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11 May 2008

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September 26, 2008

Abstract

The paper extends the analysis of price competition among capacity-constrained sellers beyond the cases of duopoly and symmetric oligopoly. We first provide some general results for the oligopoly and then focus on the triopoly, providing a complete characterization of the mixed strategy equilibrium of the price game. The region of the capacity space where the equilibrium is mixed is partitioned according to the features of the mixed strategy equilibrium arising in each subregion. Then computing the mixed strategy equilibrium becomes a quite simple task. The analysis reveals features of the mixed strategy equilibrium which do not arise in the duopoly.

1 Introduction

The issue of price competition among capacity-constrained sellers has attracted considerable interest since Levitan and Shubik's (1972) modern reappraisal of Bertrand and Edgeworth. Assume a given number of firms producing an homogeneous good at constant and identical unit variable cost up to some fixed capacity. Assume, also, a non-increasing and concave demand and that any rationing takes place according to the surplus maximizing rule. Then there are a few well-established facts about equilibrium of the price game. First, at any pure strategy equilibrium the firms earn the competitive profit. However, a pure strategy equilibrium need not exist, unless the capacity of the largest firm is small enough compared to total capacity (see,

[0] We are grateful to Jiawei Chen, Giuseppe Freni, Daisuke Hirata, and Attila Tasnadi for valuable comments and suggestions on an earlier version of this paper. The responsibility for any remaining errors rests entirely with the authors.

for instance, Vives, 1986). When a pure strategy equilibrium does not exist, existence of a mixed strategy equilibrium is guaranteed by Theorem 5 of Dasgupta and Maskin (1986) for discontinuous games.

Under the aforementioned assumptions on demand and cost, a characterization of mixed strategy equilibrium was provided by Kreps and Scheinkman (1983) for the duopoly within a two-stage capacity and price game. That model was subsequently extended to allow for non-concavity of demand (Osborne and Pitchik,1986) or differences in unit cost among the duopolists (Deneckere and Kovenock, 1996). This led to the discovering of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and non-identical for the duopolists.

Yet, there is still much to be learned about mixed strategy equilibria under oligopoly, even with constant and identical unit cost and concave demand, where a complete characterization of the mixed strategy equilibrium is only available for the case of equal capacities (see, among others, Vives, 1986). More recently Hirata (2008) showed how some features of the duopolistic mixed strategy equilibrium need not hold in the triopoly. The present paper differs in scope from Hirata's since we provide a complete characterization of the mixed strategy equilibria in the triopoly: our analysis will reveal any qualitative features possibly arising in the triopoly, including the facts highlighted by Hirata (2008).[1] Differences between our contribution and Hirata's will be further clarified below and in the following sections.

We first point out a number of properties of a mixed strategy equilibrium under oligopoly. Further progress requires a taxonomy, something which seems hard to manage in general oligopoly. We turned to the triopoly, whose analysis is in itself challenging and can provide insights for subsequent generalizations to oligopoly. The aim proved to be worth pursuing. Unlike in the duopoly, the equilibrium distributions need not have identical supports for all the firms. For one thing, the maximum as well as the minimum of the supports need not be the same for all the firms.[2] Furthermore, supports need not be connected. As another difference from the duopoly, the equilibrium is not unique when the capacity of the largest firm is sufficiently large, in which case there is actually one degree of freedom in the determination of the equilibrium distributions of the other firms. This result extends straightforwardly to n-firm oligopoly: under that condition,

[1] Our own research and Hirata's have been made independently from each other. (Results were made publicly available in Hirata, 2008, and De Francesco and Salvadori, 2008.)

[2] That minima may differ has also been recognized by Hirata (2008).

the equilibrium distributions of the $n-1$ firms other than the largest one are determined up to $n-2$ degrees of freedom. (On the possibility of a continuum of equilibria, see also Hirata, 2008).

The paper is organized as follows. Section 2 contains definitions and the basic assumptions of the model along with a few basic results on equilibrium payoffs in oligopoly. Section 3 shows that several features of mixed strategy equilibrium extend from duopoly to the oligopoly. Most notably, as far as the largest firm is concerned: the minimum of the support of its equilibrium strategy is determined as in the duopoly; the maximum - also determined as in the duopoly - is charged with positive probability if its capacity is strictly higher than for any other firm.

Sections 4 to 6 are devoted to the triopoly. In Section 4 the region of the capacity space involving a mixed strategy equilibrium is partitioned into several subsets according to the features of the resulting equilibrium. This leads to a classification theorem which characterizes equilibrium profits and bounds the supports of the equilibrium strategies throughout the region of mixed strategy equilibria. Section 5 addresses the event of the support being disconnected for some firm: we clarify which type of gaps can in principle arise and how gaps would be determined should they arise. Having done this, we are able to complement our classification theorem with a uniqueness theorem: the equilibrium is either unique or not fully determined, and we identify the two complementary subsets of the region of mixed strategy equilibrium where the former and the latter hold true, respectively. The event of a gap in some support is established in Section 6. In that section we compute the mixed strategy equilibrium in one of the subsets where the supports of the equilibrium strategies have the same bounds for all the firms. That subset is in its turn partitioned into two subsets according to the nature of the equilibrium: in one the supports are connected for all the firms, in the other, there is a gap in the support of the smallest firm. To show that gaps are a more general phenomen, in Section 6 we also look elsewhere in the region of mixed strategy equilibria and provide an example with a gap in the support of the intermediate-size firm. Section 7 briefly concludes.

2 Preliminaries

There are n firms, $1, 2, ..., n$, supplying a homogeneous good. The firms are assumed to produce at the same constant unit cost, normalized to zero, up to capacity. The demand is denoted as $D(p)$ and its inverse as $P(x)$. When positive, $D(p)$ is assumed to be decreasing and concave. Without loss of generality, we consider the subset of the capacity space $(K_1, K_2, ...K_n)$ such that $K_1 \geq K_2 \geq ... \geq K_n$, and we define $K = K_1 + ... + K_n$. As already said, the firms are charging the competitive price, $p^c = \max\{0, P(K)\}$ at any pure strategy equilibrium of the price game. Thus such an equilibrium fails to exist when $\arg\max p(D(p) - \Sigma_{j\neq 1} K_j) > p^c$, or, to put it more throroughly, when either

$$\Sigma_{j\neq 1} K_j < D(0),\ p^c = 0; \tag{1}$$

or

$$K_1 > -p^c \left[D'(p)\right]_{p=p^c},\ p^c > 0. \tag{2}$$

It is assumed throughout that either (1) or (2) holds, so that we are in the region of mixed strategy equilibria.

We henceforth denote by Π_i^* firm i's equilibrium payoff (expected profit), by $\Pi_i(p)$ firm i's expected profit when charging p and the rivals are playing their equilibrium profile of distributions, $\phi_{-i}(p)$, by $\phi_i(p) = \Pr(p_i < p)$ firm i's equilibrium (cumulative) distribution, where $\Pr(p_i < p)$ is the probability of i charging less than p, by S_i the support of ϕ_i, and by $p_M^{(i)}$ and $p_m^{(i)}$ the maximum and the minimum of S_i, respectively. More specifically, we say that $p \in S_j$ when $\phi_j(\cdot)$ is increasing at p, that is, when $\phi_j(p+h) > \phi_j(p-h)$ for any $h > 0$, whereas $p \notin S_j$ if $\phi_j(p+h) = \phi_j(p-h)$ for some $h > 0$. Of course, any $\phi_i(p)$ is non-decreasing and everywhere continuous except at $p^\circ : \Pr(p_i = p^\circ) > 0$, where it is left-continuous $(\lim_{p\to p^{\circ -}} \phi_i(p) = \phi_i(p^\circ))$, but not continuous.

Obviously, $\Pi_i^* \geq \Pi_i(p)$ everywhere and $\Pi_i^* = \Pi_i(p)$ almost everywhere in S_i. Some more notation is needed to go deeper through the properties of $\Pi_i(p)$. Let $N = \{1, ..., n\}$ be the set of firms, $N_{-i} = N - \{i\}$, and $\mathcal{P}(N_{-i}) = \{\psi\}$ be the power set of N_{-i}. Further, let

$$Z_i(p; \phi_{-i}) := p \sum_{\psi \in \mathcal{P}(N_{-i})} q_{i,\psi} \Pi_{r\in\psi} \phi_r \Pi_{s \in N_{-i}-\psi} (1 - \phi_s), \tag{3}$$

where $q_{i,\psi} = \max\{0, \min\{D(p) - \sum_{r\in\psi} K_r, K_i\}\}$ is firm i's output when any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[3] It is immediately recognized that for each p in which all functions $\phi_j(p)$

[3] Note that $\Pi_{r\in\psi}\phi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\Pi_{s\in N_{-i}-\psi}(1 - \phi_s) = 1$ when $\psi = N_{-i}$.

$(j \neq i)$ are continuous, then

$$\Pi_i(p) = Z_i(p; \phi_{-i}(p)),$$

whereas

$$Z_i(p^\circ; \phi_{-i}(p^\circ)) \geq \Pi_i(p^\circ) \geq \lim_{p \to p^\circ +} Z_i(p; \phi_{-i}(p))$$

if $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$. This is enough to state the following

Lemma 1 *(i) $Z_i(p; \phi_{-i})$ is continuous in p. For every p and every ϕ_{-i} there exist $\epsilon > 0$ and $\epsilon' > 0$ such that $Z_i(p; \phi_{-i})$ is concave in p in the intervals $[p, p+\epsilon]$ and $[p-\epsilon', p]$: as a consequence, $Z_i(p; \phi_{-i})$ is locally concave in p whenever it is differentiable.*

(ii) For given ϕ_{-i} and for any $\psi \in P(N_{-i})$, $Z_i(p; \phi_{-i})$ is kinked at $p = P(\sum_{r \in \psi} K_r)$ and locally convex there if $\Pi_{r \in \psi} \phi_r \Pi_{s \in N_{-i} - \psi}(1 - \phi_s) > 0$.

(iii) $Z_i(p; \phi_{-i})$ is continuous and differentiable in ϕ_j (each $j \neq i$) and $\partial Z_i / \partial \phi_j \leq 0$. More precisely $\partial Z_i / \partial \phi_j < 0$ if there exists some ψ containing j such that

$$\Pi_{r \in \psi'} \phi_r \Pi_{s \in N_{-i} - \psi \}}(1 - \phi_s) > 0. \tag{4}$$

and

$$0 < D(p) - \sum_{h \in \psi'} K_h < K_i + K_j. \tag{5}$$

where $\psi' = \psi - \{j\}$.

(iv) For any $i \in N$, $\Pi_i^* \geq \Pi_i(p)$ with $\Pi_i^* = \Pi_i(p)$ for p in the interior of S_i.

Proof. (i) $Z_i(p; \phi_{-i})$ is a linear combination of functions which are concave in the intervals $[p, p+\epsilon]$ and $[p-\epsilon', p]$ for ϵ and ϵ' sufficientely small.

(ii) The left derivative of $Z_i(p; \phi_{-i})$ at $p = P(\sum_{r \in \psi} K_r)$ equals the right derivative plus the negative quantity $pD'(p)\Pi_{r \in \psi} \phi_r \Pi_{s \in N_{-i} - \psi}(1 - \phi_s)$.

(iii) Differentiation of $Z_i(p; \phi_{-i})$ with respect to ϕ_j yields, after rearrangement,[4]

$$\frac{\partial Z_i}{\partial \phi_j} = p \sum_{\psi \in \mathcal{P}(N_{-i})} (q_{i,\psi} - q_{i,\psi'}) \Pi_{r \in \psi'} \phi_r \Pi_{s \in N_{-i} - \psi}(1 - \phi_s) \tag{6}$$

Then, $\partial Z_i / \partial \phi_j \leq 0$ since $q_{i,\psi} - q_{i,\psi'} \leq 0$. Clearly, $\partial Z_i / \partial \phi_j < 0$ if and only if there exists ψ such that inequality (4) holds and $q_{i,\psi} - q_{i,\psi'} < 0$, which leads to inequalities (5).

[4]Note that $q_{i,\psi} - q_{i,\psi'} = 0$ whenever $j \notin \psi$. This allows to simplify the notation.

(iv) Suppose contrariwise that $\Pi_i^* > \Pi_i(p^\circ)$ for some p° internal to S_i. This reveals that p° is not charged by i: it is $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$ and $Z_i(p^\circ; \phi_{-i}(p^\circ)) > \Pi_i(p^\circ) > \lim_{p \to p^\circ +} Z_i(p; \phi_{-i}(p))$. As a consequence there is a right neighbourhood of p° in which $\Pi_i^* > \Pi_i(p)$: a contradiction.

■

Let $p_M = \max_i p_M^{(i)}$ and $p_m = \min_i p_m^{(i)}$, $M = \{i : p_M^{(i)} = p_M\}$ and $L = \{i : p_m^{(i)} = p_m\}$. Moreover, if $\#M < n$, then we define $\widehat{p}_M = \max_{i \notin M} p_M^{(i)}$ and, with an abuse of language, if $\#M = n$, then we say that $\widehat{p}_M = p_M$. Similarly, if $\#L < n$, then we define $\widehat{p}_m = \min_{i \notin L} p_m^{(i)}$ whereas $\widehat{p}_m = p_m$ if $\#L = n$. We henceforth write $\phi_{i\alpha}(p)$ and $\Pi_{i\alpha}(p)$ to refer to $\phi_i(p)$ and $\Pi_i(p)$, respectively, over some range α; then, $\phi_i(p)$ does not need to equal $\phi_{i\alpha}(p)$ outside α. Finally, $\lim_{p \to h+} \Pi_{i\alpha}(p)$ and $\lim_{p \to h-} \Pi_{i\alpha}(p)$ are denoted as $\Pi_{i\alpha}(h^+)$ and $\Pi_{i\alpha}(h^-)$, respectively.

Since Kreps and Scheinkman it is known that $p_M = p_M^{(1)} = p_M^{(2)} = \arg\max p(D(p) - K_2)$ in a duopoly with $K_1 \geq K_2$; also, $\phi_1(p_M) < \phi_2(p_M) = 1$ if $K_1 > K_2$, while $\phi_1(p_M) = \phi_2(p_M) = 1$ if $K_1 = K_2$. Therefore $\Pi_i^* = p_M(D(p_M) - K_2)$ for any i such that $K_i = K_1$. The next proposition summarizes some generalizations of these results to oligopoly that have been made recently.

Proposition 1 *$p_M = \arg\max p(D(p) - \Sigma_{j \neq 1} K_j)$ and, for any $i : K_i = K_1$, $\Pi_i^* = \max p(D(p) - \Sigma_{j \neq 1} K_j)$; furthermore, $p_M^{(i)} = p_M$ for any $i : K_i = K_1$ and $\phi_j(p_M) = 1$ for any $j : K_j < K_1$.*

Proof. This statement is an obvious consequence of the statement that $p_M^{(i)} = p_M$ for *some* $i : K_i = K_1$ and that $\Pi_i^* = \max p(D(p) - \Sigma_{j \neq 1} K_j)$ for *any* $i : K_i = K_1$.(A complete proof of this statement is in De Francesco (2003); see also Boccard and Wauthy (2001) and, for a more recent proof, Loertscher (2008).) ■

According to this result, in the region of mixed strategy equilibria, the equilibrium payoff of the largest firm is decreasing in the capacity of any rival and is independent on its own capacity. The fact that $\Pi_i^* = \max p(D(p) - \Sigma_{j \neq 1} K_j)$ for any $i : K_i = K_1$ has a nice interpretation. Note that, in the region of the capacity space where the equilibrium is in mixed strategies, $\max p(D(p) - \Sigma_{j \neq 1} K_j)$ is nothing but the minimax payoff for any $i : K_i = K_1$.[5] Thus, what Proposition 1 is actually saying is that the equilibrium payoff of (any of) the largest firm(s) equals its minimax payoff.

[5] Let σ_{-i} denote any mixed strategy profile on the part of firm i's rivals, where $i : K_i = K_1$ and let $p(\sigma_{-i})$ denote any of firm i's best response to σ_{-i}. It is immediately

Since Kreps and Scheinkman it is also known that, in a duopoly, $\#L = 2$ and $\Pr(p_i = p_m) = 0$ for $i = 1, 2$, so that $\Pi_1^* = p_m \min\{D(p_m), K_1\}$. This implies that $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$, where $\widehat{p} \equiv \Pi_1^*/K_1$ and $\widehat{\widehat{p}}$ is the smallest solution of the equation in p

$$pD(p) = \Pi_1^*.$$

Finally, firm 2's equilibrium payoff is $\Pi_2^* = p_m K_2$. Since $S_1 = S_2 = [p_m, p_M]$, then $\phi_1(p)$ and $\phi_2(p)$ are found straightforwardly by solving the two-equation system $\Pi_i^* = Z_i(p; \phi_{-i}(p))$. It will be seen below to which extent these results generalize beyond duopoly.

3 Some properties of equilibrium for the oligopoly

In this section we establish a number of general properties of mixed strategy equilibria under oligopoly. The following proposition presents a number of basic properties, which represent generalizations of analogous results holding for duopoly.

Proposition 2 *(i)* $\#M \geq 2$ *and* $\#L \geq 2$.

(ii) At any $p^\circ \in (p_m, p_M)$, *it cannot be* $\#\{i : p^\circ \in S_i\} = 1$.

(iii) For any $p^\circ \in (p_m, p_M)$, $p^\circ > P(\Sigma_{i:p_m^{(i)} < p^\circ} K_i)$.

(iv) $i \in L$ *for any* $i : K_i = K_1$.

(v) Let $i \in N_{-1}$ *and* $j \in N_{-1} - \{i\}$. *At any* $p \in (p_m, p_M)$:

(v.a) $\partial Z_1/\partial \phi_i < 0$ *and* $\partial Z_i/\partial \phi_1 < 0$ *for any* i;

(v.b) if $p \geq P(K_1)$, $\partial Z_i/\partial \phi_j = 0$;

(v.c) if $p < P(K_1)$ *and n=3, then* $\partial Z_i/\partial \phi_j < 0$; *if* $p < P(K_1)$ *and* $n > 3$, *then, for each* $i \in R(p)$ *(for each* $j \in R(p)$*), there is some* $j \in R(p)$ *(resp., some* $i \in R(p)$*) such that* $\partial Z_i/\partial \phi_j < 0$, *where* $R(p) = \{r : p_m^{(r)} \leq p\}$.

(vi) $p_m > P(\Sigma_{j \in L} K_j)$.

(vii) For any $p^\circ \in (p_m, p_M)$, $\Pr(p_j = p^\circ) = 0$ *for any* j.

(viii) $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$.

Proof. (i) This is so because $Z_i(\cdot)$ is concave in p on a right neighbourhood of p_m and on a left neighbourhood of p_M. Suppose contrariwise that $\#L = 1$ and let $L = \{i\}$. Then $\phi_j' = 0$ for all $j \neq i$ in a neighbourhood of p_m. Hence $d\Pi_i(p)/dp = \partial Z_i/\partial p$, contrary to the fact that $\Pi_i(p) = \Pi_i^*$ in a right neighbourhood of p_m. A similar argument rules out the event of $\#M = 1$.

understood that $\Pi_i(p(\sigma_{-i})) \geq p_M(D(p_M) - \Sigma_{j \neq 1} K_j)$ with strict equality holding for some σ_{-i}.

(ii) The proof is similar to the previous one, given the fact that $Z_i(\cdot)$ is concave on a right neighbourhood of any p and a left neighbourhood of any p.

(iii) Otherwise for $i : p_m^{(i)} < p$ it would be $\Pi_i(p) = pK_i$ for $p \in S_i \cap [p_m, p^\circ]$: a contradiction.

(iv) Since $D(p_M) > \sum_{j \neq 1} K_j$, if $p_m < p_m^{(i)}$ for some $i : K_i = K_1$, then *a fortiori* $D(p) > \sum_{j \in L} K_j$ for $p \leq p_M$: as a consequence, for any $j \in L$, $\Pi_j(p)$ is increasing for $p \in [p_m, p_m^{(i)})$: a contradiction.

(v.a) A crucial role is played here by statements (i) to (iv) above and the fact that $D(p) > \sum_{j \neq 1} K_j$. To see that $\partial Z_1(p)/\partial \phi_i < 0$ one must check that at least one product on the right-hand side of (4) is strictly negative. This is so for $\psi = R(p) - \{1\}$ if $j \in R(p)$ and $\psi = R(p) \cup \{j\} - \{1\}$ if $j \notin R(p)$: in fact, $q_{1,\psi} - q_{1,\psi'} < 0$ since $0 < q_{1,\psi} < K_1$ and, at the same time, $\Pi_{r \in \psi'} \phi_r \Pi_{s \in N_{-i} - \psi}(1 - \phi_s) > 0$. One can similarly see that $\partial Z_i(p)/\partial \phi_1 < 0$: in fact, we take $\psi = R(p)$ if $i \notin R(p)$ and $\psi = R(p) - \{i\}$ if $i \in R(p)$, and see that $q_{1,\psi} - q_{1,\psi'} < 0$.

(v.b) Now $q_{i,\psi} - q_{i,\psi'} = K_i - K_i = 0$ for any $\psi \in N_{-i}$ such that $1 \notin \psi$, while $q_{i,\psi} - q_{i,\psi'} = 0 - 0 = 0$ for any $\psi \in N_{-i}$ such that $1 \in \psi$.

(v.c) Note that the first inequality (6) holds for $\psi = \{1, j\}$ since $p < P(K_1)$, whereas the second inequality (6) holds for $\psi = N_{-i}$ since $D(p) < K$: therefore $\partial Z_i/\partial \phi_j < 0$ for $n = 3$, since then $\{1, j\} = N_{-i}$. Turning to the oligopoly, note that, by statement (iii), the second inequality (6) also holds for $\psi = R(p) - \{i\}$. Thus $\partial Z_i/\partial \phi_j < 0$ if $\#R(p) = 3$, since then $\{1, j\} = R(p) - \{i\}$. Finally, with $\#R(p) > 3$, let Ψ_1 (Ψ_2) be the set of the subsets ψ of N_{-i} which satisfy the first (resp., the second) inequality (6): neither Ψ_1 nor Ψ_2 are empty. If $\Psi_1 \cap \Psi_2 \neq \emptyset$, then $\partial Z_i/\partial \phi_j < 0$. If instead $\Psi_1 \cap \Psi_2 = \emptyset$, then for any $\psi \in \Psi_1$,

$$D(p) - \sum_{h \in \psi} K_h \geq K_i > -K_j,$$

while, for any $\psi \in \Psi_2$,

$$D(p) - \sum_{h \in \psi} K_h \leq -K_j < K_i.$$

Of course, there is some $\psi_l \in \Psi_1$ such that $\psi_l \cup \{l\} \in \Psi_2$ where $l \in R(p) - \{i, j\}$ and therefore

$$-K_j \geq D(p) - \sum_{h \in \psi_l} K_h - K_l \geq K_i - K_l.$$

Thus $K_l \geq K_i + K_j$. But this cannot hold if either i or j is the largest firm in $R(p)$ apart for firm 1. This completes the proof of the claim.

(vi) If $\#L = n$, then inequality $p_m \leq P(\Sigma_{j\in L}K_j)$ implies that each firm earns no more than its competitive profit, contrary to Proposition 1. Suppose next $\#L < n$. If $p_m < P(\Sigma_{j\in L}K_j)$, then $\Pi_j(p)$ would be increasing over the range $[p_m, \min\{\widehat{p}_m, P(\Sigma_{j\in L}K_j)\}]$ for any $j \in L$. To rule out the event of $p_m = P(\Sigma_{j\in L}K_j)$ when $\#L < n$, it will be shown that otherwise it would be $\lim_{p\to p_m^+} \phi_i'(p) < 0$ for each $i \in L$. Note that $\Pi_i^* = p_m K_i$ and

$$\begin{aligned}\Pi_i^* = \Pi_i(p) &= p[D(p) - \Sigma_{j\in L-\{i\}}K_j]\Pi_{j\in L-\{i\}}\phi_j + pK_i(1 - \Pi_{j\in L-\{i\}}\phi_j) \\ &= p[D(p) - D(p_m)]\Pi_{j\in L-\{i\}}\phi_j + pK_i\end{aligned}$$

in a neighborhood of p_m. Therefore

$$\Pi_{j\in L-\{i\}}\phi_j = \frac{(p_m - p)K_i}{p[D(p) - D(p_m)]}.$$

Then

$$\frac{d\Pi_{j\in L-\{i\}}\phi_j}{dp} = K_i\frac{-p_m[D(p) - D(p_m) + pD'(p)] + p^2D'(p)}{p^2[D(p) - D(p_m)]^2},$$

and

$$\lim_{p\to p_m^+}\frac{d\Pi_{j\in L-\{i\}}\phi_j}{dp} = K_i\frac{p_mD''(p) + 2D'(p)}{2p_m^2[D'(p)]^2} < 0.$$

This in its turn implies that $\lim_{p\to p_m^+} \phi_i'(p) < 0$ for each $i \in L$ since, in the present case, $\phi_i(p)K_i = \phi_j(p)K_j$ for each $i, j \in L$.

(vii) A distinction is drawn according as to whether $p^\circ \in S_1$ or $p^\circ \notin S_1$. In the former case, if contrariwise $\phi_j(p^\circ) < \lim_{p\to p^{\circ+}} \phi_j(p)$ for some $j \neq 1$, then $\lim_{p\to p^{\circ+}} \Pi_1(p) < \lim_{p\to p^{\circ-}} \Pi_i(p)$ since $\partial Z_1/\partial\phi_j < 0$ because of statement (v): a contradiction. In a similar way it is also proved that $\phi_1(p^\circ) = \lim_{p\to p^{\circ+}} \phi_1(p)$. Assume now that $p^\circ \notin S_1$. It must preliminarily be noted that such an event might only arise (if ever) when $p^\circ < P(K_1)$. Indeed, if $p^\circ \geq P(K_1)$ and $p^\circ \notin S_1$, then, as a consequence of statement (v) above, $d\Pi_i(p)/dp = \partial Z_i/\partial p$ in a neighbourhood of p°, contrary to the fact that $\Pi_i(p)$ is constant in a neighbourhood of p° for any i such that $p^\circ \in S_i$. If, on the other hand, $p^\circ < P(K_1)$, then, according to statement (v.c), $\partial Z_i/\partial\phi_j < 0$ for some i such that $p_m^{(i)} \leq p$. Therefore, if $\phi_j(p^\circ) < \lim_{p\to p^{\circ+}} \phi_j(p)$ for some j, it would be $\lim_{p\to p^{\circ+}} \Pi_i(p) < \lim_{p\to p^{\circ-}} \Pi_i(p)$: a contradiction.

(viii) Since $\Pi_1(p) \leq p \min\{D(p), K_1\}$, then $\Pi_1(p) < \Pi_1^*$ for $p < \max\{\widehat{p}, \widehat{\widehat{p}}\}$. At the same time it cannot be that $p_m > \max\{\widehat{p}, \widehat{\widehat{p}}\}$: if it were, then it would be $\Pi_1(p_m^-) = p_m \min\{D(p_m), K_1\} > \Pi_1^*$. ■

Note that, since $\widehat{p}$ is decreasing in K_1, the event of $\widehat{\widehat{p}} \geq \widehat{p}$ arises at relatively large levels of K_1. An immediate consequence of statement (viii) is

Corollary 1. $p_m \geq P(K_1)$ if and only if $\widehat{\widehat{p}} \geq \widehat{p}$.

Note that if $\Pi_j^* = p_m K_j$ for all $j \neq 1$ and $S_i = [p_m, p_M]$ for all i, then the equilibrium distributions would be found, as in duopoly, by solving the n-equation system $\Pi_i^* = Z_i(p, \phi_{-i}(p))$ throughout $[p_m, p_M]$. But there is no guarantee that the above features hold, hence we are not yet in a position to the determine the equilibrium. Yet, we can make some remarks regarding p_M.

Proposition 3 *(i) Let $K_1 > K_2$. Then $\phi_1(p_M) < 1$. (ii) If $K_r = K_1$, then $\phi_r(p) = \phi_1(p)$ for $p \in [p_m, p_M]$, and $\phi_r(p_M) = \phi_1(p_M) = 1$. Furthermore, if at the same time $K_j < K_1$ for some j, then $p_M^{(j)} < p_M$.*

Proof. (i) Suppose contrariwise that $\phi_1(p_M) = 1$. As a consequence, $\Pi_i^* = \Pi_i(p_M^-) = p_M \max\{D(p_M) - \sum_{j \neq i} K_j, 0\}$ for $i \in M - \{1\}$. If $D(p_M) \leq \sum_{j \neq i} K_j$, then $\Pi_i^* = 0$ while $\Pi_i(p_m^-) = p_m K_i > 0$: a contradiction. If, instead, $D(p_M) - \sum_{j \neq i} K_j > 0$, then, since $\arg\max p[D(p) - \sum_{j \neq i} K_j] \in (0, p_M)$ for $i \in M - \{1\}$, it would be $\Pi_i(p) > \Pi_i(p_M^-)$ for some p: a contradiction. Thus it must be $\Pr(p_1 = p_M) > 0$ and $\Pi_i(p_M^-) > \Pi_i(p_M)$.

(ii) Since $D(p_M) > \sum_{j \neq 1} K_j$, we can write $\Pi_r^* = \Pi_r(p) = p\phi_1 E(x_r \mid p_1 < p) + p(1 - \phi_1)K_r = p\phi_1[E(x_r \mid p_1 < p) - K_r] + pK_r$, where p is internal to S_r and $E(x_r \mid p_1 < p)$ denotes r's expected output at p conditional on firm 1 charging less than p. Similarly, we can write $\Pi_1^* = \Pi_1(p) = p\phi_r[E(x_1 \mid p_r < p) - K_1] + pK_1$ for p internal to S_1. Obviously, $E(x_r \mid p_1 < p) = E(x_1 \mid p_r < p)$, so that $\Pi_r(p) = \Pi_1(p)$ - as required by Proposition 1 - if and only if $\phi_r = \phi_1$. Further, it cannot be $\phi_r(p_M) = \phi_1(p_M) < 1$, otherwise $\Pi_r(p_M^-) > \Pi_r(p_M)$ contrary to the presumption that p_M is quoted with positive probability by firm r. Nor can it be $p_M^{(j)} = p_M$ for any $j : K_j < K_1$. By arguing as in the proof of the previous statement we would obtain that $\Pi_j(p) > \Pi_j(p_M)$ at some $p < p_M$. ■

4 The triopoly: upper and lower bounds of the supports of equilibrium strategies

In the preceding sections we have seen how there are a number of properties which generalize from the duopoly to oligopoly. Equipped with these results and in order to get further insights for the oligopoly, in the remainder of the paper we provide a comprehensive study of mixed strategy equilibria in the triopoly. Compared to the duopoly, the triopoly will be seen to allow for much wider diversity throughout the region of mixed equilibria, the equilibrium being affected on several grounds by the ranking of p_m and p_M relative to the demand prices of different aggregate capacities, namely, $P(K_1 + K_2)$, $P(K_1 + K_3)$, and $P(K_1)$.

Without loss of generality, in the region of mixed strategy equilibria of the (K_1, K_2, K_3)-space we restrict ourselves to the subset where $K_1 \geq K_2 \geq K_3$. As soon as one set out to construct the equilibrium it emerges that there may be significant differences in some equilibrium features at the different points in that subset. The following partition of that subset gives a full account of the diversity in the bounds of the equilibrium supports and on the degree of determinateness of the equilibrium.[6] (Note that, because of Proposition 1 and statement (viii) of Proposition 2, p_M and p_m are known once K_1, K_2, and K_3 are given.)

[6] The results presented by Hirata (2008) refer to a less fine partition of that subset. Most notably, that partition does not fully distinguish according as to whether $p_M \lesseqgtr P(K_1)$ nor according as to whether $p_M \leq P(K_1 + K_3)$ or $P(K_1 + K_3) < p_M \leq P(K_1)$. As a consequence, the possibility of a continuum of equilibria is not addressed at full lenght nor is it the determination of M.

$$A = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, p_m \leq P(K_1 + K_2), p_M \leq P(K_1 + K_3)\}$$
$$B_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$p_m \leq P(K_1 + K_2), P(K_1 + K_3) < p_M \leq P(K_1)\}$$
$$E_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, p_m \leq P(K_1 + K_2), p_M > P(K_1)\}$$
$$C_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, P(K_1 + K_2) < p_m < P(K_1 + K_3)\}$$
$$C_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, P(K_1 + K_3) \leq p_m, p_M \leq P(K_1)\}$$
$$C_3 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$P(K_1 + K_3) \leq p_m < \frac{K_1 - K_3}{K_1} P(K_1), p_M > P(K_1)\}$$
$$F = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$\max\{P(K_1 + K_3), \frac{K_1 - K_3}{K_1} P(K_1)\} \leq p_m < P(K_1), p_M > P(K_1)\}$$
$$D = \{(K_1, K_2, K_3) : K_1 \geq K_2 \geq K_3, p_m \geq P(K_1)\}$$
$$B_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 = K_3, p_m < P(K_1), p_M \leq P(K_1)\}$$
$$E_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 = K_3, p_m < P(K_1), p_M > P(K_1)\}$$

It is easily checked that it is actually $K_1 > K_2 + K_3$ whenever $p_M \geq P(K_1)$, hence at any $(K_1, K_2, K_3) \in C_3 \cup D \cup E_1 \cup E_2 \cup F$, and $K_1 > K_2$ whenever $p_M \geq P(K_1 + K_3)$, hence at any $(K_1, K_2, K_3) \in B_1 \cup C_2$.

The following theorem collects most of the results to be achieved in this section.

Theorem 1. (a) *In A, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 2, 3\}$ and $M = \{1, 2\}$.*

(b) *In $B_1 \cup B_2$, $\Pi_i^* = p_m K_i$ for all i and $L = M = \{1, 2, 3\}$.*

(c) *In $C_1 \cup C_2 \cup C_3$, $\Pi_i^* = p_m K_i$ for $i \neq 3$ and $\Pi_3^* > p_m K_3$; $L = M = \{1, 2\}$; $p_M^{(3)} < P(K_1)$.*

(d) *In D, $\Pi_1^* = p_m D(p_m)$ and $\Pi_j^* = p_m K_j$ for $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that $pK_2\phi_2 + pK_3\phi_3 = pD(p) - \Pi_1^*$, $\phi_j(p_m) = 0$ and $\phi_j(p_M) = 1$ for $j \neq 1$.*

(e) *In $E_1 \cup E_2$, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 2, 3\}$ and $\#M \geq 2$ with $\widehat{p}_M \geq P(K_1)$. Over $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$, and $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that $pK_2\phi_2 + pK_3\phi_3 = pD(p) - \Pi_1^*$, $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$ and $\phi_j(p_M) = 1$ for $j \neq 1$.*

(f) *In F, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 3\}$ and $p_m^{(2)} \geq P(K_1)$. Over the range $[P(K_1), p_M]$ distributions are determined like in $E_1 \cup E_2$.*

In addition, we will see how to determine $p_m^{(3)}$ and Π_3^* when $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$. The route leading to the results listed in Theorem 1 begins with the determination of $\#L$ in the various subsets making up the partition of the region of mixed strategy equilibria. Then we will address the determination of L and the $\Pi_i' s$. Finally, we will determine M in each subset of the partition. In connection to the first task an intermediate step is made by the following Lemma.

Lemma 2. *If $\#L = 2$, then $\Pr(p_j = p_m) = 0$ for each $j \in L$; if $\#L = 3$ and $\Pr(p_i = p_m) > 0$ for some i, then $\Pr(p_j = p_m) = 0$ for each $j \neq i$.*

Proof. Let $L = \{i, j\}$. If $\Pr(p_j = p_m) > 0$, then, taking account of statement (vi) of Proposition 2, $\Pi_i^* = \Pi_i(p_m^+) < p_m \min\{D(p_m), K_i\}$ while $\Pi_i(p_m^-) = p_m \min\{D(p_m), K_i\}$: a contradiction. A similar argument establishes the second part of the statement, relating to the event of $L = \{i, j, k\}$.

■

We are now ready to address the determination of $\#L$. First of all note that if $\#L = 3$ then equilibrium distributions constitute a solution of system

$$\Pi_i^* = Z_i(p, \phi_{-i}(p)), \phi_i > 0, \phi_i' \geq 0 \text{ for each } i, \tag{7}$$

in an open to the left right neighbourhood of p_m, where Π_2^* and Π_3^* are constants to be determined. Note, furthermore, that $\Pr(p_i = p_m) = \phi_i(p_m^+)$. The following result addresses the determination of $\#L$ in the whole region of mixed strategy equilibria except set D along with the determination of $\Pr(p_i = p_m)$ throughout the partition. In this connection, it must be noted that subset $B_2 \cup E_2$ can be partitioned into two subsets, one in which $p_m \leq P(K_1 + K_2) = P(K_1 + K_3)$ and one in which $P(K_1 + K_2) = P(K_1 + K_3) < p_m < P(K_1)$. It is shown that whether $\#L = 2$ or $\#L = 3$ depends on the size of p_m relative to $P(K_1 + K_2)$ and $P(K_1)$, as well as on whether $K_2 > K_3$ or $K_2 = K_3$.

Proposition 4 *(i) Let $p_m \leq P(K_1 + K_2)$ or, equivalently, let $(K_1, K_2, K_3) \in A \cup B_1 \cup E_1$ or (K_1, K_2, K_3) fall in the subset of $B_2 \cup E_2$ where $p_m \leq P(K_1 + K_2)$. Then $\#L = 3$ and $\Pr(p_i = p_m) = 0$ for each i.*

(ii) Let (K_1, K_2, K_3) fall in the subset of $B_2 \cup E_2$ where $P(K_1 + K_2) = P(K_1 + K_3) < p_m < P(K_1)$. Then $\#L = 3$ and $\Pr(p_i = p_m) = 0$ for each i.[7]

[7] That $L = 3$ in the circumstances of statements (i) and (ii) has independently been discovered also by Hirata (2008, see Claims 2 and 5). Hirata does not address the issue of $\Pr(p_i = p_m)$.

(iii) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3 \cup F$, or, equivalently, $P(K_1+K_2) < p_m < P(K_1)$ and $K_2 > K_3$. Then $\#L = 2$.

(iv) Let $(K_1, K_2, K_3) \in D$, that is, $p_m \geq P(K_1)$. Then $\Pr(p_i = p_m) = 0$ for each i.

(v) $\Pr(p_i = p_m) = 0$ for each $i \in L$.

(vi) $\Pi_i^* = p_m K_i$ for each $i \in L$, except that $\Pi_1^* = p_m D(p_m)$ in set D.

Proof. (i) The first part is an obvious consequence of statement (vi) of Proposition 2. The second part of the statement is proved by showing that $\phi_i(p_m^+) = 0$ for each i at any solution of system (7). Suppose first that $p_m < P(K_1 + K_2)$. Then the equations in system (7) read

$$\begin{aligned}
\Pi_1^* &= p\phi_2\phi_3[D(p) - K] + pK_1, \\
\Pi_2^* &= p\phi_1\phi_3[D(p) - K] + pK_2, \\
\Pi_3^* &= p\phi_1\phi_2[D(p) - K] + pK_3.
\end{aligned}$$

Hence $\left[dZ_i(p, \phi_{-i}(p))/dp\right]_{p=p_m^+} = 0$ for each i if and only if

$$\begin{aligned}
(D-K)[\phi_2\phi_3 + p_m(\phi_2'\phi_3 + \phi_2\phi_3')] + D'p_m\phi_2\phi_3 + K_1 &= 0, \\
(D-K)[(\phi_1\phi_3 + p_m(\phi_1'\phi_3 + \phi_1\phi_3')] + D'p_m\phi_1\phi_3 + K_2 &= 0, \\
(D-K)[(\phi_1\phi_2 + p_m(\phi_1'\phi_2 + \phi_1\phi_2')] + D'p_m\phi_1\phi_2 + K_3 &= 0,
\end{aligned}$$

where $D, D', \phi_1, \phi_2, \phi_3, \phi_1', \phi_2'$, and ϕ_3' are all to be undertood as limits for $p \to p_m^+$. Now, suppose contrariwise that, say, $\phi_1(p_m^+) > 0$ (one might as well suppose either $\phi_2(p_m^+) > 0$ or $\phi_3(p_m^+) > 0$). Then, according to Lemma 2, $\phi_2(p_m^+) = \phi_3(p_m^+) = 0$, and the system above becomes

$$\begin{aligned}
p_m(D-K)(\phi_2'\phi_3 + \phi_2\phi_3') &= -K_1, \\
p_m(D-K)(\phi_1'\phi_3 + \phi_1\phi_3') &= -K_2, \\
p_m(D-K)(\phi_1'\phi_2 + \phi_1\phi_2') &= -K_3.
\end{aligned}$$

But this system cannot hold. Indeed, in order for the first equation to hold it must be either $\phi_2' = \infty$ or $\phi_3' = \infty$ (or both): then, either the third equation or the second equation (or both) cannot hold. The same logic applies when $p_m = P(K_1+K_2)$, regardless of whether $K_2 > K_3$ or $K_2 = K_3$. For example, in the former case, the equations in system (7) read

$$\begin{aligned}
\Pi_1^* &= p\phi_2[D(p) - K_1 - K_2] - p\phi_2\phi_3K_3 + pK_1, \\
\Pi_2^* &= p\phi_1[D(p) - K_1 - K_2] - p\phi_1\phi_3K_3 + pK_2 \\
\Pi_3^* &= p(1 - \phi_1\phi_2)K_3,
\end{aligned}$$

in a right neighbourhood of p_m and the same procedure proves the statement in this case too.

(ii) Assume contrariwise that $p_m^{(1)} = p_m^{(2)} < p_m^{(3)}$. Then

$$\Pi_2^* = Z_2(p, \phi_{-2}(p)) = p\phi_1(D(p) - K_1) + p(1 - \phi_1)K_2$$

$$\Pi_3(p) = Z_3(p, \phi_{-3}(p)) = p\phi_1(1 - \phi_2)(D(p) - K_1) + p(1 - \phi_1)K_2$$

for $p \in (p_m, p_m^{(3)}]$. It is immediately seen that $Z_3(\cdot) < Z_2(\cdot)$ for any $\phi_1, \phi_2 > 0$. Consequently, $\Pi_3^* = Z_3(p_m^{(3)+}, \phi_{-3}(p_m^{(3)})) < \Pi_2^*$: firm 3 has not made a best response since it can guarantee itself Π_2^* by charging p_m. To establish the second part of the statement, assume contrariwise that $\phi_i(p_m^+) > 0$ for some i. Then $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$: a contradiction.

(iii) The statement is proved by showing that, if $\#L = 3$, then either $Z_i(p_m^+, \phi_{-i}(p_m^+)) < Z_i(p_m, \phi_{-i}(p_m))$ for some i - a clear contradiction - or system (7) has no solution. The proof runs somewhat differently according as to whether $P(K_1+K_2) < p_m < P(K_1+K_3)$ or $P(K_1+K_3) \leq p_m < P(K_1)$.

(iii.a) $P(K_1 + K_2) < p_m < P(K_1 + K_3)$.

There are three cases to consider: either $\phi_i(p_m^+) > 0$ for some $i \in \{1, 2\}$, or $\phi_3(p_m^+) > 0$, or $\phi_i(p_m^+) = 0$ for each i. In the first case $\Pi_j^* = \Pi_j(p_m^+) < \Pi_j(p_m^-) = p_m K_j$ for $j \in \{1, 2\}$ and $j \neq i$. In both the second and third case the equations in system (7) read

$$\begin{aligned}
\Pi_1^* &= p\phi_2[D(p) - K_1 - K_2] - p\phi_2\phi_3K_3 + pK_1, \\
\Pi_2^* &= p\phi_1[D(p) - K_1 - K_2] - p\phi_1\phi_3K_3 + pK_2, \\
\Pi_3^* &= p(1 - \phi_1\phi_2)K_3,
\end{aligned}$$

over range $(p_m, P(K_1+K_3))$. Then $\left[dZ_i(p, \phi_{-i}(p))/dp\right]_{p=p_m^+} = 0$ if and only if

$$\begin{aligned}
p_m[\phi_2'\phi_3K_3 + \phi_2\phi_3'K_3 - \phi_2'(D - K_1 - K_2)] &= K_1, \\
p_m[\phi_1'\phi_3K_3 + \phi_1\phi_3'K_3 - \phi_1'(D - K_1 - K_2)] &= K_2, \\
p_m(\phi_1'\phi_2 + \phi_1\phi_2') &= 1.
\end{aligned}$$

Since $\phi_i' \geq 0$, the first two equations cannot hold unless ϕ_2' and ϕ_1' are both finite, whereas the third equation requires that at least one of them is not.

(iii.b) $P(K_1 + K_3) \leq p_m < P(K_1)$.

The equations in system (7) read

$$\begin{aligned}\Pi_1^* = p[\phi_2(D(p) - K_1 - K_2) - \phi_2\phi_3(D(p) - K_1) \\ +\phi_3(D(p) - K_1 - K_3) + K_1], \\ \Pi_2^* = p[\phi_1(D(p) - K_1 - K_2) - \phi_1\phi_3(D(p) - K_1) + K_2], \\ \Pi_3^* = p[\phi_1(D(p) - K_1 - K_3) - \phi_1\phi_2(D(p) - K_1) + K_3],\end{aligned}$$

for $p \in (p_m, \min\{\widehat{p}_M, P(K_1)\}]$. We consider a partition of four cases. In the first case, $P(K_1 + K_3) < p_m$ and $\phi_i(p_m^+) > 0$ for some i. If $i = 1$, then $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$; if $i \in \{2, 3\}$, then $\Pi_1^* = Z_1(p_m^+, \phi_{-1}(p_m^+)) < \Pi_1(p_m^-) = p_m K_1$. A similar contradiction is obtained in the second case, in which $P(K_1 + K_3) = p_m$ and $\phi_i(p_m^+) > 0$ for some $i \in \{1, 2\}$. Then, $\Pi_j^* = Z_j(p_m^+, \phi_{-1}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$ and $j \in \{1, 2\}$. As third case, assume that $P(K_1 + K_3) = p_m$ and $\phi_1(p_m^+) = \phi_2(p_m^+) = 0$. Then the proof follows as in the last two cases inspected in (iii.a). The partition is completed by the case where $P(K_1 + K_3) < p_m$ and $\phi_i(p_m^+) = 0$ for each i. Arguing as before it is now obtained

$$\begin{aligned}p_m\left[\phi_2'\phi_3(D - K_1) + \phi_2\phi_3'(D - K_1) - \phi_2'(D - K_1 - K_2) + \right. \\ \left. -\phi_3'(D - K_1 - K_3)\right] = K_1, \\ p_m[\phi_1'\phi_3(D - K_1) + \phi_1\phi_3'(D - K_1) - \phi_1'(D - K_1 - K_2)] = K_2, \\ p_m[\phi_1'\phi_2(D - K_1) + \phi_1\phi_2'(D - K_1) - \phi_1'(D - K_1 - K_3)] = K_3.\end{aligned}$$

On close scrutiny, a *necessary* condition for such equations to hold is that $0 < \phi_i' < \infty$ for each i. Granted this, the last two equations become

$$\begin{aligned}-p_m\phi_1'(D - K_1 - K_2) &= K_2, \\ -p_m\phi_1'(D - K_1 - K_3) &= K_3,\end{aligned}$$

which cannot simultaneously hold since $K_2 > K_3$ and $D(p_m) > K_1$.

(iv) Under the present circumstances, equation $\Pi_1^* = Z_1(p, \phi_{-1})$ reads

$$\Pi_1^* = p_m[D(p_m) - \phi_2 K_2 - \phi_3 K_3].$$

If either $\phi_2(p_m^+) > 0$ or $\phi_3(p_m^+) > 0$, then $\Pi_1^* = Z_1(p_m^+, \phi_{-1}(p_m^+)) < \Pi_1(p_m^-) = p_m D(p_m)$: a contradiction. To dispose of the event of $\phi_1(p_m^+) > 0$, note that $Z_j(p, \phi_{-j}(p)) = p(1 - \phi_1)K_j$ for $j \neq 1$: then, if $\phi_1(p_m^+) > 0$, it would be $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \in L - \{1\}$.

(v) It is a consequence of statements (i)-(iv) and Lemma 2.

(vi) It is a consequence of previous statement and Corollary 1. ■

We know from Sections 2 and 3 that p_m and p_M are determined just as in the duopoly. Unlike in duopoly, however, the supports S_i need not be the same for all i, as is immediately revealed by the fact that $\#L = 2$ may hold. One group of related questions is then whether $L = \{1, 2\}$ or $L = \{1, 3\}$ and how $\widehat{p}_m$ is determined under the circumstances of statement (iii) of Proposition 4. According to the following proposition, $L = \{1, 2\}$ in $C_1 \cup C_2 \cup C_3$ and $L = \{1, 3\}$ in F. Furthermore, the proposition points the indeterminacy affecting the equilibrium at $p > P(K_1)$ when $\widehat{p}_M > P(K_1)$. Figure 1 illustrates statement (a.ii) of the following proposition and statement (i) of Proposition 6 when set $C_1 \cup C_2 \cup C_3$ is concerned.

Proposition 5 *(a) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$. Then: (a.i) $L = \{1, 2\}$ and $\Pi_i^* = p_m K_i$ for $i \neq 3$; (a.ii) Let $\alpha = [p_m, p_m^{(3)}]$, so that $\Pi_1^* = \Pi_{1\alpha}(p) = Z_1(p, \phi_{2\alpha}, 0)$ and $\Pi_2^* = \Pi_{2\alpha}(p) = Z_2(p, \phi_{1\alpha}, 0)$.[8] Then $\Pi_3^* = \max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) > p_m K_3$ and $p_m^{(3)} = \arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p)$,[9] where $\Pi_{3\alpha}(p) = Z_3(p, \phi_{1\alpha}, \phi_{2\alpha})$, $\widetilde{\alpha} = [p_m, p_M^*]$ and p_M^* is such that $\phi_{2\alpha}(p_M^*) = 1$.*

(b) If $(K_1, K_2, K_3) \in D$, then $\Pi_1^* = p_m D(p_m)$ and $\Pi_j^* = p_m K_j$ for $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that [10]

$$\phi_2 = \frac{pD(p) - \Pi_1^* - pK_3\phi_3}{pK_2}, \tag{8}$$

$\phi_j(p_m) = 0$ and $\phi_j(p_M) = 1$ for $j \neq 1$. Equation (8) is consistent with $L = \{1, 2, 3\}$, $L = \{1, 2\}$ and $L = \{1, 3\}$, as well as $M = \{1, 2, 3\}$, $M = \{1, 2\}$ and $M = \{1, 3\}$, and even with (non-overlapping) gaps in S_2 and S_3. Among the infinite solutions, there exists a simmetric one in ϕ_2 and ϕ_3.

(c) If $(K_1, K_2, K_3) \in F$, then $L = \{1, 3\}$, $p_m^{(2)} \geq P(K_1)$ and $\Pi_i^* = p_m K_i$ for all i. Over the range $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$ while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions meeting (8) and such that $\phi_3(P(K_1^+) = \phi_3(P(K_1^-)$ and $\phi_2(P(K_1^+) = 0$. It is $\phi_3(P(K_1)) < 1$ unless $\frac{K_1 - K_3}{K_1} P(K_1) = p_m$: in this special case, $S_2 \cap S_3 = \{P(K_1)\}$ and the

[8] We take for granted that $[p_m, \widehat{p}_m] \in S_1 \cap S_2$. For the sake of simplicity the proof that there is no gap in the range $[p_m, \widehat{p}_m]$ is postponed to next section.

[9] The necessity, in what is here called C_1, or the possibility, in $C_2 \cup C_3 \cup F$, of $p_m^{(3)} > p_m$ and $\Pi_3^* > p_m K_3$ has also been recognized by Hirata (2008, see, respectively, Claims 3 and 5). However, Hirata is not concerned with how $p_m^{(3)}$ and Π_3^* are actually determined in that event.

[10] That there is a continuum of equilibria in this region has also been proved by Hirata (2008, see Claim 1).

equilibrium is determined.[11]

Proof. (a.i) Given statement (iii) of Proposition 4, we just need to rule out the event of $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$. Consider first $(K_1, K_2, K_3) \in C_1$. Under that event $\Pi_3^* = Z_3(p, \phi_{-3}(p)) = pK_3$ for $p \in (p_m, \min\{p_m^{(2)}, P(K1 + K3)\}]$: an obvious contradiction. Next let $(K_1, K_2, K_3) \in C_2$. If it were $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$, then, for $i, j \in L$ it would be $\Pi_i^* = p\phi_j(D(p) - K_j) + p(1 - \phi_j)K_i$ over the range $[p_m, p_m^{(2)}]$, and, as a consequence,

$$\phi_j = \frac{(p_m - p)K_i}{p[D(p) - K_i - K_j]} \tag{9}$$

over that range. By charging a price there firm 2 would get

$$\Pi_2(p) = Z_2(p, \phi_{-2}(p)) = p\phi_1(1 - \phi_3)[D(p) - K_1] + p(1 - \phi_1)K_2,$$

which is lower than $p_m K_2$ at any $p < P(K_1)$. As a consequence, if $p_M < P(K_1)$, $\Pi_2^* = \Pi_2(p_m^{(2)}) < \Pi_2(p_m)$: a contradiction. If instead $p_M = P(K_1)$, then one can avoid the same contradiction only by taking $p_m^{(2)} = p_M^{(2)} = p_M$, that is, $\Pr(p_2 = p_M) = 1 > 0$, contrary to Proposition 1. Finally, let $(K_1, K_2, K_3) \in C_3$. Now, with $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$ it should be $p_m^{(2)} \geq P(K_1)$, to avoid the previous contradiction; but then, according to (9), $\phi_3(P(K_1)) > 1$ since $p_m K_1 < (K_1 - K_3)P(K_1)$.

(a.ii) It is easily checked that $\Pi_{3\alpha}(p_m) = p_m K_3 = \Pi_{3\alpha}(P(K_1))$ and, if $P(K_1) > p_M^*$, $\Pi_{3\alpha}(p_M^*) < p_m K_3$; furthermore, $\Pi'_{3\alpha}(p)_{p=p_m} > 0$.[12] It follows immediately that $\Pi_3^* > p_m K_3$, since firm 3 will earn more than $p_m K_3$ at a price higher than and sufficiently close to p_m, and that $\Pi_{3\alpha}(p)$ has an internal maximum over the range $[p_m, \min\{P(K_1), p_M^*\}]$. Thus it cannot be $p_m^{(3)} > \arg\max \Pi_{3\alpha}(p)$, otherwise $\Pi_3^* = \Pi_{3\alpha}(p_m^{(3)}) < \max \Pi_{3\alpha}(p)$, while firm 3 can earn $\max \Pi_{3\alpha}(p)$ by charging $\arg\max \Pi_{3\alpha}(p)$. This being so, let $\beta = [p_m^{(3)}, \min\{P(K_1), \widehat{p}_M\}]$. To rule out the event of $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$, note that, on a right neighbourhood of $p_m^{(3)}$, $\Pi_i^* = \Pi_{i\beta}(p) = Z_i(p, \phi_{-i\beta}(p))$ for all i and $\Pi_i^* = \Pi_{i\alpha}(p) = Z_i(p, \phi_{-i\alpha}(p))$ for $i \in \{1, 2\}$. Thus, taking account of statement (v) of Proposition 2, $\phi_{2\beta} < \phi_{2\alpha}$ and $\phi_{1\beta} < \phi_{1\alpha}$ since

[11] See also Hirata (2008, Claim 4) for a proof of a similar result However, Hirata omits that these results require, among other things, that $p_M > P(K_1)$.

[12] In C_1, $\Pi'_{3\alpha}(p)_{p=p_m} = K_3$, in $C_2 \cup C_3$, $\Pi'_{3\alpha}(p)_{p=p_m} = p_m [\phi'_{1\alpha}]_{p=p_m} [D(p_m) - K_1 - K_3] + K_3$, where $[\phi'_{1\alpha}]_{p=p_m} = -\frac{K_2}{p_m[D(p_m) - K_1 - K_2]}$.

$\phi_{3\beta} > \phi_{3\alpha} = 0$), implying that $Z_3(p, \phi_{-3\beta}(p)) > Z_3(p, \phi_{-3\alpha}(p))$. Hence if $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$, we get a contradiction since $Z_3(p, \phi_{-3\alpha}(p)) > Z_3(p_m^{(3)}, \phi_{-3\alpha}(p_m^{(3)})) = \Pi_3^* = Z_3(p, \phi_{-3\beta}(p))$ on a right neighbourhood of $p_m^{(3)}$.[13]

(b) It is immediately checked that $\Pi_j^* = p\,(1 - \phi_1)\,K_j$ for $j \neq 1$ and $p \in S_j$. Since $\#L > 1$, $\Pi_j^* = p_m K_j$ for some $j \neq 1$, in its turn implying $\phi_1(p) = 1 - \frac{p_m}{p}$. Further, equation $\Pi_1^* = Z_1(p, \phi_{-1}(p))$ reads

$$\begin{aligned}\Pi_1^* &= p\phi_2\phi_3[D(p) - K_2 - K_3] + p\phi_2(1 - \phi_3)[D(p) - K_2] \\ &\quad + p(1 - \phi_2)\phi_3[D(p) - K_3] + p\,(1 - \phi_2)\,(1 - \phi_3)D(p).\end{aligned}$$

This leads to equation (8), leaving one conditional degree of freedom in the determination of ϕ_2 and ϕ_3, additional constraints being, of course, $\phi_j' \geq 0$ throughout $[p_m, p_M]$ for all $j \neq 1$, $\phi_j(p_m) = 0$, and $\phi_j(p_M) = 1$.[14]

As one can easily check, these constraints are met at the symmetric solution of (8), namely,

$$\phi_j(p) = \frac{pD(p) - \Pi_1^*}{p(K_2 + K_3)} \; for \; j \neq 1. \tag{8'}$$

(c) Again taking account of statement (iii) of Proposition 4, we just need to rule out the event of $p_m^{(1)} = p_m^{(2)} < p_m^{(3)}$. Under such an event, $\Pi_3(p) = Z_3(p, \phi_{-2}(p)) = p\phi_1(1 - \phi_2)[D(p) - K_1] + p(1 - \phi_1)K_3$ in a neighbourhood of p_m, where $\phi_1(p)$ and $\phi_2(p)$ are given by equations (9). It is easily checked that $\Pi_3(p) > p_m K_3$ in an open to the left neighbourhood of p_m. This implies, first, that $\Pi_3^* > p_m K_3$ and, second, that $p_M^{(3)} < P(K_1)$, otherwise $p\,(1 - \phi_1)\,K_3 = \Pi_3^* > p_m K_3$ and $\Pi_2^* = p\,(1 - \phi_1)\,K_2 = p_m K_2$ for $p \in [P(K_1), p_M^{(3)}]$. As a result, $\Pi_1^* = p\phi_2[D(p) - K_2 - K_3] + p(1 - \phi_2)[D(p) - K_3]$ over the range $(p_M^{(3)}, P(K_1))$ and $\phi_2 = \frac{p[D(p) - K_3] - \Pi_1^*}{pK_2}$.[15] But then $\phi_2(P(K_1)) \leq 0$ since $p_m K_1 \geq (K_1 - K_3)P(K_1)$: an obvious contradiction. Thus it must be $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$. Further, it cannot

[13] One might wish to account for the event of $\Pi_{3\alpha}(p)$ reaching its maximum more than once in $\widetilde{\alpha}$. Arguing as in the text, it is established that $\phi_{3\beta} = 0$ for any $p \leq \max\{\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p)\}$, hence $p_m^{(3)} = \max\{\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p)\}$.

[14] By the way, holding equation (9), $\phi_2(p_m) = 0$ if $\phi_3(p_m) = 0$ and $\phi_2(p_M) = 1$ if $\phi_3(p_M) = 1$.

[15] In the assumption that $(p_M^{(3)}, P(K_1)) \subset S_1 \cap S_2$. Assuming otherwise that this range belongs neither to S_1 nor to S_2 would lead to a contradiction. See below, Proposition 7(ii).

be $p_m^{(2)} < P(K_1)$, otherwise - as shown in the proof of statement (a.i) - $\Pi_2^* = \Pi_2(p_m^{(2)}) < p_m K_2$. Thus ϕ_1 and ϕ_3 are given by equations (9) over the range $[p_m, P(K_1)]$. Over the range $(P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$, whereas ϕ_2 and ϕ_3 are any pair of non-decreasing functions meeting equation (8) and such that $\phi_3(P(K_1)^+ = \phi_3(P(K_1)^-$. (Note that $\phi_2(P(K_1)^+) = 0$ whenever $\phi_3(P(K_1)^+ = \phi_3(P(K_1)^-$.) Quite interestingly, it can be $p_m^{(2)} > P(K_1)$ rather than $p_m^{(2)} = P(K_1)$. In the former case, $\phi_3 = \frac{pD(p)-\Pi_1^*}{pK_3}$ over the range $[P(K_1), p_m^{(2)}]$ and it would still be $\phi_2(p_m^{(2)})^+) = 0$. Finally, $\phi_3(P(K_1)) = 1$ if and only if $\frac{K_1-K_3}{K_1}P(K_1) = p_m$; in this special case, $\phi_2 = \frac{p[D(p)-K_3]-\Pi_1^*}{pK_2}$ over range $[P(K_1), p_M]$. ■

A few remarks are in order as regards the regions of indeterminacy of equilibrium. For example, as far as region D is concerned, one can generate solutions with any of the qualitative features claimed in statement (b) of Proposition 5, by slightly perturbing $\phi_3(p)$ around $\phi_j(p)$ (the symmetric solution in ϕ_2 and ϕ_3) over some $[p^\circ, p^{\circ\circ}] \subseteq [p_m, p_M]$. For example, one can construct infinitely many equilibria with $L = \{1, 2\}$ and $M = \{1, 2, 3\}$, or $L = \{1, 3\}$ and $\#M \geq 2$, or even equilibria such that $\phi_j' = 0$ for some $j \neq 1$, over a subset of $[p_m^{(j)}, p_M^{(j)}]$: in other words, S_j need not be connected.[16] Finally, it is worth looking at what underlies the indeterminacy of equilibrium. Except in a duopoly, this feature can arise when K_1 is sufficiently large. With $n \geq 3$, the output of any firm $i \neq 1$ when charging $p > P(K_1)$ does not depend on prices quoted by all other firms except firm 1: the demand forthcoming to i being zero whenever $p_1 < p$ and higher than K_i whenever $p_1 > p$. (Recall that $D(p) > \sum_{i \neq 1} K_i$ at any $p \leq p_M$.) Thus ϕ_i (each $i \neq 1$) only affects firm 1's payoff at any $p \in (P(K_1), p_M)$: consequently, there is one degree of freedom in the determination of ϕ_2 and ϕ_3.

Two remarks are in order about statement (a.ii). If $\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) \neq P(K_1 + K_3)$, then $[\phi'_{3\beta}]_{p=p_m^{(3)}} = 0$ and $[\phi'_{j\beta}]_{p=p_m^{(3)}+} = [\phi'_{j\alpha}]_{p=p_m^{(3)}-}$ for $j = 1, 2$; whereas if $\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) = P(K_1 + K_3)$, then $[\phi'_{3\beta}]_{p=p_m^{(3)}} > 0$ and $[\phi'_{j\beta}]_{p=p_m^{(3)}+} < [\phi'_{j\alpha}]_{p=p_m^{(3)}-}$ for $j = 1, 2$. (We omit the proof, which can be derived straightforwardly.)

We still have to determine M in all regions but D and F.

[16] Not dissimilar considerations hold - over range $[P(K_1), p_M]$ - for (K_1, K_2, K_3) in F or, as we will see in next proposition, in E_1 or in E_2. On all this, see the earlier version of this paper (De Francesco and Salvadori, 2008).

Proposition 6 *(i) Let $(K_1, K_2, K_3) \in A \cup C_1 \cup C_2 \cup C_3$. Then $M = \{1, 2\}$ and $p_M^{(3)} < P(K_1)$.*

(ii) Let $(K_1, K_2, K_3) \in B_1 \cup B_2$. Then $\#M = 3$. Furthermore, $\phi_2(p) = \phi_3(p)$ whenever $K_2 = K_3$.

(iii) Let $(K_1, K_2, K_3) \in E_1 \cup E_2$. Then $p_M^{(j)} \geq P(K_1)$ for $j \neq 1$. For $p \in [P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$ while $\phi_2(p)$ and $\phi_3(p)$ are any non-decreasing functions consistent with equation (8) and such that $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$ and $\phi_j(p_M) = 1$ for $j \neq 1$. This is consistent with $\#M = 3$, $p_M^{(2)} < p_M$, and $p_M^{(3)} < p_M$, and even with (non-overlapping) gaps in S_2 and in S_3.

Proof. (i) In order to establish that $M = \{1, 2\}$, the set $A \cup C_1 \cup C_2 \cup C_3$ is partitioned into the following regions: region (a), where $p_M \leq P(K_1 + K_2)$; region (b), where $p_m \leq P(K_1 + K_2) < p_M < P(K_1 + K_3)$; region (c), where $p_m \leq P(K_1 + K_2) < P(K_1 + K_3) = p_M$; region (d), where $P(K_1 + K_2) < p_m < p_M < P(K_1 + K_3)$; region (e), $C_2 \cup C_3$; region (f), where $P(K_1 + K_2) < p_m < P(K_1 + K_3) \leq p_M$. This is a partition because regions (a), (b), and (c) make up set A, while regions (d) and (f) make up set C_1.

A constructive argument is provided for region (a). By statement (i) of Proposition 4, $p \in S_1 \cap S_2 \cap S_3$ in a neighbourhood of p_m. Hence, over that neighbourhood equilibrium distributions are the solution of the three-equation system

$$p_m K_i = p\phi_j \phi_r (D(p) - K_j - K_r) + p(1 - \phi_j \phi_r) K_i,$$

so that $\phi_i = (K_j/K_i)\phi_j$. Based on this, it can be neither $\#M = 3$ nor $p_M^{(2)} < p_M$. It is instead $p_M^{(3)} < p_M$ and $S_1 = S_2 = [p_m, p_M]$ and $S_3 = [p_m, p_M^{(3)}]$ at one equilibrium.

As to regions (b) through (f), we first rule out the event of $\#M = 3$ and then the event of $p_M^{(2)} < p_M$. Recall that, by Proposition 3, with $\#M = 3$ it is $\phi_1(p_M) < 1 = \phi_2(p_M) = \phi_3(p_M)$. Further, in a left neighbourhood of p_M equilibrium distributions would be the solutions of the three-equation system (7), call them $\phi^\circ{}_i$. Let us consider region (c) first. As seen more thoroughly in the following section, solving this system yields $\phi^\circ{}_1 = \sqrt{\frac{K_2}{K_1}\frac{(p-p_m)}{p}}$, $\phi^\circ{}_2 = \frac{K_1}{K_2}\phi^\circ{}_1$, and $\phi^\circ{}_3 = \frac{D(p)-K_1-K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_1$ for $p \in \beta = [P(K_1 + K_2), P(K_1 + K_3)]$. Since $\phi^\circ{}_2(P(K_1 + K_3)) = 1$, then $\phi^\circ{}_1(P(K_1 + K_3)) = K_2/K_1$; upon differentiation of $\phi^\circ{}_3$ and recalling that $D(p_M) - K_2 - K_3 + p_M \left[D'(p)\right]_{p=p_M} = 0$ and $\Pi_1^* = p_M[D(p_M) - K_2 - K_3]$, it is found $\left[\phi^{\circ\prime}{}_3(p)\right]_{p=P(K_1+K_3)^-} = \frac{[D'(p)]_{p=p_M}}{2K_3} < 0$: a contradiction.

The event $\#M = 3$ in regions (b), (d), (e), and (f) can be dismissed more easily. Under that event, $\Pi_2(p_M^-) = Z_2(p_M, \phi_{-2}(p_M)) = \Pi_2^*$ and $\Pi_3(p_M^-) = Z_3(p_M, \phi_{-3}(p_M)) = \Pi_3^*$. These two equations contradict each other since $\phi_2(p_M) = \phi_3(p_M) = 1$. For example, if the former holds, then $\Pi_3(p_M^-) < \Pi_3^*$ and the latter cannot hold. Let us see how this works in each case. Note that, both in (e) and (f), $p_M \geq P(K_1 + K_3)$. Hence, in either case, under our working assumption it would be $\Pi_2^* = p_m K_2 = p_M[1 - \phi_1(p_M)]K_2$. This yields $\phi_1(p_M) = 1 - p_m/p_M$, in its turn implying $Z_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3 = p_m K_3$, contrary to statement (a.ii) of Proposition 5. In (d), $\Pi_2^* = p_m K_2 = Z_2(p_M^-) = p_M[\phi_1(p_M)(D(p_M) - K_1 - K_3) + (1 - \phi_1(p_M))K_2]$, yielding $\phi_1(p_M) = \frac{p_M - p_m}{p_M}\frac{K_2}{K - D(p_M)}$. By substituting this into $Z_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3$ it is obtained $Z_3(p_M^-) = \frac{p_M[K_1 + K_3 - D(p_M)] + p_m K_2}{K - D(p_M)}K_3$. Note that $\frac{p_M[K_1 + K_3 - D(p_M)] + p_m K_2}{K - D(p_M)} < p_m$ since $P(K_1 + K_3) > p_M$; hence $Z_3(p_M^-) < p_m K_3$, contrary to statement (a.ii) of Proposition 5. A similar argument applies to (b).

It remains to dismiss the event of $p_M^{(2)} < p_M$ in regions (b), (c), (d), (e), and (f). This is done by showing that it would otherwise be $\Pi_2(p) > \Pi_2^*$ in a left neighbourhood of p_M. If $p_M^{(2)} < p_M$ in regions (d), (e) and (f), then $\Pi_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3 = \Pi_3^* > p_m K_3$, implying $\phi_1(p_M) = 1 - \frac{\Pi_3^*}{p_M K_3} < 1 - \frac{p_m}{p_M}$ and hence $\Pi_2(p_M^-) = p_M \phi_1(p_M) \max\{0, D(p_M) - K_1 - K_3\} + p_M[1 - \phi_1(p_M)]K_2 \geq \frac{\Pi_3^*}{K_3}K_2 > \frac{p_m K_3}{K_3}K_2 = p_m K_2$. If $p_M^{(2)} < p_M$ under (b) or (c), then $\phi_1(p) = 1 - \frac{p_m}{p}$ in a neighbourhood of p_M. Consequently, by charging a price in that neighbourhood firm 2 would earn $\Pi_2(p) = p\phi_1\phi_3[D(p) - K_1 - K_3] + p\phi_1(1 - \phi_3)[D(p) - K_1] + p(1 - \phi_1)K_2 > p(1 - \phi_1)K_2 = p_m K_2 = \Pi_2^*$.

Next we prove that $p_M^{(3)} < P(K_1)$. This is trivial when $p_M \leq P(K_1)$, i.e. in $A \cup C_2$. In $C_1 \cup C_3$, $\Pi_3^* > p_m K_3$ and if $p_M^{(3)} \geq P(K_1)$, then $\Pi_3^* = P(K_1)[1 - \phi_1(P(K_1))]K_3$ and $p_m K_2 = P(K_1)[1 - \phi_1(P(K_1))]K_2$: an obvious contradiction.

(ii) Note that, by statements (i) , (ii), and (vi) of Proposition 4, $\#L = 3$ and $\Pi_i^* = p_m K_i$. Consider first the case where $(K_1, K_2, K_3) \in B_1$. If $p_M^{(j)} < p_M$ for some $j \neq 1$, then one can easily check that $\Pi_j(p) > \Pi_j^*$ for $p \in [\max\{p_M^{(3)}, P(K_1 + K_3)\}, p_M]$. Turn next to the case where $(K_1, K_2, K_3) \in B_2$. Here $\Pi_2^* = \Pi_3^* = p_m K_2$ (recall that $K_2 = K_3$), hence $p_m K_2 = Z_2(p, \phi_{-2}(p)) = Z_2(p, \phi_{-3}(p))$ on a right neighbourhood of p_m. Therefore, $\phi_2(p) = \phi_3(p)$ and, of course, $\#M = 3$.

(iii) According to statements (i), (ii) and (vi) of Proposition 4, $\Pi_j^* = p_m K_j$ for any $j \neq 1$. Also, $\Pi_j(p) = p\,(1 - \phi_1)\,K_j$ for $p \in [P(K_1), p_M]$:

this leads to $\phi_1(p) = 1 - p_m/p$ since $p_M^{(j)} = p_M$ for some $j \neq 1$. Now, if it were $p_M^{(r)} < P(K_1)$ then it would be $\Pi_r(p) > \Pi_r^*$ for $p \in [p_M^{(r)}, P(K_1)]$, as one can easily check. Also, the argument in the proof of statement (ii) of Proposition 5 leads to the stated relationship between ϕ_2 and ϕ_3 over $[P(K_1), p_M]$. Any ϕ_2 and ϕ_3 consistent with equation (8) constitutes a pair of equilibrium distributions so long as $\phi_j' \geq 0$, $\phi_j(p_M) = 1$ for $j \neq 1$, and $\phi_j(p)$ is continuous in $P(K_1)$. ■

Once Propositions 4, 5, and 6 are proved, Theorem 1 is proved too. Further, Statement (a) of Proposition 5 provides also for $p_m^{(3)}$ and Π_3^* in the regions where $L = \{1, 2\}$. In these regions $p_M^{(3)}$ is easily determined once Π_3^* has been computed. Let $\gamma = [p_M^{(3)}, p_M]$ so that we can refer to the equilibrium distributions of firms 1 and 2 over this range as $\phi_{1\gamma}$ and $\phi_{2\gamma}$: clearly, $Z_2(p, \phi_{1\gamma}, 1) = \Pi_2^*$ and $Z_1(p, \phi_{2\gamma}, 1) = \Pi_1^*$. Next consider $Z_3(p, \phi_{1\gamma}, \phi_{2\gamma})$ on any left neighbourhood of p_M. On reflection, $p_M^{(3)}$ is such that $Z_3(p_M^{(3)}, \phi_{1\gamma}(p_M^{(3)}), \phi_{2\gamma}(p_M^{(3)})) = \Pi_3^*$, $Z_3(p, \phi_{1\gamma}, \phi_{2\gamma}) > \Pi_3^*$ on a left neighbourhood of $p_M^{(3)}$ and $Z_3(p, \phi_{1\gamma}, \phi_{2\gamma}) \leq \Pi_3^*$ for $p \in (p_M^{(3)}, p_M]$.

5 The triopoly: gaps in the supports and uniqueness of equilibrium strategies

If the supports of equilibrium strategies are connected, then they are determined by Theorem 1 and the equilibrium strategies are the solutions to the appropriate equations in system (7). However, supports of equilibrium strategies need not be connected. For example, if $p_M > P(K_1)$ then $\phi_2(p)$ and $\phi_3(p)$ are not uniquely determined and we have seen that equation (8) allows for gaps in S_2 or S_3 and even non-overlapping gaps in both, within range $[P(K_1), p_M]$. We have also seen that no gap may exist in S_1 within range $[P(K_1), p_M]$. In this section we will be able to see which type of gaps can in principle arise for $p < P(K_1)$; in the next section, that type of gap will be seen to be a concrete possibility in subsets B_1 and C_1. Our analysis here will allow us to determine S_1, S_2 and S_3 and to conclude with a uniqueness result.

Lemma 3. (i) $Z_1(p; \phi_2, \phi_3)$ is concave and increasing in p throughout $[p_m, p_M]$.

(ii) If $p_m < P(K_1 + K_3)$, then $Z_2(p; \phi_1, \phi_3)$ is concave in p over ranges $[p_m, P(K_1+K_3))$ and $(P(K_1+K_3), P(K_1)]$, but locally convex at $P(K_1+K_3)$ if $\phi_3 > 0$; otherwise it is is concave over range $(P(K_1 + K_3), P(K_1)]$. If

$p_m < P(K_1+K_2)$, $Z_3(p;\phi_1,\phi_2)$ is concave in p over ranges $[p_m, P(K_1+K_2))$ and $(P(K_1+K_2), P(K_1)]$, but locally convex at $P(K_1+K_2)$; otherwise it is is concave over range $(P(K_1+K_2), P(K_1)]$.

Proof. (i) For each ϕ_2 and ϕ_3, function $Z_1(p;\phi_2,\phi_3)$ is a weighted arithmetic average of functions of p which are concave and increasing over the range $[p_m, p_M]$.

(ii) A similar argument applies to establish concavity; as to local convexity, see Lemma 1(ii). ■

In this section we denote by ${\phi_1}^\circ(p), {\phi_2}^\circ(p)$, and ${\phi_3}^\circ(p)$ the solutions to the equations in system (7), whether they are the equilibrium strategies or not. The following result holds.

Lemma 4. (i) *$({\phi_1}^\circ, {\phi_2}^\circ, {\phi_3}^\circ)$ is unique at any $p \leq P(K_1)$.*

(ii) In A, $B_1 \cup B_2$ and $C_1 \cup C_2 \cup C_3$, if ${\phi_1}^\circ, {\phi_2}^\circ$, and ${\phi_3}^\circ$ are increasing over the range $(\widehat{p}_m, \widehat{p}_M)$, then ${\phi_1}^\circ, {\phi_2}^\circ$, and ${\phi_3}^\circ$ are the equilibrium distributions throughout $(\widehat{p}_m, \widehat{p}_M)$.

Proof. (i) Let contrariwise $(\widehat{\phi}^\circ{}_1, \widehat{\phi}^\circ{}_2, \widehat{\phi}^\circ{}_3)$ be another solution and let, without loss of generality, $\widehat{\phi}^\circ{}_1(p) < \phi^\circ{}_1(p)$ at some p. Then, since $\partial Z_3/\partial\phi_2 < 0$ and $\partial Z_2/\partial\phi_3 < 0$, it should be $\widehat{\phi}^\circ{}_2(p) > \phi^\circ{}_2(p)$ in order for $Z_3(p, \widehat{\phi}^\circ{}_{-3}) = \Pi_3^*$ and it should be $\widehat{\phi}^\circ{}_3(p) > \phi^\circ{}_3(p)$ in order for $Z_2(p, \widehat{\phi}^\circ{}_{-2}) = \Pi_2^*$. Consequently, since $\partial Z_1/\partial\phi_j < 0$ for $j \neq 1$, it would be $Z_1(p, \widehat{\phi}^\circ{}_{-1}) < \Pi_1^*$: a contradiction.

(ii) It must preliminarily be noted that $p < P(K_1)$ at any $p \in (\widehat{p}_m, \widehat{p}_M)$ in A, $B_1 \cup B_2$, and $C_1 \cup C_2 \cup C_3$; hence we are in the circumstances of statement (i). The statement is violated if and only if there is a gap $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m, \widehat{p}_M]$ in S_j for some j, so that $\phi_j(\widetilde{\widetilde{p}}) = \phi_j(\widetilde{p}^+)$. On the other hand, ${\phi_j}^\circ(\widetilde{\widetilde{p}}^+) > \phi_j(\widetilde{p}) = {\phi_j}^\circ(\widetilde{p})$: consequently, either $\widetilde{p}$ or $\widetilde{\widetilde{p}}$ or both are charged with positive probability, contrary to statement (vii) of Proposition 2. ■

Taking account of statement (ii) of Proposition 2, there are three conceivable types of gaps: some subset of $[\widehat{p}_m, \widehat{p}_M]$ is a gap in a single support; some subset of $[\widehat{p}_m, \widehat{p}_M]$ is a gap in all supports; some subset of $[p_m, \widehat{p}_m]$ or of $[\widehat{p}_M, p_M]$ is a gap in two supports. We will show that only gaps of the first type are feasible. This means that gaps may only arise within the range $[\widehat{p}_m, \widehat{p}_M]$, and only in one support at a time. As a consequence the union of all supports equals $[p_m, p_M]$.

Proposition 7 *(i) Assume that some interval $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_m, min(\widehat{p}_M, P(K_1))]$ is a gap for S_i while belonging to S_j and S_r. Then ${\phi_i}^\circ(p) > \phi_i(p)$. As a consequence ${\phi_i}^\circ(p)$ is decreasing in a left neighborhood of $\widetilde{\widetilde{p}}$.*

(ii) $S_1 \cup S_2 \cup S_3 = [p_m, p_M]$

Proof. (i) In $(\widetilde{p}, \widetilde{\widetilde{p}})$ we have

$$\Pi_i^* > Z_i(p, \phi_j(p), \phi_r(p)) \tag{10}$$

$$\Pi_j^* = Z_j(p, \phi_i(p), \phi_r(p)) \tag{11}$$

$$\Pi_r^* = Z_r(p, \phi_i(p), \phi_j(p)). \tag{12}$$

Because of inequality (10), either $\phi_j(p) > \phi_j{}^{\circ}(p)$ or $\phi_r(p) > \phi_r{}^{\circ}(p)$, or both. Assume $\phi_j(p) > \phi_j{}^{\circ}(p)$, then equation (12) implies $\phi_i(p) < \phi_i{}^{\circ}(p)$. Thus $\phi_i{}^{\circ}(p)$ is decreasing in a left neighborhood of $\widetilde{\widetilde{p}}$ since it must be $\phi_i(\widetilde{\widetilde{p}}^{+}) = \phi_i(\widetilde{\widetilde{p}})$. Note that then equation (11) implies $\phi_r(p) > \phi_r{}^{\circ}(p)$.

(ii) To make our point we have to rule out the event of some interval $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_m, \widehat{p}_M]$ being a gap in all supports, and the events of $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m, \widehat{p}_m]$ or $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_M, p_M]$ being a gap in two supports. Let us take the first case first. Arguing *ab absurdo*, let $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_m, min(\widehat{p}_M, P(K_1))]$ be the largest interval constituting a gap in S_1, S_2, and S_3. It must preliminarily be noted that the gap in S_1 must extend on the left of $\widetilde{p}$. In fact, if $\widetilde{p} \in S_1$ so that $\Pi_1^* = \Pi_1(\widetilde{p})$, it would be $\Pi_1(p) > \Pi_1^*$ at p slightly higher than $\widetilde{p}$ - a contradiction - since $dZ_1/dp = \partial Z_1/\partial p$ on a right neighbourhood of $\widetilde{p}$ and, by statement (i) of Lemma 3, $\partial Z_1/\partial p > 0$. To avoid a similar contradiction for firm 2 and 3, it must be $\partial Z_3/\partial p \leq 0$ and $\partial Z_3/\partial p \leq 0$ in a right neighbourhood of $\widetilde{p}$. We will use this fact to prove that $\widetilde{p}$ cannot be in any subset of $[p_m, P(K_1 + K_3)]$. In $[p_m, P(K_1 + K_2)]$,

$$Z_2(p, \phi_1, \phi_3) = p \left\{ \phi_1\phi_3(D(p) - K_1 - K_3) + (1 - \phi_1\phi_3) K_2 \right\}.$$

Then

$$\begin{aligned}
&\tfrac{\partial Z_2}{\partial p} = K_2 + \phi_1\phi_3(D(p) - K + pD'(p)) \geq \\
&\geq K_2 + \tfrac{P(K_1+K_2)-p_m}{P(K_1+K_2)} \tfrac{K_2}{K_3}(D(p) - K + pD'(p)) \geq \\
&\geq K_2\{1 + \tfrac{P(K_1+K_2)-p_m}{P(K_1+K_2)} \tfrac{1}{K_3}[-K_3 + P(K_1 + K_2)D'(p)_{p=P(K_1+K_2)}]\} = \\
&= \tfrac{K_2}{K_3 P(K_1+K_2)}\{p_m K_3 + P(K_1 + K_2)D'(p)_{p=P(K_1+K_2)}[P(K_1 + K_2) - p_m]\} > \\
&\tfrac{K_2}{K_3 P(K_1+K_2)}[\Pi_1^* - (K_1 - K_3)P(K_1 + K_2)] > 0
\end{aligned}$$

The equalities derive from simple manipulation. The first inequality follows from the requirement that $Z_2(p, \phi_1, \phi_3) = p_m K_2$ on a left neighbourhood of

$\widetilde{p}$, implying $\phi_1\phi_3 = \frac{p-p_m}{p}\frac{K_2}{K-D(p)}$ as we are stipulating that $\widetilde{p} \in [p_m, P(K_1 + K_2)]$: thus $\phi_1\phi_3$ is increasing in p and hence not higher than $\frac{P(K_1+K_2)-p_m}{P(K_1+K_2)}\frac{K_2}{K_3}$. The second inequality holds since $(D(p) - K + pD'(p))$ is a decreasing function. The third inequality follows since $pD' + (D - K_2 - K_3) > 0$ throughout $[p_m, p_M)$; the last inequality follows since $\Pi_1^* > p[D(p) - K_2 - K_3]$ throughout $[p_m, p_M]$. Nor can it be $\widetilde{p} \in [P(K_1 + K_2), P(K_1 + K_3)]$, since $\partial Z_3/\partial p = K_3(1 - \phi_1\phi_2) > 0$ over that range.

We must still rule out the event of $\widetilde{p} \in (P(K_1 + K_3), P(K_1))$. If $\widetilde{p}$ is in $[P(K_1 + K_3), P(K_1)]$, also $\widetilde{\widetilde{p}}$ is and either $\widetilde{\widetilde{p}} \in S_2$ or $\widetilde{\widetilde{p}} \in S_3$, or both. Suppose $\widetilde{\widetilde{p}} \in S_3$. From the requirement that $\partial Z_3/\partial p = 0$ at $p = \widetilde{p}$ (otherwise an immediate contradiction obtains) it follows that $\partial Z_3/\partial p < 0$ at $p = \widetilde{\widetilde{p}}$ since Z_3 is concave in p and since $\phi_1(\widetilde{\widetilde{p}}) = \phi_1(\widetilde{p})$ and $\phi_2(\widetilde{\widetilde{p}}) = \phi_2(\widetilde{p})$). But this violates the requirement that $dZ_3/dp = 0$ on a right neighbourhood of $\widetilde{\widetilde{p}}$. A similar contradiction arises if $\widetilde{\widetilde{p}} \in S_2$. Hence no interval $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_m, \widehat{p}_M]$ may be a gap in all supports.

To complete the proof we must rule out the event of $(\widetilde{p}, \widetilde{\widetilde{p}})$ being a gap in S_1 and S_j when $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m, \widehat{p}_m]$ and $L = \{1, j\}$ or $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [\widehat{p}_M, p_M]$ and $M = \{1, j\}$. Similarly as before, in that event the gap in S_1 should extend on the left of $\widetilde{p}$. As a consequence, statement (ii) of Proposition 2 does not hold: a contradiction. ■

We can now complete the analysis to see how equilibrium strategies are determined when gaps arise.

Proposition 8 *Let $N = \{i, j, r\}$ and suppose ${\phi^\circ}_i$ is decreasing on a left neighbourhood of $\widetilde{\widetilde{p}} > p_m^{(3)}$, where $[\widetilde{\widetilde{p}}, p_M^{(3)}]$ is the largest (possibly degenerate) neighbourhood of $p_M^{(3)}$ where ${\phi^\circ}_i$, ${\phi^\circ}_j$, and ${\phi^\circ}_r$ are increasing. Denote by $\widetilde{p}$ the largest solution of ${\phi_i}^\circ(p) = {\phi_i}^\circ(\widetilde{\widetilde{p}})$ in the range $(p_m^{(3)}, \widetilde{\widetilde{p}})$.*

(a) Equilibrium distributions are ${\phi^\circ}_i, {\phi^\circ}_j$, and ${\phi^\circ}_r$ over $[\widetilde{\widetilde{p}}, p_M^{(3)}]$, S_j and S_r are connected throughout $(\widetilde{p}, p_M^{(3)}]$ while $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap in S_i.

(b) Over $(p_m^{(3)}, \widetilde{p})$ if ${\phi^\circ}_i$, ${\phi^\circ}_j$, and ${\phi^\circ}_r$ are increasing, they are the equilibrium distributions. Otherwise there is a gap to be determined as in (a).

(c) If a gap emerges at step (b), a similar statement as in (b) holds for the right neighbourhood of $p_m^{(3)}$ still left to analyze and so on and so forth.[17]

Proof. By construction, each firm gets its equilibrium payoff at any $p \in [\widetilde{\widetilde{p}}, p_M^{(3)}]$ and the same holds for j and r at any $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$, where

[17]By necessity, at some step ${\phi^\circ}_i$, ${\phi^\circ}_j$, and ${\phi^\circ}_r$ are increasing on the right neighbourhood of $p_m^{(3)}$ still left to analyze.

$Z_j(p, {\phi_i}^\circ(\widetilde{p}), \phi_r(p)) = \Pi_j^*$ and $Z_r(p, {\phi_i}^\circ(\widetilde{p}), \phi_j(p)) = \Pi_r^*$. Further, it does not pay for firm i to charge any $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$: $Z_i(p, \phi_j, \phi_r) < \Pi_i^* = Z_i(p, {\phi_j}^\circ, {\phi_r}^\circ)$ since $\phi_j > {\phi^\circ}_j$ and $\phi_r > {\phi^\circ}_r$ throughout $(\widetilde{p}, \widetilde{\widetilde{p}})$. One can argue likewise while moving on the left of $\widetilde{p}$ and up to $p_m^{(3)}$: thus the strategy profile under consideration constitutes an equilibrium.

To check uniqueness, we begin by noting that, by statement (i) of Proposition 7, none of ϕ_i, ϕ_j and ϕ_r can be constant over any interval in $[\widetilde{\widetilde{p}}, p_M^{(3)}]$. By the same token we can dismiss any strategy profile with any subset of $[\widetilde{p}, p_M^{(3)}]$ other than $(\widetilde{p}, \widetilde{\widetilde{p}})$ constituting a gap in S_i. Nor can there be equilibria with a gap $(\overline{p}, \overline{\overline{p}})$ in S_j such that $\overline{\overline{p}} \in (\widetilde{p}, \widetilde{\widetilde{p}})$. This would restrict the gap in S_i to $(q, \widetilde{\widetilde{p}})$, where $q \in [\overline{\overline{p}}, \widetilde{\widetilde{p}})$, so that $\phi_i(\widetilde{\widetilde{p}}) = {\phi^\circ}_i(\widetilde{\widetilde{p}}) = {\phi^\circ}_i(q)$, contrary to the fact that ${\phi^\circ}_i(q) > {\phi^\circ}_i(\widetilde{\widetilde{p}})$. ■

The results of this section allow to supplement Theorem 1 with a uniqueness result.

Theorem 2. *In A, $B_1 \cup B_2$, and $C_1 \cup C_2 \cup C_3$, the equilibrium is unique throughout $[p_m, p_M]$. In F and $E_1 \cup E_2$, all equilibria share the same ϕ_i over range $[p_m, P(K_1)]$.*

6 On the event of a disconnected support

Based on the results above one can compute the mixed strategy equilibrium once the demand function and the firm capacities are fixed. To illustrate how this task is accomplished, in this section we will determine the equilibrium for $(K_1, K_2, K_3) \in B_1$. This region is of special interest because S_3 turns out to be disconnected under well-specified circumstances. But the possibility of gaps is by no means restricted to that region. This will be proved at the end of the section, by means of a numerical example yielding a gap in S_2 for $(K_1, K_2, K_3) \in C_1$. The example also shows that range $[\widetilde{\widetilde{p}}, p_M^{(3)}]$ may in fact be degenerate, as acknowledged in Proposition 8.

In region B_1 we partition the range $[p_m, p_M]$ into three subsets: $\alpha = [p_m, P(K_1 + K_2)$, $\beta = [P(K_1 + K_2), P(K_1 + K_3))$, and $\gamma = [P(K_1 + K_3), p_M]$. In α the equations in system (7) read

$$\begin{cases} \Pi_1^* = p[\phi_{2\alpha}\phi_{3\alpha}(D(p) - K_2 - K_3) + (1 - \phi_{2\alpha}\phi_{3\alpha})K_1] \\ \Pi_2^* = p[\phi_{1\alpha}\phi_{3\alpha}(D(p) - K_1 - K_3) + (1 - \phi_{1\alpha}\phi_{3\alpha})K_2] \\ \Pi_3^* = p[\phi_{1\alpha}\phi_{2\alpha}(D(p) - K_1 - K_2) + (1 - \phi_{1\alpha}\phi_{2\alpha})K_3], \end{cases}$$

and the solution is

$$\phi^\circ{}_{1\alpha} = \sqrt{\frac{K_2}{K_1}\frac{(p_m - p)K_3}{p(D(p) - K)}}, \phi^\circ{}_{2\alpha} = \frac{K_1}{K_2}\phi^\circ{}_{1\alpha}, \phi^\circ{}_{3\alpha} = \frac{K_1}{K_3}\phi^\circ{}_{1\alpha}. \tag{13}$$

In β, the equations in system (7) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\beta}\phi_{3\beta}(D(p) - K_2 - K_3) + \phi_{2\beta}\left(1 - \phi_{3\beta}\right)(D(p) - K_2) + \left(1 - \phi_{2\beta}\right)K_1\right], \\ \Pi_2^* = p[\phi_{1\beta}\phi_{3\beta}(D(p) - K_1 - K_3) + \phi_{1\beta}\left(1 - \phi_{3\beta}\right)(D(p) - K_1) + \left(1 - \phi_{1\beta}\right)K_2], \\ \Pi_3^* = p[\phi_{1\beta}\left(1 - \phi_{2\beta}\right) + \left(1 - \phi_{1\beta}\right)]K_3, \end{cases}$$

and the solution is

$$\phi^\circ{}_{1\beta} = \sqrt{\frac{K_2}{K_1}\frac{(p - p_m)}{p}}, \phi^\circ{}_{2\beta} = \frac{K_1}{K_2}\phi_{1\beta}, \phi^\circ{}_{3\beta} = \frac{D(p) - K_1 - K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_{1\beta}. \tag{14}$$

In γ, the equation in system (7) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\gamma}\phi_{3\gamma}(D(p) - K_2 - K_3) + p\phi_{2\gamma}\left(1 - \phi_{3\gamma}\right)(D(p) - K_2)\right. \\ \qquad\qquad \left. + \left(1 - \phi_{2\gamma}\right)\phi_{3\gamma}(D(p) - K_3) + \left(1 - \phi_{2\gamma}\right)(1 - \phi_{3\gamma})K_1\right] \\ \Pi_2^* = p\left[\phi_{1\gamma}\left(1 - \phi_{3\gamma}\right)(D(p) - K_1) + \left(1 - \phi_{1\gamma}\right)K_2\right] \\ \Pi_3^* = p\left[\phi_{1\gamma}\left(1 - \phi_{2\gamma}\right)(D(p) - K_1) + \left(1 - \phi_{1\gamma}\right)K_3\right], \end{cases}$$

and the solution is

$$\begin{aligned} \phi^\circ{}_{1\gamma} &= \sqrt{\frac{K_2K_3(p - p_m)^2}{p^2(D(p) - K_1 - K_2)(D(p) - K_1 - K_3) + (p - p_m)K_1p(D(p) - K_1)}}, \\ \phi^\circ{}_{2\gamma}(p) &= 1 - \frac{K_3}{K_2} + \frac{K_3}{K_2}\phi^\circ{}_{3\gamma} \\ \phi^\circ{}_{3\gamma} &= \frac{(p - p_m)K_2 + p\phi^\circ{}_{1\gamma}(p)(D(p) - K_1 - K_2)}{p\phi^\circ{}_{1\gamma}(D(p) - K_1)}. \end{aligned}$$

In range α, $\phi^{\circ\prime}_{i\alpha} > 0$. (If $\phi^{\circ\prime}_{i\alpha} \leq 0$ for some i, then $\phi^{\circ\prime}_{j\alpha} \leq 0$ for all $j \neq i$, thereby violating the requirement that $\Pi_i' = 0$ since Lemma 3 holds.) On the other hand, while $\phi^\circ{}_{1\alpha}(P(K_1 + K_2)) < 1$ and $\phi^\circ{}_{2\alpha}(P(K_1 + K_2)) < 1$ (the latter is checked by simple manipulation and using the fact that $\Pi_1^* > p(D - K_2 - K_3)$) throughout $[p_m, p_M)$) it might be $\phi^\circ{}_{3\alpha}P(K_1 + K_2) \geq 1$ (as illustrated by the third example below), which would obviously prevent the equilibrium distributions from coinciding with the ${\phi^\circ}_{i\alpha}$'s throughout α. In range γ, $\phi_{1\gamma}(p_M) < 1 = \phi_{2\gamma}(p_M) = \phi_{3\gamma}(p_M)$ and $\phi'_{i\gamma} > 0$ in the interior of γ, with $\phi'_{3\gamma} = \phi'_{2\gamma} = 0$ at $p = p_M$.[18] As to range β, ${\phi^\circ}_{i\beta}(P(K_1+K_3)) < 1$ for all i. This is seen almost immediately as far as ${\phi^\circ}_{1\beta}$ is concerned. As to ${\phi^\circ}_{j\beta}$ $(j \neq 1)$, by simple computations it is found that ${\phi^\circ}_{j\beta}(P(K_1+K_3)) < 1$ if and only if $\Pi_1^* > (K_1 - K_2)P(K_1 + K_3)$, which certainly holds since $\Pi_1^* > p(D - K_2 - K_3)$ throughout $[p_m, p_M)$.

It might be ${\phi^\circ}'_{3\beta} < 0$ in a left neighbourhood of $P(K_1+K_3)$. Note that

$$
{\phi^\circ}'_{3\beta} = \frac{D'(p)}{K_3} + \frac{K_1}{K_3}{\phi^\circ}'_{1\beta} = \frac{D'(p)}{K_3} + \frac{1}{2}\left(\frac{K_2}{K_1}\frac{(p-p_m)}{p}\right)^{-1/2}\frac{K_2}{K_3}\frac{p_m}{p^2}.
$$

Since ${\phi^\circ}'_{3\beta}$ is decreasing, it will be ${\phi^\circ}'_{3\beta} > 0$ throughout β if and only if $[{\phi^\circ}'_{3\beta}]_{p=P(K_1+K_3)} \geq 0$. This in its turn amounts to

$$
K_2 p_m \geq -2\left[D'(p)\right]_{p=P(K_1+K_3)} \times [P(K_1+K_3)]^2 \sqrt{\frac{K_2}{K_1}\left(1 - \frac{p_m}{P(K_1+K_3)}\right)}. \tag{15}
$$

If this inequality holds, then equilibrium distributions are actually the ${\phi^\circ}_{i\beta}$'s throughout β.(Note that in this case, ${\phi^\circ}_{3\alpha}(P(K_1+K_2)) < 1$ since ${\phi^\circ}''_{3\beta} < 0$ throughout β.) If not, then, by Proposition 8, there is a gap $[\widetilde{p}, P(K_1+K_3)]$ in S_3. Two cases are possibile according as to whether ${\phi^\circ}_{3\beta}(P(K_1+K_3)) \geq {\phi^\circ}_{3\beta}(P(K_1+K_2))$ or ${\phi^\circ}_{3\beta}(P(K_1+K_3)) < {\phi^\circ}_{3\beta}(P(K_1+K_2))$. In the former case $\widetilde{p}$ is such that ${\phi^\circ}_{3\beta}(\widetilde{p}) = {\phi^\circ}_{3\beta}(P(K_1+K_3))$, in the latter it is such that ${\phi^\circ}_{3\alpha}(\widetilde{p}) = {\phi^\circ}_{3\beta}(P(K_1+K_3))$. In the former case, the equilibrium distributions are provided by equations (13) throughout α and by equations (14) over subset $[P(K_1+K_2), \widetilde{p}]$ of β, the remaining subset $[\widetilde{p}, P(K_1+K_3)]$ being the gap in S_3: here $\phi_3 = {\phi^\circ}_{3\beta}(P(K_1+K_3))$, $\phi_1 = \frac{\Pi_2^* - pK_2}{p[D(p) - K_1 - K_2 - \phi_3 K_3]}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p[D(p) - K_1 - K_2 - \phi_3 K_3]}$. In the latter case, equations (13) provide the equilibrium distributions over subset $[p_m, \widetilde{p}]$ of α and $\phi_3 = {\phi^\circ}_{3\beta}(P(K_1+K_3))$ throughout range $[\widetilde{p}, P(K_1+K_3)]$, the gap in S_3. Now $\phi_1 = \frac{\Pi_2^* - pK_2}{p\phi_3(D(p) - K)}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p\phi_3(D(p) - K)}$ over subset $[\widetilde{p}, P(K_1+K_2)]$ of the gap and $\phi_1 = \frac{\Pi_2^* - pK_2}{p[D(p) - K_1 - K_2 - \phi_3 K_3]}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p[D(p) - K_1 - K_2 - \phi_3 K_3]}$ over the remaining subset $[P(K_1+K_2), P(K_1+K_3)]$.

[18] On all this, see the appendix in De Francesco and Salvadori (2008).

We provide one example for each of the three cases which can arise for $(K_1, K_2, K_3) \in B_1$: no gap in any S_i, a gap in S_3 with $\widetilde{p} \in \beta$, a gap in S_3 with $\widetilde{p} \in \alpha$.

First example: $D(p) = 10 - p$, $K_1 = 5.98$, $K_2 = 1$, and $K_3 = 0.97$. Then $p_M = 4.015$, $p_m = 4.015^2/5.98$, and $\Pi_i^* = p_m K_i$ for each i. Condition (15) is met, hence $S_i = [p_m, p_M]$ for all i.

Second example: $D(p) = 10 - p, K_1 = 23/4$, $K_2 = 3$, $K_3 = 2$. Then $p_M = 2.5$, $p_m = 25/23$, and $\Pi_i^* = \Pi_i^* = p_m K_i$ for each i. Condition (15) is violated, hence ϕ_3 is constant over range $[\widetilde{p}, P(K_1 + K_3)]$, where $P(K_1+K_3) = 2.25$. It is easily found that $\widetilde{p} \approx 1.57358 > P(K_1+K_2) = 1.25$.

Third example: $D(p) = 10 - p$, $K_1 = 5.45$, $K_2 = 3$, and $K_3 = 2.2$. Then $p_M = 2.4$, $p_m = 2.4^2/5.45$, and $\Pi_i^* = p_m K_i$ for each i. Condition (15) is violated, hence ϕ_3 is constant over range $[\widetilde{p}, P(K_1 + K_3)]$, where $P(K_1+K_3) = 2.35$. It is easily found that $\widetilde{p} \approx 1.48165 < P(K_1+K_2) = 1.55$. In fact, one can also easily check that $\phi^\circ{}_{3\alpha} P(K_1 + K_2) \approx 1.036$.

Finally, to get further insights on gaps we work out an example for region C_1. Let $D(p) = 20 - p$ and $(K_1, K_1, K_1) = (15, 4, 0.5)$. Then, $p_M = 7.75$, $\Pi_1^* = 60.0625$, $p_m = 4.0041\overline{6}$, and $\Pi_2^* = 16.01\overline{6}$. Note that $(15, 4, 0.5) \in C_1$ since $P(K_1 + K_2) = 1 < p_m = 4.0041\overline{6} < P(K_1 + K_3) = 4.5$. We partition $[p_m, p_M]$ into $\alpha = [p_m, p_m^{(3)})$, $\beta = [p_m^{(3)}, p_M^{(3)})$, and $\gamma = [p_M^{(3)}, p_M]$. In α, $\phi_{1\alpha} = \frac{4(4.0041\overline{6}-p)}{p(1-p)}$ and $\phi_{2\alpha} = \frac{15(4.0041\overline{6}-p)}{p(1-p)}$. One can easily check that $\arg\max_{p\in[p_m, P(K_1)]} Z_3(p, \phi_{1\alpha}, \phi_{2\alpha}) = P(K_1 + K_3)$, hence $p_m^{(3)} = 4.5$ and $\Pi_3^* = \Pi_3(p_m^{(3)}) \approx 2.11620$. To find $p_M^{(3)}$, note that, in γ, $\phi_{1\gamma} = 1 - (p_m/p) = 1 - (4.0041\overline{6}/p)$ and $\phi_{2\gamma} = \frac{p(D(p)-K_3)-\Pi_1^*}{pK_3} = 2\frac{p(19.5-p)-60.0625}{p}$. Then the equation $Z_3(p, \phi_{1\gamma}, \phi_{2\gamma}) = \Pi_3^*$ over range $[p_m, P(K_1)]$ yields $p_M^{(3)} \approx 4.66038$. Turning to range β, denote the relevant[19] solutions of the equations in system (7) by $\phi^\circ{}_{1\beta}, \phi^\circ{}_{2\beta}$, and $\phi^\circ{}_{3\beta}$. One can check that $\left[\phi^{\circ\prime}{}_{2\beta}(p)\right]_{p=p_M^{(3)}} < 0$. Therefore, there is a gap $[\widetilde{p}, \widetilde{\widetilde{p}}]$ in S_2, with $\widetilde{\widetilde{p}} = p_M^{(3)}$. As to $\widetilde{p}$, this is found by solving $\phi^\circ{}_{2\beta}(p) = \phi^\circ{}_{2\beta}(\widetilde{\widetilde{p}}) = .487931$ over $(p_m^{(3)}, p_M^{(3)})$, which yields $\widetilde{p} \approx 4.57316$. Further, one can check that $\phi^\circ{}_{1\beta}(p)$, $\phi^\circ{}_{2\beta}(p)$, and $\phi^\circ{}_{3\beta}(p)$ are all increasing throughout $[p_m^{(3)}, \widetilde{p}]$, so there are no further gaps. To sum up: $S_1 = [4.0041\overline{6}, 7.75]$, $S_2 = [4.0041\overline{6}, 4.57316] \cup [4.66038, 7.75]$, and $S_3 = [4.5, 4.66038]$.

[19] The equations in system (7) brings to a second degree algebraic equation, only one of the solutions for $\phi^\circ{}_{2\beta}$ being nonnegative.

7 Concluding remarks

In this paper we have extended the analysis of price competition among capacity-constrained sellers beyond the duopoly and symmetric oligopoly cases. We have first derived some general results on the mixed strategy equilibrium under oligopoly - among them, the fact that the minimum of the support of the equilibrium strategy is determined for the largest firm like in duopoly (a similar result was recently provided as for the maximum). It emerged in the course of our investigation that mixed strategy equilibria might look quite different depending on the firm capacities: supports of the equilibrium strategies may or may not coincide across all the firms, the equilibrium need not be fully determined as far as the firms other than the largest one are concerned, and equilibrium payoffs may or may not be proportional to capacities.[20] Thus a complete characterization of mixed strategy equilibrium requires a taxonomy, and we have provided it for the case of triopoly. We have partitioned the region of the capacity space where the equilibrium is mixed into several subregions according to the set of properties of the equilibrium which turns out to be specific to each subregion. Another novel feature - in the context of concave demand, constant and identical unit cost and efficient rationing - revealed by our analysis is the possibility of some support of equilibrium strategies being disconnected, and we have showed how gaps are actually determined in that event. Having made the taxonomy of mixed strategy equilibria - in terms to the determination of the minima and the maxima of the supports, the equilibrium payoffs of the firms, and the degree of determinateness of the equilibrium - and having seen how any gap is determined, computing the mixed strategy equilibrium is an easy task, as exemplified in Section 6.

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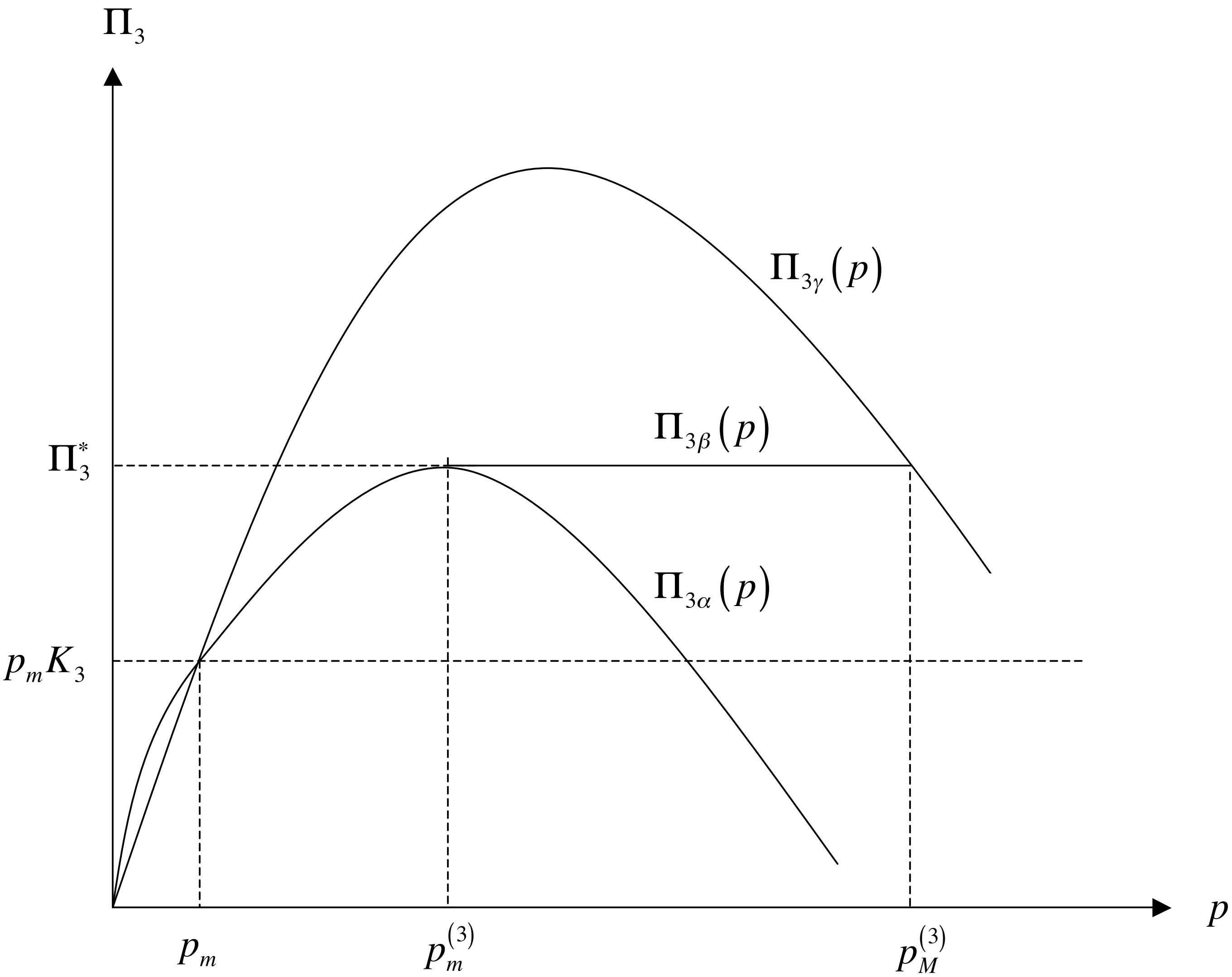

Figure 1: $\Pi_{3\alpha}(p) := Z_3(p, \phi_{1\alpha}(p), \phi_{2\alpha}(p))$, where $p_m K_1 = Z_1(p, \phi_{2\alpha}(p), 0)$ and $p_m K_2 = Z_2(p, \phi_{1\alpha}(p), 0)$; $\Pi_{3\gamma}(p) := Z_3(p, \phi_{1\gamma}(p), \phi_{2\gamma}(p))$, where $p_m K_1 = Z_1(p, \phi_{2\gamma}(p), 1)$ and $p_m K_2 = Z_2(p, \phi_{1\gamma}(p), 1)$