned}$$

where $u_h' = \lim u_{hn}' e^{-(p_1-p)n}$ *and* $u_l' = \lim u_{\ln}' e^{-(p_1-p)n}$, $\Delta b = \lim \Delta b_n e^{-pn}$.

7.2 Allais-anonymity

As in [Mabrouk 2005], I try to characterize consensus optima when the consensual criterion treats all generations the same way. The label "egalitarianism" in [Mabrouk 2005] is replaced by "Allais-anonymity" because on the one hand indifference to permutations is rather called anonymity in the literature, on the other hand the present concept of anonymity is distinct from other versions in the literature and, to my knowledge, it was first suggested in [Allais 1947], as said in sections 1 and 2.

Definition 15 *Ψ is Allais-anonymous at a point G if and only if $\Psi_i'(G) = 0$ for all $i \geq 1$.*

Proposition 16 *Let s be a one-to-one mapping on N^* and define $\widehat{s}$ the transformation on $l_{\infty}^{p_1}$ such that, for $G = (g_i) \in l_{\infty}^{p_1}$,, $\widehat{s}(G) = (g_{s(i)})$. Then, Ψ is Allais-anonymous at a point G if and only if:*

$$\delta\Psi(G) \cdot \Delta G = \delta\Psi(G) \cdot \widehat{s}(\Delta G) \tag{1}$$

for all s and for all $\Delta G \in c$ in a given neighborhood of G.

Proposition 16 shows that definition 15 is equivalent to indifference to the permutations of "small" gains and losses of utility. That corresponds, at the infinitesimal level, to the quotation of Allais in section 2.

Example 17 *Consider the linear functional* $\delta_\infty^{p_1}$ *defined on* c_{p_1} *in section 4 and take as consensual criterion* Ψ*, a positive Hahn-Banach extension of* $\delta_\infty^{p_1}$ *on* $l_\infty^{p_1}$ *(existence is guaranteed by the theorem of Krein, see [Naimark 1970] p.63). For* $i \geq 1$*, denote* e_i *the sequence* (e_{ij}) *where* $e_{ij} = 0$ *if* $i \neq j$ *and* $e_{ii} = 1$*. For* $G \in l_\infty^{p_1}$ *and* $h \in \mathbb{R}$*, we have*

$$\begin{aligned} \Psi(G + he_i) - \Psi(G) &= \Psi(G + he_i - G) \\ &= \Psi(he_i) = h\delta_\infty^{p_1}(e_i) = 0 \end{aligned}$$

Thus,

$$\Psi_i'(G) = \lim_{h \to 0} \frac{\Psi(G + he_i) - \Psi(G)}{h} = 0$$

and Ψ *is Allais-anonymous.*

Proposition 18 *Let* s *be a permutation on* $\mathbb{N}^*$ *such that the set*

$$\{i/i \neq s(i)\}$$

is finite. Denote $\widehat{s}$ *the transformation on* $l_\infty^{p_1}$ *such that, for* $G = (g_i) \in l_\infty^{p_1}$*,* $\widehat{s}(G) = (g_{s(i)})$*. Then, if* Ψ *is everywhere Allais-anonymous we have:*

$$\Psi(\widehat{s}(G)) = \Psi(G) \tag{2}$$

Equation (2) means that if we change the order of a finite number of components in G, it does not change the consensual value. Thus, if Ψ is everywhere Allais-anonymous, it is finitely-anonymous. As said in the introduction and section 2, we easily see that Ψ is not paretian. In the sequel, the condition "everywhere Allais-anonymous" will simply be referred as "Allais-anonymous".

7.3 First order condition

Suppose that

$$\Psi \text{is Allais-anonymous} \tag{3}$$

and that

$$\Psi'_{p_1\infty} \succ 0 \tag{4}$$

The condition (4) means that Ψ is sensitive to long run interest.

Let K be a steady state in $\overset{\circ}{D}$ and $\Delta B \in c_p$. According to proposition 14, we can write

$$\delta[\Psi G](K) \cdot \Delta B = \Psi'_{p_1\infty}(u_h' e^{-p} + u_l')\Delta b$$

If K is an interior maximizer of $[\Psi G](K)$, then $\delta [\Psi G](K) \cdot \Delta B = 0$ for all ΔB in l^p_∞. Consequently, we have necessarily

$$u'_h e^{-p} + u'_l = 0 \tag{5}$$

We can then state:

Theorem 19 *Under assumptions (3) and (4) on the consensual criterion, if a steady state K in $\overset{\circ}{D}$ is a consensual optimum then*

$$u'_h e^{-p} + u'_l = 0$$

Equation (5), which will be henceforth named "bequest-rule", can be written

$$\frac{u'_h}{-u'_l} = e^p \tag{6}$$

The right hand-side of (6) is the marginal rate of substitution between heritage and bequest. Thus, **at the consensual optimum, the asymptotic marginal rate of substitution between heritage and bequest is equal to the maximum capital growth rate**.

That implies that the more the capital growth rate anticipated by the agent is high, the less the agent will bequeath to his heir. All occurs as if the growth of the capital and technical progress compensate for the fall of heritage. This idea will be better developed in the paper "Allais-anonymity II", of which a significant part will be devoted to the analysis and the economic interpretation of the optimum.

7.4 Is a bequest-rule plan consensus-optimal?

Henceforth let's call a "bequest-rule plan" a plan K such that the bequest-rule holds. Since the differential of ΨG at K is shown to be equal to zero only for increments ΔB in c_p, the bequest-rule cannot yet be considered as a sufficient condition of consensual optimality.

Even if sufficiency held, if K is not in $\overset{\circ}{D}$, we would have problems for G's differentiability. Indeed, if K is not in $\overset{\circ}{D}$, linearity at infinity is no longer warranted even in standard cases since in the sphere $S(K,r)$ for a given $b' \succ 0$ there is always $n \geq 1$ and $B \in S(K,r)$ such that $b_n \leq b'$. Hence, second derivatives of U_n don't tend necessarily to 0 on $S(K,r)$. Even if linearity at infinity held, it would not imply differentiability of G. Indeed, if we look to the proof of G's differentiability (8.3), we can see that in that case, K' and K'' are not necessarily in D.

However, these issues seem to be exclusively of a mathematical nature for it is difficult to give to them an economic meaning. Perhaps with better mathematics, one could establish the sufficiency of the bequest-rule.

8 Proofs

8.1 Proof of the decomposition lemma on l_∞^{r*}

Let $\mathbb{R}$ be the real line, l_∞ the set of bounded real sequences, l_∞^* its dual,

$l_1 = \left\{B = (b_1, b_2, ...)/\ b_i \in \mathbb{R} \text{ and } \sum_{i=1}^{+\infty} |b_i| \prec +\infty\right\}$,

$c_0 = \{B = (b_1, b_2, ...)/\ b_i \in \mathbb{R} \text{ and } \lim b_i = 0\}$ and

$c = \{B = (b_1, b_2, ...)/\ b_i \in \mathbb{R} \text{ and } b_i \text{ converges}\}$.

l_∞ is a Banach space for the norm $\|x\| = \sup\limits_{n\geq 1} |x_n|$ and l_1 is a Banach space for the norm $\|x\| = \sum_{i=1}^{+\infty} |x_i|$ and

- $l_1 \subset l_\infty^*$
- $l_1^* = l_\infty$(see [Luemberger 1997])
- $c_0^* = l_1$(see appendix A)

Let E be the linear operator from l_∞ to l_∞^r defined as follows:

$$E((y_1, y_2, ...)) = (y_1 e^r, y_2 e^{2r}, ...)$$

E is one-to-one and continuous. Denote its inverse E^-. It is also continuous, and its adjoint is the inverse of the adjoint of E. We have the relations (see [Luemberger 1997]):

$$E^{-*} = E^{*-1}$$

and

$$E(l_\infty) = l_\infty^r,\ E(l_1^r) = l_1,\ E(c_0) = c_0^r \text{ and } E(c) = c_r$$

We first show the relation

$$c_0^{r*} = l_1^r$$

Proof of relation $c_0^{r*} = l_1^r$

We have $E^-(c_0^r) = c_0$. Moreover, E^- is one-to-one and continuous from c_0^r to c_0. E^{-*}, the adjoint of E^-, maps c_0^* onto c_0^{r*} . Therefore we have: $E^{-*}(c_0^*) = c_0^{r*}$. But, as proved in the detailed version of [Mabrouk 2005], $c_0^* = l_1$(the proof is also given in appendix A), so $E^{-*}(l_1) = c_0^{r*}$. Now take $f \in l_1$. By definition of the adjoint, for all $y \in c_0^r$, $\langle E^{-*}(f) \mid y\rangle = \langle f \mid E^-(y)\rangle = \sum_1^{+\infty} f_i y_i e^{-r.i} = \sum_1^{+\infty} (f_i e^{-r.i}) y_i = \langle E^-(f) \mid y\rangle$. Hence, $E^{-*}(f) = E^-(f)$ for all $f \in l_1$ so we can write $E^-(l_1) = c_0^{r*}$. On the other hand, $E^-(l_1) = l_1^r$. Therefore, $c_0^{r*} = l_1^r$■

We now proof the lemma:

According to the above proof, $E^{-*}(x_1) = E^{-}(x_1)$ for all $x_1 \in l_1$. It is also clear that $E^{-}(x_1) \in l_1^r$.

Take $y \in l_\infty^{r*}$. Since E^* maps l_∞^{r*} onto l_∞^*, $E^*(y) \in l_\infty^*$. According to the decomposition lemma on l_∞^* , proved in the detailed version of [Mabrouk 2005] (the proof is also given in appendix B), we can write:

$$E^*(y) = x_1 + x_2 \text{ where } x_1 \in l_1 \text{ and } x_2 \in c_0^\perp$$

This gives

$$\begin{aligned} E^{*-1}(E^*(y)) &= E^{*-1}(x_1 + x_2) = E^{*-1}(x_1) + E^{*-1}(x_2) \\ &= E^{-*}(x_1) + E^{*-1}(x_2) = E^{-}(x_1) + E^{*-1}(x_2) \end{aligned}$$

$E^{-}(x_1) \in l_1^r$, denote it y_1.

On the other hand, $E^{*-1}(c_0^\perp) = [E(c_0)]^\perp = c_0^{r\perp}$, where $c_0^{r\perp} = \{y \in l_\infty^{r*} / \text{ for all } x \in c_0^r,\ \langle y \mid x\rangle = 0\}$(see [Luemberger 1997]). This proves that $y_2 = E^{*-1}(x_2) \in c_0^{r\perp}$.

We show now that $c_0^{r\perp}$ is the set B_r of bounded linear functionals on l_∞^r, which restriction to c_r is proportional to δ_∞^r. Let $b_r \in c_0^{r\perp}$. We have $E^{-*}(c_0^\perp) = E^{*-1}(c_0^\perp) = c_0^{r\perp}$. Then, there is $b \in c_0^\perp$ such that $E^{-*}(b) = b_r$. Since $b \in c_0^\perp$, there is $\alpha \in \mathbb{R}$ such that the restriction of b to c is equal to $\alpha\delta_\infty$ (see appendix B), what we write $b\lfloor_c = \alpha\delta_\infty$. For all $y \in c_r$, $E^{-}(y) \in c$. Then, $\langle b_r \mid y\rangle = \langle E^{-*}(b) \mid y\rangle = \langle b \mid E^{-}(y)\rangle = \alpha.\delta_\infty\left[E^{-}(y)\right] = \alpha.\delta_\infty^r(y)$. This shows that $c_0^{r\perp} \subset B_r$. The inverse inclusion is evident.

It remains to prove that the decomposition of $y \in l_\infty^{r*}$ is unique. Suppose we can write $y = y_1 + y_2$ and $y = y_1' + y_2'$ with y_1 , $y_1' \in l_1^r$ and y_2 , $y_2' \in c_0^{r\perp}$. For all $x \in c_0^r$, we have $\langle y \mid x\rangle = \langle y_1 \mid x\rangle = \langle y_1' \mid x\rangle$. Since $l_1^r = c_0^{r*}$ (see above), y_1 , $y_1' \in c_0^{r*}$. This shows that $y_1 = y_1'$, which implies $y_2 = y_2'$ ■

Calculus of $\frac{\partial f}{\partial r\infty}(x_0)$

Here is a formula to calculate $\frac{\partial f}{\partial r\infty}(x_0)$. Let $h \in c_r$ and let $r_n(h)$ be the sequence of c_r obtained by setting to 0 the n first terms of h. Since f is a function from l_∞^r to $\mathbb{R}$, Frechet-differentiable at $x_0 \in l_\infty^r$, for all $\varepsilon \succ 0$ there is $\alpha \succ 0$ such that:

$$\|h\| \prec \alpha \Longrightarrow \frac{|f(x_0+h) - f(x_0) - \delta f(x_0).h|}{\|h\|} \prec \varepsilon$$

But $\|h\| \prec \alpha \Longrightarrow \|r_n(h)\| \prec \alpha$ for all $n \geq 1$, then

$$|f(x_0 + r_n(h)) - f(x_0) - \delta f(x_0).r_n(h)| \prec \varepsilon \|r_n(h)\|$$

Thus

$$\left| f(x_0 + r_n(h)) - f(x_0) - \sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i - \frac{\partial f}{\partial r\infty}(x_0).\delta_\infty^r(h) \right| \prec \varepsilon \|r_n(h)\|$$

Moreover, $\|r_n(h)\| = \sup_{i \succ n} |h_i| e^{-ri}$. It is a positive and decreasing sequence converging to $|\delta^r_\infty(h)|$. We have also $\sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i \longrightarrow 0$ when $n \longrightarrow +\infty$. Then

$$\left|\limsup_n f(x_0 + r_n(h)) - f(x_0) - \frac{\partial f}{\partial r\infty}(x_0).\delta^r_\infty(h)\right| \leq \varepsilon \left|\delta^r_\infty(h)\right|$$

which gives

$$\left|\frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta^r_\infty(h)} - \frac{\partial f}{\partial r\infty}(x_0)\right| \leq \varepsilon$$

This proves that

$$\frac{\partial f}{\partial r\infty}(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c_r, \delta^r_\infty(h) \neq 0} \frac{\limsup\limits_n f(x_0 + r_n(h)) - f(x_0)}{\delta^r_\infty(h)}$$

We prove similarly the same formula for lim inf ■

8.2 Proofs for interiority

Proposition 20 *Let $B \in l^p_\infty$.*

1-$0 \leq \underline{\pi}(B) \leq \overline{\pi}(B) \leq p$.
2-If $\overline{\pi}(B) = p$ then $\overline{\omega}(B) \prec +\infty$.
3-*If B is homogeneous then $\underline{\omega}(B) \leq \overline{\omega}(B)$.*

Proof: 2 and 3 are immediate and so are $0 \leq \underline{\pi}(B)$ and $\overline{\pi}(B) \leq p$. To prove $\underline{\pi}(B) \leq \overline{\pi}(B)$ suppose $\underline{\pi}(B) \succ \overline{\pi}(B)$. Let $x \in]\overline{\pi}(B), \underline{\pi}(B)[$. First, observe that if $\alpha \in \{\alpha \geq 0 / \limsup |b_n| e^{-n\alpha} \prec +\infty\}$ then for all $\beta \geq \alpha$, $\beta \in \{\alpha \geq 0 / \limsup |b_n| e^{-n\alpha} \prec +\infty\}$.

Since $x \succ \overline{\pi}(B)$, this implies that $\limsup |b_n| e^{-nx} \prec +\infty$.

Take $\varepsilon \succ 0$ such that $x + \varepsilon \in]\overline{\pi}(B), \underline{\pi}(B)[$. We have $\limsup |b_n| e^{-n(x+\varepsilon)} \leq (\limsup |b_n| e^{-nx})(\limsup e^{-n\varepsilon}) = 0$. Then $\liminf |b_n| e^{-n(x+\varepsilon)} = 0$. We see easily that it is contrary to $x + \varepsilon \prec \underline{\pi}(B)$■

Lemma 21 *s^p_∞ is open in l^p_∞.*

Proof: Let $K \in s^p_\infty$ and α in $]0, \underline{\omega}(K)[$.

$\underline{\omega}(K) = \liminf |k_n| e^{-np} \succ 0 \Longrightarrow$ for all $\varepsilon \succ 0$ there is N such that $n \succ N \Longrightarrow |k_n| e^{-np} \succ \underline{\omega}(K) - \varepsilon$. Denote $S(K, \alpha)$ the closed sphere of center K and radius α. $B \in S(K, \alpha) \Longrightarrow ||k_n| - |b_n|| e^{-np} \leq |k_n - b_n| e^{-np} \leq \alpha$. Then $|b_n| e^{-np} \geq |k_n| e^{-np} - \alpha \succ \underline{\omega}(K) - \varepsilon - \alpha$. Thus $\underline{\omega}(B) = \liminf |b_n| e^{-np} \geq \underline{\omega}(K) - \alpha \succ 0$. This shows that $S(K, \alpha) \subset s^p_\infty$■

Lemma 22 $s^p_{\infty++} = \overset{\circ}{l^p_{\infty+}}$.

Proof: If $B \notin s^p_\infty$, $\liminf |b_n| e^{-np} = 0$. For all $\alpha \succ 0$, we can find n such that the interval $[b_n e^{-np} - \alpha, b_n e^{-np}]$ contains some strictly negative μ_n. Thus the sequence Y defined as follows: $y_j = b_j$ for $j \neq n$ and $y_n = \mu_n e^{np}$, is in l^p_∞ but not in $l^p_{\infty+}$. This proves that B is not in $\overset{\circ}{l^p_{\infty+}}$. Consequently, $\overset{\circ}{l^p_{\infty+}} \subset s^p_\infty$. We can then write $\overset{\circ}{l^p_{\infty+}} \subset s^p_\infty \cap \overset{\circ}{l^p_{\infty+}}$. But since it is clear that $s^p_\infty \cap \overset{\circ}{l^p_{\infty+}} \subset s^p_{\infty++}$, we have $\overset{\circ}{l^p_{\infty+}} \subset s^p_{\infty++}$.

Conversely, let's show first that $s^p_{\infty++}$ is open. Let K be in $s^p_{\infty++}$ and α in $]0, \underline{\omega}(K)[$. There is N such that $n \geq N \Longrightarrow k_n e^{-np} \geq \underline{\omega}(K) - \alpha \succ 0$. If B is in $S(K, \frac{\underline{\omega}(K)-\alpha}{2})$ and $n \geq N$, $b_n e^{-np} \geq k_n e^{-np} - \frac{\underline{\omega}(K)-\alpha}{2} \geq \frac{\underline{\omega}(K)-\alpha}{2} \succ 0$. Denote $k = \inf \{k_n / n \in [1, N-1]\}$ and take $\beta = \inf \left(\frac{\underline{\omega}(K)-\alpha}{2}, \frac{1}{2} k e^{-Np} \right)$. Take B in $S(K, \beta)$. Since $S(K, \beta) \subset S(K, \frac{\underline{\omega}(K)-\alpha}{2})$, for $n \geq N$, $b_n e^{-np} \geq \frac{\underline{\omega}(K)-\alpha}{2} \succ 0$. For $n \leq N-1$, $b_n e^{-np} \geq k_n e^{-np} - \beta \geq k_n e^{-np} - \frac{1}{2} k e^{-Np} \geq \frac{1}{2} k e^{-Np} \succ 0$. This shows that $S(K, \beta) \subset s^p_{\infty++}$ and $s^p_{\infty++}$ is open. Furthermore, it is clear that $s^p_{\infty++} \subset l^p_{\infty+}$. Since $s^p_{\infty++}$ is open, it is also included in $\overset{\circ}{l^p_{\infty+}}$.■

Proof of proposition 5: Since $D \subset l^p_{\infty+}$ we have $\overset{\circ}{D} \subset \overset{\circ}{l^p_{\infty+}} = s^p_{\infty++}$■

8.3 Proofs for the differentiability of G

Proof of lemma 9: Let $\Delta K \in l^p_\infty$ such that $S(K, \|\Delta K\|) \subset D$. For all $n \geq 1$, since D is convex, the segment $[K, K + \Delta K]$ is in D and

$[(k_{n-1}, k_n), (k_{n-1} + \Delta k_{n-1}, k_n + \Delta k_n)]$ is in D_n, (Δk_0 is taken equal to 0). We can apply Taylor's formula : there is $t_n \in [0,1]$ such that

$$\begin{aligned} & U_n(k_{n-1} + \Delta k_{n-1}, k_n + \Delta k_n) - U_n(k_{n-1}, k_n) \qquad (7) \\ = \; & U'_{nh}(k_{n-1}, k_n) \Delta k_{n-1} + U'_{nl}(k_{n-1}, k_n) \Delta k_n + \\ & \frac{1}{2} \begin{pmatrix} \Delta k_{n-1} & \Delta k_n \end{pmatrix} \begin{pmatrix} U''_{nh^2}(k^n_{n-1}, k^n_n) & U''_{nlh}(k^n_{n-1}, k^n_n) \\ U''_{nhl}(k^n_{n-1}, k^n_n) & U''_{nl^2}(k^n_{n-1}, k^n_n) \end{pmatrix} \begin{pmatrix} \Delta k_{n-1} \\ \Delta k_n \end{pmatrix} \end{aligned}$$

where $k^n_n = k_n + t_n \Delta k_n$ and $k^n_{n-1} = k_{n-1} + t_n \Delta k_{n-1}$.

Denote

$$\begin{aligned} K' &= (k^2_1, k^2_2, k^4_3, k^4_4, ...) = \left[\left(k^{2p}_{2p-1}, k^{2p}_{2p} \right) \right]_{p \geq 1} \\ K'' &= (k^1_1, k^3_2, k^3_3, k^5_4, ...) = \left[\left(k^{2p-1}_{2p-1}, k^{2p+1}_{2p} \right) \right]_{p \geq 1} \end{aligned}$$

and take $k''_0 = k'_0 = k_0$.

K' and $K''$are in $S(K, \|\Delta K\|)$ and, according to the definition of $\theta_n(B, X)$, we can write:

$$\begin{aligned} & U_n(k_{n-1} + \Delta k_{n-1}, k_n + \Delta k_n) - U_n(k_{n-1}, k_n) \\ = \; & U'_{nh}(k_{n-1}, k_n) \Delta k_{n-1} + U'_{nl}(k_{n-1}, k_n) \Delta k_n + \frac{1}{2} \theta_n(K', \Delta K) \end{aligned}$$

for n even and

$$\begin{aligned} & U_n\left(k_{n-1}+\Delta k_{n-1}, k_n+\Delta k_n\right)-U_n\left(k_{n-1}, k_n\right) \\ = \; & U_{nh}^{'}\left(k_{n-1}, k_n\right)\Delta k_{n-1}+U_{nl}^{'}\left(k_{n-1}, k_n\right)\Delta k_n+\frac{1}{2}\theta_n(K'', \Delta K) \end{aligned}$$

for n odd.

G fulfills the condition of linearity at infinity at K means that there is $\alpha \succ 0$ and $M \geq 0$ such that for all $X \in l_\infty^p$, $\|X\|_p \leq 1$ (with $x_0 = 0$ and $X = (x_n)_{n\geq 1}$) and for all $B \in S(K, \alpha)$, we have:

$$\|\Theta(B, X)\|_{p_1} \leq M.$$

Take $\alpha \succ 0$ such that $S(K, \alpha) \subset D$ and $\Delta K \in l_\infty^p$ such that $\|\Delta K\| \leq \alpha$.

$K' \in S(K, \alpha)$ and $\left\|\frac{\Delta K}{\|\Delta K\|_p}\right\|_p \leq 1$ implies that $\left\|\Theta(K', \frac{\Delta K}{\|\Delta K\|_p})\right\|_{p_1} \leq M$. Thus

$$\left\|\Theta(K', \frac{\Delta K}{\|\Delta K\|_p})\right\|_{p_1} \|\Delta K\|_p^2 \leq M \|\Delta K\|_p^2$$

But

$$\Theta(K', \Delta K) = \Theta(K', \frac{\Delta K}{\|\Delta K\|_p}) \left[\|\Delta K\|_p\right]^2$$

thus

$$\|\Theta(K', \Delta K)\|_{p_1} \leq M \|\Delta K\|_p^2$$

Similarly

$$\|\Theta(K'', \Delta K)\|_{p_1} \leq M \|\Delta K\|_p^2$$

Denote $\Theta = (\theta_n)_{n\geq 1}$where $\theta_n = \theta_n(K', \Delta K)$ if n is even and $\theta_n = \theta_n(K'', \Delta K)$ if n is odd. We can then write $\|\Theta\|_{p_1} \leq M \|\Delta K\|_p^2$.

Equation (7) gives

$$G(K+\Delta K) - G(K) - \delta G(K) \cdot \Delta K = \frac{1}{2}\Theta$$

thus

$$\|G(K+\Delta K) - G(K) - \delta G(K) \cdot \Delta K\|_{p_1} \leq \frac{1}{2} M \|\Delta K\|_p^2 . \tag{8}$$

But

$$\begin{aligned} & \left| \|G(K+\Delta K) - G(K)\|_{p_1} - \|\delta G(K) \cdot \Delta K\|_{p_1} \right| \\ \leq \; & \|G(K+\Delta K) - G(K) - \delta G(K) \cdot \Delta K\|_{p_1} \end{aligned}$$

which gives

$$\left| \|G(K+\Delta K) - G(K)\|_{p_1} - \|\delta G(K) \cdot \Delta K\|_{p_1} \right| \leq \frac{1}{2} M \|\Delta K\|_p^2$$

and since $G(K+\Delta K)-G(K)\in l_{\infty}^{p_1}$, we have also $\delta G(K)\cdot\Delta K\in l_{\infty}^{p_1}$.

For all $X\in l_{\infty}^{p}$, $\frac{\alpha}{\|X\|_p}X\in S(0,\alpha)$ and then

$$\delta G(K)\cdot\frac{\alpha}{\|X\|_p}X=\frac{\alpha}{\|X\|_p}\delta G(K)\cdot X\in l_{\infty}^{p_1}$$

Thus, $\delta G(K)\cdot X\in l_{\infty}^{p_1}$and $\delta G(K)$ is a linear transformation from l_{∞}^{p} to $l_{\infty}^{p_1}$.

In addition, equation (8) implies

$$\lim_{\|\Delta K\|_p\longrightarrow 0}\frac{\|G(K+\Delta K)-G(K)-\delta G(K)\cdot\Delta K\|_{p_1}}{\|\Delta K\|_p}=0$$

We then see easily that G is Gateaux-differentiable at K in every direction $\Delta K\in l_{\infty}^{p}$ and its Gateaux-differential at K with the increment ΔK is $\delta G(K)\cdot\Delta K$.

Moreover, if G is continuous at K, equation (8) proves that

$$\lim_{\|\Delta K\|_p\longrightarrow 0}\delta G(K)\cdot\Delta K=0$$

This implies that $\delta G(K)$ is a continuous linear transformation. Hence, $\delta G(K)$ is the Frechet-differential of G at K.■

Proof of proposition 10: According to lemma 9, for all $B\in l_{\infty}^{p}$, the sequence
$\left[U_{nh}^{'}(k_{n-1},k_n)\,b_{n-1}+U_{nl}^{'}(k_{n-1},k_n)\,b_n\right]_{n\geq 1}$ is in $l_{\infty}^{p_1}$. Thus, there is $M(B)\geq 0$ such that

$$\left|U_{nh}^{'}(k_{n-1},k_n)\,b_{n-1}+U_{nl}^{'}(k_{n-1},k_n)\,b_n\right|e^{-np_1}\leq M(B)$$

Let $M\succ 0$. Take $B\in l_{\infty}^{p}$ such as $|b_n|\,e^{-np}=\|B\|$ for all $n\geq 1$, with $b_1\geq 0$, b_2 of the sign of $b_1U_{2l}^{'}(k_1,k_2)$ and so on ...so that

$$\begin{aligned}&\left|U_{nh}^{'}(k_{n-1},k_n)\,b_{n-1}+U_{nl}^{'}(k_{n-1},k_n)\,b_n\right|\\=\;&\left|U_{nh}^{'}(k_{n-1},k_n)\right||b_{n-1}|+\left|U_{nl}^{'}(k_{n-1},k_n)\right||b_n|\end{aligned}$$

for all $n\geq 1$. Then

$$\begin{aligned}&\left|U_{nh}^{'}(k_{n-1},k_n)\,b_{n-1}+U_{nl}^{'}(k_{n-1},k_n)\,b_n\right|e^{-np_1}\\=\;&\left[\left|U_{nh}^{'}(k_{n-1},k_n)\right||b_{n-1}|+\left|U_{nl}^{'}(k_{n-1},k_n)\right||b_n|\right]e^{-np_1}\\=\;&\left[\begin{array}{c}\left|U_{nh}^{'}(k_{n-1},k_n)\right|e^{-p_1}e^{(n-1)(p-p_1)}\left|b_{n-1}e^{-(n-1)p}\right|\\+\left|U_{nl}^{'}(k_{n-1},k_n)\right|e^{n(p-p_1)}\left|b_ne^{-np}\right|\end{array}\right]\\=\;&\|B\|\left[\begin{array}{c}\left|U_{nh}^{'}(k_{n-1},k_n)\right|e^{-p_1}e^{(n-1)(p-p_1)}\\+\left|U_{nl}^{'}(k_{n-1},k_n)\right|e^{n(p-p_1)}\end{array}\right]\leq M(B)\end{aligned}$$

Thus
$$\left[\begin{array}{c} e^{-p_1}\left|U'_{nh}\left(k_{n-1},k_n\right)\right|e^{(n-1)(p-p_1)} \\ +\left|U'_{nl}\left(k_{n-1},k_n\right)\right|e^{n(p-p_1)}\end{array}\right]\leq\frac{M(B)}{\|B\|}$$

This shows that the sequences $\left(\left|U'_{nh}\left(k_{n-1},k_n\right)\right|\right)_{n\geq1}$ and $\left(\left|U'_{nl}\left(k_{n-1},k_n\right)\right|\right)_{n\geq1}$ are in $l_\infty^{p_1-p}$■

Proof of proposition 11: Given lemma 9, it is enough to prove the continuity of G at K. As in the proof of lemma 9, take $\alpha_1\succ0$ such that $S\left(K,\alpha_1\right)\subset\overset{\circ}{D}$. Let $\Delta K\in l_\infty^p$ be such that $K+\Delta K$ is in $S\left(K,\alpha_1\right)$. For all $n\geq1$, the segment $[(k_{n-1},k_n),(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n)]$ is in D_n, (Δk_0 is taken equal to 0). We apply Taylor's formula : there is $t_n\in[0,1]$ such that

$$\begin{array}{rl} & U'_{nh}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U'_{nh}\left(k_{n-1},k_n\right) \\ = & U''_{nh^2}(k_{n-1}^n,k_n^n)\Delta k_{n-1}+U''_{nhl}(k_{n-1}^n,k_n^n)\Delta k_n\end{array}$$

where $k_n^n=k_n+t_n\Delta k_n$ and $k_{n-1}^n=k_{n-1}+t_n\Delta k_{n-1}$.

Denote

$$\begin{array}{rcl} K' & = & (k_1^2,k_2^2,k_3^4,k_4^4,...)=\left[\left(k_{2p-1}^{2p},k_{2p}^{2p}\right)\right]_{p\geq1} \\ K'' & = & (k_1^1,k_2^3,k_3^3,k_4^5,...)=\left[\left(k_{2p-1}^{2p-1},k_{2p}^{2p+1}\right)\right]_{p\geq1}\end{array}$$

and take $k''_0=k'_0=k_0$.

K' and $K''$are in $S\left(K,\|\Delta K\|\right)$.

G fulfills the condition of linearity at infinity at K means that there is $\alpha_2\succ0$ and $M\geq0$ such that for all $X\in l_\infty^p$, $\|X\|_p\leq1$ (with $x_0=0$ and $X=(x_n)_{n\geq1}$) and for all $B\in S(K,\alpha_2)$ we have:

$$\|\Theta(B,X)\|_{p_1}\leq M.$$

Take $\Delta K\in l_\infty^p$ such that $\|\Delta K\|\leq\alpha=\inf(\alpha_1,\alpha_2)$. For all $X\in l_\infty^p$ such that $\|X\|_p\leq1$, we then have $\|\Theta(K',X)\|_{p_1}\leq M$. This implies

$$\left|\begin{array}{c} x_{n-1}^2U''_{nh^2}(k'_{n-1},k'_n)+x_nx_{n-1}U''_{nhl}(k'_{n-1},k'_n) \\ +x_n^2U''_{nl^2}(k'_{n-1},k'_n)\end{array}\right|\leq Me^{np_1}$$

for all $n\geq1$.

Since the functions $(U_n)_{n\geq1}$ are concave, $U''_{nh^2}(k'_{n-1},k'_n)\leq0$ and $U''_{nl^2}(k'_{n-1},k'_n)$ ≤0.

Take $x_1=e^p$,

$x_2=e^{2p}sign\left[-x_1U''_{2hl}(k'_1,k'_2)\right]...$,

$x_n = e^{np} sign\left[-x_{n-1}U''_{nhl}(k'_{n-1},k'_n)\right]\ldots$ Then, $X=(x_n)_{n\geq 1}\in l^p_\infty$ and $\|X\|_p=1$. Hence, for all $n\geq 1$, $x_n x_{n-1}U''_{nhl}(k'_{n-1},k'_n)\leq 0$. We then have

$$\begin{array}{cc} & -x_{n-1}^2U''_{nh^2}(k'_{n-1},k'_n)-x_nx_{n-1}U''_{nhl}(k'_{n-1},k'_n) \\ \leq & \begin{array}{c} -x_{n-1}^2U''_{nh^2}(k'_{n-1},k'_n)-x_nx_{n-1}U''_{nhl}(k'_{n-1},k'_n) \\ -x_n^2U''_{nl^2}(k'_{n-1},k'_n) \end{array} \\ = & \left|\begin{array}{c} x_{n-1}^2U''_{nh^2}(k'_{n-1},k'_n)+x_nx_{n-1}U''_{nhl}(k'_{n-1},k'_n) \\ +x_n^2U''_{nl^2}(k'_{n-1},k'_n) \end{array}\right| \\ \leq & Me^{np_1} \end{array}$$

But for $n\geq 2$,

$$\begin{array}{cc} & -x_{n-1}^2U''_{nh^2}(k'_{n-1},k'_n)-x_nx_{n-1}U''_{nhl}(k'_{n-1},k'_n) \\ = & e^{(n-1)p}\left[e^{(n-1)p}\left|U''_{nh^2}(k'_{n-1},k'_n)\right|+e^{np}\left|U''_{nhl}(k'_{n-1},k'_n)\right|\right] \end{array}$$

Thus

$$\left[e^{(n-1)p}\left|U''_{nh^2}(k'_{n-1},k'_n)\right|+e^{np}\left|U''_{nhl}(k'_{n-1},k'_n)\right|\right]\leq Me^{n(p_1-p)}e^p$$

Similarly, we have

$$\left[e^{(n-1)p}\left|U''_{nh^2}(k''_{n-1},k''_n)\right|+e^{np}\left|U''_{nhl}(k''_{n-1},k''_n)\right|\right]\leq Me^{n(p_1-p)}e^p$$

On the other hand

$$\begin{array}{cc} & U'_{nh}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U'_{nh}\left(k_{n-1},k_n\right) \\ = & U''_{nh^2}(k'_{n-1},k'_n)\Delta k_{n-1}+U''_{nhl}(k'_{n-1},k'_n)\Delta k_n \end{array}$$

for n even and

$$\begin{array}{cc} & U'_{nh}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U'_{nh}\left(k_{n-1},k_n\right) \\ = & U''_{nh^2}(k''_{n-1},k''_n)\Delta k_{n-1}+U''_{nhl}(k''_{n-1},k''_n)\Delta k_n \end{array}$$

for n odd. Thus, for n even

$$\begin{array}{cc} & \left|U'_{nh}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U'_{nh}\left(k_{n-1},k_n\right)\right| \\ \leq & \left|U''_{nh^2}(k'_{n-1},k'_n)\right|\left|\Delta k_{n-1}\right|+\left|U''_{nhl}(k'_{n-1},k'_n)\right|\left|\Delta k_n\right| \\ \leq & \left|U''_{nh^2}(k'_{n-1},k'_n)\right|\left\|\Delta K\right\|e^{(n-1)p}+\left|U''_{nhl}(k'_{n-1},k'_n)\right|\left\|\Delta K\right\|e^{np} \\ \leq & \left\|\Delta K\right\|Me^pe^{n(p_1-p)} \end{array}$$

and similarly for n odd. We show the same way that

$$\begin{array}{cc} & \left|U'_{nl}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U'_{nl}\left(k_{n-1},k_n\right)\right| \\ \leq & \left\|\Delta K\right\|Me^pe^{n(p_1-p)} \end{array}$$

Denote respectively the sequences $\left(U'_{nh}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)\right)_{n\geq 1}$and $\left(U'_{nl}\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)\right)_{n\geq 1}$ by $U'_h\left(K+\Delta K\right)$ and $U'_l\left(K+\Delta K\right)$. Previous inequalities give $\left\|U'_h\left(K+\Delta K\right)-U'_h\left(K\right)\right\|_{p_1-p}\leq\left\|\Delta K\right\|Me^p$ and

$\left\|U'_l\left(K+\Delta K\right)-U'_l\left(K\right)\right\|_{p_1-p}\leq\left\|\Delta K\right\|Me^p$. Thus, since $U'_h\left(K\right)$ and $U'_l\left(K\right)$ are in $l_\infty^{p_1-p}$ (see proposition 10), we have

$$\begin{aligned}\left\|U'_h\left(K+\Delta K\right)\right\|_{p_1-p} &\leq \left\|\Delta K\right\|_p Me^p+\left\|U'_h\left(K\right)\right\|_{p_1-p}\\ &\leq \alpha Me^p+\left\|U'_h\left(K\right)\right\|_{p_1-p}\end{aligned}$$

We obtain the same inequality for $\left\|U'_l\left(K+\Delta K\right)\right\|_{p_1-p}$.

Denote $\Gamma=\sup\left[\alpha Me^p+\left\|U'_h\left(K\right)\right\|_{p_1-p},\alpha Me^p+\left\|U'_l\left(K\right)\right\|_{p_1-p}\right]$.

Now apply Taylor formula to U_n and build K'_1 and K''_1 in $S(K,\Delta K)$ as above. We have respectively for n even and odd:

$$\begin{aligned}& U_n\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U_n\left(k_{n-1},k_n\right)\\ =\ & U'_{nh}(k'_{1n-1},k'_{1n})\Delta k_{n-1}+U'_{nl}(k'_{1n-1},k'_{1n})\Delta k_n\end{aligned}$$

and

$$\begin{aligned}& U_n\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U_n\left(k_{n-1},k_n\right)\\ =\ & U'_{nh}(k''_{1n-1},k''_{1n})\Delta k_{n-1}+U'_{nl}(k''_{1n-1},k''_{1n})\Delta k_n\end{aligned}$$

thus

$$\begin{aligned}& \left|U_n\left(k_{n-1}+\Delta k_{n-1},k_n+\Delta k_n\right)-U_n\left(k_{n-1},k_n\right)\right|e^{-np_1}\\ \leq\ & \Gamma\left(e^{-p}\left|\Delta k_{n-1}\right|e^{-(n-1)p}+\left|\Delta k_n\right|e^{-np}\right)\leq\left\|\Delta K\right\|\Gamma\left(1+e^{-p}\right)\end{aligned}$$

then

$$\left\|G(K+\Delta K)-G(K)\right\|\leq\left\|\Delta K\right\|\Gamma\left(1+e^{-p}\right)$$

and G is continuous at K.$\blacksquare$

8.4 Proofs for consensual optimality

Proof of proposition 14: We have

$$\delta\left[\Psi G\right](K)\cdot\Delta B=\delta\Psi(G(K))\cdot\left[\delta G(K)\cdot\Delta B\right]$$

and

$$\delta G(K)\cdot\Delta B=(u'_{hi}\Delta b_{i-1}+u'_{li}\Delta b_i)_{i\geq 1}$$

for all ΔB in l_∞^p.

Using lemma 2, write $\delta\Psi(G(K))=\delta\Psi_1(G(K))+\delta\Psi_2(G(K))$ where $\delta\Psi_1(G(K))\in l_1^{p_1}$and $\delta\Psi_2(G(K))$ is such that its restriction to $c_{p_1}$is proportional to $\delta_\infty^{p_1}$.

For $\Delta X \in c_{p_1}$, denote $\Delta x = \lim_n \Delta x_n e^{-p_1 n}$. We have

$$\delta\Psi_2(G(K)) \cdot \Delta X = \left[\frac{\partial\Psi}{\partial p_1\infty}(G(K))\right]\Delta x$$

Since K is a steady state and $\Delta B \in c_p$, then $(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1} \in c_{p_1}$. Denote $\Delta b = \lim_n \Delta b_n e^{-pn}$and $\Delta b_0 = 0$. We then have:

$$\begin{aligned}
& \delta\left[\Psi G\right](K) \cdot \Delta B \\
= \; & \delta\Psi_1(G(K)) \cdot [(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}] \\
& +\delta\Psi_2(G(K)) \cdot [(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}] \\
= \; & \sum_{i=1}^{+\infty} \Psi'_i(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i) + \Psi'_{p_1\infty} \lim_i \left(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i\right) e^{-p_1 i} \\
= \; & \sum_{i=1}^{+\infty}(\Psi'_i u'_{li} + \Psi'_{i+1}u'_{hi+1})\Delta b_i + \Psi'_{p_1\infty}(u'_h e^{-p} + u'_l)\Delta b
\end{aligned}$$

$\blacksquare$

Proof of proposition *16:* It is similar to the proof of proposition 8 in [Mabrouk 2005]

Proof of proposition *18:* Let $n = \sup\{i/i \neq s(i)\}$. Let $x \in \mathbb{R}^n$. For $G \in l_\infty^{p_1}$ denote $xG = (x_1, x_2, ..., x_n, g_{n+1}, g_{n+2}, ...)$. Let $f(x) = \Psi(xG)$. We have

$$f'_i(x) = \Psi'_i(xG) = 0$$

for all $x \in \mathbb{R}^n$ and $G \in l_\infty^{p_1}$.

f is then constant on $\mathbb{R}^n$. There is $x \in \mathbb{R}^n$ such that $\widehat{s}(G) = xG$. Thus, $\Psi(\widehat{s}(G)) = \Psi(G)$$\blacksquare$

A Proof of $c_0^* = l_1$

For $i \geq 1$, let e_i be the element of l_∞ such that all its components are zero except the i^{th} which is 1. Let $x \in c_0$ and $f \in c_0^*$. We have $\sum_1^n x_i e_i \longrightarrow x$, so $f\left(\sum_1^n x_i e_i\right) \longrightarrow f(x)$, then $\sum_1^{+\infty} x_i f(e_i) = f(x)$. One the other hand, f *continuous* $\Leftrightarrow \frac{|f(x)|}{\|x\|} \leq \|f\|$ for all $x \in c_0$ (see appendix1). $\|e_i\| = 1$ gives $|f(e_i)| \leq \|f\|$ for all $i \geq 1$. Let $\alpha \in]0,1[$. Take $x_n = sign(f(e_n)) \cdot \frac{1}{n^\alpha}$ then $x = (x_n)_{n\geq 1} \in c_0$. We have

$$\sum_1^{+\infty} \frac{|f(e_n)|}{n^\alpha} = |f(x)| \leq \|x\| . \|f\| = \|f\|$$

Now, let $\varphi(\alpha) = \sum_1^{+\infty} \frac{|f(e_n)|}{n^\alpha}$. Then φ is bounded and decreasing on $]0,1[$. Hence, it has a finite limit as $\alpha \longrightarrow 0$. We can show easily that this limit is $\sum_1^{+\infty} |f(e_n)|$. Thus the sequence $(f(e_n))_{n\geq 1}$ is in l_1. Owing to the equality $\sum_1^{+\infty} x_i f(e_i) = f(x)$, we can identify it to f.

So $c_0^* \subset l_1$. The inverse inclusion is evident■

B The decomposition lemma on l_∞^*

B.1 The lemma

Lemma 23 *Let $y \in l_\infty^*$. Then we can write in a unique manner:*

$$y = y_1 + y_2$$

where y_1 verifies:

$$\sum_{i=1}^{+\infty} |y_{1i}| \prec +\infty$$

and y_2 is such as its restriction to c is proportional to δ_∞.

B.2 Proof

B.2.1 Projection from l_∞^* on l_1

For $i \geq 1$, let e_i be the element of l_∞ such that all its components are zero except the i^{th} which is 1.

Let $y \in l_\infty^*$. Consider the sequence: $(y \mid e_i)_{i\geq 1}$. This sequence is in l_1.[7] Denote Φ the mapping from l_∞^* to l_1 which associates to y the sequence $(y \mid e_i)_{i\geq 1}$. Φ is a projection from l_∞^* to l_1. Indeed, it is a linear transformation and, considering $l_1$as a subset of l_∞^*, if $y \in l_1$then $\Phi(y) = y$.

B.2.2 Decomposition of an element $y \in l_\infty^*$ by Φ

Consider the mapping Identity I from c_0 to l_∞:

$$I : c_0 \underset{x \longrightarrow x}{\longrightarrow} l_\infty$$

We can verify easily that Φ is the adjoint operator of I, what we write:

$$\Phi = I^*$$

I being linear and continuous, we deduce that Φ is a continuous[8] linear operator.

[7] We show this like we have shown that $(f(e_n))_{n\geq 1}$is in l_1. See appendix A.

[8] The adjoint of a continuous linear operator is continuous too.

Furthermore, we have:[9]

$$R(I)^{\perp} = N(I^{*})$$

where

$$R(I) = \{y \in l_{\infty} \diagup \exists x \in c_0 : I(x) = y\} = c_0$$

and

$$N(I^{*}) = \{x \in c_0 \diagup I^{*}(x) = 0\}$$

which means:

$$N(\Phi) = c_0^{\perp}$$

For $y \in l_{\infty}^{*}$, define $k = \Phi(y) - y$. We can write:

$$y = \Phi(y) + k.$$

with $\Phi(y) \in l_1$ and $k \in c_0^{\perp}$.

We have decomposed an element y of l_{∞}^{*} as a sum of an element of l_1 and an element of $c_0^{\perp}$. We easily show that this decomposition is unique.

B.2.3 Study of $c_0^{\perp}$

We have:

$$\|\delta_{\infty}\| = \sup_{x \in c} \frac{|\lim x_n|}{\|x\|} = \sup_{x \in c} \frac{|\lim x_n|}{\sup |x_n|} = 1$$

and

$$\forall \alpha \in R : \|\alpha \delta_{\infty}\| = |\alpha| \, \|\delta_{\infty}\| = |\alpha|$$

so we can apply Hahn-Banach theorem, and extend $\alpha\delta_{\infty}$ with an element of l_{∞}^{*}, say β.

Denote B the set of such linear functionals. We now show that $c_0^{\perp} = B$. We see easily that B is a vector subspace of l_{∞}^{*} included in $c_0^{\perp}$. Reciprocally, let $\beta \in c_0^{\perp}$ and $x \in c$. Denote $e = (1, 1 \cdots)$. We have $x - (\delta_{\infty} \mid x)e \in c_0$, so $\langle \beta \mid (x - (\delta_{\infty} \mid x)e) \rangle = 0$. Thus $\beta \mid x = (\beta \mid e)(\delta_{\infty} \mid x)$. This proves that the restriction of β to c is proportional to δ_{∞}. Then $\beta \in B$ and $c_0^{\perp} \subset B$.■

[9] See [Luemberger 1997] p155.

References

[Allais 1947] M. Allais, Economie et intérêt, Clément Juglar($2^{ième}$ edition), Paris, 1998.

[Arrow 1995] K. Arrow, Intergenerational equity and the rate of discount in long-term social investment, IEA World Congress, December 1995.

[Asheim 1991] G.B. Asheim, Unjust intergenerational allocations, Journal of economic theory 54(1991)350-371.

[Asheim-Buchholz-Tungodden 2001] G.B. Asheim, W. Buchholz, B. Tungodden, Justifying sustainability, Journal of environmental economics and management 41(2001)252-268.

[Asheim-Mitra-Tungodden 2006] G.B. Asheim, T. Mitra, B. Tungodden, Sustainable recursive social welfare functions, working paper, University of Oslo, Cornell University, Norwegian School of Economics and Business Administration, March 2006.

[Asheim-Buchholz 2003] G.B. Asheim, W. Buchholz, The malleability of undiscounted utilitarianism as a criterion of intergenerational justice, Economica 70, 405-422, 2003.

[Balasko-Shell 1980] Y. Balasko, K. Shell, The OLG model, I: the case of pure exchange without money, Journal of Economic Theory 23(1980)281-306.

[Basu-Mitra 2003] K. Basu, T. Mitra, Aggregating infinite utility streams with intergenerational equity: the impossibility of being paretian, Econometrica vol71, n°5, September 2003, 1557-1563

[Bayer-Cansier 1998] S. Bayer, D. Cansier, Intergenerational discounting: a new approach, Tübinger Diskussionsbeitrag Nr145, September 1998.

[Beltratti-Chichilnisky-Heal 1994] A. Beltratti, G. Chichilnisky, G. Heal, The environment and the long run: a comparison of different criteria, Ricerche Economiche (1994) 48,319-340.

[Blackorby-Bossert-Donaldson 2005] C. Blackorby, W. Bossert, D. Donaldson, Intertemporal social evaluation, Département des sciences économiques, Université de Montréal, CIREQ 06-2005, January 2005.

[Bossert-Sprumont-Suzumura 2004] W. Bosser, Y. Sprumont, K. Suzumura, The possibility of ordering infinite utility streams, Working paper July 2004, Université de Montréal.

[Boyd 1990] J.H. Boyd, Recursive utility and the Ramsey problem, Journal of Economic Theory 50(1990)326-345.

[Cass1965] D. Cass, On the existence of weakly maximal programmes in a multisector economy, Review of Economic Studies 37, 275-280.

[Cass 1972] D. Cass, On capital overaccumulation in the aggregative, neoclassical model of economic growth: a complete characterization, Journal of Economic Theory 4(1972)200-223.

[Chichilnisky 1996] G. Chichilnisky, An axiomatic approach to sustainable development, Social Choice and Welfare 14(1996)231-257.

[Cowen-Parfit 1992] T. Cowen, D. Parfit, Against the social discount rate, in Justice between age groups and generations, P. Laslett and J. Fishkin eds, Yale University press (1992), New Haven, 162-168.

[Diamond 1965] P.A. Diamond, The evaluation of infinite utility streams, Econometrica, vol.33, n°1(January, 1965)

[Farmer-Lahiri 2005] R.E.A. Farmer, A. Lahiri, Recursive preferences and balanced growth, Journal of Economic Theory (in press).

[Ferejohn-Page 1978] J. Ferejohn, T. Page, On the foundations of intertemporal choice, American Journal of Agricultural Economics 60(1978)269-275.

[Fleurbaey-Michel 2003] M. Fleurbaey, P. Michel, Intertemporal equity and the extension of the Ramsey criterion, Journal of Mathematical Economics 39(2003)777-802.

[Fleurbaey-Suzumura-Tadenuma 2005] M. Fleurbaey, K. Suzumura, K. Tadenuma, Arrovian aggregation in economic environments: how much should we know about indifference surfaces? Journal of Economic Theory 124(2005)22-44.

[Frederick-Loewenstein-O'Donoghue 2002] S. Frederick, G. Loewenstein, T. O'Donoghue, Time discounting and time preference: a critical review, Journal of Economic Literature 40(2002)351-401.

[Gollier 2004] C. Gollier, Maximizing the expected net future value as an alternative strategy to gamma discounting, Finance research letters 1(2)(2004)85-89.

[Gollier 2002] C. Gollier, Time horizon and the discount rate, Journal of economic theory 107(2)(2002) 463-473.

[Groth 2003] C. Groth, Course of advanced macroeconomics 2003, Institute of Economics, University of Copenhagen, www.econ.ku.dk.

[Guesnerie 2004] R. Guesnerie, Calcul économique et développement durable, DELTA wp n°2004-2

[Hara-Shinotsuka-Suzumura-Xu 2005] C. Hara, T. Shinotsuka, K. Suzumura, Y. Xu, On the possibility of continuous paretian and egalitarian evaluation of infinite utility streams, Working paper, Institute of economic Research, Hitotsubashi University 26/5/2005.

[Heal 2001] Optimality or sustainability?, G. Heal, paper prepared for presentation at the EAERE 2001 Conference, Southampton June 2001.

[Heidjra-van der Poeg 2002] B. Heijdra, R. van der Poeg, Foundations of modern macroeconomics, Oxford, 2002.

[Koopmans 1960] T.C. Koopmans, Stationnary ordinal utility and impatience, Econometrica 28(1960)287-309.

[Koopmans-Diamond-Williamson 1964] T. C. Koopmans, P.A. Diamond, R.E Williamson, Stationnary utility and time perspective, Econometrica, vol32,n°1,(1964)82-100.

[Lauwers 1998] L. Lauwers, Intertemporal objective functions: Strong Pareto versus anonymity, Mathematical Social Sciences 35(1998)37-55.

[Lauwers 2000] L. Lauwers, Topological social choice, Mathematical social sciences 40(2000)1-39.

[Le Van-Vailakis 2005] C. LeVan, Y Vailakis, Recursive utility and optimal growth with bounded or unbounded returns, Journal of Economic Theory (in press).

[Le Van-Saglam 2004] C. Le Van, H.C. Saglam, Optimal growth models and the Lagrange multiplier, Journal of Mathematical Economics 40(2004)393-410.

[Luemberger 1997] D. Luenberger, Optimization by vector spaces, Wiley-Interscience, New Ed edition 1997.

[Mabrouk 2005] M. Mabrouk, Optimal growth path in an OLG economy without time-preference assumptions, Annales Maghrébines de l'Ingénieur ENIT-Tunis (in press), available at http://econwpa.wustl.edu/eps /ge/papers/0510/0510006.pdf

[Mankiw 2001] G.N. Mankiw, Macroéconomie (French translation), De Boeck Université, Bruxelles, 2001.

[Michel 1990] P. Michel, Criticism of the social time-preference hypothesis in optimal growth, CORE discussion paper 9039, Louvain-la-Neuve.

[Mihara 1999] H. R. Mihara, Arrow's theorem, countably many agents, and more visible invisible dictators, Journal of Mathematical Economics 32(1999)267-287.

[Mitra-Basu 2005] T. Mitra, K. Basu, On the existence of paretian welfare relations for infinite utility streams with extended anonymity, CAE working-paper ♯05-06, Cornell University, May 2005.

[Naimark 1970] M.A. Naimark, Normed rings, Wolters-Noordhoff, Groningen, 1970.

[Rasmussen 2001] T.Rasmussen, Modeling the economics of greenhouse gas abatement: infinite horizon or overlapping generations?, Discussion papers n°2001-2, Center for Economic and Business Research, Copenhagen, March 2001.

[Rawls 1971] J. Rawls, A theory of justice, Harvard University Press, 1971

[Sakai 2003a] T. Sakai, An axiomatic approach to intergenerational equity, Social Choice and Welfare, vol20, issue1(2003)167-176.

[Sakai 2003b] T. Sakai, Intergenerational preferences and the sensitivity to the present, Economics Bulletin, vol4, n°26(2003)1-5.

[Samuelson 1958] P.A. Samuelson, An exact consumption-loan model of interest with or without the social contrivance of money, Journal of Political Economics. 66(1958)467-482.

[Schelling 1995] T.C. Schelling, Intergenerational discounting, Energy policy, 23(4/5)(1995)395-401.

[Sidgwick 1907] H. Sidgwick, The methods of Ethics, Macmillan, London.

[Solow 1956] R. Solow, A contribution to the theory of optimal growth, Quarterly Journal of Economics 70(1956) 65-94.

[Solow 1973] R. Solow, Intergenerational equity and exhaustible resources, discussion paper n°103, M.I.T.

[Svensson 1980] L.G. Svensson, Equity among generations, Econometrica 48(1980)1251-1256.