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Abstract

In a Yaari-Blanchard overlapping generations endowment economy, and drawing on the equivalence between Radner (R) and Arrow-Debreu (AD) equilibria, we prove that equilibrium AD prices have an explicit representation as a double integral equation. This allows for an analytic characterization of the relationship between life-cycle and cohort heterogeneity and asset prices. For a simple distribution, we prove that bubbles may exist, and derive conditions for ruling them out.

Keywords: overlapping generations, asset pricing, bubbles, integral equations, LambertW function.

JEL classification: D51, G12, J0.

1 Introduction

The main difficulty in modelling overlapping generation equilibrium economies are related to the consistent modelling of the dimensions of the income distribution, along the life-cycle, along age-income profiles for every moment in time, and between different cohorts. This poses difficulties in defining a consistent Arrow-Debreu equilibrium, as was already noted by Shell (1971).

In the continuous time overlapping generations literature there are two main strands of models, following Yaari (1965)- Blanchard (1985) uncertain lifetime model, and Cass and Yaari (1967) finite lifetime model.

As in most continuous time OLG literature, we follow Yaari (1965)- Blanchard (1985) framework. It solves the consistency problem by dealing with a Radner, or sequential market equilibrium, economy, and by assuming a particular demographic structure. The consequence of this is that the age-structure of endowments has no effect on asset prices. We try to overcome this difficulty by generalising demographics, though still assuming a constant population, and by drawing on the equivalence between Radner (R) equilibria with complete asset markets and Arrow-Debreu (AD) equilibria.

In both R and AD economies agents are heterogeneous by age, and perform intertemporal allocations of resources by trading with members of other generations. In a R economy there is a continuum of sequential spot asset markets, and in an AD economy there is a system of simultaneous forward real markets. In the last economy, consistency between contracts available to different cohorts are achieved by assuming that all prices are formed at an "Archimedian" time $t = 0$ [1]. We prove that the general equilibria in both economies are equivalent. We also prove that equilibrium AD prices have a representation as an integral equation. By assuming a simple particular case of heterogeneous age-distribution of endowments, where all endowments are distributed at a single age α, we prove that equilibrium prices exist, but may display rational bubbles depending on the relationship of α and a critical value which depends on the elasticity of intertemporal substitution [2]

Finite lifetime models of the Cass and Yaari (1967) contribution have been recently revived. For example, Demichelis and Polemarchakis (2007) present an Arrow-Debreu (AD) endowment equilibrium with a structure similar to the one presented here, but within the finite lifetime framework. Though we reach a similar solution for AD prices, our approach allows for a simpler characterization of equilibrium prices, and, in particular for an explicit derivation of conditions for existence of speculative bubbles.

In section 2 we present the R and the AD equilibria and proves their equivalence, in section 3 we prove the existence of AD prices, and in section 4 we characterise AD prices and present conditions for ruling out bubbles.

2 Radner and Arrow-Debreu equilibria

Consider an overlapping-generations (OLG) economy with an age-structured demography and a constant aggregate population. At each point in time people are

[1] See Geanakoplos and Polemarchakis (1991) and Geanakoplos (2008).

[2] We deal with CRRA utility. For logarithmic utility Brito and Dilão (2006) proves that this approch can be extended to realistic Mincerian distribution functions.

distributed among homogeneous cohorts. There is inter-cohort heterogeneity because the representative members of different cohorts are in different phases of their life-cycles. The dimension of a cohort t_0 at time $t = t_0 + a$ is denoted by $n(a, t) = n(a, t_0 + a)$, where t_0 is the time of birth, and $a \in [0, \infty)$ is the age of the representative member of a cohort. In a closed economy, the dimension of cohort t_0 is $n(0, t_0)$ at the time of birth, and decays along the rest of the life-cycle by mortality. Then $n(a, t) = n(0, t - a)e^{-\int_0^a \mu(s,t)ds}$, where $\mu(a, t) > 0$, is the instantaneous probability of death for people with age a at time t. Total population at time t is $N(t) = \int_0^\infty n(a, t)da$.

We assume that both the number of new-borns and the probability of death are time-independent, $n(0, t_0) = n_0$ and $\mu(a) > 0$. However, we do not introduce the Blanchard (1985) simplifying assumptions: $\mu(s) = \mu = 1/n_0$ constant.

The previous assumptions have two implications: First, they imply that the decay factor of a cohort along the lifetime, $\pi(a) \equiv e^{-\int_0^a \mu(s)ds}$, only depends on age. Given intra-cohort homogeneity then $\pi(a)$ also represents the probability of survival at age a for individual agents. Second, they imply that $n(a, t) = n_0\pi(a)$ is time-independent but age-dependent, and the aggregate population $n_0 \int_0^\infty \pi(a)da = N$ is constant.

We hypothesise a single-good endowment economy, in which the representative member of cohort t_0 is entitled to the lifetime exogenous stream of endowments $y(t_0) = \{y(a, t_0 + a), a \in \mathbf{R}_+\}$, and there are no intergenerational transfers (in particular, no bequests). Therefore, the total endowment for the economy at time t, $Y(t) = \int_0^\infty n(a, t)y(a, t)da$, is also exogenous.

The representative member of cohort t_0 has preferences of the Yaari (1965)-Blanchard (1985) type: she/he has an uncertain lifetime, has an instantaneous probability of survival at age a equal to $\pi(a)$, and has an additive expected lifetime utility functional, over the lifetime path of consumption $c(t_0) = \{c(a, t_0 + a), a \in \mathbf{R}_+\}$,

$$U[c(t_0)] = E_{t_0}\left[\int_0^\infty u(c(a, t_0 + a))R(a)da\right] = \int_0^\infty u(c(a, t_0 + a))R(a)\pi(a)da, \quad (1)$$

where $R(a) \equiv e^{-\int_0^a \rho(s)ds}$ is the psychological discount factor with $\rho(t) > 0$ for all $t \geq 0$.

To complete the characterization of the economy we need to specify the structure of markets allowing for the intertemporal allocation of resources. We consider two alternative structures of markets leading to two alternative but equivalent economies: Radner and Arrow-Debreu.

In a Radner (R) economy there are financial assets, and the institutional structure is characterised by the existence of a sequence of spot markets for the good and for financial assets. In addition to a financial asset, paying a rate of return r, there is also a Yaari (1965) insurance market.

Let $w(a, t)$ denote the real stock of financial wealth for an agent with age a at time $t = t_0 + a$, using the spot price of the good as a *numéraire*. The instantaneous budget constraint is given by the partial differential equation along a characteristic,

$$\frac{\partial w}{\partial a} + \frac{\partial w}{\partial t} = z(a, t) + (r(t) + \mu(a))w(a, t), \quad (2)$$

where the excess of the endowment over consumption, $z(a, t) \equiv y(a, t) - c(a, t)$, and the rate of return $r(t)$ are perfectly anticipated. The no bequests assumption imposes the following boundary constraints

$$w(0, t_0) = \lim_{a \to \infty} \pi(a) w(a, t_0 + a) e^{-\int_0^a r(t_0+s)ds} = 0. \tag{3}$$

Definition: *The General Equilibrium for the OLG Radner economy is defined by the pair of densities and trajectories* $(c^*, r^*) = \left(\{c^*(a,t),\ (a,t) \in \mathbf{R}_+^2\}, \{r^*(t), t \in \mathbf{R}_+\}\right)$, *such that* $\{c^*(a, t_0 + a) :\ a \in \mathbf{R}_+\}$ *maximises the intertemporal utility function (1) subject to restrictions (2) and (3), for all cohorts* $t_0 \in \mathbf{R}_+$, *and the rate of return on the financial asset,* $r^*(t)$, *clears the good market for every* $t \in \mathbf{R}_+$.

The equilibrium condition is

$$\int_0^\infty c^*(a,t) n(a,t) da = \int_0^\infty y(a,t) n(a,t) da. \tag{4}$$

In an Arrow-Debreu (AD) economy there are, instead, forwards real markets in which contracts for future delivery of the good are traded. In particular, agents belonging to cohort t_0 perform contracts, at the time of birth, for delivery of the good at every future moment along their lifetimes, such that the intertemporal budget constraint holds

$$\int_0^\infty \pi(a) p(t_0, t_0 + a) z(a, t_0 + a) da = 0, \tag{5}$$

where $p(t_0, t_0 + a)$ is the price in the market operating at time t_0 for delivery of the good at time $t = t_0 + a$, for $a \in [0, \infty)$. In this AD economy, $z(a, t_0 + a)$ is the net supply of the representative agent of cohort t_0 in the market for delivery at time $t = t_0 + a$.

In a non-OLG AD economy one would assume that all markets would operate simultaneously at time $t = 0$, that is at the moment of birth of the representative dynasty, and there would be an infinite number of markets for delivery in every future date.

In our OLG AD economy there is a well know consistency problem, between the simultaneity of the operation of the forward markets and the need for agents to contract for all delivery of goods on all future moments along their lifetimes. If contracts are made at the time of birth, the simultaneity of operation of markets at a single point in time will not be possible because cohorts are continuously being born. This problem has been solved in the literature (see Geanakoplos and Polemarchakis (1991) and Geanakoplos (2008), that attributes this idea to I. Fisher) by assuming that prices faced by every cohort are consistent to those set at an "Archimedian time" $t = 0$. At this date all prices $p(0, t) = p(t)$, for $t \in \mathbf{R}_+$ are determined. Then, prices available to cohort t_0 should verify $p(t_0, t) = p(t)/p(t_0)$, for $t \geq t_0$.

Definition: *The General Equilibrium for the OLG Arrow-Debreu economy, in which all markets open only at time $t = 0$, is defined by the pair $(c^*, p^*) = \left(\{c^*(a, t),\ (a, t) \in \mathbf{R}_+^2\}, \{p^*(t), t \in \mathbf{R}_+\}\right)$, where $\{c^*(a, t_0 + a) :\ a \in \mathbf{R}_+\}$ maximises the intertemporal utility function (1) subject to restriction (5), for all cohorts $t_0 \in \mathbf{R}_+$, where $p^*(t_0, t) = p^*(t)/p^*(t_0)$, and the AD price $p^*(t)$ clears the market for every future $t \in \mathbf{R}_+$.*

Note that the equilibrium condition is formally equation (4).

Proposition 1 *Let*

$$p(t) = e^{-\int_0^t r(s)ds}, \; t \in \mathbf{R}_+. \tag{6}$$

then the OLG-Radner and the OLG-Arrow-Debreu equilibria are equivalent.

Proof The solution of equation (2) with conditions (3) is the intertemporal budget constraint $\int_0^\infty e^{-\int_0^a r(t_0+s)ds}\pi(a)z(a,t_0+a) = 0$. This constraint is equivalent to the constraint for the representative member of a cohort in the AD economy, (5), if and only if $p(t_0, t_0+a) = e^{-\int_0^a r(t_0+s)ds}$ or $p(t_0,t) = e^{-\int_0^{t-t_0} r(t_0+s)ds} = e^{-\int_{t_0}^t r(s)ds}$, or $p(t)/p(t_0) = e^{-\int_0^t r(s)ds}e^{\int_{t_0}^0 r(s)ds}$. Then, the problems for the representative members of cohorts t_0 are equivalent in both R and AD economies. As the market equilibrium conditions are formally identical (see equation (4)) for every $t \in \mathbf{R}_+$, then the two equilibria are equivalent. $\square$

3 Determination of equilibrium AD prices

As $r(t) = -d\ln p(t)/dt$ for any $t \in \mathbf{R}_+$, from equation (6), we can establish existence, uniqueness, and characterise the OLG-R equilibrium from the equivalent OLG-AD equilibrium. Brito and Dilão (2006) study a similar OLG-AD equilibrium but assume a logarithmic utility function and derive a representation of the equilibrium AD prices as a linear double integral equation. Here, we assume an isoelastic utility function and get a representation of AD prices as a non-linear double integral equation.

Proposition 2 *Let the Bernoulli utility function be* $u(c) = (1-\sigma)^{-1}c^{1-\sigma}$*, where* $\sigma > 0$*. Then equilibrium AD prices,* $p(t)$ *for* $t \in \mathbf{R}_+$*, follow the non-linear double* *integral equation*

$$p(t)^{1/\sigma} = \frac{\int_0^\infty K(t-a)R(a)^{1/\sigma}\pi(a)da}{\int_0^\infty y(a,t)\pi(a)da},\ t \in \mathbf{R}_+ \tag{7}$$

where

$$K(t-a) \equiv \frac{\int_0^\infty y(s, t-a+s)p(t-a+s)\pi(s)ds}{\int_0^\infty p(t-a+s)^{(\sigma-1)/\sigma} R(s)^{1/\sigma}\pi(s)ds}.$$

Proof The optimal consumption for cohort t_0 in the OLG-AD economy, when its representative member has age a, is

$$c^*(a, t_0+a) = \left(\frac{R(a)}{p(t_0,t)}\right)^{1/\sigma} \left(\frac{\int_0^\infty p(t_0, t_0+s)y(s, t_0+s)\pi(s)ds}{\int_0^\infty p(t_0, t_0+s)^{(\sigma-1)/\sigma} R(s)^{1/\sigma}\pi(s)ds}\right),\ a \in \mathbf{R}_+$$

If we substitute $t_0 = t - a$, aggregate across all cohorts, impose the consistency condition $p(\tau_0)p(\tau_0, \tau_1) = p(\tau_1)$, and substitute into the equilibrium condition, $C(t) = Y(t)$, then we get equation (7). □

In order to solve the integral equation (7) we consider the case of a balanced growth path with a constant population. In particular, we assume that endowments are separable, but are age-structured, and that the other demographic and behavioural parameters are age-independent and constant.

Proposition 3 *Assume that the endowment density follows* $y(a,t) = \phi(a)e^{\gamma t}$*, where* $\phi(a) \neq 0 \in L^1(\mathbf{R}_+)$*, and* $\gamma \geq 0$*. Further assume* $\mu(a) = \mu > 0$*,* $\rho(a) = \rho > 0$*. Assume that the set* $X \equiv \{x:\ x+\eta > 0,\ \text{and } x(1-\sigma)+\eta > 0\}$ *is non-empty,* *where* $\eta \equiv \rho + \mu + (\sigma - 1)(\gamma + \mu)$. *Define*

$$S(x) \equiv \int_0^\infty \phi(a) e^{-\mu a} \left(1 - \frac{\eta + x(1-\sigma)}{\eta + x} e^{xa}\right) da. \tag{8}$$

Then, equation (7) has a solution of the form $p(t) = \sum_{j=1}^{n} k_j e^{(x_j - \gamma)t}$, *where* $x_j \in \{x \in X : S(x) = 0\}$, *for* $j = 1, \ldots, n$.

Proof As in Polyanin and Manzhirov (1998, p.325), and in Brito and Dilão (2006), let the general solution of equation (7) be of type $f(t) = ke^{(x-\gamma)t}$, where k is an arbitrary constant. If we substitute this candidate solution into equation (7) and assume that $x + \eta > 0$ and $x(1-\sigma) + \eta > 0$, we find it is equivalent to equation $S(x) = 0$, where $S(x)$ is in (7). □

We call characteristic equation to $S(x) = 0$. It does not have a closed form solution for any admissible value of the parameters. However, we can prove existence of a solution $x = 0 \in X$, with very mild conditions:

Proposition 4 *Assume that* $\phi(a) \neq 0$, *for all* $a \in \mathbf{R}_+$ *and that* $\sigma > \sigma_f \equiv \max\{0, (\gamma - \rho)/(\mu + \gamma)\}$. *Then the characteristic equation has at least one root* $x = 0$.

Proof If $\phi(a) \neq 0$, we see by simple inspection that $x = 0 \in X$ is a root of the $S(x) = 0$ only if $\eta > 0$. But $\eta > 0$ if and only if $\sigma > \sigma_f$. □

If $x = 0$ is the unique solution of the characteristic equation then the equilibrium AD price is $p(t) = ke^{-\gamma t}$, and the real interest rate is $r(t) = \gamma$, for all $t \in \mathbf{R}_+$, as in an analogous dynastic non-OLG model. Therefore, prices are asymptotically bounded, and there are no speculative bubbles.

However, the characteristic equation may have other non-zero roots, depending on the type of function $\phi(.)$. If there is no root larger than γ, then we would get $\lim_{t\to\infty} p(t) = 0$ and $\lim_{t\to\infty} r(t) > 0$: prices are bounded and the asymptotic interest rates are positive. If there is at least one root which is greater than γ then $\lim_{t\to\infty} p(t) = \infty$ and $\lim_{t\to\infty} r(t) < 0$, and we say that rational speculative bubbles would exist.

4 AD prices and age-heterogeneity of endowments

Our approach, differently from Blanchard (1985), allows for the characterization of the consequences of different age-distribution of incomes (or of age-dependent shocks upon it) on the dynamics of asset prices. In particular, as all roots different from zero depend on $\phi(a)$, we may relate the existence or not of speculative bubbles to the age-distribution of endowments.

Let us consider the simplest age-heterogeneous distribution such that endowments are only received by consumers with a specific age, $a = \alpha \geq 0$. Formally, $\phi(a) = \phi_0 \delta(a - \alpha)$ where $\delta(.)$ is Dirac's delta function.

At each moment in time, t, the age- distribution of endowments across ages is totally concentrated on consumer with age α. Therefore, the AD price $p(t)$ clears the market for time t, in which there is only one cohort in the supply side and all the other cohorts are in the demand side.

The characteristic equation is now

$$S(x, \alpha) \equiv \eta + x - (\eta + x(1 - \sigma))e^{\alpha x} = 0, \tag{9}$$

and still does not have a closed form solution. However, we can characterise solutions as far as the existence of bubbles is concerned:

Proposition 5 *Let $\sigma_f < \sigma \leq 1$, and $\alpha \geq \alpha_1$, or $1 < \sigma < \sigma_c$, and $\alpha \leq \alpha_2$, or $\sigma \geq \sigma_c$ and $\alpha \leq \alpha_1$, where*

$$\begin{aligned} \alpha_1 &\equiv \frac{1}{\gamma}\ln\left(\frac{\rho+\sigma(\mu+\gamma)}{\rho+\sigma\mu}\right), && (10) \\ \alpha_2 &\equiv \left\{\alpha:\ S\left(\frac{1}{\alpha}\left[W_0\left(\frac{e^{1+\alpha\eta/(1-\sigma)}}{1-\sigma}\right)-1+\frac{\alpha\eta}{1-\sigma}\right],\alpha\right)=0\right\} && (11) \\ \alpha_c &\equiv \frac{1}{\gamma}\left(\frac{\sigma-1}{\rho+\sigma\mu}+W_0\left(\frac{\gamma}{\rho+\sigma\mu}e^{\gamma(1-\sigma)/(\rho+\sigma\mu)}\right)\right), && (12) \\ \sigma_c &\equiv \{\sigma:\ S(\gamma,\alpha_c(\sigma))=0\} && (13) \end{aligned}$$

where W_0 is the principal branch of the Lambert-W function [3]. *Then, there are no speculative bubbles.*

Proof If $\sigma > \sigma_f$ then $\eta > 0$, and $g(x) = 0$ will only have roots in the interval $(-\eta, +\infty)$, if $\sigma_f < \sigma \leq 1$, or in the interval $(-\eta, \geq \eta/(\sigma-1))$, if $\sigma > 1$. As $\gamma < \eta/(\sigma-1))$ then roots $x > \gamma$ can exist. In the appendix we prove that if $\sigma_f < \sigma \leq 1$, and $\alpha \geq \alpha_1$, or $1 < \sigma < \sigma_c$, and $\alpha \leq \alpha_2$, or $\sigma \geq \sigma_c$ and $\alpha \leq \alpha_1$ then there are no roots $x > \gamma$. □

Figure 1 illustrates this result by presenting combinations of values for σ and α such that bubbles are ruled out, for feasible values of the parameters. We see that if $\sigma_f < \sigma \leq 1$ then endowment should be distributed after age α_1 which is a function not only of σ but also of the other parameters μ and γ. If $\sigma > 1$ it should be distributed before age α_2 or α_1, which also depend on σ, μ and γ.

[3] Corless et al. (1996).

In order to have an intuition for this result observe that the characteristic equation (9) can be written as $S(x) = \int_0^\infty \phi(a)n(a)s(x,a)da = 0$, where $s(x,a)$ is the density of net savings, which in this case is equivalent to the density of excess supply of AD contracts across all ages. Then $S(.)$ represents aggregate savings, or the aggregate excess supply, in every moment in time. Then, AD prices, or interest rates, are determined such that aggregate supply across all age profiles are zero at every moment in time. If $\phi(a)$ is a degenerate distribution then $S(x) = s(x,\alpha) = 0$ for any moment in time. Our previous result suggests that the existence of bubbles, that is values of $x > \gamma$ which clear aggregate savings, has not a single relationship with α. If the elasticity of intertemporal substitution is low (high) an increase (reduction) in α above a critical value will generate a reduction in net savings which have to be compensated by an increase in prices associated to all forward contracts to delivery in the future.

5 Conclusions

A simple generalization of the demography, and a slight change in methodology, allows us to represent Arrow-Debreu prices for a continuos time OLG economy as a integral equation. We show that this allows for the study of the effects of age-dependent endowments and demographics in asset prices. Bubbles can occur for particular profiles of the age-distribution of endowments and of the intertemporal elasticity of substitution.

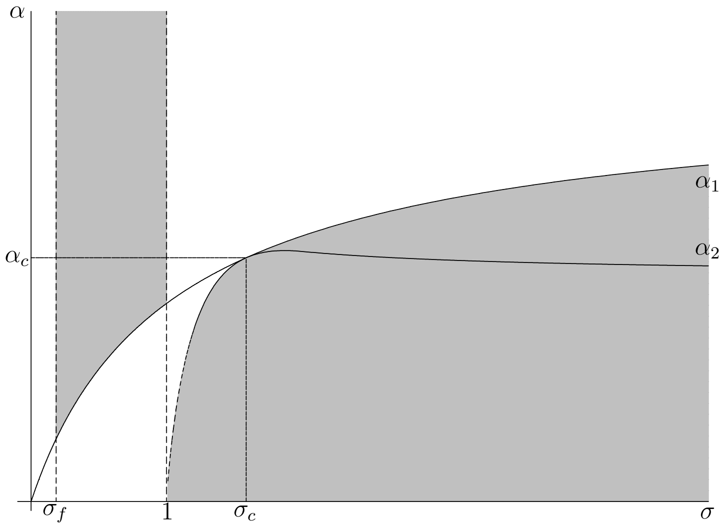

Figure 1: Shaded area: values of (σ, α) if $\gamma > \rho$ such that there are no speculative bubbles.

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Appendix

In this appendix more detailed versions of some proofs are presented.

Derivation of the instantaneous budget constraint for the OLG-R economy. The instantaneous budget constraint (2) is a first order partial differential equation over a characteristic, i.e., it is defined for values of (a, t) such that $da/dt = 1$. This is so because the cohort's 'time', a, is related to 'universal time', t, as $t = t_0 + a$, where t_0 is the date of birth of a cohort. The financial wealth of an agent belonging to cohort t_0 when it has age a at time t is denoted by $w(a, t) = w(a, t_0 + a)$. If we consider a small time interval $\Delta a = \varepsilon$ the flow budget constraint is,

$$w(a + \varepsilon, t_0 + a + \varepsilon) - w(a, t_0 + a) = \left[z(a, t_0 + a) + (r(t_0 + a) + \mu(a))w(a, t_0 + a)\right]\varepsilon.$$

This means that the variation in wealth is equal to the sum of the net inflow of the good, $z(a, t_0 + a) = y(a, t_0 + a) - c(a, t_0 + a)$, and with the income from asset holdings, resulting from the market interest rate and from the proceeds from the Yaari contracts. As

$$\lim_{\varepsilon \to 0} \frac{w(a + \varepsilon, t_0 + a + \varepsilon) - w(a, t_0 + a)}{\varepsilon} = \frac{dw(a, t_0 + a)}{da} = \frac{\partial w}{\partial a} + \frac{\partial w}{\partial t}\frac{dt}{da}$$

and $da = dt$, along a characteristic, then we get the flow budget constraint (2) for the Radner economy.

Proof of Proposition 1 Equation (2) is a linear first order partial differential equation, with has the general solution

$$w(a,t) = ke^{\int_0^a r(\tau+t_0)+\mu(\tau)d\tau} + \int_0^a e^{\int_s^a r(\tau+t_0)+\mu(\tau)d\tau} z(s, t_0+s)ds$$

where k is an arbitrary constant. If we set $a = 0$, and $t = t_0$, and use the first boundary condition (3), then we get $w(0,t_0) = k = 0$. If we substitute k, and multiply both terms by $e^{-\int_0^a r(\tau+t_0)+\mu(\tau)d\tau}$, and apply the second boundary condition (3), we finally obtain

$$\lim_{a\to\infty} w(a,t)e^{-\int_0^a r(\tau+t_0)+\mu(\tau)d\tau} = \int_0^\infty e^{-\int_0^s r(\tau+t_0)+\mu(\tau)d\tau} z(s, t_0+s)ds = 0.$$

This is the intertemporal constraint for cohort t_0 in the R economy. If is equivalent to the constraint at birth for a cohort living on the AD economy, (5), if and only if $p(t_0, t_0+a) = e^{-\int_0^a r(t_0+s)ds}$. As $p(t_0, t_0+a) = p(t_0, t) = e^{-\int_0^{t-t_0} r(t_0+s)ds} = e^{-\int_{t_0}^t r(s)ds}$, if prices available for cohort t_0 are consistent to the "Archimedian" prices, then we get equivalently $p(t) = p(t_0)p(t_0,t) = e^{-\int_0^{t_0} r(s)ds}e^{-\int_{t_0}^t r(s)ds} = e^{-\int_0^t r(s)ds}$. □

Proof of Proposition 2 To obtain a representation for the AD equilibrium in terms of the endogenous variables, $c(a,t)$ and $p(t)$, we solve the cohort's t_0 problem, aggregate consumption for all cohorts, and use the equilibrium conditions for the spot and all the forward markets for delivery at $(0,\infty)$ which open at $t = 0$. First, using the same method as in Brito and Dilão (2006)we set the Lagrangian for a representative member of cohort t_0,

$$L = \int_0^\infty \left((1-\sigma)^{-1}c(a,t_0+a)^{1-\sigma}R(a) + \lambda p(t_0,t_0+a)z(a,t_0+a)\right)\pi(a)da,$$

where $R(a) = e^{-\int_0^a \rho(s)ds}$ and $\pi(a) = e^{\int_0^a \mu(s)ds}$ and λ is a Lagrange multiplier. The first order conditions are (see Gelfand and Fomin (1963)) are

$$\begin{aligned} \frac{\delta L}{\delta c} &= \int_0^\infty \left(c(a,t_0+a)^{-\sigma}R(a) - \lambda p(t_0,t_0+a)\right)\pi(a)\psi(a)da = 0 && (14) \\ \frac{\partial L}{\partial \lambda} &= \int_0^\infty p(t_0,t_0+a)z(a,t_0+a)\pi(a)da = 0 && (15) \end{aligned}$$

where $\delta L/\delta c$ is the functional derivative for a perturbation $\psi(a) \in L^1(\mathbf{R}_+)$. As the first condition, (14), should hold for any perturbation, we get the optimal consumption for any moment along the lifetime of cohort t_0,

$$c^*(a,t_0+a) = \left(\frac{\lambda p(t_0,t_0+a)}{R(a)}\right)^{-1/\sigma}, \ a \in \mathbf{R}_+.$$

Condition (15) can be written equivalently as an equality between the mathematical expectation of the value of the lifetime optimal consumption, measured at the AD prices for t_0, and human wealth , at the moment of birth, $\tilde{c}(t_0) = h(t_0)$, where $\tilde{c}(t_0) \equiv \int_0^\infty p(t_0,t_0+a)c^*(a,t_0+a)\pi(a)da$ and $h(t_0) \equiv \int_0^\infty p(t_0,t_0+a)y(a,t_0+a)\pi(a)da$. We can write $\tilde{c}(t_0) = m(t_0)\lambda^{-1/\sigma}$, where $m(t_0) \equiv \int_0^\infty p(t_0,t_0+a)^{(\sigma-1)/\sigma}R(a)^{-1/\sigma}\pi(a)da$. Then, we get the Lagrange multiplier as $\lambda^* = \left(\frac{m(t_0)}{h(t_0)}\right)^\sigma$ and the optimal instantaneous consumption for cohort t_0 along its lifetime as a linear function of the human wealth at birth,

$$c^*(a,t_0+a) = \left(\frac{h(t_0)}{m(t_0)}\right)\left(\frac{R(a)}{p(t_0,t)}\right)^{1/\sigma}, \ a \in \mathbf{R}_+.$$

Second, aggregate demand is $C(t) = \int_0^\infty n(a,t)c^*(a,t)da$. Using the relationship

$t_0 = t-a$, the expression for the population density $n(a,t) = n_0\pi(a)$ and the optimal consumption density just derived, we get

$$C(t) = n_0 \int_0^\infty \left(\frac{h(t-a)}{m(t-a)}\right)\left(\frac{R(a)}{p(t-a,t)}\right)^{1/\sigma} \pi(a)da.$$

where $h(t-a) \equiv \int_0^\infty p(t-a, t-a+s)y(s, t-a+s)\pi(s)ds$ and $m(t-a) \equiv \int_0^\infty p(t-a, t-a+s)^{(\sigma-1)/\sigma} R(s)^{1/\sigma}\pi(s)ds$. If we introduce the consistency condition between the cohort's t_0 prices and the prices formed in the AD markets at time $t=0$, then $p(t-a,t) = p(t)/p(t-a)$ and $p(t-a, t-a+s) = p(t-a+s)/p(t-a)$, and the aggregate consumption becomes,

$$C(t) = n_0 p(t)^{-1/\sigma} \int_0^\infty K(t-a)R(a)^{1/\sigma}\pi(a)da.$$

where

$$K(t-a) = \frac{\int_0^\infty p(t-a+s)y(s,t-a+s)\pi(s)ds}{\int_0^\infty p(t-a+s)^{(\sigma-1)/\sigma} R(s)^{-1/\sigma}\pi(s)ds}.$$

At last, substituting consumption in the equilibrium condition for the AD markets $C(t) = Y(t) = n_0 \int^\infty y(a,t)\pi(a)da$, equation (7) results. $\square$

Proof of Proposition 3 If we introduce in equation (7) the assumptions regarding $R(a)$, $\pi(a)$, and $y(a,t)$, we get the non-linear double integral equation

$$p(t)^{1/\sigma} \int_0^\infty \phi(a)e^{-\mu a}da = \int_0^\infty \left(\frac{\int_0^\infty \phi(s)p(t-a+s)e^{(\gamma-\mu)s}ds}{\int_0^\infty p(t-a+s)^{(\sigma-1)/\sigma}e^{-\frac{\gamma+\mu}{\sigma}s}ds}\right) e^{-\left(\frac{\rho+\sigma(\mu+\gamma)}{\sigma}\right)a}da. \tag{16}$$

Following the method for solving similar equations in Polyanin and Manzhirov (1998, p.325), which we used in Brito and Dilão (2006), we conjecture that its general solution of equation is a sum of functions of type $f(t) = ke^{(x-\gamma)t}$ where k is an arbitrary constant. Making the substitution in equation (16), we get

$$\int_0^\infty \phi(a)e^{-\mu a} - \frac{\xi_1(x)}{\xi_2(x)} \int_0^\infty \phi(a)e^{-\mu a} da = 0$$

where $\xi_1 = \int_0^\infty e^{-(\eta+x)/\sigma a} da$ and $\xi_2 = \int_0^\infty e^{-(\eta+x(1-\sigma))/\sigma a} da$. If $\eta + x > 0$ and $\eta + x(1-\sigma) > 0$ then those functions are integrable, and

$$\frac{\xi_1}{\xi_2} = \frac{\eta + x}{\eta + x(1-\sigma)} > 0,$$

which leads to equation (9). The elimination of the dependence on t proves that our conjecture is right, and the existence of a solution equation (16) depends on the existence of roots to equation $S(x) = 0$ verifying $\eta + x > 0$ and $\eta + x(1-\sigma) > 0$. □

Proof of Proposition 4 First, observe that X is non-empty if and only if $0 < \sigma \leq 1$ and $x > -\eta$, or if $\sigma > 1$, and $-\eta < x < \eta/(\sigma - 1)$. Therefore, if $\eta > 0$ ($\eta \leq 0$) then $x = 0$ belongs (does not belong) to the set of feasible solutions of equation $S(x) = 0$. If $\eta > 0$, we see, by simple inspection, that $x = 0$ is a solution for any choice of the $\phi(a)$ function (such that $\phi(a) \neq 0$). The necessary and sufficient condition for $\eta > 0$ is $\sigma > \max\{0, (\gamma - \rho)/(\gamma + \mu)\}$. □

Proof of Proposition 5 Consider function the characteristic equation $S(x) = 0$, in equation (9).

1. Proposition 4 applies here as well: if $\sigma > \sigma_f$ then $\eta > 0$ and $x = 0$ is always a root of $S(x) = 0$.

2. We can prove further that, if $\eta > 0$, all roots of $S(x) = 0$ will verify $x > -\eta$

if $0 < \sigma \leq 1$, or $-\eta < x < \eta/(\sigma - 1)$ if $\sigma > 1$. Reasoning by contradiction: (1) if any $\sigma > \sigma_f$, and $x < -\eta \leq 0$ then $\eta + x < (\eta + x(1-\sigma))e^{\alpha x} < 0$, this implies that $S(x) < 0$ for all $x < -\eta$, and, therefore $S(x) = 0$ has no roots in this interval; (2) if $\sigma > 1$ and $x \geq \eta/(\sigma - 1)$ then $\eta + x(1-\sigma) \leq 0$ and $\eta + x \geq \eta\sigma/(\sigma - 1) > 0$, and then $S(x) > 0$, and $S(x) = 0$ has no roots in this case.

3. The former result does not exclude the possibility that there are roots of $S(x) = 0$ such that $x > \gamma$, for any $\sigma > \sigma_f$. To find conditions for ruling this case out, we consider separately cases $0 < \sigma \leq 1$ and $\sigma > 1$ and worry only about positive roots of $S(x) = 0$.

First case: if $\sigma_f < \sigma \leq 1$ then function $S(x)$ is similar to a parabola, because $\lim_{x\to\pm\infty} S(x) = -\infty$, $\lim_{x\to-\infty} \partial S(x)/\partial x = 1$, and $\lim_{x\to+\infty} \partial S(x)/\partial x = -\infty$. Therefore, it has one root $x = 0$ and can have another root in the interval $(-\eta, +\infty)$, where $\eta > 0$, from assumption 1. This is so because the characteristic equation has only a maximum at $x = x^*$ and $S(x^*) > 0$. To prove this observe that $\partial S/\partial x = 0$ if and only if $[\alpha(\eta + x(1-\sigma)) + 1 - \sigma]e^{\alpha x} = 1$, which is equivalent to $(\alpha x + y)e^{\alpha x + y} = e^y/(1-\sigma)$ where $y \equiv \alpha + \eta/(1-\sigma)$. If we solve for x we get

$$x^* = (W(z) - y)/\alpha = \frac{1}{\alpha}\left[W\left(\frac{1}{1-\sigma}e^{\frac{1-\sigma+\alpha\eta}{1-\sigma}}\right) - \frac{1-\sigma+\alpha\eta}{1-\sigma}\right]$$

where, $z \equiv e^y/(1-\sigma)$, W is the LambertW function(see Corless et al. (1996)). If $0 < \sigma < 1$ and $\eta > 0$ then $z > 0$ and $W(z) = W_0(z) > 0$ (see again Corless et al. (1996)). And finally we get $S(x^*) > 0$ from the properties of the Lambert W function. In order to have a root $0 < x < \gamma$ we only need to determine conditions under which $S(\gamma) = S(x)|_{x=\gamma} \leq 0$. We readily see that $\alpha \geq \alpha_1 \equiv \frac{1}{\gamma} \ln \left(\frac{\rho+\sigma(\mu+\gamma)}{\rho+\sigma\mu} \right)$ is a necessary and sufficient condition for $S(\gamma) = \eta + \gamma - (\eta + \gamma(1-\sigma))e^{\alpha\gamma} \leq 0$.

Second, if $\sigma > 1$ then function $S(x)$ is similar to a cubic polynomial. The characteristic equation has one, two, or three roots, because $\lim_{x\to-\infty} S(x) = -\infty$, $\lim_{x\to\infty} S(x) = \infty$, $\lim_{x\to-\infty} \partial S(x)/\partial x = 1$, and $\lim_{x\to+\infty} \partial S(x)/\partial x = +\infty$. The other roots, in addition to $x = 0$ should belong to the interval $(-\eta, \eta/(\sigma-1))$. As $\gamma < \eta/(\sigma-1) = \gamma + (\rho+\sigma\mu)/(\sigma-1)$ then we may have roots belonging to the interval $(\gamma, \eta/(\sigma-1))$. Clearly, a necessary condition ruling out roots $x > \gamma$ is $S(x^*) < 0$ and $x^* < \gamma$, or $S(x^*) > 0$ and $x* > \gamma$, and $S(\gamma) \geq 0$.

In order to determine x^* and $S(x^*)$, observe that if $\sigma > 1$ then $y < 0$ and $e^y/(1-\sigma) < 0$. In Corless et al. (1996) it is proven that if $z < 0$ then $W(z)$ has two branches, the principal branch $W_0(z)$ and the branch $W_{-1}(z)$ for $0 > z \geq -e^{-1}$, and has not real values in the domain $z < -e^{-1}$. Also, $W_0(z) \in (0, -1)$, $W_{-1}(z) < -1$ and that $W_0(-e^{-1}) = W_{-1}(-e^{-1}) = -1$. Then, there are no local maxima and minima if $e^y/(1-\sigma) < -e^{-1}$ which is equivalent to $\alpha < (\sigma-1)(\ln(\sigma-1)-2)/\eta$. This is illustrated in the next figure 2 by area lines below curve $W(-1/e)$. If $\alpha > (\sigma-1)(\ln(\sigma-1)-2)/\eta$ then there are two local maximum or minimum $x^* = (W_0(e^y/(1-\sigma)) - y)/\alpha$ and $x^*_{-1} = (W_{-1}(e^y/(1-\sigma)) - y)/\alpha$ with $x^* > x^*_{-1}$. If, we have values of α such that $S(x^*) = 0$, then the highest value of a local extremum will coincide with a root of $S(x) = 0$. In particular $x^* = \gamma$ and the highest local extremum will coincide with root $x = 0$ if and only if $S(\gamma) = S'(\gamma) = S(x^*) = 0$. This is the case if $\alpha = \alpha_c$ and $\sigma = \sigma_c$.

If instead we have $S(\gamma) \geq 0$, $S'(\gamma) < 0$ then $x^* > \gamma$ and there will not be no

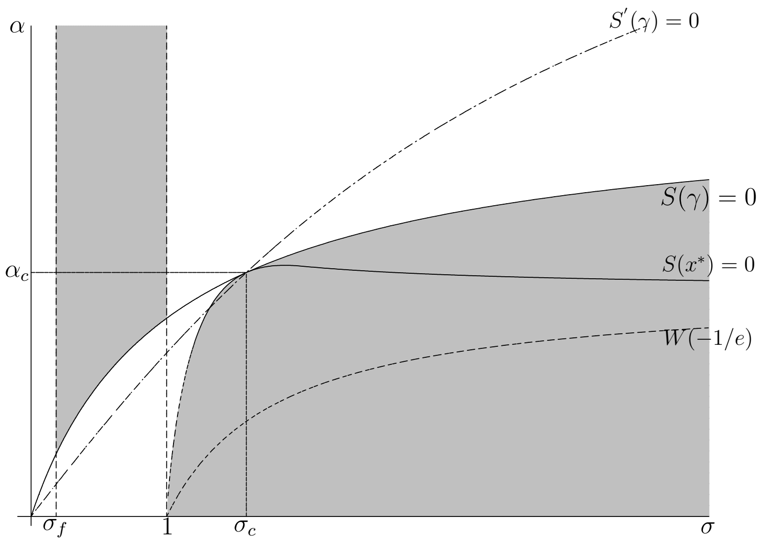

Figure 2: Curves $S(\gamma) = 0$, $S^{'}(\gamma) = 0$, $S(x^*) = 0$ and $e^{(1+\alpha\eta/(1-\sigma))}(1-\sigma)^{-1} = -e^{-1}$ and values of (σ, α) such that there are no speculative bubbles.

roots for $x > \gamma$ if $S(x^*) > 0$. This case will occur if $1 < \sigma < \sigma_c$ and $\alpha < \alpha_2 \equiv \{\alpha : S(x^*(\alpha), \alpha) > 0\}$.

At last, if $S(\gamma) \geq 0$ and $S^{'}(\gamma) > 0$ then $x^* < \gamma$ and the last condition is not binding. This case occurs if $\sigma > \sigma_c$ and $\alpha < \alpha_1$