On Meritocratic Inequality Indices

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Abstract

We establish a Theorem on Structural Inequality Indices which provides fundamental link between inequality measurement and a concept of social justice embedded in meritocracy framework by taking axiomatic approach and redefining standard properties of inequality indices in a way that incorporates meritocracy, in particular equality of opportunity concept of Roemer (1998). Taking into account recent proof Benabou(2000) that meritocracy contributes positively to growth, which break the conventional trade off between equity and efficiency, the theorem provides for their connection with the theory of inequality measurement. If an index is to be both an inequality index and meritocratic it has to be of a form given in our theorem. We then propose a two-dimensional measure of meritocratic inequality index and discuss its advantages over standard Gini index and in reflecting better the nature of inequality in a society.

JEL classification: D63, D71

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1 Introduction

The aim of this paper is to provide a formal structure for judging on what normative grounds equality of income is justified. In economic literature it has been often argued that there is a trade off between equity and efficiency. The problem is also a hotly debated issue in public discourse and often a reason for unwillingness of richer classes to transfer resources to the allegedly lazy ones. Indeed, a society usually will not consider as fair a transfer from a hard-working richer person to a lazy poor if the latter lacks

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only willingness, not possibilities, to earn more. However, as Benabou (2000) shows, the conventional trade off between equity and efficiency is no longer valid in case when meritocracy (which covers both equality of opportunity and inequality of outcomes) is taken into account.[1] Whereas economists often argue that equity does not consider the so called "size of the cake", the critique does not apply to meritocracy. In Benabou's dynamic heterogeneous agent economy setting, equalizing the young's opportunities for human-capital investment enhances both social mobility and the growth of aggregate output. Both dimensions of meritocracy contribute positively to growth.

Inequality measurement theory seems to exist in isolation of these results and modern social justice theory concepts. These links need to be established as inequality measurement is largely considered a settled issue and its applications are vast. First, this is the aim of this paper to provide for such. Secondly, a unified framework of both theories would, on one hand, enrich normative content of inequality indices and allow for a more precise picture of the nature of inequality in a society[2], on the other hand, it will support meritocracy concept with consistent inequality measures, that is, measures that are bound to fulfill standard inequality measurement axioms.

We develop a framework that unifies the theory of inequality measurement and meritocracy paradigm by taking axiomatic approach. Before we proceed to explain this, we need a definition of meritocracy we will refer to. We employ a two-dimensional definition of meritocracy similar to Benabou's (2000), however we add a significant interpretative change to Benabou's understanding of equality of opportunity (EOP). It now reflects to what extent not only talent and market luck, but also effort is a determinant of income relative to background. This is much closer to the very concept of equalizing opportunities as proposed originally by Roemer (1998). Notice that had we done it Benabou's way, there would virtually be no room for people's own decisions on how hard to try to achieve outcomes. For talent as well as market luck are given. Effort reinforces talent and all what constitutes person's merits. Therefore we define meritocracy as consisting of two main concepts: *Equality of opportunity* (EOP): the extent to which person's effort, talent or market luck rather than background is a determinant of income or rewards *Inequality of outcomes*: the extent to which effort or talent is rewarded in a society. Rewarding effort is considered fair as effort is a person's choice, thus inequality of outcomes is also a dimension of meritocracy. A person should not be rewarded or punished for a set of circumstances that were beyond his or her control and this is formulated as EOP.

[1] This will be defined later in the text.

[2] In particular, the meritocratic index of inequality should be sensitive to the kind of transfers we described earlier in a section.

In accordance with Roemer's method, such defined meritocracy involves categorizing people based on a set of circumstances and then comparing effort distribution[3] in each percentile as we consider percentile to be an accurate intertype-comparable measure of effort.

As it has been already stated, in order to link two theories we employ a general axiomatic approach. Standard properties of inequality indices are redefined in a way that incorporates group categorizing and group comparison characteristic to meritocracy concept as this is our main interest. However, since the approach is general it allows for associating inequality measurement theory with other social choice theory concepts, which have the similar underlying logical structure. We call redefined properties *structural* as they relate to the group structure characteristic of meritocracy framework.

After redefining the axioms we establish a theorem saying that structural inequality properties are met if and only if the overall inequality index is an aggregation function of the indices in groups. Structural inequality index, hence its specific example we call meritocratic inequality index, are inherently group decomposable, which is a desirable characteristic of an inequality index. Since we use axioms that are standard for inequality indices and they were redefined to ensure consistency with meritocracy, we consider the result as minimal and fundamental for the unified framework we aimed to develop. This is the main result of the article. It says that if you want an inequality index to have a coherent meritocratic content and meaning, its structure has to be the same as imposed by the theorem.

Based on the result, we propose an inequality measure that is very intuitive for both equality of opportunity and inequality of outcomes. We investigate its properties against standard inequality indices. To make it normatively significant we construct a social welfare function (SWF) connected with it. The embedded implicit value judgements are quite similar to Gini's SWF, but in our measure even greater welfare weight is associated with the income of a poorer individual.

This article is organized as follows. In Section 1 standard axioms of inequality theory are redefined and explained. Section 2 includes main result, that is, a theorem linking two above mentioned theories, we call it A Theorem on Structural Inequality Indices. In Section 3 we define a specific meritocratic inequality index and investigate its properties against Gini index. Four examples are studied, two show consistency of our measure with Gini and the other two do not. We argue that in cases of inconsistency Gini is to a large part irrelevant in picturing inequalities whereas meritocratic inequality index is not. We

[3]For those not familiar with Roemer's method, the procedure leads to justifying outcomes, that is, usually income distributions, as reflecting effort only.

also construct SWF and interpret value judgements involved. Section 4 concludes.

2 Properties of structural inequality indices

2.1 Standard axioms

Standard properties of inequality indices are the following (see e.g.: Dutta (2002)) :

- An *index of inequality* is a function $I : \mathbb{R}^n_+ \mapsto \mathbb{R}$, where $\mathbb{R}^n_+$ stands for a domain, a set of income distributions corresponding to a population of size n. I is a continuous and strictly $S-convex$ function. [4]

- Lorenz dominance: Let L denote *Lorenz curve*, that is, $L(x,p)$ is the income accruing to the $100l\%$ poorest individuals in $x \in \mathbb{R}^n_+$ for $l \in (0,1)$. For $x, y \in \mathbb{R}^n_+$, we say x *Lorenz dominates* y and denote it by xLy, if $L(x,l) \geqslant L(y,l)$ for all $l \in (0,1)$, with strict inequality for some l.

- Pigou-Dalton Transfer: Given $x, y \in \mathbb{R}^n_+$, x is obtained from y by a *progressive transfer* if $x - y = \delta(e_i - e_j)$ for $\delta > 0$, and $y_j > x_i > y_i$, where e_i denotes the i-th standard basis vector. Inequality index satisfies the *Pigou-Dalton Transfer principle* if for $x, y \in \mathbb{R}^n_+$, $I(x) < I(y)$, whenever x is obtained from y by means of a progressive transfer.

- Symmetry: Inequality index satisfies *Symmetry principle* if $I(x) = I(\pi y)$ for all x and all permutation matrices π.

- Ratio Scale Invariance: Index of inequality satisfies *Ratio Scale Invariance* if $I(x) = I(\lambda x)$ for all $\lambda > 0$

It is well known that symmetry and Pigou-Dalton Transfer are together equivalent to strict S-convexity (Sen, Foster (1997)).

2.2 Standard axioms redefined

Before we proceed with redefining the above described standard axioms of inequality indices we explain why these axioms are not suitable for our unified framework.

Let us consider S-convexity first. In order to be consistent with meritocracy concept, in particular its EOP part in accordance with Roemer, we partition people in groups based on a set of characteristics. To keep the discussion simple, let us assume there is just one

[4] A function $f : \mathbb{R}^n_+ \mapsto \mathbb{R}$ is $S-convex$ if and only if $f(Qy) \leqslant f(y)$ for all $y \in \mathbb{R}^n_+$ and all $n \times n$ bistochastic matrices Q. Strict convexity requires inequality whenever Qy is not a permutation of y.

circumstance beyond people's control which determines incomes, namely sex. Assume also that in a society we consider, being a men raises chances of higher future wages, which is a quite realistic assumption anyway. Then if we multiply an income distribution by a bi-stochastic matrix[5], this may involve transfers from women who tried hard to earn a lot (using our intertype-effort measure they are in a higher centile of the effort distribution) to men who did not and therefore earn less. We do not want then *meritocratic* index of inequality to show less inequalities in this society! Thus we need to develop a definition of S-convexity which excludes such transfers. We call it *structural S-convexity* since it is supposed to be relating to the underlying structure of group categorizing. This example is studied thoroughly in Section 3. It is pathological with respect to the Lorenz dominance too. We call a new criterion *structural Lorenz dominance*. Analysis concerning Pigou-Dalton Transfer and Symmetry follows the same lines and is left to the reader.

Last axiom, Ratio Scale Invariance is also irrelevant in meritocracy context. A proper meritocratic measure should be invariant with respect to group scaling. For instance, if we scale up wages of all women, the inequality measure should be invariant to scaling in this particular group, namely women. However, one can say, definitely women enjoy now better or less worse position than men. This can be reflected by other measures which are invariant to other group categorization e.g. percentiles instead of categorization based on sex.

With normative justification for rejection of standard definitions of axioms, we are now equipped well enough to construct new ones. These are given below.

Denote P a finite partition of X (usual conditions are fulfilled), x_p will denote a distribution of $x \in X$ corresponding to $p \in P$.

(S1) I is a *strictly structural S-convex* function compatible with P if

$$\forall_{x \in \mathbb{R}^n_+} I(Mx) \geqslant I(x), \tag{1}$$

for any stochastic matrix M such that $i \in p_1 \wedge j \in p_2 \wedge p_1 \neq p_2 \implies M_{ij} = 0$ and the inequality is sharp if M is not a permutation matrix.

[5]There is a bi-stochasting matrix in the definition of S-convexity.

(S2) I fulfills *structural ratio scale invariance* compatible with P if

$$I(y) = I(x), \tag{2}$$

whenever for all $p \in P$ there exist $\lambda_p > 0$ such that $y_p = \lambda_p x_p$.

Next axiom is not a standard property of inequality axioms. It is added for technical reason to exclude pathological cases when ordering of income distributions in one group would inevitably impose ordering in others.[6] This axiom is independent of other axioms.

(S3) I fulfills *structural consistency* compatible with P if for any $p \in P$

$$I(x) \leqslant I(\tilde{x}) \Leftrightarrow I(y) \leqslant I(\tilde{y}), \tag{3}$$

whenever $x_p = y_p$, $\tilde{x}_p = \tilde{y}_p$ and for all $q \neq p$ $x_p = \tilde{x}_p$ and $y_p = \tilde{y}_p$

Below redefined are next two standard axioms of inequality measurement.

(S4) I fulfills *structural symmetry* compatible with P if

$$\forall_x I(Hx) = I(x), \tag{4}$$

for any permutation matrix H such that $i \in p_1 \wedge j \in p_2 \wedge p_1 \neq p_2 \implies H_{ij} = 0$.

(S5) We say that x *Lorenz dominate in a structural sense* compatible with P y and denote it by $yL_s x$ if

$$\forall_{p \in P} y_p L x_p, \tag{5}$$

where L denotes the standard Lorentz dominance.

(S6) We say that x was obtained from y by means of a *structural progressive transfer* compatible with P if for all $p \in P$, x_p can be obtained by means of a progressive transfer from y_p.

I fulfills *structural Pigou-Dalton transfer* principle compatible with P if for $I(x) < I(y)$ whenever x was obtained from y by a structural progressive transfer.

3 Theorem on Structural Inequality Indices

The first three of redefined axioms impose a specific form on inequality indexes, for which reason we call these indices *structural*.

[6] Such cases of contradictory orderings are often treated and resolved in social choice theory, however this is not of our prime interest here.

Theorem 3.1 (On Structural Inequality Indices). *Let $P = \{p_1, p_2, \ldots, p_n\}$. Assume that a continuous function I fulfills (S1),(S2),(S3) then I can be represented as*

$$I(x) = f\left(i_{p_1}(x_{p_1}), i_{p_2}(x_{p_2}), \ldots, i_{p_n}(x_{p_n})\right), \tag{6}$$

where i_{p_k} are ratio scale invariant indices and f is a strictly increasing function with respect to each coordinate. i_{p_k} provided by such decomposition are unique up to an increasing transformation.
Conversely, if I is of the form (6) *then it fulfils (S1),(S2),(S3).*

Proof. Fix $z \in \mathbb{R}^n_+$ and for $p \in P$ define

$$i_p(y) = I(z|_p y), \tag{7}$$

where $z|_p y$ is z with p part replaced by y. By assumptions (S1),(S2) it is straightforward to check that i_p is a ratio scale invariant index. From assumption (S3) it follows that the indices $\hat{i}_p$'s defined by the above procedure for any reference vector $\hat{z} \neq \tilde{x}$ are consistent with i_p's in a sense that each of them is an increasing transformation of the corresponding i_p.
Assumption (S3) ensures that for i_p's defined above

$$f\left(i_{p_1}(x_{p_1}), i_{p_2}(x_{p_2}), \ldots, i_{p_n}(x_{p_n})\right) = I(x) \tag{8}$$

is a valid definition of f on the domain $D = Im(i_{p_1}) \times Im(i_{p_2}) \times \ldots \times Im(i_{p_n})$ (where Im denoted the image of a function). This will be shown once we prove that for any a, b such that $\forall_{p_k \in P} \wedge i_{p_k}(a_{p_k}) = i_{p_k}(b_{p_k})$ we have $I(a) = I(b)$. Denote $x := z|_{p_1} a_{p_1}$, $\tilde{x} := z|_{p_1} b_{p_1}$, $y := a$, $\tilde{y} := a|_{p_1} b_{p_1}$. Definition of i_{p_1} implies that $I(x) = I(\tilde{x})$ and consequently by (S3) $I(y) = I(\tilde{y})$. Next we define a sequence $\tilde{y}_k := \tilde{y}|_{p_k} b_{p_k}$ taking for $\tilde{y}_1 = \tilde{y}$. (S3) implies that $I(\tilde{y}_{k+1}) = I(\tilde{y}_k)$, notice that $\tilde{y}_n = b$, hence finally $I(a) = I(b)$.
Assume that we have another decomposition

$$I(x) = g\left(j_{p_1}(x_{p_1}), j_{p_2}(x_{p_2}), \ldots, j_{p_n}(x_{p_n})\right)$$

consider i_{p_1} and j_{p_1}. By assumption that f, g are strictly increasing for any x, y we have $i_{p_1}(x) \geqslant i_{p_1}(y) \Leftrightarrow j_{p_1}(x) \geqslant j_{p_1}(y)$. Hence $h(j_{p_1}) = i_{p_1}$ is a valid definition of an increasing function.
The converse part is obvious and left to the reader. ☐

The theorem establishes an elegant characterization and representation of a function that is both an inequality index and is meritocratic. The application of a meritocratic index to inequality measurement broadens the knowledge about the nature of inequality in a society, since the index also uses the information about inequalities within and between groups that are categorized in accordance with some normatively significant concepts, in our case it is meritocracy. Axiom (S6) is fulfilled by structural S-convexity and axiom (S5) by Atkinson's theorem (Atkinson (1970)).

4 Meritocracy in opportunities and meritocracy in outcomes

4.1 Meritocratic inequality indices

Based on the result of Theorem on Structural Inequality Indices we propose a specific structural inequality index that reflects well the concept of meritocracy as described in the Introduction. We employ a two-dimensional measure, each dimension representing one part of the definition of meritocracy. Following Benabou (2000), we call them *meritocracy in opportunities* and *meritocracy in outcomes.*

The first definition concerns equality of opportunities. In accordance with John Roemer's method we categorize groups based on a set of circumstances, which a society views as being beyond people's control, a sort of background characteristics. Based on EOP literature we claim that equalizing opportunities means that people who tried the same should be treated the same. We are interested with a measure which judges a degree of equalizing opportunities in a society. We call this measure *meritocracy in opportunities* index.

In below by $G(x)$ we will denote the standard Gini index of income distribution x.

Defintion 4.1. ***Meritocracy in opportunities*** *Let $x \in \mathbb{R}^n$ denotes income distribution in a society partitioned according to P. Let $y_p = \{q_p^i : i \in P\}$ where q_p^i is p-th percentile in i-th group of P. M^{opp} denotes meritocracy in opportunities index and is defined by*

$$M^{opp}(x) := \sqrt{\int_0^1 G(y_p)^2 \mathrm{d}p}. \tag{9}$$

Remark 4.1. The equation (9) is not very handy in applications, hence we will use

$$M^{opp}(x) := \sqrt{\frac{\Sigma_{p=1}^{100} G(y_p)^2}{100}}. \tag{10}$$

which well approximates (9).

The second definition concerns inequality of outcomes. This property is desirable only if it rewards effort by strengthening incentives. For this to hold we should concentrate on groups categorized according to background characteristics and then construct an index that measures to what extent society encourages effort, or in other words, rewards effort. We call such a maesure *meritocracy in outcomes* index.

Defintion 4.2. ***Meritocracy in outcomes*** *Let $x \in \mathbb{R}^n$ denotes income distribution in a society partitioned according to P. M^{out} denotes meritocracy in outcomes index and is defined by*

$$M^{out}(x) := \sqrt{\frac{\Sigma_{i \in P}(1 - G(x_i))^2}{|P|}}, \tag{11}$$

where x_i is the distribution of income x in group i and $|P|$ is number of groups.

4.2 Properties of meritocratic inequality indices

We will now study the behavior of our measures in relation to standard Gini index, as this should give us an intuition about M^{opp} and M^{out}. We will do this by considering examples in which our measures and standard Gini are compatible and in which they are not, justifying that using meritocratic index gives a better picture of the nature of inequalities in a society. Of course since M^{opp} and $1 - M^{out}$ are both of the form as a general structural inequality index (the aggregation function takes the form $f(G_1, G_2, ..., G_{100}) = \sqrt{\Sigma_{i=1}^{100}(G_i)^2}$ in case of M^{opp} and $f(G_1, G_2, ..., G_k) = 1 - \sqrt{\Sigma_{i=1}^{k}(1 - G_i)^2}$ in case of $1 - M^{out}$) they fulfill the structural standard axioms.

Example 1 Consider two societies with two groups and the following distribution of incomes:

- Society 1: men - $\mathcal{N}(150, 10)$; women - $\mathcal{N}(100, 10)$
  Then $G = 0.107$; $M^{opp} = 0.101$; $M^{out} = 0.985$
- Society 2: men - $\mathcal{N}(170, 10)$; women - $\mathcal{N}(120, 10)$
  Then $G = 0.092$; $M^{opp} = 0.087$; $M^{out} = 0.987$

Here, M^{opp} is consistent with Gini index, which is not surprising as in Society 2 both groups have higher incomes on average and the variance is preserved. In each percentile the distribution then did not change, up to the mean. As to M^{out}, in each group inequalities are lower, which since we excluded background determinants of income, means effort is rewarded less now. This makes M^{out} go up and it is reasonable not to consider less inequalities better as it distorts people's incentives.

Example 2 Consider two societies with two groups and the following distribution of incomes:

- Society 1: men - $\mathcal{N}(150, 10)$; women - $\mathcal{N}(100, 10)$
  Then $G = 0.107$; $M^{opp} = 0.101$; $M^{out} = 0.985$
- Society 2: men - $\mathcal{N}(140, 10)$; women - $\mathcal{N}(110, 10)$
  Then $G = 0.067$; $M^{opp} = 0.067$; $M^{out} = 0.986$

Here, M^{opp} and Gini are lower than in Example 1, because we make two groups less distant and this lowers M^{opp}.

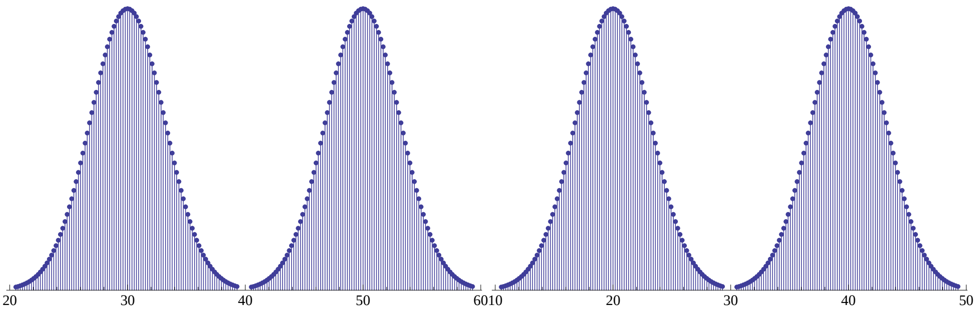
Example 3 Consider two societies with two groups and the following distribution of incomes:

- Society 1: men - $\mathcal{N}(200, 10)$; women - $\mathcal{N}(180, 10)$
  Then $G = 0.0309$; $M^{opp} = 0.0270$; $M^{out} = 0.990$
- Society 2: men - $\mathcal{N}(200, 100)$; women - $\mathcal{N}(190, 100)$
  Then $G = 0.0321$; $M^{opp} = 0.0151$; $M^{out} = 0.971$

Income variance changed a lot in both groups and this makes standard Gini go up, though the income averages in both groups are now closer. The latter effect is reflected by M^{opp}, which decreases. Thanks to using M^{opp}, we can spot otherwise unobservable phenomenon that apart from large variance, differences in outcomes between people who try the same lowered. Here we can see best that picture of inequalities as measured by Gini only is incomplete.

Example 4 Now we consider an example similar to the one described in Section 2. We have two societies with two groups: women and men and assume being a men raises chance of higher earnings. Now we transfer income from hard-working women and therefore earning more (mean 40) to lazy men (mean 30). This is shown below in pictures 1 and 2. Gini index decreased as we transferred money from rich to poor, so the transfer is progressive, in response to what Gini has to go down. However, based on our meritocratic values, we do not consider such a transfer fair! And this exactly what increasing M^{opp} informs us about.

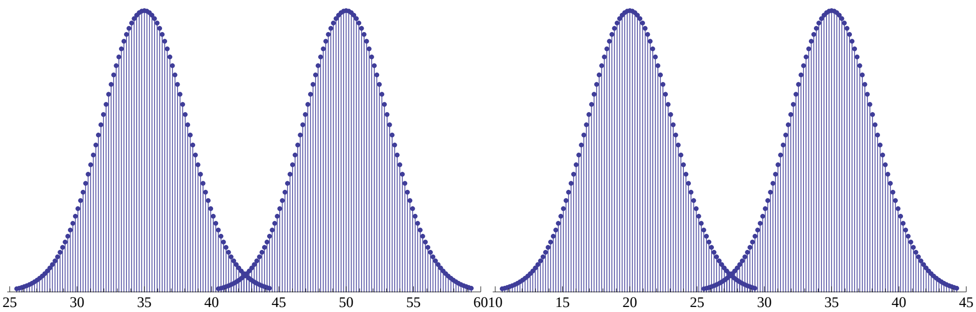
Figure 1: Distribution of income before transfer



(a) Men (b) Wemen

Then $G = 0.191$; $M^{opp} = 0.0907$; $M^{out} = 0.829$

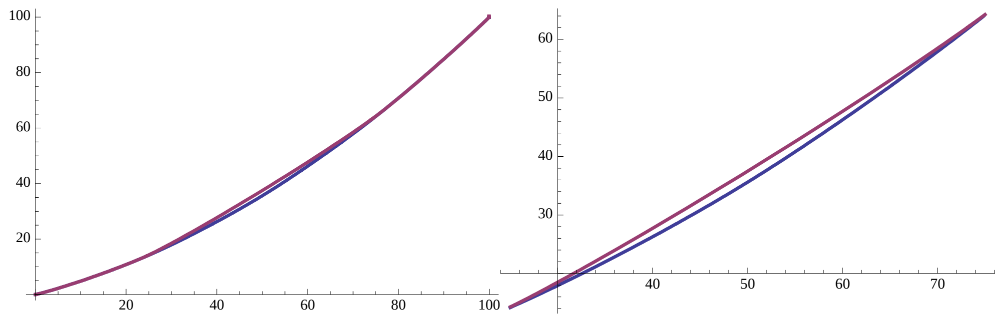
Figure 2: Distribution of income after transfer



(a) Men (b) Wemen

Then $G = 0.1796$; $M^{opp} = 0.124$; $M^{out} = 0.861$

Figure 3: Lorenz curve (blue - before transfer, purple - after transfer)



4.3 Social Welfare Function of meritocratic inequality indices

Both M^{opp} and M^{out} are normatively significant, in a sense that whenever each of them shows higher/lower values, social welfare function shows also higher/lower values. SWF connected with our measures is of the form:[7]

$$W(x) = \Phi((1 - M^{opp}(x))\mu(x)), \tag{12}$$

where Φ is an arbitrary increasing transformation, $\mu(x)$ is the mean of the income distribution x and for M^{opp} the formula is identical with one exception. Instead of M^{out}, we have M^{opp} in equation and Φ is a decreasing transformation. It is difficult to find a clear-cut interpretation of this specific SWF, however some value judgements can be traced. It is known Gini index is connected with a kind of SWF that associate greater welfare weights with poorer individuals. Since we take a quadratic transformations of Gini's in groups this will "favor" groups with larger differentiation as compared to standard Gini measure. In case of M^{opp} we can conclude that it is particularly sensitive to groups where inequalities are large comparing to others. With M^{out} the logic is reversed, it will additionally weigh down poorer individuals. This is a reasonable interpretation as they are considered as having abilities and opportunities, but unwilling to work.

5 Conclusions

The trade off between equity and efficiency is one of the cornerstones of economic literature. The recent advancements such as meritocracy and, in particular, equality of opportunity frameworks, break this traditionally unresolved tie, proving that redistribution of incomes is often efficient. On the other hand, the measurement of inequality exists out of touch with modern social justice theory. In this paper we established a theorem which imposes a functional form on inequality indices which are to be meritocratic, but can cover other concepts as well provided that their logical structure is similar to meritocracy concept. The theorem founds two theories, meritocracy and inequality measurement, with basic interrelations. Based on the theorem, we propose a two-dimensional meritocratic inequality index and and present its advantages over Gini index. Value judgements embedded in our measure are close to Gini's, but the more underlie inequalities the larger they are. Meritocratic inequality index should be further studied in relation to other standard indices.

[7] This characterization is easy to derive with very well know formula for Atkinson-Kolm-Sen index (Dutta (2002)).

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