TIME VALUE OF MONEY: APPLICATION AND RATIONALITY- AN APPROACH USING DIFFERENTIAL EQUATIONS AND DEFINITE INTEGRALS

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Abstract

The time value of money is one of the most important concepts in finance. Money that a firm has in its possession today is more valuable than future payments because today's money can be invested to earn positive returns in future. The principles of time value analysis have many applications, ranging from setting up schedules for paying off loans to decisions about whether to acquire new equipment. Problems concerning Time Value of Money, which involves calculation of these concepts, are usually solved by algebraic formulae. This paper attempts to solve such problems using differential equations & definite integral techniques and makes a comparison with the results obtained by the traditional method.

1.0 Introduction:

The Time Value of Money is one of the most important concepts in finance. The same amount of money of today is more valuable than that in the future because the value of money is always decreasing. So, the timing of cash outflows and inflows has important economic consequences. The financial decision making is based on the core concepts of interest, present value, annuities, amortization, creation of sinking funds, etc.

Problems concerning Time Value of Money which involves calculation of compound interest and future value are often solved by algebraic formulae, which involves exponents, logarithms etc.

This paper attempts to solve such problems using differential equations & definite integral. From that point of view this study is somewhat diagnostic format and tries to establish some formulae related to differential equation and definite integral regarding the solutions of the problems of time value of money.

2.0 Time Value of Money:

Money has time value. The value of Tk. 1 today is more worthy than the value of Tk. 1 tomorrow. This economic principle recognizes that the passage of time affects the value of money. This relationship between time and money is called the 'Time Value of Money'.

Everyone becomes involved in transactions where interest rates affect the amount to be paid or received. The value of the money is changed considering that interest rate together with time. Modern business relies 100 percent on the time value of money and no one can move a step without neglecting the changing rate.

3.0 Compound Interest And The Future Value:

Compound interest is the interest that is earned on a given deposit and has become part of the principal at the end of a specified period. The future value of a present amount is found by applying compound interest over a specific period of time. The consequence of compounding is that at any time ***t***, the amount in the account is ***y(t)*** which is increasing at the rate of the given interest. If the interest rate is K percent and specified period of time is very small, this equation approximately can be written as

$$\frac{dy(t)}{dt} = \frac{K}{100} y(t) \text{..........}(i)$$

whose solution is $y = A e^{\frac{Kt}{100}} \text{...............}(ii)$

Now, if the initial amount is $y(0) = y_0$ when $t = 0$,

we get $y_0 = A$, and equation (ii) becomes,

$$y = y_0 e^{\frac{Kt}{100}}$$

i.e.

Future Amount = (Principal Amount) e $^{\text{(interest rate) (time)}}$

Illustration 3.1 **If Tk. 500 is invested at 6 percent compounded monthly, what will be the future value after 30 years?**

Given Principal Amount = Tk. 500, Interest rate = 6 percent = 0.005/month,

Time = 30 years = 360 months.

According to our rule we get

$$\text{Future Amount} = 500e^{0.005(360)} = 3024.82$$

i.e. Future Amount = Tk . 3024.82

Using algebraic formula we get

$$\text{Future Amount} = \text{Tk}.\ 500\,(1 + 0.005)^{360} = \text{Tk}.\ 3011.29$$

Here error in the amount = 0.45 percent, which is not significant.

4.0 Ordinary Annuity: Future Value

An annuity is a series of equal payments made at fixed intervals for a specified number of periods. Annuities can be classified by when they begin and end, by when the payments are made, and by whether or not the payment intervals coincide with the interest intervals.

An ordinary or deferred annuity consists of series of equal payments made at the end of each period.

Now if the interest rate is i , and the first payment is $A,$ then according to the previous section the change in the amount deposited at any time t is given by

$$A(t) = Ae^{it}$$

If the interval is small then the total future value after all payments will be approximately the area under the increasing rate curve.

Now if the total period t = n

Then

$$A(n) = \int_0^n A\, e^{it}\, dt$$

which is the formula to calculate the future value for ordinary annuity.

Illustration 4.1 If Tk. 100 is deposited in an account at the end of every month for the next 5 years, how much will be in the account at the time of the final deposit if interest rate is 8 % compounded monthly?

Given , A = Tk. 100,

Interest rate i = 8 percent compounded monthly = 2/300

Total period n = 5 years= 60 months

Then we have from the formula

$$A(n) = \int_0^n A\, e^{it}\, dt = \frac{A}{i}\left[e^{in} - 1\right]$$
$$= \frac{100 \times 300}{2}\left(e^{0.4} - 1\right)$$
$$= 7377.37$$

Hence the future amount after 5 years is Tk. 7377.37

Using algebraic formula we get

Future amount = 100 [(1+ 2/300)^{60} –1] / (2/300)= TK. 7347.69

and error in the amount = .40 percent, which is not significant.

5.0 Ordinary Annuity: Sinking Fund

To generate the future value of an annuity, periodic payments are made into a sinking fund. A sinking fund is a fund into which periodic payments are made in order to accumulate a specified amount at some point in the future.

From the future value formula of ordinary annuity we have

$$A(n) = \int_0^n A\, e^{it}\, dt$$

Hence the formula for sinking fund payment is

$$A = \frac{A(n)}{\int_0^n e^{it}\,dt}$$

Illustration 5.1 **How much should be deposited in a sinking fund at the end of each month for 5 years to accumulate Tk. 10,000 if the fund earns interest 8 percent compounded monthly?**

Here, A (n) = 10,000, where n = 5 years = 60 months,

Interest rate i = 8 percent compounded monthly = 2/300

So, we have from the formula for the sinking fund payment

$$A = \frac{A(n)}{\int_0^n e^{it}\,dt} = \frac{10{,}000}{\frac{1}{i}(e^{in}-1)}$$

$$= \frac{10000}{\frac{300}{2}(e^{0.4}-1)}$$

$$= \frac{10000}{73.7737}$$

$$=135.55$$

Over the life of the sinking fund, the sum of the deposits will be 60 (135.55) = TK. 8133

This sum plus interest earned will provide the desired Tk. 10,000.

Using algebraic formula we get

$$\text{Sinking fund} = 10000\left[\frac{\frac{2}{300}}{\left(1+\frac{2}{300}\right)^{60}-1}\right] = 136.10$$

Error in the amount = .40 percent, which is insignificant.

6.0 Ordinary Annuity: Present Value

The present value formula of an annuity can be generated from the compounded present value formula together with the annuity future value formula.

Now, if we refer Principal amount =Present value =P, Future amount =F, Interest rate = i, and time t = (no. of years) x (no. of period per year)= n

then, $F = Pe^{it} = Pe^{in}$ ……..(i)

Also we know, $A(n) = \int_0^n A\,e^{it}\,dt$, where A (n) is the future value after completion of the period n,

Hence we can write, Future Value after period n is $F = \int_0^n A\,e^{it}\,dt$ …….(ii)

Combining equations (i) and (ii)

we have the formula for the present value of the ordinary annuity as

$$P = \frac{A}{e^{in}} \int_0^n e^{it}\,dt$$

Illustration 6.1 The directors of a company have voted to establish a fund that will pay a retiring accountant or his estate Tk. 1000 per month for the next 10 years, with the first payment to be made a month from now. How much should be placed in the fund if it earns interest at 7% compounded monthly? How much interest will the fund earn during its existence?

Given payment per period A = Tk. 1000, i = 7 percent compounded monthly = 0.07 / 12,

Total period n = 10 years = 120 months.

We have $\int_0^n e^{it}\,dt = \frac{1}{i}(e^{in} - 1)$

Hence according to the formula

$$P = \frac{A}{e^{\text{in}}} \int_0^n e^{\text{it}}\, dt = \frac{A}{ie^{\text{in}}}\left[e^{in} - 1\right]$$

$$= \frac{A}{\text{i}}\left[1 - e^{-in}\right]$$

$$= \frac{1000 \times 12}{0.07}\left[1 - e^{-0.7}\right]$$

$$= 86299.66$$

Hence P = Tk. 86299. 66 should be placed in the fund.

For calculating interest, we have (120 payments) x (Tk. 1000 per payment) = Tk. 120, 000

Interest earned = Tk. 120, 000 – 86299.66 = Tk. 33700.34

Using algebraic formula we get

Present Value = 1000 [1- (1+ .07/12)$^{-120}$] / (.07/12)= TK. 86126.35

Hence error in the amount = 0.20 percent, which is a negligible amount.

7.0 Ordinary Annuity: Amortization

Loan Amortization is the determination of the equal periodic loan payments necessary to provide a lender with a specified interest return and to repay the loan principal over a specified period. Prominent examples of amortization are loans taken to buy a flat or a car and amortized over a period of 5 to 10 years. Given the amount of the loan (the current principal, P), the number of periods (n) , and the interest rate (i), the quantity to be calculated is A, the amount of the period payment. The n payments of A Taka each constitute an ordinary annuity whose present value is P, and we have learned that

$$Pe^{\text{in}} = \int_0^n A\, e^{\text{it}}\, dt$$

Solving this for the unknown A, we have

$$\boxed{\text{Amortization Payment:} \qquad A = \frac{Pe^{in}}{\int_0^n e^{it}\, dt}}$$

Illustration 7.1 **If some one borrow Tk. 5000 . He will amortize the loan by half-monthly payments of Tk. A each over a period of 3 years. Find the half-monthly payment if interest is 12 percent compounded half-monthly. Also find the total amount that the person will pay.**

Given P = Tk. 5000, Interest rate i = 12 % compounded half-monthly = 0.12/ 24 = 0.005 per half month, n = (3 years)x (24 half months per year) = 72 periods.

Here $\int_0^n e^{it}\,dt = \frac{1}{i}(e^{in} - 1)$

Hence

$$A = \frac{Pe^{in}}{\int_0^n e^{it}\,dt} = \frac{iPe^{in}}{(e^{in} - 1)}$$

$$= \frac{iP}{1 - e^{-in}}$$

$$= \frac{.005 \times 5000}{1 - e^{-0.005 \times 72}} = 82.69$$

$\therefore$ The amortization payment is Tk 82.69

From algebraic formula we get A = 5000[0.005/ 1- $(1+.005)^{-72}$]= 82.86

Hence there is an error of 0.21 percent which is very insignificant.

Amortization Schedule, 12% interest compounded half-monthly.

Period	Beginning Amount (a)	Payment (b)	Interest @ 0.005 per half month (c)	Repayment of principal d = (b-c)	Remaining balance e = (a -d)
1	5000.00	82.69	25.00	57.69	4942.31
2	4942.31	82.69	24.71	57.98	4884.33
3	4884.33	82.69	24.42	58.27	4826.06
4	4826.06	82.69	24.13	58.56	4767.50
5	4767.50	82.69	23.84	58.85	4708.65
6	4708.65	82.69	23.54	59.15	4649.50
7	4649.50	82.69	23.25	59.44	4590.06
8	4590.06	82.69	22.95	59.74	4530.32
9	4530.32	82.69	22.65	60.04	4470.28
10	4470.28	82.69	22.35	60.34	4409.95

11	4409.95	82.69	22.05	60.64	4349.30
12	4349.30	82.69	21.75	60.94	4288.36
13	4288.36	82.69	21.44	61.25	4227.11
14	4227.11	82.69	21.14	61.55	4165.56
15	4165.56	82.69	20.83	61.86	4103.70
16	4103.70	82.69	20.52	62.17	4041.53
17	4041.53	82.69	20.21	62.48	3979.04
18	3979.04	82.69	19.90	62.79	3916.25
19	3916.25	82.69	19.58	63.11	3853.14
20	3853.14	82.69	19.27	63.42	3789.71
21	3789.71	82.69	18.95	63.74	3725.97
22	3725.97	82.69	18.63	64.06	3661.91
23	3661.91	82.69	18.31	64.38	3597.53
24	3597.53	82.69	17.99	64.70	3532.83
25	3532.83	82.69	17.66	65.03	3467.80
26	3467.80	82.69	17.34	65.35	3402.45
27	3402.45	82.69	17.01	65.68	3336.78
28	3336.78	82.69	16.68	66.01	3270.77
29	3270.77	82.69	16.35	66.34	3204.43
30	3204.43	82.69	16.02	66.67	3137.77
31	3137.77	82.69	15.69	67.00	3070.76
32	3070.76	82.69	15.35	67.34	3003.43
33	3003.43	82.69	15.02	67.67	2935.76
34	2935.76	82.69	14.68	68.01	2867.74
35	2867.74	82.69	14.34	68.35	2799.39
36	2799.39	82.69	14.00	68.69	2730.70
37	2730.70	82.69	13.65	69.04	2661.66
38	2661.66	82.69	13.31	69.38	2592.28
39	2592.28	82.69	12.96	69.73	2522.55
40	2522.55	82.69	12.61	70.08	2452.48
41	2452.48	82.69	12.26	70.43	2382.05
42	2382.05	82.69	11.91	70.78	2311.27
43	2311.27	82.69	11.56	71.13	2240.14
44	2240.14	82.69	11.20	71.49	2168.65
45	2168.65	82.69	10.84	71.85	2096.80
46	2096.80	82.69	10.48	72.21	2024.59
47	2024.59	82.69	10.12	72.57	1952.03
48	1952.03	82.69	9.76	72.93	1879.10
49	1879.10	82.69	9.40	73.29	1805.80
50	1805.80	82.69	9.03	73.66	1732.14
51	1732.14	82.69	8.66	74.03	1658.11
52	1658.11	82.69	8.29	74.40	1583.71
53	1583.71	82.69	7.92	74.77	1508.94
54	1508.94	82.69	7.54	75.15	1433.80
55	1433.80	82.69	7.17	75.52	1358.27
56	1358.27	82.69	6.79	75.90	1282.38
57	1282.38	82.69	6.41	76.28	1206.10
58	1206.10	82.69	6.03	76.66	1129.44
59	1129.44	82.69	5.65	77.04	1052.39
60	1052.39	82.69	5.26	77.43	974.97

61	974.97	82.69	4.87	77.82	897.15
62	897.15	82.69	4.49	78.20	818.95
63	818.95	82.69	4.09	78.60	740.35
64	740.35	82.69	3.70	78.99	661.36
65	661.36	82.69	3.31	79.38	581.98
66	581.98	82.69	2.91	79.78	502.20
67	502.20	82.69	2.51	80.18	422.02
68	422.02	82.69	2.11	80.58	341.44
69	341.44	82.69	1.71	80.98	260.46
70	260.46	82.69	1.30	81.39	179.07
71	179.07	82.69	0.90	81.79	97.28
72	97.28	82.69	0.49	82.20	15.07

8. Results And Discussions:

The formulae we used in solving problems as given in illustrations have most been newly derived. These formulae other than used algebraic procedures, utilize solution techniques of differential equations and definite integrals. These methods and techniques are new ones. This paper attempts to look at The – Time – Value – Of – Money problems with more advanced mathematical formulae which give results with some errors. These results are shown in Amortization table. This discrepancy could, presumably in near future, be properly handled by some researcher interested in the field.

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