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Abstract

We propose a model of an incomplete markets economy with production, in which the firm acts as financial innovator by issuing claims against its stock. The firm's objective is to maximize its *adjusted* value, which is the sum of the market value and the shareholders' surplus from their trades in the stock markets. If a firm maximizes its adjusted value, then its financial policy is relevant (i.e., Modigliani-Miller theorem does not hold), equilibrium outcomes are stable to shareholders' renegotiation and endogenously incomplete markets typically arise at the equilibrium. If the firm is competitive in the financial markets, the adjusted value coincides with the Grossman-Hart objective.

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1 Introduction

When the financial markets are complete, all shareholders of a perfectly competitive firm attach the same value to any given investment plan and unanimously agree that market value maximizing production plans are the optimal ones. For a firm that has market power, the price effect of a production decision may generates a conflict between the interests of shareholders as consumers and as receivers of the firm's dividends. That is because higher profits may come at the expense of higher prices for some goods that a shareholder consumes (or lower prices for the goods he/she owns and sells). This may render value maximization undesired –from shareholders' points of view– and thus an unjustified objective. Alternative objectives for the imperfectly competitive firm were defined, for example, in [1] and [8].

Market incompleteness adds a new dimension to the problem, because in such an environment a firm's production decision not only modifies the supply of consumption goods in the economy, but it may also change the asset span. The firm's equity contract as well as other securities that the firm may issue to finance its production are risk-hedging instruments that may not be replicable by a portfolio of the other traded securities. In that case, the firm-issued securities cannot be priced in the existing markets and, as a result, different shareholders may attach different values to the same investment/production plan. When evaluating a particular production plan, every shareholder assesses not only its impact on the firm's market value and the change in relative prices that it induces, but also the risk-hedging opportunities that the firm-issued securities offer. It may be that the value maximizing production plan is a riskier alternative than some other plan, which generates a lower value. Depending on their wealth, preferences and attitudes toward risk, different shareholders may favor lower risk over higher market value, or the other way round. These three effects (which we call the income, price and portfolio risk effects, respectively) may influence a shareholder's wealth differently – this is the source of shareholder's conflict of interests – and have different impact on different shareholders – which generates the disagreement among them. Note that the first two effects can arise in complete *and* incomplete markets. The third effect arises *only* in incomplete markets.

A special case of production under incomplete markets occurs when the firm's production set is contained in the asset span (a condition known in the literature as "spanning property"). In this case, the firm's decisions do not affect the asset market structure and firm's problem is then essentially the same as in complete markets (see, for example, Ekern-Wilson [11], Leland, [16], Radner, [20] and chapter 6 in Magill-Quinzii [18]). Therefore, a firm's production decision can have only an income and/or a price effect on its shareholders' wealth, but no portfolio risk effect.

The following diagram summarizes the possible effects that a firm's choice may have on its shareholders' wealth, depending on the environment in which the firm acts and its market power.

	complete markets	incomplete markets	
		spanning	no spanning
Perfectly competitive (no market power)	income	income	income+risk
Market power	income+price	income+price	income+price+risk

Shareholders unanimously approve profit maximization whenever the firm's choices have only an income effect on their wealth. Disagreement among shareholders concerning the firm's objective and incompatibility of profit maximization with the shareholders' interests may arise if two or more effects are present.

The above discussion points out to the fact that, as soon as one leaves the idealistic environment of complete and perfectly competitive markets, profit/value maximization by a privately owned firm may be incompatible with the preferences of that firm's shareholders and thus it is an unjustified and inadequate objective for firms that have market power or act in an incomplete markets environment (or both).

The first attempts to solve this problem (in a more general framework than multiplicative uncertainty or spanning) were made by Drèze [9] and later Grossman and Hart [14]. For two-period economies with uncertainty and incomplete financial markets, Drèze [9] proposed a Pareto-Nash criterion to discard "unreasonable" (from the shareholders' point of view) choices of a perfectly competitive firm. The criterion requires that the firm's decision respect the unanimity among shareholders, provided that they can make side payments in consumption good at date 0 to achieve unanimity. Shareholders are not allowed, however, to make any changes in their portfolio holdings, *including* their share holdings. Drèze interpreted that as respecting the firm's *final* shareholders' interests and proved that this requirement is equivalent to maximizing firm's value, taking as given a system of state prices that is a weighted average of the shareholders' marginal rates of substitution, with weights equal to their final shares.

The equilibrium in which firms follow Drèze's objective satisfies the first order conditions for constrained Pareto optimality. However, because of the non-convexity of the constrained feasible set, the first order conditions may not be sufficient for optimality. As a consequence, at a Drèze equilibrium firm's final shareholders may have incentives to re-trade their shares. In other words, there is a conflict between the interests of a consumer as a final shareholder of the firm and his interests as an initial shareholder[1]. This conflict creates problems for the extension of the model to environments with multiple trading periods, where one period's final shareholders become the next period's initial

[1] Dierker-Dierker-Grodal proved that when following Drèze's objective, the firm may in fact select a production plan at which the initial shareholders' surplus is <u>minimized</u> (see [6]).

shareholders[2].

As a first step toward an extension of Drèze model to an environment with more than one trading period, Grossman and Hart [14] pointed out to the necessity of investigating the problem from the perspective of the *initial* shareholders. Since initial shareholders may eventually trade their shares, they need to have some expectations, or perceptions about shares' prices, as being related to the choice of the production plans. Hence, the authors introduced the "competitive price perceptions" assumption. Under competitive price perceptions, it is assumed that every consumer uses his own present-value vector (normalized gradient of his utility) as state prices to evaluate payoffs that are not in the asset span. As pointed out by the authors themselves, competitive price perceptions make shareholders expect that the benefits they accrue from any dividend stream are exactly compensated by its price. The shareholders are therefore neglecting the spanning, or risk-hedging opportunities that a certain dividend stream offers. A shareholder that has competitive price perceptions wants his/her firm to maximize its value, computed at his/her vector of state prices, and is indifferent among the policies that finance that plan. Value maximization under the state prices that are the weighted average of shareholders' marginal rates of substitution (with weights equal to their *initial* shares) is proved to achieve efficiency, from the initial shareholders' point of view, *given their price perceptions*.

Although very elegant and appealing, the Drèze's and Grossman and Hart's formulations of a firm's objective suffer from the drawback that they completely eliminate a firm's incentives to financially innovate, even in incomplete markets. In both Drèze's and Grossman and Hart's models, a firm's financial policy is irrelevant. In Drèze's model, final shareholders make the decisions about the firm's policy *after* the financial markets close and no further trade in securities takes place. In Grossman and Hart's model, the absence of the incentives to innovate comes from the particular form of shareholders' price perceptions.

We propose a model of a two-date incomplete markets economy with production, in which the firm's decisions are taken by a group of investors, called *the control group*. We introduce, as the objective of a privately owned firm, the maximization of the *C-adjusted value* (where C stands for the set of firm's control group members). The C-adjusted value is the value of the firm as perceived by the members of the firm's control group. This does not, in general, coincide with the standard market value. Instead, the C-adjusted value is the sum of the firm's net market value and the surplus accrued by the members of the control group from their transactions in the financial markets. This surplus can come from two different sources. One is the difference in the prices of traded securities that the firm's choices might generate. The second, is the

[2]Bonnisseau and Lachiri [3] extends Dréze criterion to a multiperiod environment. The authors propose an objective for the firm that is derived from the first order conditions of the non-convex constrained Pareto optimal problem (as Dreze's objective is). However, the authors do not relate that objective to the preferences of any group of shareholders.

firm's equity contract, which may not be replicable by a portfolio of the other traded securities and thus represents a new risk-hedging instrument. By taking into account these surpluses, the C-adjusted value accounts not only for the income effect (captured by the firm's market value), but also the price and risk effects that the firm's choices have on the shareholders' wealth.

The C-adjusted value generalizes Drèze and Grossman-Hart objectives, in the sense that it coincides with those if the firm does not have market power. If, in addition, markets are complete, then the C-adjusted value coincides with profit maximization.

We show that equilibria in which the firm maximizes its C-adjusted value generate production-financial plans that are Pareto efficient from the point of view of the members of the control group. This means that the members of the control group cannot all achieve higher utilities by switching to a different production plan and making after-trade, date-0 transfers among themselves. Although this efficiency concept is based on the same idea as the criteria used by Drèze, and Grossman and Hart, it differs from those in several respects. It differs from Drèze's notion of shareholder-efficiency in that shareholders are allowed to optimally change their portfolio holdings when pondering two production-financial policies. It also differs from Grossman-Hart's criterion in the way shareholder's preferences over production plans are defined. In Grossman and Hart, shareholders use their competitive price perceptions to derive their preferences over firm's production plans. Competitive price perceptions are rational in the sense that they are fulfilled *at* the equilibrium point; however, they may be false everywhere else. In this paper, we are imposing a stronger version of rationality: the control group's price perceptions are assumed to be fulfilled at *any* point in the firm's feasible set, not only at the equilibrium one. In other words, members of the control group correctly understand (or anticipate) the effect of the firm's production-financial decisions on securities prices. By doing that, they are able to account for the benefits of the new financial instruments that their firm creates.

In contrast to Drèze and Grossman-Hart objectives, maximization of the C-adjusted value makes the financial policy of the firm relevant. Therefore, the firm acts as a financial innovator by issuing claims against its stock. It should be emphasized here that the model accommodates a large variety of types of firms. In particular, it is not restricted to production units in the usual sense. The firm can be, for example, a financial firm which does not supply any consumption goods to the market, but only has a "technology" for creating securities. Financial intermediaries, as described for example in Bisin [2], are particular firms, whose "production" set consists of the null vector and which are owned by a single individual who only cares about, and is endowed with some amount of the date 0 consumption good. More generally, a financial intermediary can be modeled as a firm with a singleton (but non-zero) production set, $Y = \{(y_0, y_1, ...y_S)\}$ where $(y_1, ...y_S)$ is the payoff of a security the intermediary owns and y_0 is the cost it incurs from selling that (or the claims against it) in the market. It should also be pointed out that the model abstracts from more sophisticated transaction cost schemes, bid-ask spreads and taxing issues that are generally associated with issuing securities and focuses instead on the investors' need for better risk-hedging opportunities and general equilibrium price effects as sources of the incentives to innovate.

It is shown that markets are, typically, incomplete at the equilibrium. However, there are special cases in which equilibria are Pareto optimal and thus markets are effectively complete[3]. Two such cases correspond to CAPM and an economy in which consumers preferences are given by linear risk tolerance expected utility functions. In general, however, one should expect an incomplete and suboptimal market structure to arise in the equilibrium. Shareholders' unanimity is obtained if the firm is competitive in the goods market and has a Diamond-type technology or if consumers have mean-variance of linear risk tolerance utilities.

The paper is organized as follows. An example that illustrates the inadequacy of value maximization as the objective of a monopolistic firm in incomplete markets is presented in section 2. Section 3 introduces the notion of C-adjusted value for a simplified model, in which firm's financial policy is restricted to issuing equity only. The full model, in which firm has access to different financial policies to finance its production is presented in section 4. Section 5 identifies sufficient conditions for obtaining shareholders' unanimity and shows that the C-adjusted value coincides with Drèze and Grossman-Hart's objectives if firm does not have market power. Finally, section 6 tackles the problems of the existence and efficiency of the equilibrium.

2 Shareholders' Interest and Firm's Objective: An Example

We start with an example which illustrates the difficulties posed by the decision problem of a privately owned firm that acts in an incomplete markets environment. The example is a slight modification of an example in Duffie, [10] (page 121).

Eample 1

Consider an economy that lasts over two periods, $t = 0, 1$, with two possible states at date 1: $s = 1, 2$. There is only one consumption good per date and state. The economy is populated by two consumers/investors and one firm with the following characteristics:

[3] There does not seem to be a standard notion of effectively complete markets in the literature. I am using here Elul's definition. As stated in [12], markets are effectively complete if every equilibrium allocation is Pareto optimal. For an alternative definition, the reader is referred to LeRoy and Werner [17].

$$u^1(c) = (c_0)^3 c_1 (c_2)^2, u^2(c) = (c_0)^3 (c_1)^2 c_2,$$
$$\omega^1 = \omega^2 = (1, 0, 0), \ \delta^1 = \delta^2 = \frac{1}{2},$$
$$Y = \left\{ y \in \mathbb{R}_- \times \mathbb{R}^2_+ \mid (y_1)^2 + (y_2)^2 \leq -y_0 \right\},$$

where ω^i and δ^i denote investor i's endowment of goods and shares of the firm's stock, respectively. The only way to transfer consumption among dates/states is through trade in firm's shares. Investors are therefore unable to obtain every payoff stream in $\mathbb{R}^2$ through stock market transactions and thus markets are incomplete.

Investors choose share holdings and consumption for each date/state. It is assumed that they behave competitively and thus maximize their utilities within their budget constraints, taking the price of the firm's shares as given.

Simple computations show that, if the firm chooses production plan $y = (y_0, y_1, y_2) \in Y$ and does not issue additional shares, a market clearing price exists if and only if $y_0 > -2$ and in that case it is equal to $2 + y_0$. At this price, investors do not trade firm's shares, and their indirect utilities are: $U^1(y) = \left(\frac{1}{2}\right)^6 (2 + y_0)^6 y_1 (y_2)^2$, $U^2(y) = \left(\frac{1}{2}\right)^6 (2 + y_0)^6 (y_1)^2 y_2$. The firm's equilibrium net market value is $V(y) = y_0 + (2 + y_0) = 2(1 + y_0)$. Figure 1 depicts the pairs of exchange equilibrium indirect utilities for every possible value of $y \in Y \cap \left((-2, 0] \times \mathbb{R}^2_+\right)$.

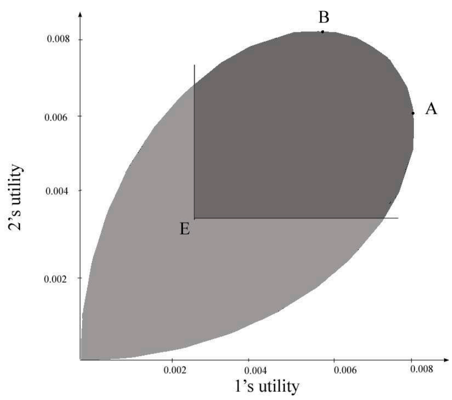

Figure 1: Exchange equilibrium utilities

As the picture shows, the two owners of the firm disagree on the most preferred production plan. If investor 1 were the only one deciding on the firm's production plan (i.e., he were the controller of the firm), then he would choose $\overline{y}_A = \left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right) = \arg\max U^1(y)$, which corresponds to point A on the graph. If, instead, investor 2 controlled the firm, she would choose $\overline{y}_B =$

$\left(-\frac{2}{3}, \frac{2}{3}, \frac{\sqrt{2}}{3}\right) = \arg\max U^2(y)$, i.e., point B. If they *both* control the firm, their disagreement poses problems for defining an appropriate objective for the firm.

If the firm's objective were to maximize its market value, the optimal choice would be $y^* = 0$. This choice gives firm's owners an utility of 0 and thus the value maximizing production plan is the *worst* possible outcome for *each* owner.

Although an unanimously approved plan does not exist, we believe that a desirable property of an equilibrium outcome in this environment is that of being, at least, Pareto undominated from the shareholders' point of view. On these grounds, one must eliminate, as possible equilibrium outcomes, all points that do not lie along the frontier between the points A and B. If, for instance, point E were the equilibrium outcome then shareholders would have an incentive to renegotiate to a better alternative in the shaded area of figure 2, and thus the firm would be vulnerable to a take-over. For this reason we view point E as an "unstable" and thus undesirable equilibrium outcome.

We are proposing here an objective for the firm which selects (some) production plans that are efficient from the shareholders' point of view. Clearly, the maximization of a weighted average of the shareholders' indirect utilities has this property. However, different utility representations of shareholders' preferences would generate different objectives for the firm and different optimal choices: a highly undesirable feature! We are therefore interested in finding an objective for the firm which is *independent* of the utility representations of shareholders' preferences, while still delivering equilibrium outcomes that are stable to shareholders' renegotiation[4]. This paper proposes a way of constructing such an objective.

3 The Simplified Model: Equity-Financed Production

Consider an economy that lasts for two periods, $t = 0, 1$. There are S possible states of nature at date 1 and only one non-storable consumption good per date and state.

The economy is populated by I consumers/investors, and there is one firm. Firm's production possibilities are described by the convex and closed subset $Y \subseteq \mathbb{R}_- \times \mathbb{R}^S_+$. A typical element of Y is of the form $y = (y_0, y_1, ...y_S)$, where $-y_0$ represents the amount of input put into production (or investment made) at date 0, and y_s is the amount of output produced at date 1 if state s occurs. Investors are characterized by their preferences over state contingent consumption plans and their endowments of goods and shares in the firm's profits. Investor i is endowed with the vector $\omega^i = (\omega^i_0, \omega^i_1, ..., \omega^i_S) \in \mathbb{R}^{S+1}_{++}$

[4] Profit (or value) maximization depends only on the market prices and thus is utility-independent. However, it cannot be a candidate for such an objective because, as proved by example 1, it is not consistent with the shareholders' preferences.

of state-contingent consumption goods and $\delta^i \in [0,1]$ shares in firm's profits. His/Her preferences over consumption streams $c = (c_0, c_1, ...c_S) \in \mathbb{R}_+^{S+1}$ are represented by the continuously differentiable, increasing in every argument and strictly quasi-concave utility function $u^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$.

Let $C \subseteq I$[5] be the set of investors who control firm's decisions; we call them the *control group*. One could think of C as being, for example, the Board of Directors. Members of the control group can be shareholders as well as non-shareholders of the firm (for example, a non-shareholder employee of the firm can be a member of the control group). Let $\delta^C \overset{def}{=} \sum_{i \in C} \delta^i$ be the aggregate share holdings by members of the control group.

The only way of transferring consumption among dates/states is through trade in firm's shares. They are available for trade at time 0 and pay dividends at time 1. Short sale of firm's shares is allowed for up to L units, where $L \in [0, +\infty]$ is exogenously given. $L = +\infty$ means that unlimited short sales are allowed.

For now, it is assumed that the firm's sole decision is to select a feasible production plan $y \in Y$ (i.e., it is implicitly assumed that the firm finances its production/investment only by issuing equity). In section 4 we relax this assumption and analyze the situation in which the firm takes not only production, but also more complex financial decisions.

Investors choose their share holdings and the amount of goods they want to consume in every date/state such that to maximize their utilities and satisfy their budget constraints at every date and state. It is assumed that the number of investors, I, is large enough, so that their behavior can be approximated by the familiar price taking hypothesis. This means that they act in the belief that their portfolio and consumption decisions do not affect the price of the security available for trade in the assets market. However, those investors that are members of the control group do understand the effect of *firm*'s production choices on the share price. We assume that, for every production plan that the firm may choose, they are able to anticipate correctly (some of) the corresponding equilibrium share price(s). We say that members of the control group have *rational price perceptions*.

Given some production plan, y, and price of shares, v, consider agent i's optimization problem as an investor,

$$\begin{cases} \max_{\theta,c} u^i(c) \\ \text{s.t.} \quad c_0 + \theta v = \omega_0 + \delta^i (v + y_0) \\ \qquad c_s = \omega_s + \theta y_s \quad s = 1, .., S \\ \qquad \theta \geq -L, c_0, c_1, ...c_S \geq 0, \end{cases} \tag{1}$$

and let $\left(c^i(y,v), \theta^i(y,v)\right)$ be its solution.

An equilibrium price of the stock-exchange economy corresponding to y is

[5] We are making the usual abuse of notation by using the same symbol to denote a finite set and the number of its elements. Hence we assume that $I = \{1, 2, ..., I\}$.

any real number, v, that solves: $\sum_{i=1}^{I} \theta^i(y, v) = 1$. Let $\widetilde{\prod}(y)$ denote the set of all solutions. $\widetilde{\prod}$ is thus a correspondence from Y to $\mathbb{R}$. Let $\widehat{Y}$ be the set of production plans for which $\widetilde{\prod}(y)$ is non-empty.

To simplify matters, we assume here that the control group's (rational) price perception is a particular measurable selection, Π, from $\widetilde{\prod}$. [In the next section we will consider the situation in which the control group holds non-degenerate beliefs over possible equilibrium prices.] Therefore $\Pi : \widehat{Y} \to \mathbb{R}$ is such that $\Pi(y) \in \widetilde{\prod}(y)$, $\forall y \in \widehat{Y}$. Let $c^i(y) \stackrel{def}{=} c^i(y, \Pi(y))$ and $\theta^i(y) \stackrel{def}{=} \theta^i(y, \Pi(y))$.

The price perception Π and utilities $(u^i)_{i \in C}$ induce members' preferences over firm's production plans as represented by the indirect utilities $V_\Pi^i : \widehat{Y} \to \mathbb{R}$, where $V_\Pi^i(y) \stackrel{def}{=} u^i(c^i)$, for every $i \in C$.

The next definition introduces our notion of C-efficiency, i.e., efficiency from the control group members' point of view. The basic idea is the following. Suppose that the economy is at a status-quo at which the firm's production plan is $\bar{y}$. If there exists an alternative plan, y, and a system of date-0, after-trade side payments that improve every member's utility, then the control group has an incentive to move away from the status-quo plan by adopting y and implementing the transfers. We are interested in production plans $\bar{y}$ that are stable to such arrangements. Those production plans are called C-efficient.

Definition 3.1 *A production plan $\overline{y} \in Y$ is C-efficient (given the price perception Π) if there does not exist a vector $\left(y, (\tau^i)_{i \in C}\right)$ consisting of a production plan $y \in Y$ and date-0 transfers $(\tau^i)_{i \in C} \in \mathbb{R}^C$ satisfying:*

i) $\sum \tau^i \leq 0$,

ii) $u^i(c^i(y) + \tau^i e_0) \geq V_\Pi^i(\overline{y})$ *for every* $i \in C$,

iii) $u^j(c^i(y) + \tau^i e_0) > V_\Pi^i(\overline{y})$ *for some* $j \in C$,

where $e_0 = (1, 0, ..., 0) \in \mathbb{R}^{S+1}$.

Note that this concept of C-efficiency is different from Drèze's original concept in that investors are allowed to adjust their shares in response to a proposed change in the production plan of the firm. It also differs from Grossman-Hart's concept in the type of price perceptions used for defining members' preferences over production plans.

In the remaining of this section we construct an objective of the firm that generates C-efficient production plans in equilibrium. To do that we measure – in units of date-0 consumption – the change in each investor's welfare due to a change of the production plan from $\widetilde{y}$ to y.

Let $\varepsilon^i(\widetilde{y}, y)$ denote the minimal amount of date-0 consumption good that investor i must be compensated with, in order to accept a change from $\widetilde{y}$ to y (i.e., to accept a change from the consumption plan $c^i(\widetilde{y}, \Pi(\widetilde{y}))$ to $c^i(y, \Pi(y))$). Thus $\varepsilon^i(\widetilde{y}, y)$ satisfies:

$$u^i(c_0^i(y) + \varepsilon^i(\widetilde{y}, y), c^{i1}(y)) = u^i(c_0^i(\widetilde{y}), c^{i1}(\widetilde{y})). \tag{2}$$

Clearly, $\varepsilon^i(\widetilde{y}, y) \geq 0$ if and only if investor i weakly prefers production plan $\widetilde{y}$ to y, i.e., $V^i_\Pi(\widetilde{y}) \geq V^i_\Pi(y)$.

Since u^i is quasi-concave, relation (2) implies:

$$\left((c^i_0(y) + \varepsilon^i(\widetilde{y}, y), c^{i1}(y) - (c^i_0(\widetilde{y}), c^{i1}(\widetilde{y}))\right) \nabla u^i(c^i_0(\widetilde{y}), c^{i1}(\widetilde{y})) \geq 0, \tag{3}$$

where ∇u^i denotes the gradient of u^i. Using investor i's budget constraints and rearranging terms, (3) becomes:

$$\begin{aligned}\varepsilon^i(\widetilde{y}, y) \geq & \left[\delta^i\left(\Pi(\widetilde{y}) + \widetilde{y}_0\right) + \theta^i(\widetilde{y})\left(MRS^i(\widetilde{y})\widetilde{y}^1 - \Pi(\widetilde{y})\right)\right] - \\ & - \left[\delta^i\left(\Pi(y) + y_0\right) + \theta^i(y)\left(MRS^i(\widetilde{y})y^1 - \Pi(y)\right)\right],\end{aligned}$$

where $MRS^i(\widetilde{y}) \stackrel{def}{=} MRS^i\left(c^i(\widetilde{y})\right) = \left(\frac{\partial_1 u^i(\widetilde{c}^i)}{\partial_0 u^i(\widetilde{c}^i)}, ..., \frac{\partial_S u^i(\widetilde{c}^i)}{\partial_0 u^i(\widetilde{c}^i)}\right)$ is investor i's marginal rate of substitution between consumption at date 1 and consumption at date 0.

Let $\mathcal{W}^i_{\widetilde{y}}(y) \stackrel{def}{=} \delta^i\left(\Pi(y) + y_0\right) + \theta^i(y)\left(MRS^i(\widetilde{y})y^1 - \Pi(y)\right)$. If $\mathcal{W}^i_{\widetilde{y}}(\widetilde{y}) \geq \mathcal{W}^i_{\widetilde{y}}(y)$ then $\varepsilon^i(\widetilde{y}, y) \geq 0$ and thus investor i weakly prefers[6] production plan $\widetilde{y}$ over y.

Definition 3.2 *For a given price perception* Π *and every* $\widetilde{y}, y \in \widehat{Y}$ *let*

$$\mathcal{V}^C_{\widetilde{y},\Pi}(y) \stackrel{def}{=} \sum_{i\in C} \mathcal{W}^i_{\widetilde{y}}(y) = \delta^C\left(\Pi(y) + y_0\right) + \sum_{i\in C} \theta^i(y)\left(MRS^i(\widetilde{y})\, y^1 - \Pi(y)\right),$$

where $\delta^C \stackrel{def}{=} \sum_{i\in C} \delta^i$ *is the initial aggregate share holding of the control group.* $\mathcal{V}^C_{\widetilde{y},\Pi}(y)$ *is called the* C*-adjusted value of* y *at* $\widetilde{y}$.

The C-adjusted value is the firm's value *as perceived* by the members of the control group. It is the sum of the market value of the control group's initial share of the firm, $\delta^C\left(\Pi(y) + y_0\right)$, and the value that the control group attributes, at $\widetilde{y}$, to trading in firm's common stock *as a risk-hedging instrument,*

$$\sum_{i\in C} \theta^i(y)\left(MRS^i(\widetilde{y})\, y^1 - \Pi(y)\right). \tag{4}$$

For every i, $MRS^i(\widetilde{y})\, y^1$ is an estimate of the investor i's valuation of the security y^1, and thus $MRS^i(\widetilde{y})\, y^1 - \Pi(y)$ is a measure of the surplus, to investor i, from purchasing one unit of the security y^1. Therefore, expression (4) represents the control group's aggregate surplus from trading firm's equity.

[6] Note, however, that $\mathcal{W}^i_{\widetilde{y}} : \widehat{Y} \to \mathbb{R}$ is not a utility representation of investor i's preferences over production plans since $V^i_\Pi(\widetilde{y}) \geq V^i_\Pi(y)$ does not necessarily imply $\mathcal{W}^i_{\widetilde{y}}(\widetilde{y}) \geq \mathcal{W}^i_{\widetilde{y}}(y)$.

Accounting for this surplus in the firm's objective is essential for obtaining C-efficient production plans at the equilibrium and the significance of the firm's financial policy (see section 6). Grossman-Hart's assumption of "competitive price perceptions" implies that investors behave *as if* trading in firm's common stock generates no surplus (or loss) for them. Therefore, term (4) is neglected from the computation of the firm's objective. As we will show in section 5, this is an entitled omission if the firm is competitive. However, if the firm has some market power, the surplus given by formula (4) is different than zero and should be taken into account.

If $C = I$, the C-adjusted value is

$$\mathcal{V}^I_{\overline{y},\Pi}(y) = y_0 + \left[\sum_{i \in I} \theta^i\left(y\right) MRS^i\left(\overline{c^i}\right)\right] y^1.$$

This is the firm's market value computed under a system of state prices equal to the average of the shareholders' marginal rates of substitutions weighted by their final share holdings. This is similar to Drèze objective, except that the weights vary with y. Thus, in contrast to Drèze and Grossman-Hart objectives[7], the C-adjusted value is not a linear function of y.

Definition 3.3 *An equilibrium of the production economy consistent with the control group's price perception* Π *is any vector* $\left(\overline{y}, \left(\overline{c}^i, \overline{\theta}^i\right)_{i \in I}, \overline{v}\right) \in \widehat{Y} \times \left(\mathbb{R}^S_+ \times \mathbb{R}\right)^I \times \mathbb{R}_+$ *that satisfies the following conditions:*

i. $\left(\overline{c}^i, \overline{\theta}^i\right)_{i \in I}$ *solves* (1) *given the price* $\overline{v}$,

ii. $\overline{v} = \Pi\left(\overline{y}\right)$,

iii. $\overline{y} \in \arg\max_{y \in \widehat{Y}} \mathcal{V}^C_{\overline{y},\Pi}(y)$,

iv. $\sum_{i \in I} \overline{c}^i = \overline{y} + \sum_{i \in I} \omega^i$.

Theorem 3.4 *In every equilibrium the firm's production plan is* C*-efficient.*

The proof of this proposition is given in section 4.4 in a more general context.

For $C = I$ theorem 3.4 is a first welfare theorem type of result. It says that the equilibria of the production economy are *minimally* constrained efficient[8], in the sense that the equilibrium allocations cannot be improved upon by a social

[7] With the notation of this paper, Drèze's objective is $\mathcal{D}_{\overline{y}}(y) = y_0 + \left[\sum_{i \in I} \theta^i\left(\widetilde{y}\right) MRS^i\left(\widetilde{y}\right)\right] y^1$, while Grossman-Hart's is $\mathcal{GH}_{\overline{y}}(y) = y_0 + \left[\sum_{i \in I} \delta^i MRS^i\left(\widetilde{y}\right)\right] y^1$.

[8] The notion of minimal constrained efficiency was first introduced (to the best of my knowledge) by Dierker, Dierker and Grodal in [7]

planner who can choose the production plan and redistribute consumption at date 0, but has to use the markets to purchase the date-1 consumption for every investor. The formal definition of this efficiency concept follows.

Definition 3.5 *An allocation* $\left(\overline{y}, \left(\overline{c}_0^i, \overline{c}^{i1}\left(\overline{y}\right)\right)_{i\in I}\right)$ *is minimally constrained efficient given* Π *if and only if:*

1. $\sum_{i\in I} \overline{c}_0^i = \sum_{i\in I} \omega_0^i + \overline{y}_0$,
2. *there does not exist a Pareto superior allocation* $\left(y, \left(c_0^i, c^{i1}\left(y\right)\right)_{i\in I}\right)$ *such that* $\sum_{i\in I} c_0^i = \sum_{i\in I} \omega_0^i + y_0$.

The next corollary is an immediate consequence of theorem 3.4.

Corollary 3.6 *If* $C = I$*, every equilibrium is minimally constrained efficient.*

Note that minimal constrained efficiency is weaker than constrained Pareto optimality. A planner could, potentially, improve upon a minimally constrained efficient allocation by redistributing shares (or, equivalently, using *before* trade date-0 transfers). Therefore, even when $C = I$, equilibria in which the firm maximizes its C-adjusted value are not, in general, constrained Pareto optimal.

Example 2 To illustrate the results of this section, consider again the example of section 2. If $C = \{1\}$ (or $C = \{2\}$), the unique C-efficient outcome is point A (respectively B). This corresponds to the production plan $\overline{y}_A = \left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right)$ (respectively $\overline{y}_B = \left(-\frac{2}{3}, \frac{2}{3}, \frac{\sqrt{2}}{3}\right)$). The firm's C-adjusted value (at $\overline{y}_A$) is $\mathcal{V}_{\overline{y}_A}^{\{1\}}(y) = 2\left(1 + y_0\right) + \frac{1}{3}\left(\sqrt{2}y_1 + 2y_2\right)$, which is maximized at $\left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right)$. Thus $\overline{y}_A$ is the unique equilibrium production plan if $C = \{1\}$.

Suppose now that $C = \{1, 2\}$. An allocation $\left(\overline{y}, \left(\overline{c}_0^1, \frac{1}{2}\overline{y}_1, \frac{1}{2}\overline{y}_2\right), \left(\overline{c}_0^2, \frac{1}{2}\overline{y}_1, \frac{1}{2}\overline{y}_2\right)\right)$ is minimally constrained efficient (MCE in the sequel) if and only if $\left(\overline{y}, \overline{c}_0^1, \overline{c}_0^2\right)$ solves

$$\max\left\{\lambda u^1\left(c_0^1, \frac{1}{2}y_1, \frac{1}{2}y_2\right) + \left(1-\lambda\right)u^2\left(c_0^2, \frac{1}{2}y_1, \frac{1}{2}y_2\right) \;\middle|\; \begin{array}{c} c_0^1 + c_0^2 = 2 + y_0 \\ c_0^1, c_0^2 \geq 0 \end{array}\right\}$$

for some $\lambda \in [0, 1]$. The parametric equation of the MCE frontier (in utility coordinates) is given by

$$\left(\left(\tfrac{8}{3} - 6\left(y_1\right)^2\right)^3 \tfrac{y_1}{2}\left(\tfrac{1}{6} - \tfrac{\left(y_1\right)^2}{4}\right), \left(6\left(y_1\right)^2 - \tfrac{4}{3}\right)^3 \tfrac{\left(y_1\right)^2}{4}\tfrac{\sqrt{\frac{2}{3} - \left(y_1\right)^2}}{2}\right).$$

As figure 2 illustrates, the MCE frontier intersects the exchange equilibrium utility frontier in one point, D. Thus D is the only C-efficient outcome,

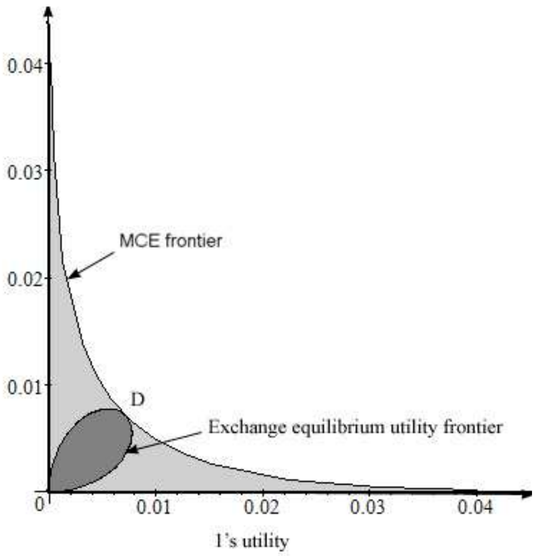

Figure 2: Minimally constrained efficient frontier

and therefore the only candidate for an equilibrium production plan. Straightforward computations show that $\overline{y_D} = \left(-\frac{2}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ maximizes the adjusted value $\mathcal{V}^{\{1,2\}}_{\overline{y}_D}$ and thus it is an equilibrium production vector.

Although in this example $C = I$, the minimally C-efficient production plans are not necessarily constrained Pareto optimal. Figure 3 illustrates that. However, since the two frontiers intersect in point D the equilibrium corresponding to $C = \{1, 2\}$ is *constrained* Pareto optimal. In section 5 we will show that this is a more general result, that applies to economies in which the firm has no market power in the financial markets. Indeed, in this example, the firm does not influence the markets by making its equity contract available for trade, because in every exchange equilibrium, consumers choose *not* to trade firm's shares.

4 The General Model

In this section we present our general model, which extends the previous one in two directions. First, it allows for more complex financial policies. The firm can finance its production plan not only by selling equity, but also by borrowing in the market and issuing new securities. Second, we allow for more general price perceptions. The control group may be unsure about the exact equilibrium asset prices and assign, instead, positive probabilities to several possible prices.

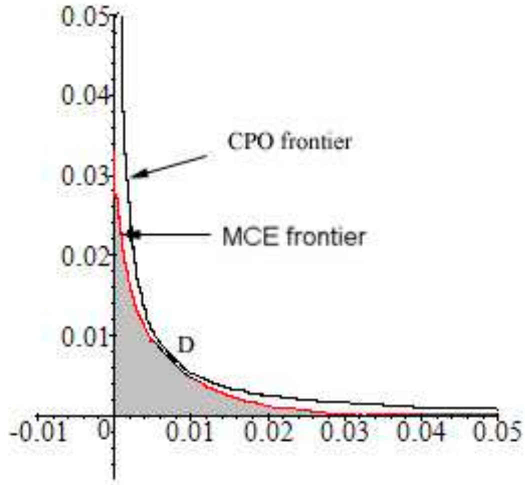

Figure 3: CPO versus MCE

4.1 The Economy

The market participants have the same characteristics as described in section 3. The market structure is different in that there are $J < S$ exogenously given securities available for trade at date 0. Security A^j pays a_{js} units of the state-contingent consumption good if state s occurs. The matrix $A \stackrel{def}{=} (a_{js})_{\substack{j=1..J \\ s=1..S}} \in \mathcal{M}_{S\times J}(\mathbb{R})$ is called the payoff matrix.

The firm can finance its production plan by borrowing in the existing security markets and issuing new shares and/or securities[9]. We assume that the firm cannot issue more than M new shares and is allowed to design exactly N firm-specific securities[10]. N is taken to be large enough[11] so that the firm has the capacity to complete the markets (if that is optimal for its control group).

The payoffs of the securities that the firm may issue to finance a production plan y are constrained to lie in some exogenously given set $\mathcal{K}(y) \subseteq \mathbb{R}^S$. The set $\mathcal{K}(y)$ encompasses the restrictions that firm faces when issuing new securities. For example, the firm may be constrained to issue only securities with positive payoff in every state, in which case $\mathcal{K}(y) \subseteq \mathbb{R}^S_+$. Alternatively, $\mathcal{K}(y) \subseteq \{(f(y_1), ..., f(y_S)) \mid f : \mathbb{R} \to \mathbb{R}\}$ means that the firm can issue only derivatives on its equity (such as options on equity). It is assumed that $\mathbf{0}_S \in \mathcal{K}(y)$ and $y^1 \in \mathcal{K}(y)$, so that the firm can always choose the trivial finan-

[9] Examples of such firm-issued securities are convertibles, warrants, floating-rate debt, zero-coupons, primes and scores, etc.

[10] This assumption is made to simplify the technicalities of the model. However, since the firm is allowed to issue securities with zero payoff in all states, the constraint merely imposes an upper bound on the number of new securities the firm may issue.

[11] If unlimited short sales are allowed and A is full rank this means $N \geq S - J$.

cial structure $\{\mathbf{0}_S, ..., \mathbf{0}_S, y^1\}$, where $\mathbf{0}_S$ is the zero vector in $\mathbb{R}^S$. Let also M be the exogenously given upper bound on new equity issues.

For every $j \in \{1, ..., J+N+1\}$ investors are allowed to sell short up to $L_j \in [0, +\infty]$ units of the security j. The same constraints are faced by the firm for its trades in the exogenously given securities. Let $L \stackrel{def}{=} (L_1, ..., L_{N+J+1})$ denote the vector of portfolio bounds.

The firm has to decide on: (a). a production plan $y \in Y$, (b). a stream of dividends $D \in \mathbb{R}^S$ to be paid to its shareholders at date 1, (c). a matrix of payoffs $X \in \mathcal{M}_{S\times N}(\mathbb{R})$ for the N securities to be issued, (d). a portfolio[12] $b^f \in \mathbb{R}^J$ of asset holdings and (e). a number $\theta^f \in \mathbb{R}$ of new shares to issue.[13] A vector $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in Y \times \mathbb{R}^S \times \mathcal{M}_{S\times N}(\mathbb{R}) \times \mathbb{R}^{J+1}$ is called a *production-financial plan/policy.*

Definition 4.1 *A production-financial plan* $\left(y, D, X, b^f, \theta^f\right)$ *is called feasible if:*

i. $y \in Y$

ii. $D, X^n \in \mathcal{K}(y)$, *for every* $n \in \{1, ..., N\}$, *where* X^n *denotes the n-th column of* X,

iii. $\theta^f \in [-1, M]$ *and* $b^f_j \geq -L_j$ *for every* $j \in \{1, ..., J\}$,

iv. $y^1 = \left(1+\theta^f\right) D + Ab^f + X\mathbf{1}_N$, *where* $\mathbf{1}_N = (1, ..., 1)^t \in \mathbb{R}^N$.

The set of all feasible plans is denoted by $\mathcal{F}$.

If the firm chooses production-financial plan $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$ and $(q, p, v) \in \mathbb{R}^J \times \mathbb{R}^N \times \mathbb{R}$ are the market prices of the already existing securities, the firm-issued securities and the dividend stream, then the firm's profit at date 0 is

$$D_0 \stackrel{def}{=} y_0 + qb^f + p\mathbf{1}_N + \theta^f v.$$

This is distributed to firm's shareholders according to their initial shares[14]. Therefore, if the firm chooses policy $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$, having an initial endowment of δ shares is equivalent, in terms of wealth generated, to receiving δy_0 units of the date-0 consumption good and being endowed with a portfolio $\left(\delta b^f, \delta\mathbf{1}_N, \delta\theta^f\right)$ of the $J+N+1$ traded securities.

Hence, every feasible production-financial plan $\mathcal{P}$, generates the following (artificial) stock-exchange economy:

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[12] To simplify notation we are assuming that portfolio holdings are column vectors.

[13] A negative value of θ^f indicates that the firm repurchases $-\theta^f$ of its outstanding shares. A positive value indicates that firm is issuing θ^f additional shares.

[14] The situation described in section 3 corresponds to the case in which $D_0 = y_0$, $D^1 = y^1 \in \mathbb{R}^{S+1}$, $b^f = 0 \in \mathbb{R}^J$, $\theta^f = 0 \in \mathbb{R}$, $N = 0$.

$$\mathcal{E}_{\mathcal{P}} = ((u^i, \left(\omega_0^i + \delta^i y_0, \omega^{1i}\right), \delta^i b^f, \delta^i \mathbf{1}_N, \delta^i \theta^f)_i, A, X, D).$$

In $\mathcal{E}_{\mathcal{P}}$ every investor i takes the asset prices as given and chooses his/her consumption and portfolio by solving:

$$\begin{cases} \max_{c^i, b^i, r^i, \theta^i} u^i\left(c^i\right) \\ \text{s.t. } c_0^i + qb^i + pr^i + v\theta^i = \omega_0^i + \delta^i\left(y_0 + qb^f + p\mathbf{1}_N + v\theta^f\right) \\ \qquad c^{i1} = \omega^{i1} + Ab^i + Xr^i + D\theta^i \\ \qquad c^i \geq 0, \left(b^i, r^i, \theta^i\right) \geq -L. \end{cases} \tag{5}$$

Definition 4.2 *An equilibrium of the stock-exchange economy $\mathcal{E}_{\mathcal{P}}$ consists of prices, $(\overline{q}, \overline{p}, \overline{v})$, consumption allocations, $\left(\overline{c}^i\right)_{i=1\ldots I}$, and securities holdings, $\left(\overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i=1\ldots I}$, such that the following conditions are satisfied:*

1. *$\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i=1\ldots I}$ solves (5) given the prices $(\overline{q}, \overline{p}, \overline{v})$,*

2. *all markets clear, i.e.,*

$$\begin{aligned} \sum_{i=1}^{I} \overline{b}^i &= b^f, \; \sum_{i=1}^{I} \overline{r}^i = 1, \; \sum_{i=1}^{I} \overline{\theta}^i = 1 + \theta^f \\ \sum_{i=1}^{I} \overline{c}^i &= \sum_{i=1}^{I} \omega^i + y. \end{aligned}$$

4.2 Control Group's Price Beliefs

The control group's beliefs are positive probability measures over the space of asset prices. The beliefs are *rational* in the sense that they assign zero probability to all prices that are not among the equilibrium ones. Their construction is shown in the sequel.

For every $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in \mathcal{F}$ let $\widetilde{\prod}(\mathcal{P}) \subseteq \mathbb{R}^{J+N+1}$ be the set of equilibrium asset prices of $\mathcal{E}_{\mathcal{P}}$. Define $\widehat{Y} \stackrel{def}{=} \left\{y \in Y \mid y_0 \geq -\min \frac{\omega_0^i}{\delta^i} + \varepsilon\right\}$ and $\widehat{\mathcal{F}} \stackrel{def}{=} \left\{\mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in \mathcal{F} \mid y \in \widehat{Y}\right\}$, where $\varepsilon > 0$ is arbitrarily small. Then $\widetilde{\prod}(\mathcal{P}) \neq \emptyset$ for every $\mathcal{P} \in \widehat{\mathcal{F}}$ (see the appendix for a proof). We restrict[15] the firm's choices to the feasible production-financial plans in $\widehat{\mathcal{F}}$.

Let $\mathcal{M}$ be the space of all measurable selections from $\widetilde{\prod}$. We endow $\mathcal{M}$ with the product topology and consider the associated Borel σ-algebra, $\mathcal{B}(\mathcal{M})$.

Definition 4.3 *Any probability measure, μ, over $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ is called a rational belief for the control group.*

[15] This restriction can be relaxed considerably by enlarging the set of "permissible" production plans to a superset of $\widehat{Y}$. However, that level of generality is beyond the scope of this paper.

4.3 Firm's Objective and the Equilibrium Concept

The firm's objective is to maximize the *expected* C-adjusted value, given control group's beliefs over the set of equilibrium prices. As in the model of section 3, the C-adjusted value will be defined as the sum of the fraction of firm's market value received by the members of the control group as initial shareholders and a measure of the members' surplus from their transactions in the stock markets. Since investors can trade in many securities, the surplus acquired from their trade in *every* marketed security must be taken into account.

We introduce first some notation.

Let μ be control group's price belief, $\Pi = (q, p, v) \in \text{supp}(\mu)$ an arbitrary price functional in the support of μ and $\mathcal{P} \in \widehat{\mathcal{F}}$ an arbitrary financial-production plan in the firm's action set. Then:

1. $\Pi_{\mathcal{P}}(l)$ is the price of security $l \in \left\{ (A^j)_{j=1,\dots,J}, D, (X^n)_{n=1,\dots,N} \right\}$ given by the pricing functional Π. For example, if $l = D$, then $\Pi_{\mathcal{P}}(l) \overset{def}{=} v(\mathcal{P})$,
2. $Z^i_{\mathcal{P}}(l)$ is consumer i's optimal holding of security l at prices $\Pi(\mathcal{P})$,
3. $Z^C_{\mathcal{P}}(l) \overset{def}{=} \sum_{i \in C} Z^i_y(l)$,
4. $c^i(\mathcal{P})$ is investor i's optimal consumption in the economy $\mathcal{E}_{\mathcal{P}}$, at prices $\Pi(\mathcal{P})$.

Definition 4.4 *For $\mathcal{P}$, $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$ and a given $\Pi \in$supp(μ) we define the C-adjusted value of $\mathcal{P}$ at $\overline{\mathcal{P}}$ as:*

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) \overset{def}{=} \delta^C \left(\Pi_{\mathcal{P}}(D) + D_0 \right) + \mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P}), \tag{6}$$

where

$$\begin{aligned} D_0 &= y_0 + q(\mathcal{P})\, b^f + p(\mathcal{P})\, \mathbf{1}_N + v(\mathcal{P})\, \theta^f \quad \textit{and} \\ \mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P}) &= \sum_{l \in (A_j)_j, (X_n)_n, D} \mathcal{W}^l_{\overline{\mathcal{P}}}(\mathcal{P}), \\ \mathcal{W}^l_{\overline{\mathcal{P}}}(\mathcal{P}) &= \sum_{i \in C} Z^i_{\mathcal{P}}(l)\, MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right) l - Z^C_{\mathcal{P}}(l)\, \Pi_{\mathcal{P}}(l). \end{aligned}$$

Similarly to our discussion in section 3, we interpret the expression

$$Z^i_{\mathcal{P}}(l) \left(MRS^i\left(\overline{c}^i\right) l - \Pi_{\mathcal{P}}(l) \right)$$

as investor i's surplus from trading security l. Thus $\mathcal{W}^l_{\overline{\mathcal{P}}}(\mathcal{P})$ measures the control group's surplus from trading l and $\mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P})$ is the surplus from all its transactions in the stock markets.

According to definition 4.4, the C-adjusted value depends on every member's optimal portfolio. If there are redundant securities in the market, the optimal portfolios may not be unique. Investors who trade in at least one redundant security are, in fact, indifferent among a continuum of portfolios. The C-adjusted value is well-defined if and only if it is invariant to the choice of optimal portfolios. Next proposition shows that this is indeed the case.

Proposition 4.5 *The C-adjusted value is well-defined.*

Proof. Let $\Pi \in \text{supp}(\mu)$, $(\mathcal{P}, \overline{\mathcal{P}}) \in \widehat{\mathcal{F}} \times \widehat{\mathcal{F}}$ and $\left((Z^i_{\mathcal{P}}(l))_{l \in (A_j)_j, (X_n)_n, D} \right)_{i \in C}$ a vector of optimal portfolios for the members of the control group. Using every investor i's budget constraint ($i \in C$) and rearranging terms in (4.4) we obtain:

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) = \sum_{i \in C} \left(c^i_0(\mathcal{P}) - \omega^i_0 \right) + \sum_{i \in C} MRS^i \left(c^{i1}\left(\overline{\mathcal{P}} \right) \right) \left(c^{i1}(\mathcal{P}) - \omega^{i1} \right). \quad (7)$$

Since u^i is strictly quasi-concave, the optimal consumption stream $c^i(\mathcal{P})$ is unique and thus the value of $\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})$ is independent of the choice of the members' optimal portfolios. ■

The firm's objective is to maximize its expected C-adjusted value, given its beliefs. This means, $\max_{\mathcal{P}} E^{\mu}(\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})) = \int_{\mathcal{M}} \mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\, d\mu(\Pi)$. Note that the integral $\int_{\mathcal{M}} \mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\, d\mu(\Pi)$ is well-defined because, given the product topology on $\mathcal{M}$, the mapping $\Pi \longmapsto \Pi(\mathcal{P})$ from $\mathcal{M}$ to $\mathbb{R}^{J+N+1}$ is continuous (and thus integrable) for every $\mathcal{P}$. Moreover, consumers' demands for goods are continuous functions of prices.

Let μ denote the control group's rational beliefs over the set of equilibrium asset prices.

Definition 4.6 *An equilibrium of the production economy consists of: (1). some rational belief, μ, (2). a production-financial plan $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$ for the firm, (3). a consumption-portfolio allocation for the investors, $\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i \right)_{i=1\ldots I}$, and (4). a system of prices, $(\overline{q}, \overline{p}, \overline{v})$, such that:*

- *i. consumption-portfolio allocations are optimal within each investor's budget constraint, i.e., $\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i \right)_{i=1\ldots I}$ solves (5) given $(\overline{q}, \overline{p}, \overline{v})$,*
- *ii. beliefs are consistent with the equilibrium prices, i.e., there exists $\overline{\Pi} \in$ supp(μ) such that $(\overline{q}, \overline{p}, \overline{v}) = \overline{\Pi}\left(\overline{\mathcal{P}} \right)$,*
- *iii. the firm maximizes its expected C-adjusted value at $\overline{\mathcal{P}}$, given the beliefs μ, i.e., $\overline{\mathcal{P}} \in \arg\max_{\mathcal{P}} \int_{\mathcal{M}} \mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\, d\mu(\Pi)$,*
- *iv. all markets clear.*

4.4 C-efficient Policies

We say that a production-financial plan is *not* C-efficient if for *every* price in the support of the control group's beliefs, there exists another production-financial plan and a system of date-0 transfers that lead to an improvement in everyone member's utility. The formal definition follows.

Definition 4.7 *Let μ be the control group's beliefs over $\mathcal{M}$. A production-financial plan $\overline{\mathcal{P}}$ is* <u>*not*</u> *C-efficient (given μ) if there exists a production-financial plan $\mathcal{P}$ and a system of date 0 transfers $(\tau^i)_{i\in C}$ such that, $\sum \tau^i \leq 0$ and, for every $\Pi \in supp(\mu)$,*
a) $u^i\left(c^i(\mathcal{P}) + \tau^i e_0\right) \geq u^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)$ for every $i \in C$,
b) $u^j\left(c^i(\mathcal{P}) + \tau^i e_0\right) > u^j\left(c^i\left(\overline{\mathcal{P}}\right)\right)$ for some $j \in C$.

Theorem 4.8 *In every equilibrium of the production economy the firm's production-financial plan is C-efficient.*

Proof. Let $\overline{\mathcal{P}}$ be an equilibrium production-financial plan corresponding to belief μ. If it is not C-efficient then there exists an alternative plan, $\mathcal{P}$, and a system of transfers at time $0, (\tau^i)_{i\in C}$, such that $\sum_{i\in C}\tau^i \leq 0$ and, for every $\Pi \in \text{supp}(\mu)$, $u^i\left(c^i(\mathcal{P}) + \tau^i e_0\right) \geq u^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)$ $\forall i \in C$, with at least one strict inequality for some $j \in C$.

Strict quasi-concavity of u^i implies that:

$$c_0^i(\mathcal{P}) + \tau^i - c_0^i\left(\overline{\mathcal{P}}\right) + MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i1}(\mathcal{P}) - c^{i1}\left(\overline{\mathcal{P}}\right)\right] \geq 0,$$

for every $i \in C$, with strict inequality for some $j \in C$.

Adding up over $i \in C$ and using $\sum_{i\in C}\tau^i \leq 0$ yields

$$\sum_{i\in C}\left(c_0^i(\mathcal{P}) - c_0^i\left(\overline{\mathcal{P}}\right)\right) + \sum_{i\in C} MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i1}(\mathcal{P}) - c^{i1}\left(\overline{\mathcal{P}}\right)\right] > 0,$$

or, equivalently,

$$\sum_{i\in C}\left(c_0^i(\mathcal{P}) - \omega_0^i\right) + \sum_{i\in C} MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i1}(\mathcal{P}) - c^{i1}\left(\overline{\mathcal{P}}\right)\right] > \sum_{i\in C}\left(c_0^i\left(\overline{\mathcal{P}}\right) - \omega_0^i\right). \tag{8}$$

Using (6), inequality (8) becomes $\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) > \mathcal{V}^C_{\overline{\mathcal{P}}}\left(\overline{\mathcal{P}}\right)$. Integrating this last inequality over $\Pi \in \text{supp}(\mu)$ we obtain $E^\mu\left(\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\right) > E^\mu\left(\mathcal{V}^C_{\overline{\mathcal{P}}}\left(\overline{\mathcal{P}}\right)\right)$ which is a contradiction with the equilibrium condition (ii). ■

5 Shareholders Unanimity and Firm's Market Power

A firm that acts in an incomplete market environment may affect the market prices through two channels: its production choices and its financial policy.

In this section we restrict attention to firms whose production choices have no impact on the market prices. We call such firm *competitive in the goods markets.*

If, in addition, the spanning condition holds (i.e., $\mathcal{K}(y) \subseteq span(A)$ for every $y \in Y$) any security that the firm may issue is replicable by a portfolio of the exogenously given assets, and thus a change in the firm's financial policy has no influence on the market equilibrium either. Thus the firm is competitive in both, the goods *and* the financial markets. In this case, its C-adjusted value coincides with the (unambiguously defined) market value and is independent of the choice of the financial policy. Modigliani-Miller theorem holds and the firm's shareholders unanimously agree on the value-maximizing production plan.

If spanning does not hold (i.e., $\mathcal{K}(y) \nsubseteq span(A)$ for at least some $y \in Y$), a firm that is competitive in the goods market may still influence the equilibrium prices through its issuing of new securities. As shown by Diamond in [5], shareholders' unanimity can still be obtained in this case if the firm's production set projects on $\mathbb{R}^S$ along an 1-dimensional subspace.

This section identifies more general conditions under which unanimity of the firm's shareholders on the most preferred production-financial plan obtains. We show that, when concerned with the shareholders' unanimity for all profiles of preferences, spanning and the Diamond-type technology are necessary and sufficient conditions. For particular subclasses of preferences (i.e., linear risk tolerant and mean-variance), less restrictive conditions are sufficient to guarantee the shareholders' unanimity.

Definition 5.1 *A firm is said to be competitive in the goods markets if and only if, for any feasible plans $\mathcal{P}$ and $\mathcal{P}'$ for which $span(A, X, D) = span(A, X', D')$, it is true that $\Pi_{\mathcal{P}}(l) = \Pi_{\mathcal{P}'}(l)$ for every $l \in span(A, X, D)$.*

Let $X \subseteq \mathbb{R}_- \times \mathbb{R}_+$ be an arbitrary set with $0 \in X$ and $\vartheta \in \mathbb{R}^S_+$ an arbitrary vector. We say that a firm has a *Diamond-type technology* if its production set is of the form $Y \overset{def}{=} \{(y_0, \vartheta k) \mid (y_0, k) \in X\}$, and $\mathcal{K}(y) = \{0, \vartheta k\}$ for any $y = (y_0, \vartheta k) \in Y$. Therefore, a firm with Diamond-type technology is restricted to finance its production by issuing only equity, and may expand the asset span along a single direction.

Proposition 5.2 *For a firm that is competitive in the goods markets and has a Diamond-type technology, the maximization of the (expected) C-adjusted value coincides with the maximization of the (expected) market value, which is unanimously supported by all shareholders.*

Proof. Since the firm is restricted to issuing equity only, its C-adjusted value is:

$$\mathcal{V}^C_{\overline{y}}(y) = \delta^C \left(\Pi\left(y^1\right) + y_0\right) + \sum_{i \in C} \theta^i(y) \left(MRS^i(\overline{y})\, y^1 - \Pi\left(y^1\right)\right).$$

Since $\overline{y}^1$ and y^1 are colinear and the firm is competitive in the goods markets, $MRS^i(\overline{y})\,y^1 - \Pi(y^1) = 0$ and thus $\mathcal{V}_{\overline{y}}^C(y) = \delta^C(\Pi(y^1) + y_0)$. ■

As proved above, shareholders' unanimity is obtained if the firm's technology satisfies the spanning property or is of the Diamond-type. These restrictions on the firm's technology are also necessary for obtaining the shareholders' unanimity within the entire class of rational preference profiles. In other words, if the firm's technology does not satisfy the spanning property and is not of Diamond-type then, by appropriately choosing the shareholders' preferences, one can construct economies in which shareholders disagree on their most preferred production-financial plan.

Therefore, obtaining shareholders' unanimity for all arbitrary profiles of their preferences imposes strong restrictions on the firm's technology. By contrast, within the class of market economies in which consumers have either mean-variance or linear risk tolerance utilities, shareholders' unanimity is obtained for *any* technology. In such economies, the firm's objective renders its financial policy trivial and the maximization of the C-adjusted value coincides with Grossman and Hart' value maximization.

Definition 5.3 *Let $y \in \widehat{Y}$ and $(c^i)_{i\in I}$ be a feasible consumption allocation for the exchange economy corresponding to y. (i.e., $\sum_i c^i = y + \sum_i \omega^i$). Then $(c^i)_{i\in I}$ is called distribution-efficient if it is Pareto optimal in the exchange economy corresponding to y.*

Proposition 5.4 (LRT utilities) *Suppose that the exogenously given asset market contains the risk-free bond, as well as every investor's initial endowment of goods. If investors' preferences can be represented by expected utilities with linear risk tolerance and positive identical slopes, then the firm's financial policy is irrelevant and the equilibrium allocation is distribution-efficient. If, moreover, $C = I$, the equilibrium production plan, $\overline{y}$ satisfies:*

$$\overline{y} \in \arg\max \left\{ y_0 + MRS(\overline{y})y^1 \right\},$$

where $MRS(\overline{y})$ denotes investors' common intertemporal marginal rate of substitution.

Proof. If investors' utilities have linear risk tolerance with the same slope, then every distribution-efficient allocation can be achieved by holding a portfolio composed of the risk-free and the market security (i.e., the aggregate endowment). Since for every financial policy that finances a production plan y the asset span contains both the risk-free bond and the market security, all distribution-efficient allocations corresponding to y are in the asset span. Moreover, since every feasible allocation (for the exchange economy corresponding to y) is weakly dominated by a distribution-efficient allocation corresponding to y, it follows that every exchange equilibrium in the associated stock-market economy has the two-fund spanning property and thus all financial policies generate the same equilibrium consumption stream as the equity-financed plan. Therefore, the financial policy is irrelevant in this case (consequence of proposition 4.5) and the equilibrium allocation is distribution-efficient.

Since the equilibrium is distribution-efficient, investors' intertemporal marginal rates of substitutions are equal, and thus $MRS^i(\overline{y}) = MRS^j(\overline{y}) \stackrel{def}{=} MRS(\overline{y})$ for every $i, j \in I$. If $C = I$ then $\theta^C(y) = 1$ and thus $\overline{y} \in \arg\max \{y_0 + MRS(\overline{y})y^1\}$.
■

Suppose now that investors' utilities u^i are quasi-concave and satisfy:

$$u^i\left(c^i\right) = h^i\left(c_0^i, E\left(c^{i1}\right), var\left(c^{i1}\right)\right), \qquad \forall i \in I$$

with h^i differentiable, increasing in the first two variables and decreasing in the third. Suppose, as in the previous example that $C = I$ and the risk-free bond, as well as investors' initial endowments of goods are in the asset span.

Proposition 5.5 (CAPM Equilibrium) *Under the above stated assumptions, the equilibrium allocations are Pareto optimal and firm's equilibrium production plan satisfies:*

$$\overline{y} \in \arg\max\left\{y_0 + MRS(\overline{y})y^1\right\}. \tag{9}$$

Proof. Given the particular preferences, every equilibrium production plan $\overline{y}$ generates distribution-efficient consumption allocations (see Magill-Quinzii [18], pp. 181-183 for a proof). Then, as before, investors' marginal rates of substitution are equalized at the equilibrium and firms' objective becomes the one in formula (9). ■

5.1 Competitiveness in Financial Markets

As emphasized before, if the spanning property holds, a firm's financial policy alone has no influence on the market equilibrium. We identify here more general conditions that guarantee firm's competitiveness in the financial markets and analyze their equilibrium implications. We show that if a firm is competitive in the financial markets, then its C-adjusted value is unaffected by the financial policy and thus it is irrelevant how the firm finances its investment. If the firm is competitive in both, the financial and the goods markets, its C-adjusted value coincides with the Grossman-Hart objective. If, in addition, there is no production-specific risk, i.e., if $Y \subseteq span\,(A)$ (one instance in which this happens is if markets are complete, i.e., $span\,(A) = \mathbb{R}^S$), then the C-adjusted value coincides with the (unambiguously defined) firm's market value. The production plans that maximize the firm's market value are the most preferred production plans by *all* shareholders.

We assume, throughout this section that *unlimited* short sales are allowed.

For every $y \in Y$ consider the artificial stock-exchange economy $\mathcal{E}_y^0$ in which the market structure is given by the payoff matrix A, investors's preferences are the same as before, their endowments of goods are $\omega^i + \delta^i y$ and their endowments of securities are 0. $\mathcal{E}_y^0$ is therefore an imaginary stock-exchange economy in which firm has committed to producing y but its stock is not traded in the market. For every $l \in \mathbb{R}^S$, let $\mathcal{E}_y^l$ be the economy with the same characteristics as $\mathcal{E}_y^0$, except that its market structure is given by $(A, l) \in \mathcal{M}_{S\times(J+1)}$.

Let $\widehat{Y}$ be the set of production plans for which $\mathcal{E}_y^0$ has a stock-exchange equilibrium and assume that the equilibrium is unique[16]. Let that be $\left(c^i(y), b^i(y), q(y)\right)_{i\in I}$. Define

$$\mathcal{D}_y \stackrel{def}{=} \left\{ l \in \mathbb{R}^S \mid MRS^i\left(c^i(y)\right) l = MRS^j\left(c^j(y)\right) l,\ \forall i \neq j \in I \right\}.$$

The set $\mathcal{D}_y$ is called the *extended asset span* corresponding to y. Clearly, $span\,(A) \subseteq \mathcal{D}_y$, but the inclusion may be strict. If $span\,(A) \subsetneqq \mathcal{D}_y$ and the equilibrium consumption allocation $(c^i(y))_{i\in I}$ is interior, then any security whose payoff lies in $\mathcal{D}_y \setminus span\,(A)$ has the property that, if introduced into the market, it does not change the equilibrium prices or allocations. The following proposition states this property formally.

Proposition 5.6 *Let $l \in \mathcal{D}_y \setminus span\,(A)$ and $p(y) = MRS^i\left(c^i(y)\right) l$, $i \in I$. If $\left(c^i(y)\right)_{i\in I}$ is interior, then consumption allocation $\left(c^i(y)\right)_{i\in I}$, portfolio holdings $\left(b^i(y), 0\right)_{i\in I}$ and prices $\left(q(y), p(y)\right)$ form an equilibrium of the economy $\mathcal{E}_y^l$.*

Proof. Since $c^i(y) \gg 0\ \forall i \in I$ and $p(y) = MRS^i\left(c^i(y)\right) l = MRS^j\left(c(y)^j\right) l$, $\forall i \neq j \in I$, the first order conditions for the consumption choice in $\mathcal{E}_y^l$, given $(q(y), p(y))$, are satisfied at $\left(c^i(y)\right)_{i\in I}$. Quasi-concavity of investors' utility functions imply that the first order conditions are sufficient and thus

$$\left(\left(c^i(y), \left(b^i(y), 0\right)\right)_{i\in I}, \left(q(y), p(y)\right) \right)$$

is an equilibrium in $\mathcal{E}_y^l$. ■

Definition 5.7 *We say that the firm is competitive (or has no market power) in the financial markets if $\mathcal{K}(y) \subseteq \mathcal{D}_y$ for every $y \in \widehat{Y}$.*

If the firm is competitive in the financial markets, any security it may issue has no effect on the equilibrium prices of the other traded securities or on the equilibrium consumption allocations. Therefore investors are indifferent between having the markets for the firm-specific securities open or closed. Being competitive in the financial markets does *not* mean that the firm's decisions do not influence the markets. It means only that its *financial* policy has no impact on the markets. However, changes in the production plan can still affect prices.

[16]This assumption is made only to simplify the exposition and it is not necessary for deriving the results of this section.

If the firm is competitive in the financial markets then, for every feasible production-financial plan $\mathcal{P} = \left(y, X, D, \theta^f, b^f\right)$, there exists an exchange equilibrium of $\mathcal{E}_{\mathcal{P}}$ in which investor i's holdings of firm's securities are: $Z^i_{\mathcal{P}}(D) = \delta^i\left(1+\theta^f\right)$ and $Z^i_{\mathcal{P}}(l) = \delta^i, \forall l \in \left\{X^1, ...X^N\right\}$. Since holding a share δ^i of the firm is equivalent to holding that fraction of the firm's portfolio, the last condition actually says that, if the firm is competitive, investors' net trades in the securities issued by the firm are zero.

Competitiveness in the financial markets implies that the C-adjusted value is invariant to changes in the financial policies. This is an immediate consequence of formula (7) and proposition 4.5. The C-adjusted value of a firm that is competitive in the financial markets is:

$$\mathcal{V}^C_{\mathcal{P}}(\mathcal{P}) = \delta^C y_0 + \sum_{i \in C} \delta^i MRS^i\left(c^i\left(\overline{y}\right)\right) y^1 + b^C(y)\left(q\left(\overline{y}\right) - q(y)\right).$$

Definition 5.8 *The firm is called competitive in all markets (or simply, competitive) if it is competitive in both, goods and financial markets*[17]*.*

The following proposition shows that the C-adjusted value can be seen as a generalization of Grossman-Hart's objective, since it coincides with that if the firm is competitive.

Proposition 5.9 *If the firm is competitive, its C-adjusted value coincides with the Grossman-Hart objective.*

If, in addition, $\widehat{Y}^1 \subseteq span(A), \forall y \in \widehat{Y}$ then the C-adjusted value coincides with firm's market value. Production plans that maximize firm's market value are every shareholder's most preferred plans.

Proof. Let μ be the control group's price beliefs and $\Pi \in \mathrm{supp}(\mu)$ arbitrary. As proved above, competitiveness in the financial markets implies

$$\mathcal{V}^C_{\mathcal{P}}(\mathcal{P}) = \delta^C y_0 + \sum_{i \in C} \delta^i MRS^i\left(c^i\left(\overline{y}\right)\right) y^1 + b^C(y)\left(q\left(\overline{y}\right) - q(y)\right),$$

while competitiveness in the goods market implies $q\left(\overline{y}\right) - q(y) = 0$ and thus

$$\mathcal{V}^C_{\mathcal{P}}(\mathcal{P}) = \delta^C y_0 + \sum_{i \in C} \delta^i MRS^i\left(c^i\left(\overline{y}\right)\right) y^1.$$

If $span\left(\mathcal{K}(y)\right) \supseteq \widehat{Y}^1$, then, in particular, $MRS^i\left(\overline{c}^i\right) y^1 = MRS^j\left(\overline{c}^j\right) y^1$ for every $i, j \in I$ and all $y \in \widehat{Y}$. Therefore, investors unanimously support the maximization of the C-adjusted value, which becomes: $\mathcal{V}^C_{\mathcal{P}}(\mathcal{P}) = y_0 + MRS^i\left(\overline{c}^i\right) y^1 = y_0 + MRS^j\left(\overline{c}^j\right) y^1, \forall i, j \in I.$ ■

[17]The definition of competitiveness adopted here differs from Makowski's, [19] which requires that investors' holdings of the firm-issued securities be 0. For strictly convex preferences, Makowski's definition of competitiveness implies spanning; ours does not.

The firm in example 1 of section 2 is competitive in financial markets and $C = I$. Therefore, for that example, the C-adjusted value and Grossman-Hart objective coincide.

It should be emphasized, that firm's competitiveness does *not* imply that Grossman-Hart competitive price perceptions are correct everywhere. They only generate some preferences for the control group's members whose peaks coincide with their actual most preferred production plans.

Whenever the firm is not competitive, following Grossman-Hart or Drèze objective can generate less social welfare for the members of the control group than by maximizing the C-adjusted value.

6 Existence and Pareto Optimality of Equilibria

Theorem 6.1 *If $L \ll +\infty$, the mapping $y \mapsto \mathcal{K}(y)$ is upper hemi-continuous and compact-valued and investors' utilities are strictly increasing in every component, then an equilibrium in which the firm maximizes the expected C-adjusted value exists.*

The proof of the theorem is delegated to the appendix[18].

An important characteristic of the equilibrium in which firm maximizes its C-adjusted value is that the financial policy is no longer irrelevant (as it was in Drèze's and Grossman & Hart's models). Two financial plans that finance the same production plan but generate different asset spans may give different utilities to the members of the control group and thus may be ranked differently. Hence, Modigliani-Miller theorem does not necessarily hold.

Since members of the control group decide the optimal production-financial plan based on their attitudes toward risk and their needs for risk hedging opportunities, a natural question arises: Is the optimal plan the one that creates a complete asset structure? The answer is in general negative, as shown by the following example. Thus, we should expect an incomplete asset structure to arise at the equilibrium. Given the endogenous market incompleteness, it is not surprising to see that equilibria are, in general, Pareto suboptimal. However, particular types of preferences for the investors do generate Pareto optimal allocations at the equilibrium and thus effectively complete markets (in the terminology of Elul [12]).

Example 3

Consider a two-date economy with three states of nature at date $t = 1$. There are two investors and one firm. Firm's production set consists of only one

[18] Note that the existence theorem in Kelsey-Milne [15] cannot be applied here because firm's preference relation fails to satisfy the required convexity property.

plan: $Y = \{(-1,0,-1,1)\}$. This firm can be interpreted as being a financial intermediary that owns the security $(0,-1,1)$ and incurs a cost of 1 unit of date-0 consumption for trading it in the market. Investors' characteristics are:

$$\begin{gathered} u^1(c) = c_0 + \log c_1 + 2\log c_2,\ \omega^1 = (2,1,2,4)\,,\ \delta^1 = 1 \\ u^2(c) = c_0 + \log c_{3,}\ \omega^2 = (1,5,0,0)\,,\ \delta^2 = 0. \end{gathered}$$

There are no exogenously given assets that can be traded in the market.

Suppose first that the firm chooses to finance its plan by issuing only equity. Denote this financial plan by $\mathcal{P}_0$. Given $\mathcal{P}_0$, the first investor solves

$$\begin{cases} \max\{c_0 + \log c_1 + 2\log c_2\} \\ \text{s.t. } c_0 + \theta v = 2 + v - 1 \\ \qquad c_1 = 1, c_2 = 2 - \theta, c_3 = 4 + \theta \\ \qquad c_i \geq 0,\ i = 0,..,3 \end{cases}$$

and thus chooses $\theta^1 = -4$.

The second investor solves

$$\begin{cases} \max\{c_0 + \log c_3\} \\ \text{s.t. } c_0 + \theta v = 1 \\ \qquad c_1 = 5, c_2 = 5 - \theta, c_3 = \theta \\ \qquad c_i \geq 0,\ i = 0,..,3 \end{cases}$$

and chooses $\theta^2 = \frac{1}{v}$ for $0 < v \leq \frac{1}{5}$.

The stock market equilibrium dictates then $v = \frac{1}{5}$ and thus $c_0^1 = 2$, $c_1^1 = 1$, $c_2^1 = 6$, $c_3^1 = 0$, which gives investor 1 an utility $U_{im}^1 = 2 + 2\log 6$. The firm's market value is $-1 + \frac{1}{5} = -\frac{4}{5}$. Using formula (7) to compute the C-adjusted value we obtain

$$\mathcal{V}_{\mathcal{P}_0}^{\{1\}}(\mathcal{P}_0) = 2 - 2 + \left(1, \frac{1}{3}, 0\right)\left(\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right) = \frac{4}{3}.$$

Assume now that the firm chooses a financial plan that completes the markets. Denote that financial plan by $\mathcal{P}_c$. Given $\mathcal{P}_c$, investors solve

$$\begin{cases} \max\{c_0 + \log c_1 + 2\log c_2\} \\ \text{s.t. } c_0 + p_1c_1 + p_2c_2 + p_3c_3 = 2 + p_1 + 2p_2 + 4p_3 + (-1 - p_2 + p_3) \\ \qquad c_i \geq 0,\ i = 0,..,3 \end{cases}$$

and

$$\begin{cases} \max\{c_0 + \log c_3\} \\ \text{s.t. } c_0 + p_1c_1 + p_2c_2 + p_3c_3 = 1 + 5p_1 + 5p_2 \\ \qquad c_i \geq 0,\ i = 0,..,3. \end{cases}$$

In equilibrium, $p_1 = \frac{6}{45}$, $p_2 = \frac{12}{45}$, $p_3 = \frac{1}{5}$ and $c_0^1 = 0$, $c_1^1 = 6$, $c_2^1 = 6$, $c_3^1 = 0$ which gives investor 1 a utility $U_{cm}^1 = 3\log 6 < 2 + 2\log 6 = U_{im}^1$. Hence, the owner of the firm does not want its firm to issue enough securities to complete the markets.

If $\mathcal{P}_c$ is chosen, the market value of the firm is $-1 - \frac{12}{45} + \frac{1}{5} = -\frac{16}{15}$; the C-adjusted value of $\mathcal{P}_c$ at $\mathcal{P}_0$ is:

$$\mathcal{V}_{\mathcal{P}_0}^{\{1\}}(\mathcal{P}_c) = 0 - 2 + \left(1, \frac{1}{3}, 0\right)\left(\begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right) = \frac{4}{3} - 7 < \frac{4}{3}.$$

The example is robust in the sense that the results still hold if we slightly perturb the endowments.

7 Appendix

Proof of theorem 6.1

The proof is based on the observation that, for a given rational belief of the control group, μ, there is a bijection between the set of equilibria of the production economy which are consistent with μ and the set of Nash equilibria of a two-player imitation game that will be constructed here.

We start by proving some preparatory lemmas.

Lemma 7.1 *The economy $\mathcal{E}_{\mathcal{P}}$ has an equilibrium for every $\mathcal{P} \in \widehat{\mathcal{F}}$.*

Proof. Notice first that the economy $\mathcal{E}_{\mathcal{P}}$ is equivalent to a standard stock-exchange economy, $\mathcal{E}_{\mathcal{P}}^0$, in which

1. consumers' endowments of goods are $\left(\omega^i + \delta^i y\right)_{i \in I}$,
2. asset structure is given by (A, X, D),
3. there is no initial endowment of assets,
4. consumers face "personalized" short-sale bounds, $L^{i,\mathcal{P}} \in \mathbb{R}^{J+N+1}$, given by:
   (a) $L_j + \delta^i b^f$ for every exogenously given security j,
   (b) $L_n + \delta^i$ for every firm-issued security n,
   (c) $L_d + \delta^i\left(1 + \theta^f\right)$ for the firm's shares.

Since $y \in \widehat{Y}$, every consumer's endowment of goods in $\mathcal{E}^0_{\mathcal{P}}$ is strictly positive.

The proof of the existence of an equilibrium for $\mathcal{E}^0_{\mathcal{P}}$ is similar to the standard Arrow-Debreu existence proof. The reader is referred to Geanakoplos and Polemarchakis[19], [13] for the details. An important step in the proof is the construction of an appropriate convex and compact price space. For our specification of the model, that set is defined as follows. Let λ_0 be the price of date-0 consumption in some units of account and let π the price vector of the $J+N+1$ assets (expressed in the same units). Let

$$Q \overset{def}{=} \left\{ (\lambda_0, \pi) \in \mathbb{R}_+ \times \mathbb{R}^{J+N+1} \mid \exists \lambda \in \mathbb{R}^S_+ \text{ s.t. } \pi = \lambda (A, X, D) \right\}.$$

Clearly, Q is a convex and closed cone. If Q does not contain a full line then there exists a hyperplane $H \subseteq \mathbb{R}^{J+N+2}$ (of dimension $J+N+1$) such that $0 \neq (\lambda_0, \pi q) \in Q$ if and only if $\alpha (\lambda_0, \pi) \in Q \cap H$ for some $\alpha > 0$. If Q contains a full line we take H to be half the unit sphere in $\mathbb{R}^{J+N+2}$, centered at origin. Let $Q^0 \overset{def}{=} Q \cap H$ be the price space. Then Q^0 is a convex and compact set (or an acyclic absolute neighborhood retract if H is the half sphere). ■

Let $\widetilde{\prod}^0 (\mathcal{P})$ be the set of normalized equilibrium prices of $\mathcal{E}^0_{\mathcal{P}}$. Thus $\widetilde{\prod}^0 : \widehat{\mathcal{F}} \rightrightarrows Q^0$ and $\widetilde{\prod} (\mathcal{P}) = \left\{ \left(1, \frac{\pi}{\lambda_0}\right) \mid (\lambda_0, \pi) \in \widetilde{\prod}^0 (\mathcal{P}) \right\}$.

Lemma 7.2 *The equilibrium price correspondence* $\widetilde{\prod}^0 : \widehat{\mathcal{F}} \rightrightarrows Q^0$ *is upper hemi-continuous, with compact values.*

Proof.

It is enough to show that $\widetilde{\prod}^0$ has closed graph.

For that, notice first that the equilibrium portfolios are bounded, due to the short sale constraints. Let K be a cube in $\mathbb{R}^{J+N+1}$, large enough so that it contains all the portfolio bounds. Consider the truncated portfolio demands $Z^i_K : \widehat{\mathcal{F}} \times Q^0 \rightrightarrows K$. Then Z^i_K has non-empty, convex and compact values, and is upper hemi-continuous at every $(\mathcal{P}, \pi) \in \widehat{\mathcal{F}} \times Q^0$ with $\lambda_0 \left(\omega_0 + \delta^i y_0\right) + \pi L^{i,\mathcal{P}} \neq 0$.

To overcome the possible discontinuity of the demand at points $(\mathcal{P}, \pi) \in \widehat{\mathcal{F}} \times Q^0$ for which $\lambda_0 \left(\omega_0 + \delta^i y_0\right) + \pi L^{i,\mathcal{P}} = 0$, we construct a smoothed demand correspondence, $\widehat{Z^i_K}$, and a quasi-equilibrium as in [4]. It can be shown that every quasi-equilibrium of $\mathcal{E}^0_{\mathcal{P}}$ is an equilibrium and that the smoothed demand correspondence is upper hemi-continuous everywhere.

The closed graph property of $\widetilde{\prod}^0$ follows now immediately from the upper-hemi-continuity of the smoothed aggregate demand. Since Q^0 is compact, this implies that $\widetilde{\prod}^0$ is upper hemi-continuous with compact values. ■

The rest of the proof will proceed in 3 steps.

[19] Their proof is given for economies with unlimited short-sales. Portfolio constraints only simplify the problem, as it is enough to prove existence of an equilibrium for the truncated economy.

Step 1. Construction of the game.

For a given $\mu \in \mathcal{M}$ we construct a normal form two-player game, Γ_μ, as follows:

- The strategy set of each player is

$$\widehat{\mathcal{F}}' = \left\{ \mathcal{P} = \left(y, X, D, b^f, \theta^f\right) \in \widehat{\mathcal{F}} \mid \left(b^f, \mathbf{0}_N, 0\right) \in K \right\}.$$

- The first player's payoff function is

$$\Phi_\mu^1 \left(\mathcal{P}_1, \mathcal{P}_2\right) = -\left\| \mathcal{P}_1 - \mathcal{P}_2 \right\|,$$

  where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{2S+SN+J+2}$ (we are considering $\mathcal{P}_1$ and $\mathcal{P}_2$ as $(2S+SN+J+2)$-dimensional vectors here).

- The second player's payoff function is

$$\Phi_\mu^2 \left(\mathcal{P}_1, \mathcal{P}_2\right) = \int_{\mathcal{M}} \mathcal{V}_{\mathcal{P}_1}^C \left(\mathcal{P}_2\right) d\mu \left(\Pi\right).$$

It is easy to see that $\mathcal{P}$ is an equilibrium production-financial plan consistent with the belief μ if and only if $(\mathcal{P}, \mathcal{P})$ is a Nash equilibrium of the game Γ_μ.

Step 2: The strategy space $\widehat{\mathcal{F}}'$ is compact.

We prove first that $\widehat{Y}$ is compact. Since $\widehat{Y}$ is a closed subset of $\mathbb{R}^{S+1}$, it is enough to prove that it is bounded. Suppose it is not. Then there exists a sequence $(y^n)_n \subseteq \widehat{Y}$ such that $\|y^n\| > n$, $\forall n \geq 1$. Convexity of $\widehat{Y}$ together with $0 \in \widehat{Y}$ implies then:

$$\frac{1}{\|y^n\|} y^n + \left(1 - \frac{1}{\|y^n\|}\right) 0 \in \widehat{Y}, \quad \forall n \geq 1.$$

Since $\left\| \frac{1}{\|y^n\|} y^n \right\| = 1$, we can assume, without loss of generality, that $\frac{1}{\|y^n\|} y^n \rightarrow y \in \mathbb{R}^{S+1}$, with $\|y^0\| = 1$. $\widehat{Y}$ closed implies then that $y \in \widehat{Y}$.

On the other hand, $y^n \in \widehat{Y} \Longrightarrow y^n \geq \left(-\min_i \frac{\omega_0}{\delta^i} + \varepsilon, 0, ...0\right)$ and therefore

$$lim_{n \to \infty} \frac{1}{\|y^n\|} y^n \geq -lim_{n \to \infty} \frac{\left(-\min_i \frac{\omega_0}{\delta^i}, 0, ...0\right)}{\|y^n\|} = 0.$$

Hence, $y = 0$, which contradicts $\|y\| = 1$.

$\widehat{\mathcal{F}}'$ compact follows now immediately from $\widehat{Y}$ and K being compact and $y \longmapsto \mathcal{K}(y)$ being upper hemi-continuous with compact values.

Step 3: There exists a probability measure μ such that the game Γ_μ has a Nash equilibrium.

To prove that we show that the family of games $(\Gamma_\mu)_{\mu \in \mathcal{M}}$ induces a game with endogenous sharing rules that satisfies all the hypotheses of the main theorem in [21].

Define the payoff correspondences:

$$Q^1, Q^1 : \widehat{\mathcal{F}}' \times \widehat{\mathcal{F}}' \rightrightarrows \mathbb{R}^2$$
$$Q^1 (\mathcal{P}_1, \mathcal{P}_2) = - \|\mathcal{P}_1 - \mathcal{P}_2\|,$$
$$Q^2 (\mathcal{P}_1, \mathcal{P}_2) = \left\{ \int_{\mathcal{M}} \widetilde{\mathcal{V}}_{\mathcal{P}_1} (\mathcal{P}_2)\, d\mu (\Pi) \mid \mu = \text{probability on } \mathcal{M} \right\}.$$

The game satisfy the hypotheses of the main theorem in [21] if (a) the strategy sets are compact metric spaces, and (b) correspondences Q^1 and Q^2 are upper hemi-continuous with compact and convex values.

a) The strategy space $\widehat{\mathcal{F}}'$ is a metric space with the metric induced by the Euclidian metric of $\mathbb{R}^{2S+SN+J+2}$. According to step 2, it is also compact.

b) Q^1 is a continuous function and thus upper hemi-continuous as a correspondence. Clearly, it has compact and convex values. Upper hemi-continuity of Q^2 (as well as compactness of its values) follows immediately from the upper hemi-continuity and compactness of the values of $\widetilde{\prod}^0$, together with the continuity of the optimal consumption as a function of prices and endowments. Convexity of values follows from the linearity of the integral with respect to μ.

Therefore, there exists a probability measure μ such that the game Γ_μ has a Nash equilibrium.

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