Auctions with Dynamic Populations: Efficiency and Revenue Maximization

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ABSTRACT: We examine an environment where goods and privately informed buyers arrive stochastically to a market. A seller in this setting faces a sequential allocation problem with a changing population. We characterize the set of incentive compatible allocation rules and provide a generalized revenue equivalence result. In contrast to a static setting where incentive compatibility implies that higher-valued buyers have a greater likelihood of receiving an object, in this dynamic setting, incentive compatibility implies that higher-valued buyers have a greater likelihood of receiving an object sooner.

We also characterize the set of efficient allocation rules and show that a dynamic Vickrey-Clarke-Groves mechanism is efficient and dominant strategy incentive compatible. We then derive an optimal direct mechanism. We show that the revenue-maximizing direct mechanism is a pivot mechanism with a reserve price.

Finally, we consider sequential ascending auctions in this setting, both with and without a reserve price. We construct memoryless equilibrium bidding strategies in this indirect mechanism. Bidders reveal their private information in every period, yielding the same outcomes as the direct mechanisms. Thus, the sequential ascending auction is a natural institution for achieving either efficient or optimal outcomes. Interestingly, this is not the case for sequential second-price auctions, as the bids in a second-price auction do not reveal sufficient information to realize either the efficient or optimal allocation.

KEYWORDS: Dynamic mechanism design, Random arrivals, Revenue equivalence, Indirect mechanisms, Sequential ascending auctions.

JEL CLASSIFICATION: C73, D44, D82, D83.

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1. Introduction

In this paper, we study the problem of a seller faced with a stochastic sequential allocation problem. At each point in time, a random number of buyers and objects arrive to a market. Buyers are risk-neutral and patient, while objects are homogeneous and perishable. Each buyer desires a single unit of the good in question; however, valuations for the good vary across buyers. The mechanism designer must elicit the private information of these buyers in order to achieve a desirable allocation—one that is either efficient or revenue maximizing, depending on the designer's objective function.

We show that the properties of the Vickrey-Clarke-Groves (VCG) mechanism in the static world carry through to our dynamic setting. In particular, by using a dynamic analogue of the VCG mechanism, the social planner ensures that truth-telling is a dominant strategy for all agents, resulting in an efficient allocation. Moreover, we show that the optimal (revenue-maximizing) mechanism in this dynamic setting is essentially a pivot mechanism with a reserve price. Finally, we characterize a simple indirect mechanism, the sequential ascending auction, that serves as a natural institution for achieving both the efficient and the optimal outcome.

Sequential allocation problems arise naturally in a wide variety of economic contexts. Consider, for example, the problem faced by an airport with a limited number of take-off and landing slots available at any time. If the airport wishes to use its limited resources efficiently, it must allocate slots to airlines based on their relative value. However, these values are typically the private information of the airlines. Therefore, the airport must elicit this information. Without the appropriate incentives for truthful revelation, however, an efficient allocation will be unattainable. As has recently been proposed for New York's LaGuardia airport, auctions can play an important role in providing these incentives. An important mechanism design consideration, however, is one of market dynamics. New airlines may wish to start serving routes to the airport in question, while established carriers may consolidate routes or merge with one another. Such events alter the competitive environment and the demand for the airport's limited runway space (and time), and should be taken into account.

Similar problems occur in other settings. Electricity markets must be able to dynamically adjust to fluctuations in supply (as new generation facilities come online or unexpected outages occur in existing power plants, for instance) as well as in demand (for example, when consumption changes in response to weather conditions). The addition or removal of nodes from a communications network requires the reallocation of bandwidth in response to capacity constraints and the relative importance or value of network traffic. Likewise, computational tasks submitted to a supercomputing facility are placed in a queue for future processing in response to the submission of higher-priority tasks, while sponsored search platforms such as Google or Yahoo! need to dynamically assign advertisements to ranked "slots" when internet users search various keywords and new advertisers join the platform.

The model we consider here abstracts away many of the details of these various situations and markets, focusing instead on what we view are their essential features. In particular, demand is not constant, as the set and number of buyers change over time, with patient buyers entering and exiting the market according to a stochastic process. Similarly, supply is random. In some periods there may be many units available, while in others none. Finally, each buyer's valuation—her willingness to pay—is private information. Therefore, a welfare- or revenue-maximizing seller must provide appropriate incentives for information revelation to this dynamic population. The seller then makes use of this information to dynamically allocate goods to buyers.

It is well known that, in static environments, the VCG mechanism is efficient. By choosing a transfer payment for each buyer that equals the externality imposed by her report on other participants in the mechanism, the VCG mechanism aligns the incentives of the buyer with those of a welfare-maximizing social planner. This leads to efficiency and dominant-strategy incentive compatibility, as truthful reporting now maximizes both the planner's and the buyer's objective functions. In the dynamic environment we consider, the arrival of a new buyer imposes an externality on her competitors by reordering the (anticipated) schedule of allocations to those buyers currently present, as well as to those buyers expected to arrive in the future. We show that by charging each agent, upon her arrival, a price equal to this expected externality, the buyer's incentives are aligned with those of the forward-looking planner. Therefore, this dynamic version of the VCG mechanism is efficient and dominant-strategy incentive compatible.

In addition, we are able to construct a revenue-maximizing direct mechanism for this setting. Making use of the risk-neutrality of buyers, we show that the optimal policy for a revenue-maximizing seller is equivalent to that of a social planner who wishes to maximize allocative efficiency, except that buyers' values are replaced by their *virtual* values. Each buyers' incentives may then be aligned with those of the seller by a variation of the VCG mechanism where buyers face discriminatory reserve prices. By providing each newly arriving buyer with an expected payoff equal to her expected marginal contribution to the *virtual* surplus, the seller is able to induce truthful reporting of private information in dominant strategies. This allows the seller to discriminate between buyers in such a way as to maximize revenue.

Both of the mechanisms discussed above are dynamic direct revelation mechanisms, requiring buyers to report their values to the mechanism upon their arrival to the market. In practice, however, direct revelation mechanisms may be difficult to implement. For instance, the multi-unit Vickrey auction—the (static) mutli-unit generalization of the standard VCG mechanism—is a direct revelation mechanism in which truth-telling is a dominant strategy. Despite this, Ausubel (2004) points out that it lacks simplicity and transparency, explaining that "many [economists] believe it is too complicated for practitioners to understand." Moreover, Rothkopf, Teisberg, and Kahn (1990) explain that concerns about privacy or the potential for future misuse of information revealed in a direct mechanism may preclude the real-world use of direct mechanisms. These criticisms are corroborated by experimental evidence. According to Kagel, Harstad, and Levin (1987), who examined single-unit auctions with affiliated private values, the predictions of auction theory for bidding behavior are significantly more accurate in ascending price-auctions than in second-price auctions, despite the existence of a dominant-strategy equilibrium in the second-price (Vickrey) auction. In another study examining the efficiency properties of several mechanisms in a resource allocation problem similar to the one we consider here, Banks, Ledyard, and Porter (1989) find that "the transparency of a mechanism . . . is important in achieving more efficient allocations." In their experiments, a simple ascending auction dominated both centralized administrative allocation processes as well as decentralized markets in terms of both efficiency and revenues.

With these criticisms and "real-world feasibility" constraints in mind, we turn to the design of simple, transparent indirect mechanisms. In particular, we consider the possibility of achieving efficient or revenue-maximizing outcomes via a sequence of auctions. Despite the resemblance of our direct mechanisms to their single-unit static counterparts, we find that this relationship does not hold for the corresponding auction formats. Recall that, in the canonical auction environment, the analogue of the VCG mechanism is either the second-price auction or the ascending auction. In our dynamic environment, however, a sequence of second-price auctions cannot yield outcomes equivalent to those of the dynamic VCG mechanism. In a sequential auction, there is an "option value" associated with losing in a particular period, as buyers have the possibility of winning an auction in a future period. The value of this option depends, in general, on the private information of all other competitors, as the expected price in the future will depend on their values—despite the assumption of independent private information, the dynamics of the market create an environment with interdependent values.

Therefore, a standard second-price auction does not reveal sufficient information for the correct determination of buyers' option values. On the other hand, the ascending auction is a simple open auction format that *does* allow for the gradual revelation of private information. We use this fact to construct intuitive equilibrium bidding strategies for buyers in a sequence of ascending auctions. In each period, buyers bid up to the price at which they are indifferent between winning an object and receiving their expected marginal contribution to the social welfare in the future. As buyers drop out of the auction, they (indirectly) reveal their private information to their competitors, who are then able to condition their bidding on this information. When this process of information revelation is repeated in *every* period, newly arrived buyers are able to learn about their competitors without being privy to the events of previous periods. This information renewal is crucial for providing the appropriate incentives for new entrants to also reveal their private information, leading to prices and allocations identical to the dominant-strategy equilibrium of the efficient direct mechanism. Moreover, these strategies form a periodic *ex post* equilibrium: given her expectations about future competition, each buyer's behavior in any period remains optimal even after observing her current opponents' values.

Similar arguments apply when considering revenue-maximizing indirect mechanisms. When buyers' values are drawn from the same distribution, the sequential ascending auction with a reserve price admits an equilibrium that is equivalent to truth-telling in the optimal direct mechanism. Thus, the sequential ascending auction is a natural institution for achieving either efficient or optimal outcomes.

The present work contributes to a recent literature exploring dynamic allocation problems and dynamic mechanism design. Bergemann and Välimäki (2008) develop the dynamic pivot mechanism, a dynamic generalization of the Vickrey-Clarke-Groves mechanism that yields efficient outcomes when agents' private information evolves stochastically over time. Athey and Segal (2007) characterize an efficient dynamic mechanism that is budget-balanced and incentive compatible, again in the presence of evolving private information. In a similar setting, Pavan, Segal, and

Toikka (2008) consider the more general question of characterizing incentive-compatible mechanisms. While these papers study dynamic mechanisms for a fixed set of buyers whose types may change over time, we examine a setting where the number and set of buyers may change over time but types are fixed.

This paper also relates to work on dynamic auctions and revenue management. For instance, Mierendorff (2008) characterizes an auction mechanism that efficiently allocates a single storable object when buyers arrive over the course of the auction. Pai and Vohra (2008) derive the revenue-maximizing mechanism for allocating a finite number of storable objects to buyers whose arrival to and departure from the market is also private information. Vulcano, van Ryzin, and Maglaras (2002) also examine optimal mechanisms for selling identical objects to randomly arriving buyers. When the objects are heterogeneous but commonly-ranked, Gershkov and Moldovanu (2008a) and (2008b) derive revenue maximizing and efficient mechanisms. In contrast to the present work, the buyers in these models are impatient and there are a fixed number of storable objects to be allocated.

Finally, our analysis of indirect mechanisms is linked to the sequential auctions literature. The seminal work is Milgrom and Weber (2000), which examines the properties of a variety of auction formats for the (simultaneous or sequential) sale of a fixed set of objects to a fixed set of buyers. However, they allow for neither discounting nor the entry of new buyers, features that play a central role in our model. Said (2008) examines the role of random entry in a model of sequential second-price auctions when objects are stochastically equivalent; that is, when values are independently and identically distributed across both buyers and objects. The computer science literature, motivated in part by the emergence of online auction sites such as eBay, has also turned attention towards sequential ascending auctions. Lavi and Nisan (2005) and Lavi and Segev (2008) examine the "worst-case" performance of sequential ascending auctions with dynamic buyer populations. Their prior-free, non-equilibrium analysis provides a lower bound on the efficiency of the allocations achieved via sequential ascending auctions.

The remainder of this paper is structured as follows. Section 2 develops the general model, introduces dynamic mechanisms, and extends the static payoff- and revenue-equivalence results of Myerson (1981) and Maskin and Riley (1989) to a dynamic environment. We provide an extended example in Section 3 of both the direct and the indirect mechanisms we study, providing an preview of our efficient implementation results. In Section 4, we fully characterize the efficient allocation rule and show that truth-telling is dominant-strategy incentive compatible in dynamic versions of the Vickrey-Clarke-Groves mechanism. We then construct an efficient equilibrium of the sequential ascending auction in Section 5 and show that it is outcome equivalent to the dynamic pivot mechanism. Section 6 parallels the development in Sections 4 and 5, characterizing the revenue-maximizing allocation policy and constructing an optimal direct mechanism for the dominant-strategy implementation of that policy. We then show that revenue maximization is achievable via the sequential ascending auction with a reserve price, in a manner analogous to the relationship between the static Myerson (1981) optimal auction and the second-price auction. We return in Section 7 to our example to illustrate these results. Finally, Section 8 concludes. All proofs may be found in Appendix A.

2. Model

2.1. *Buyers, Objects, and Random Arrivals*

We consider an infinite-horizon discrete-time environment; time periods are indexed by t, where $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. There is a countable set $\mathcal{I}$ of buyers, where each agent $i \in \mathcal{I}$ desires a single unit of a homogeneous, indivisible good. Each buyer i's valuation v_i for this good is private information, where v_i is independently distributed according to the distribution F_i. We assume that F_i has a strictly positive and continuous density f_i and support $\mathbf{V} := [0, \bar{v}]$. We assume that each buyer's virtual valuation

$$\varphi_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

is a strictly increasing function of v_i. Moreover, we assume that buyers are risk neutral, and that their preferences are quasilinear and time separable. The future is discounted exponentially with the (common) discount factor $\delta \in (0,1)$.

Buyers are not assumed to be present in each period. Rather, buyers arrive stochastically to the market. In particular, the set $\mathcal{I}$ of buyers is partitioned into disjoint subsets $\{\mathcal{I}_t\}_{t\in\mathbb{N}_0}$, where $\mathcal{I}_t$ is the finite subset of agents who may arrive in period t. The arrival of agent $i \in \mathcal{I}_t$ in period t is governed by an independent draw from a Bernoulli distribution, where $\pi_i \in [0,1]$ denotes the probability that i is present. In addition, buyers may depart from the market after each period, where the (common) probability of any buyer i "surviving" to the following period is denoted by $\gamma \in [0,1]$. Otherwise, buyers remain present in the market until they receive an object. Note that, unlike the probability of arriving to the market, the survival rate is identical across agents.

Thus, the arrivals and departures of buyers yield a stochastic process $\{\alpha_t\}_{t\in\mathbb{N}_0}$, where $\alpha_t : \mathcal{I} \to \{0,1\}$ is an indicator function of the presence of each agent in period t, and

$$\mathcal{A}_t := \{i \in \mathcal{I} : \alpha_t(i) = 1\}$$

is the subset of agents present in period t. We assume that buyers cannot conceal their presence, and so α_t (equivalently, $\mathcal{A}_t$) is commonly known to the agents present at time t.

In addition to the random arrival of buyers, several units of a homogeneous, indivisible, and non-storable good may also arrive on the market. Let $k_t \in \mathcal{K} := \{0, 1, \ldots, \overline{K}\}$ denote the number of objects that arrive in period t, where $\overline{K}$ is the maximal number of objects potentially available in any given period. As with buyers, the arrival of objects is governed by a stochastic process, where $\mu_t(k) \in [0,1]$ denotes the probability that exactly $k \in \mathcal{K}$ objects are available in period t. Moreover, these objects are non-storable; any unallocated objects "expire" at the end of each period, and hence cannot be carried over to future periods. This assumption plays an important role in the determination of the efficient policy, providing a great deal of tractibility. As with the buyer arrival process, we assume that the arrival of objects is publicly observed, and so k_t is commonly known to those agents present on the market at time t.

Thus, at the beginning of each period, new buyers arrive to the market (and old buyers may depart). Simultaneously, new objects arrive, replacing any unallocated objects left over from the previous period. The realizations of these arrival and departure processes are publicly observed by all agents present on the market. The mechanism designer may then allocate objects to agents, and we move on to the following period.

2.2. *Dynamic Direct Mechanisms*

In this setting, a dynamic direct mechanism asks each agent i to make a *single* report, upon arrival to the market, of her type v_i.[1] We denote by $\emptyset$ the "report" of an agent who has not arrived to the market. Thus, the mechanism designer has available to her in each period a collection of reports $\mathbf{r}_t : \mathcal{I}_t \to \mathbf{V} \cup \{\emptyset\}$, where $\mathcal{R}$ is the set of all such reports. Note that the report $r_i \in \mathbf{V}$ of an agent i who *has* arrived need not be truthful, as this will depend upon the incentives provided by the mechanism.

Let $\mathcal{H}_t$ denote the set of period-t histories, where each history $h_t \in \mathcal{H}_t$ is a sequence of arrivals and departures (of buyers and objects), agent reports, and allocations up to, and including, period $t-1$. Thus, we have

$$h_t = (\alpha_0, k_0, \mathbf{r}_0, \mathbf{x}_0, \alpha_1, k_1, \mathbf{r}_1, \mathbf{x}_1, \ldots, \alpha_{t-1}, k_{t-1}, \mathbf{r}_{t-1}, \mathbf{x}_{t-1}),$$

where $\mathbf{x}_s = \{x_{i,s}\}_{i \in \mathcal{I}} \in \mathbf{X} := \{0,1\}^{\mathcal{I}}$ is the allocation in period s.

A *dynamic direct mechanism* is then a sequence of feasible allocations and feasible monetary transfers $\mathcal{M} = \{\mathbf{x}_t, \mathbf{p}_t\}_{t \in \mathbb{N}_0}$, where we abuse notation and denote by

$$\mathbf{x}_t : \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R} \to \Delta(\mathbf{X})$$

a collection of allocation probabilities for each agent, and denote by

$$\mathbf{p}_t : \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R} \to \mathbb{R}^{\mathcal{I}}$$

a collection of monetary transfers *from* each agent. The period-t allocation $\mathbf{x}_t = \{x_{i,t}\}_{i \in \mathcal{I}}$ is a *feasible allocation* if, and only if,

$$\sum_{i \in \mathcal{I}} x_{i,t} \leq k_t$$

and

$$x_{i,t} = 0 \text{ for all } i \notin \mathcal{A}_t.$$

These two conditions require, respectively, that no more objects than are available in period t are allocated at that time, and that objects are only allocated to agents that are present on the market. Notice that we have implicitly ruled out the possibility of allocating multiple objects to any agent as a consequence of the single-unit demand assumption. Similarly, $\mathbf{p}_t = \{p_{i,t}\}_{i \in \mathcal{I}}$ is a *feasible monetary transfer* if, and only if,

$$p_{i,t} = 0 \text{ for all } i \notin \mathcal{A}_t;$$

that is, agents who are not present on the market cannot make or receive payments.

We assume that, upon her arrival to the market in period t, agent $i \in \mathcal{I}_t$ observes only the set $\mathcal{A}_t$ of agents present on the market (equivalently, the indicator α_t) and the number k_t of objects available at time t. Agent i does not observe the history of arrivals and departures in previous periods or the history of allocative decisions, nor does she observe the reports of agents who have

[1] The revelation principle applies in this setting, and so the restriction to direct mechanisms is without loss of generality.

arrived before her. Thus, a reporting strategy for agent i, conditional on having arrived to the market, is simply a mapping

$$r_i : \mathbf{V} \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \to \mathbf{V}.$$

Let $\mathbf{r}_{-i}$ denote the reports of all agents other than agent $i \in \mathcal{I}_t$. The expected payoff to i when she reports $r_i \in \mathbf{V}$ to the mechanism $\mathcal{M}$ and all other agents report according to $\mathbf{r}_{-i}$ is then

$$\mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t}\Big(x_{i,s}\left(h_s, \alpha_s, k_s, (r_i, \mathbf{r}_{-i})\right) v_i - p_{i,s}\left(h_s, \alpha_s, k_s, (r_i, \mathbf{r}_{-i})\right)\Big)\right],$$

where the expectation is taken with respect to the arrival and departure processes of buyers and sellers, as well the history h_t and the reports of all other agents that may be present on the market. Note that we have dropped the dependence of reporting strategies on histories and market presence to simplify notation.

2.3. *Incentive Compatibility and Individual Rationality*

Consider a direct mechanism $\mathcal{M} = \{\mathbf{x}_t, \mathbf{p}_t\}_{t\in\mathbb{N}_0}$ and fix an arbitrary period t and an arbitrary agent $i \in \mathcal{I}_t$. Suppose that all other agents $j \neq i$ are reporting truthfully; that is, suppose that

$$r_j(v_j, \alpha_s, k_s) = v_j$$

for all $s \in \mathbb{N}_0$, all $j \in \mathcal{I}_s \setminus \{i\}$, and every $(v_j, \alpha_s, k_s) \in \mathbf{V} \times \{0,1\}^{\mathcal{I}} \times \mathcal{K}$. For notational convenience, we will denote this strategy by $\mathbf{v}_{-i}$.

Recall that agent $i \in \mathcal{I}_t$, upon her arrival, observes only the set $\mathcal{A}_t$ of agents present on the market and the number k_t of objects available in period t. Thus, we may define

$$U_i(v_i', v_i, \alpha_t, k_t) := \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t}\Big(x_{i,s}\left(h_s, \alpha_s, k_s, (v_i', \mathbf{v}_{-i})\right) v_i - p_{i,s}\left(h_s, \alpha_s, k_s, (v_i', \mathbf{v}_{-i})\right)\Big)\right].$$

$U_i(v_i', v_i, \alpha_t, k_t)$ is the expected payoff of agent $i \in \mathcal{I}_t$ from reporting $v_i' \in \mathbf{V}$ when her true type is $v_i \in \mathbf{V}$ and the current state of the market is given by $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$.

The mechanism $\mathcal{M}$ is *incentive compatible* if, for all $t \in \mathbb{N}_0$ and all $i \in \mathcal{I}_t$,

$$U_i(v_i, v_i, \alpha_t, k_t) \geq U_i(v_i', v_i, \alpha_t, k_t) \text{ for all } v_i, v_i' \in \mathbf{V} \text{ and all } (\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}.$$

Thus, $\mathcal{M}$ is incentive compatible if truthful reporting by all agents, regardless of their time of entry, the agents present upon their arrival, and the number of objects available, is an equilibrium. Notice that this condition is equivalent to requiring interim incentive compatibility for each agent, for *every* realization of the arrival processes and every realization of the agent's values.

Similarly, $\mathcal{M}$ is *individually rational* if, for all $t \in \mathbb{N}_0$ and all $i \in \mathcal{I}_t$,

$$U_i(v_i, v_i, \alpha_t, k_t) \geq 0 \text{ for all } v_i \in \mathbf{V} \text{ and all } (\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}.$$

Thus, $\mathcal{M}$ is individually rational if all agents prefer to participate (truthfully) in the mechanism than not, where we have normalized the outside option of each player to zero. As with incentive compatibility, this must hold for *every* realization of the arrival processes and every realization of the agent's values.

2.4. *Revenue Equivalence*

Notice that, due to the agents' risk neutrality and the quasilinearity of payoffs, we may rewrite the payoff functions U_i as

$$U_i(v_i', v_i, \alpha_t, k_t) = q_i(v_i', \alpha_t, k_t)v_i - m_i(v_i', \alpha_t, k_t),$$

where

$$q_i(v_i', \alpha_t, k_t) := \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} x_{i,s}\left(h_s, \alpha_s, k_s, (v_i', \mathbf{v}_{-i})\right)\right] \tag{1}$$

is the expected discounted sum of object allocation probabilities, and

$$m_i(v_i', \alpha_t, k_t) := \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} p_{i,s}\left(h_s, \alpha_s, k_s, (v_i', \mathbf{v}_{-i})\right)\right]$$

is the expected discounted sum of payments. Therefore, we may rewrite the incentive compatibility and individual rationality constraints as

$$q_i(v_i, \alpha_t, k_t)v_i - m_i(v_i, \alpha_t, k_t) \geq q_i(v_i', \alpha_t, k_t)v_i - m_i(v_i', \alpha_t, k_t) \text{ for all } v_i, v_i' \in \mathbf{V}$$

$$\text{and } q_i(v_i, \alpha_t, k_t)v_i - m_i(v_i, \alpha_t, k_t) \geq 0 \text{ for all } v_i \in \mathbf{V},$$

for all $t \in \mathbb{N}_0$, all $i \in \mathcal{I}_t$, and all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$.

Define for all $t \in \mathbb{N}_0$ and $i \in \mathcal{I}_t$ the function $\widehat{U}_i : \mathbf{V} \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \to \mathbb{R}$ by

$$\widehat{U}_i(v_i, \alpha_t, k_t) := q_i(v_i, \alpha_t, k_t)v_i - m_i(v_i, \alpha_t, k_t).$$

$\widehat{U}_i$ is then the expected payoff of agent i from truthfully reporting her value v_i. By considering the properties of this function, we are able to generalize the classic Myerson (1981) characterization of incentive compatibility and expected payoffs in the static independent private values environment. In particular, we have the following characterization of implementable mechanisms.

LEMMA 1 (Implementable Mechanisms)**.**
A direct mechanism $\mathcal{M} = \{\mathbf{x}_t, \mathbf{p}_t\}_{t \in \mathbb{N}_0}$ is incentive compatible and individually rational if, and only if, the following three conditions are satisfied for all $t \in \mathbb{N}_0$ all $i \in \mathcal{I}_t$, and all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$:

(1) *$q_i(v_i, \alpha_t, k_t)$ is nondecreasing in v_i;*
(2) *$\widehat{U}_i(v_i, \alpha_t, k_t) = \widehat{U}_i(0, \alpha_t, k_t) + \int_0^{v_i} q_i(v_i', \alpha_t, k_t)\, dv_i'$ for all $v_i \in \mathbf{V}$; and*
(3) *$\widehat{U}_i(0, \alpha_t, k_t) \geq 0$.*

This lemma is the dynamic population analogue of the standard result for static allocation problems. In particular, instead of a higher type resulting in a higher probability of receiving an object (as in the static setting), incentive compatibility in this dynamic setting requires that a higher type has, roughly speaking, a higher probability of receiving an object *earlier*; that is, the discounted sum of each agent's expected allocation probabilities must be increasing in that agent's type. Moreover, the expected payoffs of a buyer in any two mechanisms with the same allocation rule can differ only by a constant. As an immediate consequence, we have the following generalization of the revenue equivalence theorem to our environment, which we state without proof.

COROLLARY 1 (Revenue Equivalence).
If the dynamic direct mechanism $\mathcal{M}$ is incentive compatible, then for all $t \in \mathbb{N}_0$ all $i \in \mathcal{I}_t$, and all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$, the expected payment of type $v_i \in \mathbf{V}$ of buyer i, conditional on entry, is

$$m_i(v_i, \alpha_t, k_t) = m_i(0, \alpha_t, k_t) + q_i(v_i, \alpha_t, k_t)v_i - \int_0^{v_i} q_i(v_i', \alpha_t, k_t)\, dv_i'.$$

If, in addition, $\mathcal{M}$ is individually rational, then $m_i(0, \alpha_t, k_t) \leq 0$.

Therefore, in any incentive compatible dynamic mechanism, the expected payment of a buyer depends (up to an additive constant) only upon the allocation rule. This fact will be particularly useful in deriving a revenue-maximizing mechanism in Section 6.

3. An Illustrative Example

Before proceeding to the analysis of efficient mechanisms in our general environment, it is instructive to examine a simple example. In each period, exactly one unit of the homogeneous good is available. In the initial period, there are $n_0 \in \mathbb{N}_0$ buyers present, and in each subsequent period, at most one new buyer may arrive to the market with probability $\rho \in [0,1]$. Buyers are symmetric: each buyer i's privately known valuation for a single object is v_i, where v_i is drawn independently and identically from the common distribution F. In addition, buyers do not exit the market until they receive an object.

Thus, in terms of the notation of Section 2, we have $|\mathcal{I}_0| = n_0$ and $\pi_i = 1$ for all $i \in \mathcal{I}_0$, while $|\mathcal{I}_t| = 1$ and $\pi_i = \rho$ for all $i \in \mathcal{I}_t, t > 0$. Since buyers do not depart the market without an object, we have $\gamma = 1$. Finally, since there is exactly one unit available in each period, for all $t \in \mathbb{N}_0$ we have $\mu_t(1) = 1$ and $\mu_t(k) = 0$ for all $k \neq 1$.

3.1. *The Efficient Allocation*

In order to achieve allocative efficiency in this simple environment, a social planner should assign each object to the highest-valued buyer present. While a formal statement and proof of this result are subsumed by Lemma 2, we will (for now) rely on an intuitive sketch of that argument. Suppose that in some period t, the planner assigns the object to an agent i with value v_i despite the availability of another agent j with $v_j > v_i$. Let $t' > t$ denote the period in which the planner eventually allocates an object to agent j, where we allow for the possibility that $t' = \infty$ (the planner never allocates an object to agent j). Notice that by "swapping" the allocations to agents i and j, the planner is able to realize a gain. In particular, consider the alternative allocation rule that does not alter the sequence of allocations to agents other than i and j, but instead assigns an object to i whenever the planner would assign it to j, and *vice versa*. This leads to a change in the planner's payoff of

$$\left(\delta^t v_i + \delta^{t'} v_j\right) - \left(\delta^t v_j + \delta^{t'} v_i\right) = \left(\delta^t - \delta^{t'}\right)\left(v_i - v_j\right) > 0.$$

Thus, allocating to a lower-valued buyer when a higher-valued buyer is available is always inefficient. By letting $v_i = 0$, this argument also demonstrates that the planner should never "discard" an object when there are any buyers present on the market, but should instead allocate it to the highest-valued buyer available.

Denote the efficient policy by $\widehat{\mathbf{x}}$. Since agent i receives an object in an arbitrary period s under policy $\widehat{\mathbf{x}}$ only if i has the highest value among all agents present at time s, agent i's expected discounted probability of receiving an object

$$\hat{q}_i(v_i, \alpha_t, k_t) = \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} \hat{x}_{i,s}(h_s, \alpha_s, k_s, (v_i, \mathbf{v}_{-i}))\right]$$

is nondecreasing in v_i. It is then immediately clear, by applying Lemma 1, that it is possible to implement the efficient policy $\widehat{\mathbf{x}}$ by designing an incentive compatible mechanism that induces buyers to truthfully reveal their private information.

3.2. *Direct Implementation*

A natural candidate for such a mechanism is one based on the static Vickrey-Clarke-Groves mechanism. In particular, it is possible to align the incentives of the buyers with those of the planner by charging buyers payments that leave them with a payoff equal to their marginal contribution to the social welfare.[2] To see why this is so, suppose there are n buyers $\{1, \ldots, n\}$ present on the market, with values

$$v_1 > v_2 > \cdots > v_n.$$

In the case of a fixed set of buyers and no entry, the efficient allocation yields a value to the planner given by $\sum_{j=1}^{n} \delta^{j-1} v_j$, as buyer 1 receives an object in the current period, buyer 2 receives an object in the next period, and so on. In the presence of entry, however, new buyers can and do alter the times at which existing agents are efficiently allocated an object.

Consider, for instance, buyer 2, the buyer with the second-highest value. If we increase her value v_2 by an infinitesimal amount, the efficient allocation in the current period is unchanged, and the planner realizes no benefit from the increase. In the following period, however, buyer 2 is allocated an object if either no new entrant arrives (which occurs with probability $1 - \rho$), or if a new entrant arrives but has a value smaller than v_2 (which occurs with probability $\rho F(v_2)$). On the other hand, if a new, higher-valued buyer *does* arrive in the next period, the efficient policy allocates the object to the entrant. The planner must then wait an additional period before potentially realizing the gains from the increase in v_2. Since the arrival process is stationary, this implies that the marginal benefit to the planner from the increase in v_2 is given by the geometric series

$$\begin{aligned} &\delta\Big[1 - \rho(1 - F(v_2)) + \rho(1 - F(v_2)) \times \delta\big[1 - \rho(1 - F(v_2)) + \rho(1 - F(v_2)) \times \delta\,[\cdots]\,\big]\Big] \\ &\qquad\qquad = \delta \sum_{t=0}^{\infty} \left[\delta\rho(1 - F(v_2))\right]^t \big(1 - \rho(1 - F(v_2))\big) = \delta \frac{1 - \delta(1 - F(v_2))}{1 - \delta\rho(1 - F(v_2))}. \end{aligned}$$

Integrating this ratio then allows us to then measure the total contribution of buyer 2 over a "null" buyer with zero value, yielding

$$\delta \int_0^{v_2} \frac{1 - \rho(1 - F(v'))}{1 - \delta\rho(1 - F(v'))}\, dv' = \delta \int_0^{v_2} \lambda(v')\, dv',$$

[2] In an early contribution to the scheduling and queueing literature, Dolan (1978) showed that a VCG-like mechanism succeeds in implementing an efficient sequence of allocations. His model is closely related to the example we consider here, but focuses on delay costs instead of discounted gains.

where we define $\lambda : \mathbf{V} \to [0,1]$ by

$$\lambda(v) := \frac{1 - \rho(1 - F(v))}{1 - \delta\rho(1 - F(v))}.$$

An identical argument may be used to show that the marginal benefit to the planner from an increase in v_k is $\delta^{k-1}\lambda^{k-1}(v_k)$, and thus the total surplus due to the presence of buyer k takes a similar integral form.

We can thus explicitly calculate the social planner's expected payoff from following the efficient policy $\widehat{\mathbf{x}}$.[3] This is given by

$$W_n(v_1, \ldots, v_n) = W_0 + \sum_{j=1}^{n} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv',$$

where the planner's expected payoff when no buyers are present on the market is

$$W_0 := \frac{\delta\rho}{1-\delta} \int_0^{\bar{v}} v'\, dF(v').$$

Similarly, the planner's payoff under the efficient policy when buyer k is removed from the market is given by

$$W_{n-1}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n) = W_0 + \sum_{j=1}^{k-1} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv' + \sum_{j=k+1}^{n} \delta^{j-2} \int_0^{v_j} \lambda^{j-2}(v')\, dv'.$$

Therefore, the marginal contribution of buyer k to the social welfare is

$$\begin{aligned} w_k(v_1, \ldots, v_n) &:= W_n(v_1, \ldots, v_n) - W_{n-1}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n) \\ &= \sum_{j=k}^{n} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv' - \sum_{j=k+1}^{n} \delta^{j-2} \int_0^{v_j} \lambda^{j-2}(v')\, dv' \\ &= \sum_{j=k}^{n} \delta^{j-1} \int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\, dv', \end{aligned}$$

where we let $v_{n+1} := 0$. Notice that removing buyer k from the market does not affect the allocation of those buyers ranked above her; however, her presence shifts back by one period the anticipated allocation to buyers ranked below her. By choosing a mechanism in which transfer payments are such that buyer k internalizes this effect, the social planner can induce her to truthfully reveal her value.

One such mechanism is the dynamic pivot mechanism of Bergemann and Välimäki (2008). This efficient mechanism prescribes transfers in each period such that every agent i's flow payoff is exactly equal to her flow marginal contribution to the social welfare. In this setting, an agent's flow marginal contribution is identically zero unless she receives an object. On the other hand, when the agent *does* receive an object, her flow marginal contribution is exactly equal to her total marginal contribution. Therefore, when n agents are present with values $v_1 > \cdots > v_n$, the dynamic pivot mechanism gives the object to buyer 1, charging her a payment $\hat{p}$ such that

$$v_1 - \hat{p} = w_1(v_1, \ldots, v_n).$$

[3] A formal statement and proof of this result may be found in Appendix B.

Rearranging this expression, we see that the transfer paid by buyer 1 is given by

$$\hat{p} = v_1 - \sum_{j=1}^{n} \delta^{j-1} \int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\, dv' = v_2 - \sum_{j=2}^{n} \delta^{j-1} \int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\, dv'. \tag{2}$$

As we will show formally in Corollary 2 in Section 4, when the social planner chooses transfers as in the equation above, the efficient policy is implemented in dominant strategies.

3.3. *Indirect Implementation*

It is important to keep in mind that the Vickrey-Clarke-Groves mechanism and its dynamic derivatives (such as the dynamic pivot mechanism) are direct revelation mechanisms, relying on a central planner to aggregate the reported values of each buyer via allocation and payment policies. It is possible, however, to implement the efficient allocation rule via a decentralized indirect mechanism. Indeed, the indirect mechanism in which a standard Milgrom and Weber (1982) "button" ascending auction is conducted in each period admits a stationary and symmetric equilibrium which is outcome equivalent to the dynamic pivot mechanism.

The following observation is central to the construction of the strategies used in this equilibrium. We may decompose the transfer paid by buyer 1 in Equation (2) into two components. The first term (v_2) is the cost imposed on society by allocating an object to buyer 1 in the current period: if she were not present, efficiency dictates that the object be assigned to buyer 2, leading to a flow benefit of v_2. The second term (the summation), however, is the expected benefit to society from postponing the allocation to buyer 2. The presence of buyer 1 implies that we anticipate allocating to buyer 2 in the next period instead of to buyer 3, and to buyer 3 in the following period instead of to buyer 4, and so on. In fact, this second term is the discounted expectation of buyer 2's *future* marginal contribution to the social welfare.[4] We therefore arrive at a natural conjecture: if, in every period, each buyer bids up to the point at which she is indifferent between receiving an object immediately and receiving her expected future marginal contribution to the social welfare, we are able to achieve the outcome of the dynamic pivot mechanism.

So, suppose that an ascending auction is conducted in every period. In this auction format, a price clock rises continuously from zero until all but one buyer exit the auction. This final buyer wins the object, paying the price at which the second-to-last buyer dropped out. The number of active bidders is common knowledge throughout the auction, as are the prices at which each bidder drops out of the bidding. (Recall that exits in a button auction are irreversible.) Therefore, bidders are able to condition their bids on the exit prices of their competitors. We assume that when there are n buyers present at the beginning of the period, each buyer i initially bids up to

$$\widehat{\beta}_{n,n}(v_i) := v_i - \delta \mathbb{E}\left[W_n(\bar{v}, \ldots, \bar{v}, v_i) - W_{n-1}(\bar{v}, \ldots, \bar{v})\right],$$

where the expectation is taken with respect to the entry process of new buyers and the possible valuation of a new entrant in the following period. In words, each buyer initially bids as though she expects that losing the current-period auction leads to receiving her marginal contribution in

[4] A formal demonstration of this property may be found in Appendix B.

the future, conditional on each of her $n-1$ competitors having the highest possible value $\bar{v}$. Thus,

$$\widehat{\beta}_{n,n}(v_i) = v_i - \delta^{n-1} \int_0^{v_i} \lambda^{n-1}(v')\, dv'.$$

Since $\lambda(v') \in [0,1]$ for all $v' \in [0, \bar{v}]$ and $\delta \in (0,1)$, the bid function $\widehat{\beta}_{n,n}$ is strictly increasing in v_i. As the price clock increases, the first buyer to exit the auction will therefore be buyer n, the one with the lowest value.

Because $\widehat{\beta}_{n,n}$ is strictly increasing, the remaining buyers are able to infer the lowest value v_n and condition their expectations about the future on this information. Therefore, when there are $n-1$ buyers remaining in the auction, we assume that each buyer $i \neq i_n$ bids according to

$$\widehat{\beta}_{n-1,n}(v_i, v_n) := v_i - \delta \mathbb{E}\left[W_n(\bar{v}, \ldots, \bar{v}, v_i, v_n) - W_{n-1}(\bar{v}, \ldots, \bar{v}, v_n)\right],$$

where the expectation is with respect to the entry and value of a new buyer in the following period, conditional on the lowest current-period value v_n. This expression implies that each buyer now bids as though she expects that losing the current-period auction leads to receiving her marginal contribution in the future, conditional on one competitor's value being v_n and the remaining $n-2$ competitors having the highest-possible value $\bar{v}$. Thus,

$$\widehat{\beta}_{n-1,n}(v_i, v_n) = v_i - \delta^{n-2} \int_{v_n}^{v_i} \lambda^{n-2}(v')\, dv' - \delta^{n-1} \int_0^{v_n} \lambda^{n-1}(v')\, dv'.$$

As with $\widehat{\beta}_{n,n}$, this bid function is strictly increasing in v_i, implying that the second buyer to exit the current-period auction will be buyer $n-1$, as she has the second-lowest value. Moreover, the remaining $n-2$ buyers will be able to infer the second-lowest value v_{n-1} and condition their expectations on this information.

Proceeding inductively, we define, for each $k = 2, \ldots, n$, the bidding function

$$\begin{aligned}\widehat{\beta}_{k,n}(v_i, v_{k+1}, \ldots, v_n) &:= v_i - \delta \mathbb{E}\left[W_n(\bar{v}, \ldots, \bar{v}, v_i, v_{k+1}, \ldots, v_n) - W_{n-1}(\bar{v}, \ldots, \bar{v}, v_{k+1}, \ldots, v_n)\right] \\ &= v_i - \delta^{k-1} \int_{v_{k+1}}^{v_i} \lambda^k(v')\, dv' - \sum_{j=k+1}^{n} \delta^{j-1} \int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\, dv'\end{aligned}$$

to be the price at which buyer i plans to exit the auction when there are k buyers remaining active in the auction, where we let $v_{n+1} := 0$. Each of these functions is strictly increasing in v_i, implying that the auction will end when the price reaches

$$\widehat{\beta}_{2,n}(v_2, \ldots, v_n) = v_2 - \sum_{j=2}^{n} \delta^{j-1} \int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\, dv'.$$

Buyer 1, the buyer with the highest value present on the market, will win the current-period object. In addition, the price she pays in the auction is exactly equal to the dynamic pivot mechanism transfer described in Equation (2). Thus, if each buyer bids according to the functions described above, the sequential ascending auction leads to outcomes identical to those of the dynamic pivot mechanism. Finally, when coupled with the appropriate off-equilibrium path beliefs, these bidding strategies do, in fact, yield a perfect Bayesian equilibrium of the sequential ascending auction mechanism.

We should point out that buyers' beliefs play no role in the strategies above other than in the expectations of future arrivals and values. More specifically, the bids in each period are "memoryless." They depend only upon information revealed in the current period, and not on any observations of past behavior. Since this process of information revelation occurs in every auction in the infinite sequence, newly arriving buyers are not at a disadvantage to their better informed competitors. On the contrary, by ignoring information revealed in the past, both informed "incumbents" and new entrants are able to behave in a symmetric manner that yields the correct incentives for information revelation by all auction participants.

4. Efficient Mechanisms

4.1. *Efficient Policy*

In order to examine the properties of efficient mechanisms or efficient auctions in this setting, it is necessary to develop an understanding of the efficient policy. Recall that, in the static single-object allocation setting, allocative efficiency is equivalent to allocating the object to the highest-valued buyer. In our dynamic setting, the structure of the environment—the nature of the arrival processes and the non-storability of objects—implies that the socially efficient policy is essentially an assortative matching. In particular, objects are ordered by their arrival time and buyers are ordered by their values, and "earlier" objects are allocated to higher-valued buyers. Of course, the feasibility constraints imposed by the dynamic nature of the agent population have an impact on the nature of the efficient policy, as the ordering of buyers by valuation need not correspond to the sequential ordering of buyers by their periods of availability. Thus, the socially efficient allocation policy is, in any given period, to allocate all available objects to the set of buyers with the highest values.

We consider a social planner who commits to a feasible direct mechanism $\mathcal{M} = \{\mathbf{x}_t, \mathbf{p}_t\}_{t\in\mathbb{N}_0}$ at time zero. The planner's goal is to maximize allocative efficiency; that is, the planner wishes to choose a mechanism $\mathcal{M}$ to maximize

$$W(\mathcal{M}) := \mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{i\in\mathcal{I}}\delta^t x_{i,t}(h_t,\alpha_t,k_t,\mathbf{v})v_i\right],$$

subject to incentive compatibility and individual rationality, where the expectation is taken with respect to the arrival and departure processes, as well the values of the agents. Recalling that $q_i(v_i,\alpha_t,k_t)$ from Equation (1) is agent i's expected discounted probability of receiving an object (conditional on entry), we may rewrite this objective function as

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{i\in\mathcal{I}_t}\delta^t\alpha_t(i)q_i(v_i,\alpha_t,k_t)v_i\right],$$

where $\alpha_t(i) = 1$ if $i \in \mathcal{I}_t$ arrives to the market and zero otherwise. (Recall that this arrival occurs with probability $\pi_i \in [0,1]$ for each agent i.)

Before a formal statement of our result, a few additional definitions are necessary. Fix any state $z_t = (h_t,\alpha_t,k_t,\mathbf{v}) \in \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R}$, where $\mathbf{v}$ denotes the truthful reporting strategy by all agents. We denote by

$$\mathcal{A}_+(z_t) := \left\{ i \in \mathcal{A}_t : \left| \left\{ j \in \mathcal{A}_t : v_j \geq v_i \right\} \right| \leq k_t \right\}.$$

the set of agents who are among the k_t highest-ranked buyers at state z_t. Similarly, the set of agents who are ranked *outside* the top k_t agents is denoted by

$$\mathcal{A}_-(z_t) := \left\{ i \in \mathcal{A}_t : \left| \left\{ j \in \mathcal{A}_t : v_j > v_i \right\} \right| \geq k_t \right\}.$$

Finally,

$$\mathcal{A}_\sim(z_t) := \mathcal{A}_t \setminus \left(\mathcal{A}_+(z_t) \bigcup \mathcal{A}_-(z_t) \right).$$

is the set of agents who are "on the boundary"—the agents who are tied for the k_t-th highest rank.

LEMMA 2 (Efficient Allocation Rules)**.**
Suppose all buyers, upon arrival, report their true values. A feasible allocation rule $\{\mathbf{x}_t\}_{t\in\mathbb{N}_0}$ is efficient if, and only if, for all states $z_t = (h_t, \alpha_t, k_t, \mathbf{v}) \in \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R}$,

$$x_{i,t}(z_t) = 1 \textit{ for all } i \in \mathcal{A}_+(z_t)$$

and

$$\sum_{i\in\mathcal{A}_\sim(z_t)} x_{i,t}(z_t) = k_t - |\mathcal{A}_+(z_t)| \textit{ if } |\mathcal{A}_+(z_t)| < k_t.$$

Note that the conditions in this lemma pin down the behavior of efficient allocation rules after almost all histories.[5] The second condition applies only in the case of "ties" among the agents, which are probability zero events. Additionally, notice that the period-t efficient allocation does not depend on past allocations or history; only the set of objects available (indicated by k_t), the set of agents present at time t (indicated by α_t), and these agents' reported values (denoted by $\mathbf{v}_t$) are relevant.

Therefore, we will henceforth restrict attention to the efficient allocation rule which breaks ties with equal probability. Thus, the efficient allocation rule is defined by

$$\hat{x}_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = \begin{cases} 1 & \text{if } i \in \mathcal{A}_+ \\ 0 & \text{if } i \in \mathcal{A}_- \\ \frac{k_t - |\mathcal{A}_+|}{|\mathcal{A}_\sim|} & \text{if } i \in \mathcal{A}_\sim \end{cases}$$

for all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$ and $\mathbf{v}_t \in \mathbf{V}^{\mathcal{A}_t}$, where we have dropped the dependence on histories.

From the perspective of any particular buyer $i \in \mathcal{I}$, the efficient policy allocates an object after a history to i if, and only if, i is among the highest-ranking buyers at that history. Thus, given the values of all other agents, $\hat{x}_{i,t}$ is nondecreasing in v_i. Since this is is true for any arbitrary history, this implies that it is true in expectation for each time period. Hence, the expected discounted probability of receiving an object q_i is nondecreasing in v_i. In light of the characterization of incentive compatibility in Lemma 1, it may then be possible to construct an incentive compatible dynamic direct mechanism which implements the efficient allocation rule.

[5]While it is straightforward to do so, we do not formally account for the zero-probability event in which a buyer's value is equal to zero. While leaving our results unchanged, this simplifies both our notation and exposition.

4.2. *Efficient Dynamic Direct Mechanisms*

A logical candidate for such a mechanism is the classic Vickrey-Clarke-Groves Mechanism. Recall that the VCG mechanism ensures that each agent's payoff is their marginal contribution to overall social welfare. In the static setting, the VCG mechanism is not only incentive compatible, but it is also truthfully implementable in dominant strategies—regardless of the reports of other agents, truth-telling is a weakly dominant strategy for each buyer. We will now show that this result extends to our setting.

For any $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$ and truthful $\mathbf{v}_t \in \mathbf{V}^{\mathcal{A}_t}$, define

$$W(\alpha_t, k_t, \mathbf{v}_t) := \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)v_j\right]$$

to be the social welfare when the efficient policy $\widehat{\mathbf{x}}$ is implemented. Denoting by α_s^{-i} the agent presence indicator in period $s \in \mathbb{N}_0$ when i is removed from the market (that is, where we impose $\alpha_s(i) = 0$), we write

$$W_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t) := \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}\setminus\{i\}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s^{-i}, k_s, \mathbf{v}_s)v_j\right].$$

for the social welfare when i is removed from the market. Thus, the arrival of agent $i \in \mathcal{I}_t$ to the market yields a marginal contribution to the social welfare of

$$w_i(\alpha_t, k_t, \mathbf{v}_t) := W(\alpha_t, k_t, \mathbf{v}_t) - W_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t).$$

Thus, the Vickrey-Clarke-Groves mechanism is the dynamic direct mechanism $\widehat{\mathcal{M}} := \{\widehat{\mathbf{x}}_t, \widehat{\mathbf{p}}_t\}_{t\in\mathbb{N}_0}$, where $\widehat{\mathbf{x}}$ is the socially efficient allocation rule, and the payment rule $\widehat{\mathbf{p}}$ is defined by

$$\hat{p}_{i,t}(\alpha_t, k_t, \mathbf{v}_t) := \hat{q}_i(\alpha_t, k_t, \mathbf{v}_t)v_i - w_i(\alpha_t, k_t, \mathbf{v}_t)$$

upon i's arrival, and $p_{i,s}(\alpha_s, k_s, \mathbf{v}_t) := 0$ for all $s \neq t$, where $\hat{q}_i$ denotes the discounted expected probability of $i \in \mathcal{I}_t$ receiving an object given her arrival at time t.

PROPOSITION 1 (VCG is Dominant Strategy IC and IR)**.**
The Vickrey-Clarke-Groves mechanism $\widehat{\mathcal{M}}$ is dominant-strategy incentive compatible and individually rational.

Note that the Vickrey-Clarke-Groves mechanism $\widehat{\mathcal{M}}$ charges each buyer a single payment upon her arrival to the mechanism, regardless of whether the buyer receives an object immediately or not. Since buyers are risk neutral, it is clearly possible to design other payment schemes with differing streams of payments that are, in terms of expected utility, equivalent to $\widehat{\mathbf{p}}$. This may be especially desirable in the case that γ, the survival probability of each agent from one period to the next, is less than 1. In this case, an agent may be charged a payment upon her arrival, but depart from the market before receiving an object in a future period. Thus, while $\widehat{\mathcal{M}}$ is individually rational from the *ex ante* perspective of an agent arriving on the market, it need not remain so upon that agent's departure.

One mechanism that does not suffer from this problem is the dynamic pivot mechanism of Bergemann and Välimäki (2008). Their direct mechanism is essentially the generalization of the Vickrey-Clarke-Groves mechanism to the case in which agents' private information may be changing over time. By choosing payments which provide agents with their *flow* marginal contribution to the social welfare in each period, the dynamic pivot mechanism obtains truth-telling as an equilibrium which implements the efficient policy.[6] Moreover, in our sequential allocation problem, the dynamic pivot mechanism imposes payments on agents only when they receive an object.

To see why this is true, note first that the flow marginal contribution of an agent i in period $t \in \mathbb{N}_0$ is simply the period-t contribution to the social welfare provided by i's presence:

$$\begin{aligned} w_i^F(\alpha_t, k_t, \mathbf{v}_t) &:= w_i(\alpha_t, k_t, \mathbf{v}_t) - \delta \mathbb{E}\left[w_i(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})\right] \\ &= W(\alpha_t, k_t, \mathbf{v}_t) - W_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t) \\ &\qquad - \delta\left(\mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})\Big] - \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})\Big]\right). \end{aligned}$$

Thus, the dynamic pivot mechanism is then the dynamic direct mechanism $\widehat{\mathcal{M}}^F := \{\widehat{\mathbf{x}}_t, \widehat{\mathbf{p}}_t^F\}_{t\in\mathbb{N}_0}$, where the payment rule $\widehat{\mathbf{p}}^F$ is defined by

$$\hat{p}_{i,t}^F(\alpha_t, k_t, \mathbf{v}_t) := \hat{x}_{i,t}(\alpha_t, k_t, \mathbf{v}_t) v_i - w_i^F(\alpha_t, k_t, \mathbf{v}_t)$$

for all $(\alpha_t, k_t, \mathbf{v}_t)$. This mechanism yields to each agent flow payoffs equal to her flow marginal contribution.

As stated above, $\hat{p}_{i,t}^F(\alpha_t, k_t, \mathbf{v}_t) = 0$ if i does not receive an object. Note that i does not receive an object only if there are sufficiently many agents $j \in \mathcal{A}_t$ with $v_j > v_i$ such that the efficient policy allocates to the same set of agents, irrespective of the presence of i. Therefore,

$$W(\alpha_t, k_t, \mathbf{v}_t) - \delta\mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \widehat{\mathbf{x}}_t(\alpha_t, k_t, \mathbf{v}_t)\Big] = \sum_{j \in \mathcal{A}_t \setminus \{i\}} \hat{x}_{j,t}(\alpha_t, k_t, \mathbf{v}_t) v_j$$

and

$$W_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t) - \delta\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \widehat{\mathbf{x}}_t(\alpha_t^{-i}, k_t, \mathbf{v}_t)\Big] = \sum_{j \in \mathcal{A}_t \setminus \{i\}} \hat{x}_{j,t}(\alpha_t, k_t, \mathbf{v}_t) v_j$$

must be equal, implying that $w_i^F(\alpha_t, k_t, \mathbf{v}_t) = 0$, and hence no payment is made by agent i. Moreover, if the efficient policy *does* allocate an object to agent i in period t, then i will not be present on the market in future periods, implying that

$$\mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})\Big] = \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})\Big].$$

Hence, i's flow marginal contribution when allocated an object is exactly equal to her *total* marginal contribution at that point in time.

Finally, note that the dynamic pivot mechanism inherits all the implementability properties of the Vickrey-Clarke-Groves mechanism in this context; in particular, $\widehat{\mathcal{M}}^F$ is both incentive compatible and individually rational, and truth-telling remains a (weakly) dominant strategy for all

[6] Bergemann and Välimäki (2008) show this result in the context of a fixed agent population. Cavallo, Parkes, and Singh (2007) are able to demonstrate that the dynamic pivot mechanism truthfully implements the socially efficient policy in the presence of an evolving agent population.

agents.[7] This follows directly from the fact that agents' payoffs are quasilinear. Hence, when agent i considers the discounted sum of a stream of payoffs, each of which is equal to i's flow marginal contribution at some point in time, this is equivalent to a single, one-time payoff equal to i's total marginal contribution. Thus, we have the following result, which we state without proof.

COROLLARY 2 (Dynamic Pivot Mechanism is Dominant Strategy IC and IR)**.** *The dynamic pivot mechanism $\widehat{\mathcal{M}}^F$ is dominant strategy incentive compatible and individually rational.*

5. AN EFFICIENT AUCTION

It is important to keep in mind that both the Vickrey-Clarke-Groves mechanism and the dynamic pivot mechanism are direct revelation mechanisms, relying on a planner to aggregate the reported values of each buyer in order to determine allocations and payments. This raises an important question: do these efficient mechanisms correspond to a familiar auction format? In the static single-object case, Vickrey (1961) provided a clear answer: the analogue of the Vickrey-Clarke-Groves mechanism for the allocation of a single indivisible good is the second-price auction. Both the sealed-bid second-price auction and the ascending (English) auction admit equilibria that are outcome equivalent to the VCG mechanism and are compelling prescriptions for "real-world" behavior.[8]

A reasonable conjecture is that a sequential auction would be useful in the context of a sequential allocation problem. But what type of auction? Clearly, no standard sealed-bid auction format can achieve the outcome of the dynamic pivot mechanism described above. To see this, consider the example of Section 3. There, the price paid by the winning buyer depended on her own value, as well as on the values of all other agents present in the market. However, it is impossible for buyers in a sealed-bid auction to submit bids that are conditioned on information that is unavailable to them. Therefore, information revelation within the course of the auction is a necessary condition for the indirect implementation of the efficient policy, and the natural candidate for an open auction format is the ascending auction. In fact, we will now demonstrate that it *is* possible to implement the efficient allocation rule via the sequential ascending auction; in particular, we will construct an equilibrium of this indirect mechanism that yields the same outcome as the dynamic pivot mechanism.[9]

We make use of a simple generalization of the Milgrom and Weber (1982) "button" model of ascending auctions. In particular, we consider a multi-unit, uniform-price variant of their model. The auction begins, *in each period*, with the price at zero and with all agents present participating

[7]Note that since agents' types do *not* change over time, we can restrict attention to dynamic direct mechanisms that require a single report upon agents' arrival to the mechanism. This differs from the mechanism considered by Bergemann and Välimäki (2008) which requires agents to make a report in *every* period.

[8]Of course, the revenue equivalence theorem applies, and several other standard auction mechanisms are able to yield efficient outcomes in the single-object static setting. However, they lack the robustness of the VCG mechanism that is provided by dominant-strategy incentive compatibility.

[9]Recall that both the Vickrey-Clarke-Groves mechanism and the dynamic pivot mechanism impose only a single payment upon each agent. However, the VCG mechanism imposes this payment upon an agent's arrival to the market, leading to a stream of payments that cannot correspond to a standard sequential auction format—a reasonable auction should require payments from buyers only when they win. The dynamic pivot mechanism, on the other hand, requires a payment by an agent only when they are allocated an object.

in the auction. Each bidder may choose any price at which to drop out of the auction. This exit decision is irreversible (in the current period), and is observable by all agents currently present. Thus, the current price and the set of active bidders is commonly known throughout the auction. When there are $m \geq 1$ objects for sale, the auction ends whenever at most m active bidders remain, with each remaining bidder receiving an object and paying the price at which the auction ended. Note that if there are fewer than m bidders initially, then the auction ends immediately at a price of zero. In addition, suppose that several bidders drop out of the auction simultaneously, leaving $m' < m$ bidders active. The auction ends at this point, and $m - m'$ of the "tied" bidders are selected with equal probability to receive an object—along with the remaining active bidders—paying the price at which the auction closed. With this in mind, each bidder's decision problem *within* a given period is not the choice of a single bid, but is instead the choice of a sequence of functions, each of which determines an exit price contingent on the (observed) exit prices of the bidders who have already dropped out of the current auction.

We now informally describe the strategies used by each player in the sequential ascending auction mechanism. In a sequential auction, there is an option value associated with losing. Buyers who do not win an object in the current period have available to them the possibility of participating in future auctions. Rational bidding therefore requires remaining active in the auction as long as the price is low enough that the benefit of winning outweighs the expected benefit of losing. As the price rises, buyers start to drop out of the auction, (indirectly) revealing their values to their competitors. This information revelation allows any remaining active buyers to update their beliefs, both about their competitors and about the option value of losing. This process continues until the auction concludes. In many cases, however, buyers will already possess information about their competitors from previous interactions. However, in the equilibrium that we construct, bidders ignore any such information and begin the information revelation process anew. This information renewal process allows *all* buyers, "incumbents" and new entrants alike, to behave symmetrically, implying that newly arrived buyers are able to learn about their current competitors without knowledge of the events of previous periods. Therefore, new entrants are provided with the appropriate incentives to reveal their own private information.

With this in mind, let $n_t := |\mathcal{A}_t| = \sum_{j \in \mathcal{I}} \alpha_t(i)$ denote the number of buyers present in period t. In addition, taking the perspective of an arbitrary bidder i, let

$$\mathbf{y}_t := \left(y_1^t, \ldots, y_{n_t-1}^t\right)$$

denote the ordered valuations of all *other* buyers present in period t, where y_1^t is the largest value, and $y_{n_t-1}^t$ is the smallest. Finally, for each $m = 1, \ldots, n_t - 1$, let

$$\bar{\mathbf{v}}^m := (\bar{v}, \ldots, \bar{v}) \in \mathbf{V}^m \text{ and } \mathbf{y}_t^{>m} := \left(y_{m+1}^t, \ldots, y_{n_t-1}^t\right).$$

If, in period t, all buyers are using symmetric strictly increasing bidding strategies, then the prices at which buyers exit the auction will reveal their values. Thus, over the course of the auction buyers will observe (in sequence) the realizations $\mathbf{y}_t^{>m}$, allowing their bids to be conditioned on this information. Finally, we define, for each $m = 1, \ldots, n_t - 1$,

$$w^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := \delta \mathbb{E}\left[w_i(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, \mathbf{y}_t^{>m})\right]$$

to be the (discounted) expected future marginal contribution of an agent $i \in \mathcal{A}_t$ with value v_i, conditional on the period-t presence of m competitors each with the highest possible value $\bar{v}$ and $n_t - m - 1$ buyers ranked below i with values $\mathbf{y}_t^{>m}$.

Define, for each $m = 1, \ldots, n_t - 1$,

$$\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := v_i - w^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}). \tag{3}$$

We will assume that, in each period $t \in \mathbb{N}_0$, each agent $i \in \mathcal{A}_t$ bids according to $\widehat{\beta}^t_{m,n_t}$ whenever she has m active competitors in the auction. Thus, each buyer i initially bids up to the point at which she is indifferent between winning the object at the current price and receiving her discounted expected marginal contribution in the next period, conditional on being the lowest-ranked of the n_t bidders currently present and all other bidders having the highest possible valuation.[10] Notice that, if $v_i > v_j$, then

$$\begin{aligned}
&w^{t+1}(\alpha_t, k_t, v_j) - w^{t+1}(\alpha_t, k_t, v_i)\\
&\quad = \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_j)\Big] - \delta \mathbb{E}\Big[W_{-j}(\alpha^{-j}_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_j)\Big]\\
&\qquad - \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_i)\Big] + \delta \mathbb{E}\Big[W_{-i}(\alpha^{-i}_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_i)\Big]\\
&\quad = \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_j)\Big] - \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_i)\Big],
\end{aligned}$$

since removing either i or j in the next period, conditional on her being the lowest-ranked agent, does not affect the order of anticipated future allocations to any other agents. Moreover, note that by treating buyer j as though her true value were v_i, we can provide a bound for the difference above. In particular, we have

$$\begin{aligned}
&\delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_j)\Big] - \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - 1}, v_i)\Big]\\
&\qquad \geq \mathbb{E}\left[\sum_{s=t+1}^{\infty} \delta^{s-t} \hat{x}_{i,s}(\alpha_s, k_s, \mathbf{v}_s) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_i)\right](v_j - v_i).
\end{aligned}$$

Thus, if $v_i > v_j$, then

$$\begin{aligned}
&\widehat{\beta}^t_{n_t-1,n_t}(\alpha_t, k_t, v_i) - \widehat{\beta}^t_{n_t-1,n_t}(\alpha_t, k_t, v_i)\\
&\qquad \geq (v_i - v_j)\left(1 - \mathbb{E}\left[\sum_{s=t+1}^{\infty} \delta^{s-t} \hat{x}_{i,s}(\alpha_s, k_s, \mathbf{v}_s) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_i)\right]\right) > 0
\end{aligned}$$

since the discounted expected probability of receiving an object in the future is bounded above by $\delta < 1$. Thus, the agent who is, in fact, the lowest-ranked buyer present in period t will be the first to drop out of the period-t auction, publicly revealing her value.

At this point, each remaining buyer i bids until she is indifferent between winning the object at the current price and receiving her discounted expected marginal contribution, conditional on the knowledge that she is the second-lowest ranked of the $n_t - 1$ bidders remaining active in the

[10] This is not strictly necessary; any beliefs about the valuations of her opponents will suffice as long as the support of those beliefs is contained in the interval $(v_i, \bar{v}]$.

auction, that all remaining active bidders have the highest possible valuation, and that the lowest-ranked buyer present has value $y_t^{n_t-1} < v_i$. In addition, suppose that buyer j with value v_j was the first to exit the auction. Then $y_t^{n_t-1} = v_j < v_i$ implies

$$\begin{aligned}
&w^{t+1}(\alpha_t, k_t, v_j) - w^{t+1}(\alpha_t, k_t, v_i, v_j) \\
&= \delta\Big(\mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_j)\Big] - \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_j)\Big]\Big) \\
&\quad - \delta\Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_j)\Big] - \mathbb{E}\Big[W_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_j)\Big]\Big).
\end{aligned}$$

However, the second difference above may be rewritten as

$$\begin{aligned}
&\Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_j)\Big] - \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_i)\Big]\Big) \\
&\quad + \Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_i)\Big] - \mathbb{E}\Big[W_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-1}, v_j)\Big]\Big).
\end{aligned}$$

Thus,

$$w^{t+1}(\alpha_t, k_t, v_j) - w^{t+1}(\alpha_t, k_t, v_i, v_j)$$

is the sum of three differences. The first is the gain in social welfare when increasing i's value from v_i to $\bar{v}$. The second is the gain in social welfare (when i is not on the market) from increasing j's value from v_j to v_i. Finally, the third difference is the loss in social welfare (when j is not present) from decreasing i's value from $\bar{v}$ to v_i. However, since $v_j < v_i$, the presence or absence of j from the market has no influence on when the efficient policy allocates to i, regardless of whether i's value is v_i or $\bar{v}$. Therefore, the gain from the first difference equals the loss from the third difference, implying that

$$\begin{aligned}
&w^{t+1}(\alpha_t, k_t, v_j) - w^{t+1}(\alpha_t, k_t, v_i, v_j) \\
&= \delta\Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_j)\Big] - \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-2}, v_i, v_i)\Big]\Big).
\end{aligned}$$

A bounding argument similar to the one previously applied may then be used to show that

$$\widehat{\beta}^t_{n_t-2,n_t}(\alpha_t, k_t, v_i, v_j) - \widehat{\beta}^t_{n_t-1,n_t}(\alpha_t, k_t, v_j) > 0.$$

Thus, there is continuity at the first drop out point, in the sense that the exit of the lowest-valued buyer does not induce the immediate exit of any buyer with a higher value. Therefore, if $\widehat{\beta}^t_{n_t-2,n_t}(\alpha_t, k_t, v_i, v_j)$ is increasing in v_i, the price at which the second exit occurs fully reveals the value of the second-lowest ranked buyer.

Similar logic may be used to show that $\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$ is strictly increasing in v_i for all m, and that the "continuity" property discussed above holds after every exit from the auction.

Lemma 3 (Bids are Fully Revealing).
The bid functions $\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$ are increasing in v_i for all $m = 1, \ldots, n_t - 1$. Moreover, if $v_i > y_t^{m+1}$, then

$$\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) > \widehat{\beta}^t_{m+1,n_t}(\alpha_t, k_t, \mathbf{y}_t^{>m}).$$

Thus, if every buyer follows these strategies in each period, the efficient allocation is achieved. Under the assumption (which we will shortly verify) that these strategies form an equilibrium of the indirect mechanism in which an ascending auction is held in each period, Corollary 1 implies that *ex ante* expected payments by buyers in this mechanism must be the same, up to a constant, as in the Vickrey-Clarke-Groves mechanism or the dynamic pivot mechanism. Since the marginal contribution of a buyer with value equal to zero is identically zero, and the bid of such a buyer in the conjectured equilibrium is also zero, this *ex ante* equivalence is exact. Analysis of the prices paid in the auction mechanism, however, show that this equivalence extends further.

PROPOSITION 2 (Sequential Ascending Auctions are Equivalent to Dynamic Pivot Mechanism)**.** *Following the bidding strategies $\widehat{\beta}^t_{m,n_t}$ in every period t in the sequential ascending auction mechanism is outcome equivalent to the dynamic pivot mechanism.*

Therefore, following the bidding strategies prescribed in Equation (3) leads to an outcome that is identical to that of truth-telling in the dynamic pivot mechanism. Moreover, we know from Corollary 2 that truth-telling is an equilibrium of the dynamic pivot mechanism. It remains to be shown, however, that the bidding strategies described in Equation (3) form an equilibrium of the sequential ascending auction mechanism.

Since the sequential ascending auction mechanism is a dynamic game of incomplete information, the equilibrium concept we will consider is that of perfect Bayesian equilibrium. This solution concept requires that behavior be sequentially rational with respect to agents' beliefs, and that agents' beliefs be updated in accordance with Bayes' rule wherever possible. Since the bidding strategies $\widehat{\beta}^t_{m,n_t}$ are strictly increasing, behavior along the equilibrium path is perfectly separating, implying that Bayesian updating fully determines beliefs. To determine optimality *off* the equilibrium path, however, we need to consider the behavior of bidders after a deviation. Since such post-deviation histories are zero probability events, we are free to choose arbitrary off-equilibrium beliefs. Therefore, we will suppose that, after a deviation, buyers disregard their previous observations, believing that the deviating agent is *currently* truthfully revealing her value in accordance with $\widehat{\beta}^t_{m,n_t}$.

This particular specification of off-equilibrium beliefs is particularly useful. Note that these beliefs are consistent with Bayes' rule even after probability zero histories. This follows immediately from the fact that, generally, this system of beliefs consists of point-mass beliefs about the types of other agents. The only agents about whom beliefs do not take this form are those that have yet to arrive to the market and those who win an object in the period of their arrival—these agents reveal only a lower bound on their value.

Moreover, this property is equivalent to the condition of *preconsistency* of beliefs in an extensive form game of incomplete information put forth by Hendon, Jacobsen, and Sloth (1996), which Perea (2002) shows to be both necessary and sufficient for the one-shot-deviation principle to hold.[11] This is an important observation, as perfect Bayesian equilibrium, in contrast to sequential equilibrium, need not satisfy the one-shot-deviation principle.[12] We can therefore prove the following result.

[11]This condition is called *updating consistency* by Perea (2002), and is also equivalent to part 3.1(1) of Fudenberg and Tirole (1991)'s definition of a *reasonable* assessment.

[12]These off-equilibrium beliefs also satisfy the "no-signaling-what-you-don't know condition" in Fudenberg and Tirole (1991). This suggests that (aside from measurability issues) one could construct a conditional probability system for

PROPOSITION 3 (Equilibrium in the Sequential Ascending Auction)**.**
Suppose that in each period, buyers bid according to the cutoff strategies given in Equation (3). This strategy profile, combined with the system of beliefs described above, forms a perfect Bayesian equilibrium of the sequential ascending auction mechanism.

Proposition 2 and Proposition 3, taken together, imply that the sequential ascending auction admits an efficient equilibrium that also yields prices identical to those of the dynamic pivot mechanism. Thus, the sequential ascending auction is a natural, intuitive institution that yields efficient outcomes.

It is interesting to note several additional properties of the equilibrium described above. First, the proof shows that deviations from the bidding strategies $\widehat{\beta}^t_{m,n_t}$ are not rational for any agent, even when conditioning on competitors' values in the current period. Thus, the strategy profile specified in Equation (3) forms a periodic *ex post* equilibrium. Introduced by Bergemann and Välimäki (2008), this notion requires that, given expectations about future behavior and arrivals of both objects and buyers, each buyer's current-period behavior is a best response to the strategies of her opponents, regardless of the history and realized values of her competitors. Since agents essentially "report" their values in each auction, the extensive-form structure of the indirect mechanism leads to a marginally weaker form of implementation (in contrast to the dominant-strategy incentive compatibility of the direct mechanisms discussed above).

Furthermore, notice that in the sequential ascending auction, buyers have the ability to drop out immediately once an auction begins. Since the bidding strategies discussed above form a periodic *ex post* equilibrium, buyers do not wish to take advantage of this possibility, even if they know their opponents' values. Thus, although we have assumed that buyers cannot conceal their presence when arriving to the market, we may conclude that, in equilibrium, they would not take advantage of that opportunity were it afforded to them. Moreover, the outcome equivalence result (Proposition 2) implies that this logic extends to the direct mechanisms considered earlier.

Finally, by using the bidding strategies described above, buyers fully reveal their private information. However, the strategy is "memoryless," as buyers make use only of information revealed in the current period, disregarding any information that may have been revealed in previous periods. The necessity of memoryless behavior arises from the informational asymmetry that develops over the course of the sequence of ascending auctions. Consider a period in which several buyers remain on the market from previous periods. Since buyers have revealed their values in the past, there is essentially no private information within this group of buyers. However, when a new buyer arrives, the "incumbent" buyers know nothing about the new entrant other than the distribution from which her value is drawn. By behaving in a symmetric way that ignores previously revealed information, incumbent buyers restore symmetry to the environment, thereby providing the appropriate incentives for the new entrant to also reveal her own private information.

this equilibrium such that it satisfies Fudenberg and Tirole's conditions for perfect extended Bayesian equilibrium. The set of all such equilibria coincides, in finite games, with the set of sequential equilibria.

6. Revenue Maximization

While the previous section provides a characterization of efficient mechanisms, we have said little about revenue and optimal mechanisms. In the static setting, Myerson (1981) showed that the optimal mechanism for selling a single indivisible unit is a Vickrey-Clarke-Groves mechanism, with the caveat that instead of allocating the good to the agent with the highest value, the seller allocates the object to the agent with the highest *virtual* value. Maskin and Riley (1989) extend Myerson's insights to the setting in which multiple identical units are offered for sale and show that, as in the single-unit case, the objects are allocated to the set of buyers with the highest virtual valuations.

In our setting, however, while the objects are individual units of a homogeneous good, from the perspective of an individual buyer, they are differentiated products. To make this clear, consider a buyer i with value v_i who is present at period t. If this buyer receives an object in period t, this yields her utility v_i. However, if she anticipates receiving an object in period $t+1$, her valuation for that object is δv_i. Thus, she does not value the two objects identically. While there does exist a literature on auctions for multiple heterogeneous objects, much of the focus has been on efficiency and not revenue maximization.[13] Thus, paralleling the development of the previous section, we will derive the optimal dynamic direct mechanism for a revenue maximizing seller. We will show that revenue maximization in our dynamic setting is achieved by an efficient mechanism applied to virtual values. We will then discuss the indirect implementation of the revenue maximizing policy via a sequence of ascending auctions.

6.1. *Optimal Direct Mechanism*

We consider a single monopolist seller who commits to a direct mechanism $\mathcal{M} = \{\mathbf{x}_t, \mathbf{p}_t\}_{t\in\mathbb{N}_0}$ at time zero.[14] The seller's expected revenue from this mechanism is the expected discounted sum of payments made by each buyer. Recalling that the expected payment of a buyer i, conditional on entry, is denoted by $m_i(v_i, \alpha_t, k_t)$, the seller's payoff may be written as

$$\Pi(\mathcal{M}) := \mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{i\in\mathcal{I}_t}\delta^t\alpha_t(i)m_i(v_i,\alpha_t,k_t)\right],$$

where $\alpha_t(i) = 1$ if $i \in \mathcal{I}_t$ arrives to the market (which occurs with probability $\pi_i \in [0,1]$) and $\alpha_t(i) = 0$ otherwise.

Conditional on the arrival of agent $i \in \mathcal{I}_t$ in period t, when the current state of the market is described by $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$, her expected payment is given by

$$\int_0^{\bar{v}} m_i(v_i,\alpha_t,k_t)f_i(v_i)\,dv_i.$$

[13] Researchers interested in keyword auctions for sponsored search *have*, however, considered the problem of designing optimal auctions for selling various advertising "slots" on a search engine, where slots are differentiated by their click-through-rate. See, for instance, Iyengar and Kumar (2006).

[14] Commitment is necessary so as to ensure that that the revelation principle applies and it is without loss of generality to consider direct mechanisms.

By applying the revenue equivalence result from Corollary 1, this may be rewritten as

$$m_i(0,\alpha_t,k_t) + \int_0^{\bar{v}} q_i(v_i,\alpha_t,k_t) v_i f_i(v_i)\, dv_i - \int_0^{\bar{v}} \int_0^{v_i} q_i(v_i',\alpha_t,k_t) f_i(v_i)\, dv_i'\, dv_i.$$

Applying the standard interchange of the order of integration of the last term above, the expected payment of buyer i, conditional on entry, is then

$$m_i(0,\alpha_t,k_t) + \int_0^{\bar{v}} q_i(v_i,\alpha_t,k_t)\varphi_i(v_i) f_i(v_i)\, dv_i,$$

where

$$\varphi_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$$

is the virtual valuation of buyer i with value v_i. Recall that we have assumed that φ_i is strictly increasing in v_i for all agents $i \in \mathcal{I}$. Thus, applying the law of iterated expectations, the revenue maximizing seller is faced with the problem of choosing a feasible mechanism $\mathcal{M}$ to maximize

$$\Pi(\mathcal{M}) = \mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{i\in\mathcal{I}_t} \delta^t \alpha_t(i) m_i(0,\alpha_t,k_t)\right] + \mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{i\in\mathcal{I}_t} \delta^t \alpha_t(i) q_i(v_i,\alpha_t,k_t)\varphi_i(v_i)\right], \tag{4}$$

subject to the incentive compatibility and individual rationality constraints discussed in Lemma 1 and Corollary 1.

Notice that the seller's objective function above is the sum of two terms: the first term is a discounted sum of expected payments, while the second is a discounted sum of weighted virtual values. Moreover, this second term is identical to the efficiency-oriented social planner's objective function in Equation (4), except that values have been replaced with virtual values. Therefore, the insights about the efficient policy carry over to this context. In particular, the revenue-maximizing policy is again an assortative matching: objects are ordered by their arrival time and buyers are ordered by their *virtual* values, and "earlier" objects are allocated to agents with higher virtual values. Again, this matching must respect the feasibility constraints placed on the allocation rule. Thus, the revenue-maximizing allocation policy is, in each period, to allocate all available objects to the buyers with the highest virtual values currently on the market. Note, however, that allocating objects to agents with negative virtual values decreases the seller's payoff. Thus, the matching described above must restrict attention to buyers with non-negative virtual values.

Before proceeding to the formal description of this policy, some definitions are necessary. For any state $z_t = (h_t, \alpha_t, k_t, \mathbf{v}) \in \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R}$, where $\mathbf{v}$ denotes the truthful reporting strategy by all agents, let

$$\mathcal{A}^{\pi}(z_t) := \{i \in \mathcal{A}_t : \varphi_i(v_i) \geq 0\}$$

denote the set of agents with non-negative virtual values. The set of agents $i \in \mathcal{A}^{\pi}(z_t)$ whose virtual value is among the k_t highest present is given by

$$\mathcal{A}^{\pi}_{+}(z_t) := \left\{i \in \mathcal{A}^{\pi}(z_t) : \left|\left\{j \in \mathcal{A}_t : \varphi_j(v_j) \geq \varphi_i(v_i)\right\}\right| \leq k_t\right\}.$$

Similarly, the set of agents $i \in \mathcal{A}^{\pi}(z_t)$ whose virtual value places them *outside* the top k_t agents is denoted by

$$\mathcal{A}^{\pi}_{-}(z_t) := \left\{i \in \mathcal{A}^{\pi}(z_t) : \left|\left\{j \in \mathcal{A}_t : \varphi_j(v_j) > \varphi_i(v_i)\right\}\right| \geq k_t\right\}.$$

Finally,

$$\mathcal{A}_{\sim}^{\pi}(z_t) := \mathcal{A}^{\pi}(z_t) \setminus \left(\mathcal{A}_{+}^{\pi}(z_t) \bigcup \mathcal{A}_{-}^{\pi}(z_t) \right).$$

is the set of agents tied for the k_t-th highest ranking virtual value. The above observations and Lemma 2 then yield the following characterization of optimal allocations.

LEMMA 4 (Revenue-Maximizing Allocation Rules)**.**
Suppose all buyers, upon arrival, report their true values. A feasible allocation rule $\{\mathbf{x}_t\}_{t\in\mathbb{N}_0}$ is optimal (revenue-maximizing) if, and only if, for all $z_t = (h_t, \alpha_t, k_t, \mathbf{v}) \in \mathcal{H}_t \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} \times \mathcal{R}$,

$$x_{i,t}(z_t) = 1 \text{ for all } i \in \mathcal{A}_{+}^{\pi}(z_t) \text{ and } \sum_{i\in\mathcal{A}_{\sim}^{\pi}(z_t)} x_{i,t}(z_t) = k_t - |\mathcal{A}_{+}^{\pi}(z_t)| \text{ if } |\mathcal{A}_{+}^{\pi}(z_t)| < k_t.$$

As with the set of efficient allocation rules described by Lemma 2, all revenue-maximizing allocation rules agree after almost all histories. The only possible variations are at probability zero histories in which multiple agents have identical (positive) virtual values.[15] Additionally, these allocation policies are independent of past history, as optimal allocations are functions only of the values of the agents *currently* present on the market and on the number of objects *currently* available.

We will therefore refer to *the* revenue-maximizing allocation rule $\tilde{\mathbf{x}}$. By this, we mean the revenue-maximizing allocation rule which breaks ties with equal probability, which is defined by

$$\tilde{x}_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = \begin{cases} 1 & \text{if } i \in \mathcal{A}_{+}^{\pi} \\ 0 & \text{if } i \in \mathcal{A}_{-}^{\pi} \\ \frac{k_t - |\mathcal{A}_{+}^{\pi}|}{|\mathcal{A}_{\sim}^{\pi}|} & \text{if } i \in \mathcal{A}_{\sim}^{\pi} \end{cases}$$

for all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$ and $\mathbf{v}_t \in \mathbf{V}^{\mathcal{A}_t}$, where we drop the dependence on h_t.

It should be clear that the revenue-maximizing allocation rule $\tilde{\mathbf{x}}$ satisfies the requirements of incentive compatibility. From the perspective of a given buyer $i \in \mathcal{I}$, $\tilde{\mathbf{x}}$ allocates an object to i after a given history if, and only if, i is among the highest-ranking (by virtual value) buyers present at that history. Since we have assumed the standard regularity condition of increasing virtual valuations, this implies that $\tilde{x}_{i,t}$ is nondecreasing in v_i, given the values of the other agents present on the market. The fact that this property holds for any arbitrary history and realization of competitors' values implies that it must also be true in expectation for each time period t; hence, the expected discounted probability of receiving an object q_i is also nondecreasing in v_i. Thus, by choosing an appropriate payment rule, it is possible to design an incentive compatible mechanism that implements the revenue-maximizing allocation policy.

Let us now examine the first term in the seller's objective function in Equation (4). Given the incentive compatibility of the revenue-maximizing allocation policy, the generalization of the revenue equivalence theorem presented in Corollary 1 implies that the individual rationality constraint faced by our seller is

$$m_i(0, \alpha_t, k_t) \leq 0$$

[15]As with the efficient policies discussed in Section 4, we (without loss of generality) disregard the zero-probability events in which an agent's virtual value is equal to zero.

for all $i \in \mathcal{I}$ and all $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$. Therefore, since m_i enters the seller's objective function additively, this constraint must be binding.

The problem of choosing a payment rule that satisfies this constraint in this dynamic setting is similar to the static optimal auction problem. In the static setting, the Myerson (1981) optimal mechanism can be reinterpreted as a Vickrey-Clarke-Groves mechanism. Instead of maximizing surplus, the revenue-maximizing single-object mechanism maximizes *virtual* surplus. When agents report their values v_i, the mechanism computes their virtual values $\varphi_i(v_i)$ and then applies the VCG mechanism to these virtual values. This yields an allocation and a "virtual price" such that the winning buyer's virtual value less this virtual price is equal to her marginal contribution to the virtual surplus. We can then invert this virtual price into a standard price, which in the single-object case is the lowest value that could have been reported by the winning buyer such that she remains the winner.

These insights can be applied in our setting; however, care must be taken to ensure that we correctly account for the heterogeneous nature of the multiple goods available. In particular, one cannot simply invert the price charged to winners via the virtual valuation functions to determine the new prices, as prices will typically be functions of several agents' values.

So, for each $i \in \mathcal{I}$, define

$$\tilde{r}_i := \varphi_i^{-1}(0).$$

This is the minimal value required for agent i to potentially receive an object under the revenue-maximizing allocation policy. Furthermore, for each $t \in \mathbb{N}_0$ and all $i \in \mathcal{I}_t$, we define for any $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$ and truthful $\mathbf{v}_t \in \mathbf{V}^{\mathcal{A}_t}$ the function

$$\Pi^i(\alpha_t, k_t, \mathbf{v}_t) := \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)\left(\varphi_i^{-1}\left(\varphi_j(v_j)\right) - \tilde{r}_i\right)\right]. \tag{5}$$

This expression is the same as the virtual surplus in the seller's objective function in Equation (4), except that instead of a weighted sum of virtual values, it is a weighted sum of the corresponding "real" values of agent i, less the reservation value $\tilde{r}_i$ applied to agent i; that is, Π^i measures the virtual surplus in the same units as i's utility function. Since we have assumed that virtual values are increasing for all agents, φ_i^{-1} is increasing. Therefore, transforming the virtual values of all agents by φ_i^{-1} preserves their ordering; moreover, $\varphi_i^{-1}(\varphi_j(v_j)) - \tilde{r}_i \geq 0$ if, and only if, $\varphi_j(v_j) \geq 0$. Therefore,

$$\widetilde{\mathbf{x}} \in \arg\max_{\{\mathbf{x}_s\}_{s=t}^{\infty}} \left\{\mathbb{E}\left[\sum_{t=0}^{\infty}\sum_{j\in\mathcal{I}_t} \delta^t \alpha_t(j) q_j(v_j, \alpha_t, k_t)\left(\varphi_i^{-1}(\varphi_j(v_i)) - \tilde{r}_i\right)\right]\right\};$$

that is, $\widetilde{\mathbf{x}}$ is an efficient allocation rule for an environment in which a social planner realizes a payoff of $\varphi_i^{-1}(\varphi_j(v_j)) - \tilde{r}_i$ when allocating an object to agent j with value v_j.

Again denoting by α_s^{-i} the agent presence indicator in period $s \in \mathbb{N}_0$ when i is removed from the market (that is, where we impose $\alpha_s(i) = 0$), we write

$$\Pi_{-i}^i(\alpha_t^{-i}, k_t, \mathbf{v}_t) := \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}\setminus\{i\}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s^{-i}, k_s, \mathbf{v}_s)\left(\varphi_i^{-1}\left(\varphi_j(v_j)\right) - \tilde{r}_i\right)\right]$$

for the virtual social welfare (in terms of i's utility) when i is removed from the market. Thus, the arrival of agent $i \in \mathcal{I}_t$ to the market yields a marginal contribution—again, in units of i's utility function—equal to

$$\widetilde{w}_i(\alpha_t, k_t, \mathbf{v}_t) := \Pi^i(\alpha_t, k_t, \mathbf{v}_t) - \Pi^i_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t).$$

We now define the *virtual Vickrey-Clarke-Groves* mechanism to be the dynamic direct mechanism $\widetilde{\mathcal{M}} := \{\widetilde{\mathbf{x}}_t, \widetilde{\mathbf{p}}_t\}_{t \in \mathbb{N}_0}$, where $\widetilde{\mathbf{x}}$ is the revenue-maximizing allocation rule and the payment rule $\widetilde{\mathbf{p}}$ is defined by

$$\tilde{p}_{i,t}(\alpha_t, k_t, \mathbf{v}_t) := \tilde{q}_i(\alpha_t, k_t, \mathbf{v}_t)v_i - \widetilde{w}_i(\alpha_t, k_t, \mathbf{v}_t)$$

upon the arrival of agent $i \in \mathcal{I}_t$ in period t, and $\tilde{p}_{i,s}(\alpha_s, k_s, \mathbf{v}_s) := 0$ for all $s \neq t$. (We denote by $\tilde{q}_i$ the discounted expected probability of i receiving an object under the allocation rule $\widetilde{\mathbf{x}}$.)

PROPOSITION 4 (Virtual VCG is Dominant Strategy IC and IR)**.**
Suppose that virtual values φ_i are increasing for all $i \in \mathcal{I}$. Then the virtual Vickrey-Clarke-Groves mechanism $\widetilde{\mathcal{M}}$ is dominant-strategy incentive compatible and individually rational.

Thus, the virtual Vickrey-Clarke-Groves mechanism $\widetilde{\mathcal{M}}$ is an optimal (revenue-maximizing) dynamic direct mechanism. However, it is subject to the same criticism that we applied to the Vickrey-Clarke-Groves mechanism: $\widetilde{\mathcal{M}}$ charges each buyer a single payment upon their arrival, even if the agent is not immediately allocated an object. In the case in which the survival probability $\gamma < 1$, this implies that agents may make a payment but depart the market before receiving an object.

We may compensate for this problem, as we did in the case of efficient mechanisms, by charging payments such that each agent's flow utility is equal to her current period contribution to the virtual surplus. This quantity is defined by

$$\begin{aligned}
\widetilde{w}_i^F(\alpha_t, k_t, \mathbf{v}_t) &:= \widetilde{w}_i(\alpha_t, k_t, \mathbf{v}_t) - \delta \mathbb{E}\left[\widetilde{w}_i(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})\right] \\
&= \Pi^i(\alpha_t, k_t, \mathbf{v}_t) - \Pi^i_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_t) \\
&\quad - \delta\left(\mathbb{E}\left[\Pi^i(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})\right] - \mathbb{E}\left[\Pi^i_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}).\right]\right)
\end{aligned}$$

We define the *dynamic virtual pivot mechanism* as the dynamic direct mechanism $\widetilde{\mathcal{M}}^F := \{\widetilde{\mathbf{x}}_t, \widetilde{\mathbf{p}}_t^F\}_{t \in \mathbb{N}_0}$, where the payment rule $\widetilde{\mathbf{p}}^F$ is defined by

$$\tilde{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) := \tilde{x}_{i,t}(\alpha_t, k_t, \mathbf{v}_t)v_i - \widetilde{w}_i^F(\alpha_t, k_t, \mathbf{v}_t)$$

for all $(\alpha_t, k_t, \mathbf{v}_t)$. As with the dynamic pivot mechanism discussed earlier, the dynamic virtual pivot mechanism does not require agents to make any transfers in periods where they are not allocated an object. If i does not receive an object, either $\varphi_i(v_i) < 0$ or there are sufficiently many agents with higher virtual values currently present on the market. In the first case, i's presence does not affect the allocation to any other agent, and hence her contribution is identically zero. If, on the other hand, the latter is true, then removing i from the market will not affect the current-period allocation, and hence i's flow contribution is zero. It should also be clear that, if agent i *does* receive an object, her flow marginal contribution to the virtual surplus is equal to her total marginal contribution, as she will no longer be present in the future.

Moreover, the expectation of a discounted sum of flow marginal contributions is equal to the the total marginal contribution. Therefore, we have the following observation, which we state without proof.

COROLLARY 3 (Dynamic Virtual Pivot Mechanism is Dominant Strategy IC and IR)**.**
Suppose that virtual values φ_i are increasing for all $i \in \mathcal{I}$. Then the dynamic virtual pivot mechanism $\widetilde{\mathcal{M}}^F$ is dominant strategy incentive compatible and individually rational.

As with the case of optimal mechanisms in static settings, there is a clear tradeoff between revenue and efficiency. The inefficiency of the revenue-maximizing mechanism stems from two sources. First, there are buyers who, despite having positive values for an object, never receive one. The second source of inefficiency enters due to the discriminatory nature of virtual values: a buyer with a higher value may have the timing of her allocation delayed because of the presence of a buyer with a lower value but higher virtual value.

This second source of inefficiency disappears in the symmetric case in which all buyers' values are drawn from the same distribution F. In this case, each buyer is faced with the same reservation value $\tilde{r}_i = \tilde{r} := \varphi^{-1}(0)$, and the ordering of virtual values agrees with the ordering of actual values. Moreover, the revenue-maximizing allocation rule becomes quasi-efficient, in the sense that it allocates objects efficiently among the subset of agents with values greater than the reserve value $\tilde{r}$. In addition, an agent's marginal contribution to the virtual surplus in this case may be reinterpreted in terms of social welfare. Recall that agent i's marginal contribution to the social welfare is the total surplus when i is present less the total surplus when she is absent. Since a buyer with value 0 does not provide (or remove) any surplus from the market, the total surplus when i is absent is unchanged when we add a "null" buyer with value zero. Therefore, i's marginal contribution may also be viewed as her "social replacement value," where instead of removing i from the market entirely, she is simply replaced with an agent whose value is 0. Analogously, i's contribution to the virtual surplus $\widetilde{w}_i$ is her replacement value, where instead of replacing i with an agent whose value is 0, we replace her with an agent whose value is equal to the reservation price $\tilde{r}$. With this in mind, the link between optimal mechanisms and efficient VCG-like mechanisms should not be surprising. In a static setting, Krishna and Perry (2000) show that the Vickrey-Clarke-Groves mechanism is revenue-maximizing among all mechanisms that are efficient, individually rational, and incentive compatible. Cavallo (2008) proves a similar result: in a dynamic setting with a fixed agent population and changing types, the dynamic pivot mechanism is revenue maximizing among the class of efficient, individually rational, and periodic *ex post* incentive compatible dynamic mechanisms. Therefore, when revenue maximization leads to quasi-efficient allocation rules, a pivot mechanism with an appropriately chosen reserve is optimal.

6.2. *Optimal Auction*

In light of the previous observation and the results of Section 4, when values are independently and identically drawn from the same distribution F, a natural candidate for a revenue maximizing auction is the sequential ascending auction. It is well-known that in a static setting with K units of a homogenous good to be allocated, efficiency is achievable by a Vickrey-Clarke-Groves mechanism. This mechanism is outcome equivalent to a K-th price sealed-bid or ascending auction. As established by Myerson (1981) in the case of a single object, and by Maskin and Riley (1989) with multiple units of a homogenous good, the revenue-maximizing mechanism is a pivot mechanism with a reserve price equal to $\tilde{r} := \varphi^{-1}(0)$. This mechanism is again outcome equivalent to a K-th price sealed bid or ascending auction with a reserve price equal to $\tilde{r}$. In our dynamic setting with randomly arriving and departing buyers, both the Vickrey-Clarke-Groves mechanism and the dynamic pivot mechanism are efficient. Moreover, the outcome of the dynamic pivot mechanism may be implemented via a sequence of ascending auctions. Reasoning by analogy, we may conclude that, since the dynamic virtual pivot mechanism is revenue maximizing and corresponds to the dynamic pivot mechanism with a reserve of $\tilde{r}$, a sequence of ascending auctions with reserve price $\tilde{r}$ is the corresponding revenue-maximizing auction.

Let us formalize this analogy. We again make use of the multi-unit, uniform-price variant of the Milgrom and Weber (1982) button model of ascending auctions. However, we introduce a reserve price equal to $\tilde{r}$. For notational convenience, we will assume that the price clock starts at zero and rises continuously.[16] When there are $m \geq 1$ units available in a given period, the auction will end whenever there are at most m bidders still active *and* the price is at least $\tilde{r}$. At that time, each remaining bidder receives an object and pays the price at which the auction ended. As before, ties are broken fairly.

Recall that we denote by $n_t := \mathcal{A}_t$ the number of buyers present in period t. Also recall that, taking the perspective of an arbitrary bidder i, we let

$$\mathbf{y}_t := (y_t^1, \ldots, y_t^{n_t-1})$$

denote the ordered valuations of all other buyers present in period t, where y_t^1 is the largest value and $y_t^{n_t-1}$ is the smallest. For each $m = 1, \ldots, n_t - 1$, we define

$$\bar{\mathbf{v}}^m := (\bar{v}, \ldots, \bar{v}) \in \mathbf{V}^m \text{ and } \mathbf{y}_t^{>m} := (y_t^{m+1}, \ldots, y_t^{n_t-1}).$$

Finally, we define, for each $m = 1, \ldots, n_t - 1$,

$$\widetilde{w}^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := \delta \mathbb{E}\left[\widetilde{w}_i(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, \mathbf{y}_t^{>m})\right].$$

This is the (discounted) expected future marginal contribution of an agent $i \in \mathcal{A}_t$ to the virtual surplus, conditional on the period-t presence of m competitors with the highest possible value $\bar{v}$ and $n_t - m - 1$ buyers ranked below i with values $\mathbf{y}_t^{>m}$. Recall from the previous discussion, however, that this is exactly i's expected contribution to the social welfare over a replacement agent with value $\tilde{r}$; moreover, note that $\widetilde{w}^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) = 0$ for all buyers with values $v_i \leq \tilde{r}$, regardless of the realization of $\mathbf{y}_t^{>m}$.

In each period $t \in \mathbb{N}_0$, we assume that each agent $i \in \mathcal{A}_t$ bids up to the cutoffs $\widetilde{\beta}^t_{m,n_t}$ whenever she has m active competitors in the auction, where

$$\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := v_i - \widetilde{w}^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}). \tag{6}$$

[16] Alternately, one could model each auction as a two-stage game in which buyers first choose to participate in the current period auction and then the price clock starts at $\tilde{r}$.

These (symmetric across agents) cutoffs are strictly increasing in v_i, implying that buyers can infer the values of those competitors that have already exited the auction. Note that when buyer i is active, and she knows the values $\mathbf{y}_t^{>m}$ of her opponents that are no longer active, the price at which she is indifferent between winning an object and receiving her discounted marginal contribution in the next period (conditional on all remaining active buyers having values greater than hers) is exactly $\widetilde{\beta}^t_{m,n_t}$.

We may then use arguments similar to those of Lemma 3, Proposition 2, and Proposition 3 to prove the following result.

PROPOSITION 5 (Revenue Maximization via Sequential Ascending Auctions)**.**
Suppose that $F_i = F$ for all $i \in \mathcal{I}$. Then following the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ in Equation (6) in every period of the sequential ascending auction mechanism with reserve price $\tilde{r} := \varphi^{-1}(0)$ is a perfect Bayesian equilibrium that is outcome equivalent to the truth-telling equilibrium of the dynamic virtual pivot mechanism.

Proposition 5 therefore implies that the sequential ascending auction with a reserve price admits an equilibrium with prices and allocations identical to those of the dynamic virtual pivot mechanism. Therefore, as was the case in Section 4, we find that a monopolist who wishes to maximize revenues while making use of a transparent, decentralized mechanism may do so by using a sequence of ascending auctions. Moreover, the method of proof of the proposition above shows that the strategy profile specified in Equation (6) forms a periodic *ex post* equilibrium. Given expectations about future arrivals and behavior, each buyer's current period bid is a best response to the strategies of her opponents, regardless of the realization of their values or the history of the mechanism.

7. An Illustrative Example, Continued

In order to make the results of the previous section more concrete, let us return to the example of Section 3. Recall that in this example, there is exactly one unit of the homogeneous good available in each period, and at most one new buyer arriving to the market in each period with probability $\rho \in [0, 1]$. In addition, buyers do not exit the market until they receive an object. Finally, each buyer i's privately known valuation for a single object is v_i, where v_i is drawn independently and identically from the common distribution F. Virtual values $\varphi(v_i)$ are strictly increasing in v_i, and we denote by $\tilde{r}$ the value corresponding to a virtual value equal to zero: $\varphi(\tilde{r}) = 0$.

Lemma 4 established that, in order to maximize revenues, the optimal policy $\widetilde{\mathbf{x}}$ assigns the object in each period to the buyer with the highest non-negative virtual value, which corresponds (in this symmetric setting) to the highest value greater than $\tilde{r}$. Moreover, the subsequent discussion (including Corollary 3) demonstrated that the dynamic virtual pivot mechanism succeeds in implementing $\widetilde{\mathbf{x}}$ in dominant strategies.

In the relatively simple setting of this example, it is possible to derive explicit expressions for the transfers in this direct mechanism. In particular, a straightforward adaptation of the arguments found in in Appendix B used to characterize the social planner's payoff may be used to calculate the virtual surplus from the perspective of an arbitrary buyer. Suppose there are n buyers

$\{1, \ldots, n\}$ present on the market, with values

$$v_1 > v_2 > \cdots > v_m > \tilde{r} > v_{m+1} > \cdots > v_n.$$

Notice that, of the n buyers present, only m buyers have values greater than $\tilde{r}$.

Consider, for instance, buyer 2, the buyer with the second-highest value. If we increase her value v_2 by an infinitesimal amount, the optimal allocation in the current period is unchanged, and the seller realizes no benefit from the increase. In the following period, however, buyer 2 is allocated an object if either no new entrant arrives (which occurs with probability $1-\rho$), or if a new entrant arrives but has a value lower than v_2 (which occurs with probability $\rho F(v_2)$). On the other hand, if a new, higher-valued buyer *does* arrive in the next period, the revenue-maximizing policy allocates the object to the entrant, and the seller must wait an additional period before potentially realizing the gains from the increase in v_2. Since the arrival process is stationary, this implies that the marginal increase in the virtual surplus due to the increase in v_2 is

$$\begin{aligned}\delta\Big[1-\rho(1-F(v_2))+\rho(1-F(v_2))\times\delta\big[1-\rho(1-F(v_2))+\rho(1-F(v_2))\times\delta\left[\cdots\right]\big]\Big]\\ =\delta\sum_{t=0}^{\infty}\left[\delta\rho(1-F(v_2))\right]^t\big(1-\rho(1-F(v_2))\big)=\delta\frac{1-\delta(1-F(v_2))}{1-\delta\rho(1-F(v_2))}.\end{aligned}$$

Integrating this expression therefore captures the total contribution of buyer 2 over a "replacement" buyer with value equal to $\tilde{r}$, yielding

$$\delta\int_{\tilde{r}}^{v_2}\lambda(v')\,dv',$$

where, as in Section 3, the function $\lambda : \mathbf{V} \to [0,1]$ is given by

$$\lambda(v) := \frac{1-\rho(1-F(v))}{1-\delta\rho(1-F(v))}.$$

Therefore, the virtual surplus from the perspective of an arbitrary buyer is (with some abuse of notation) given by

$$\Pi_n(v_1,\ldots,v_n)=\Pi_0+\sum_{j=1}^{m}\delta^{j-1}\int_{\tilde{r}}^{v_j}\lambda^{j-1}(v')\,dv',$$

where

$$\Pi_0 := \frac{\delta\rho}{1-\delta}\int_{\tilde{r}}^{\bar{v}}(v'-\tilde{r})\,dF(v')$$

is the expected virtual surplus when no buyers are present on the market. Notice that the values of buyers $m+1,\ldots,n$ do not enter the above expression. Since their values are below $\tilde{r}$, these buyers are never allocated an object, and hence make no contribution to the virtual surplus. Thus, removing any one of these agents from the market leads to no change in the virtual surplus. On the other hand, the virtual surplus when buyer i_k, where $k \leq m$, is removed from the market is given by

$$\Pi_{n-1}(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n)=\Pi_0+\sum_{j=1}^{k-1}\delta^{j-1}\int_{\tilde{r}}^{v_j}\lambda^{j-1}(v')\,dv'+\sum_{j=k+1}^{m}\delta^{j-2}\int_{\tilde{r}}^{v_j}\lambda^{j-2}(v')\,dv'.$$

Therefore, the marginal contribution of buyer k, where $k \leq m$, to the virtual surplus is

$$\begin{aligned}\widetilde{w}_k(v_1,\ldots,v_n) &:= \Pi_n(v_1,\ldots,v_n) - \Pi_{n-1}(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n)\\ &= \sum_{j=k}^{m} \delta^{j-1}\int_{\tilde{r}}^{v_j} \lambda^{j-1}(v')\,dv' - \sum_{j=k+1}^{m} \delta^{j-2}\int_{\tilde{r}}^{v_j} \lambda^{j-2}(v')\,dv'\\ &= \sum_{j=k}^{m} \delta^{j-1}\int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\,dv',\end{aligned}$$

where we abuse notation and let $v_{m+1} := \tilde{r}$. As with the efficient allocation rule discussed in Section 3, removing buyer k from the market does not affect the allocation of those buyers ranked above her; however, her presence shifts back the anticipated allocation to buyers ranked below her by one period.

By setting transfer payments such that buyer k internalizes this effect, the dynamic virtual pivot mechanism induces her to truthfully reveal her value. Recall that this mechanism makes transfers in each period such that every agent i's flow payoff is exactly equal to her flow marginal contribution to the virtual surplus. In this setting, an agent's flow marginal contribution is zero unless she receives an object. On the other hand, when the agent *does* receive an object, her flow contribution to the virtual surplus is exactly equal to her total contribution. Therefore, when there are n agents present with values $v_1 > \cdots > v_m > \tilde{r} > v_{m+1} > \cdots > v_n$, the dynamic virtual pivot mechanism gives the object to buyer 1, charging her a payment $\tilde{p}$ such that

$$v_1 - \tilde{p} = \tilde{w}_1(v_1,\ldots,v_n).$$

Rearranging this expression (and again setting $v_{m+1} := \tilde{r}$), we see that the transfer paid by buyer 1 is given by

$$\tilde{p} = v_1 - \sum_{j=1}^{m} \delta^{j-1}\int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\,dv' = v_2 - \sum_{j=2}^{m} \delta^{j-1}\int_{v_{j+1}}^{v_j} \lambda^{j-1}(v')\,dv'. \tag{7}$$

As demonstrated by Corollary 3 in Section 6, when the seller sets transfers as in the equation above, the revenue-maximizing policy is implemented in dominant strategies. Notice the resemblance of this price to the corresponding transfer in the (efficient) dynamic pivot mechanism as described in Equation (2). However, instead of the price depending on the values of all agents present, the summation ends with the agent with the lowest value above $\tilde{r}$. This is precisely due to the fact that the dynamic virtual pivot mechanism leaves buyer 1 with her marginal contribution over a replacement agent with value $\tilde{r}$, as opposed to a "null" agent with value zero.

Recall that Proposition 5 characterized an equilibrium of the sequential ascending auction (with a reserve price) that is outcome equivalent to the dynamic virtual pivot mechanism. This equilibrium may be interpreted in the same manner as that discussed in Section 3. The price paid by buyer 1 in Equation (7) can be decomposed into two components. The first term (v_2) is the loss in virtual surplus from allocating the object to buyer 1 in the current period: if she were not present, the optimal allocation rule $\tilde{\mathbf{x}}$ would assign the object to buyer 2, leading to a flow benefit of v_2. On the other hand, the summation in the expression above is the expected gain in the virtual surplus from postponing the allocation to buyer 2. The presence of buyer 1 implies that the seller anticipates allocating to buyer 2 in the next period instead of to buyer 3, and to buyer 3 in the following period instead of to buyer 4, and so on. As with the transfer of the dynamic pivot mechanism calculated in Equation (2), this second term is the discounted expectation of buyer 2's *future* contribution to the virtual surplus.[17] Thus, Proposition 5 may be restated as: if, in every period, each buyer bids up to the point at which she is indifferent between receiving an object immediately and receiving her expected future contribution to the virtual surplus, we are able to achieve the outcome of the dynamic virtual pivot mechanism.

So, suppose that an ascending auction is conducted in every period. We assume that when there are n buyers present at the beginning of the period, of which k are actively bidding in the auction, each active buyer i bids up to

$$\widetilde{\beta}_{k,n}(v_i) := v_i - \delta\mathbb{E}\left[\Pi_n(\bar{v},\ldots,\bar{v},v_i,v_{k+1},\ldots,v_n) - \Pi_{n-1}(\bar{v},\ldots,\bar{v},v_{k+1},\ldots,v_n)\right],$$

where the expectation is taken with respect to the entry process of new buyers and the possible valuation of a new entrant in the following period. (Recall that the values of buyers that have already exited the auction are revealed by their exit prices.) In words, each buyer bids as though she expects that losing the current-period auction leads to receiving her marginal contribution in the future, conditional on her remaining competitors having the highest possible value $\bar{v}$. This expectation is equal to zero for buyers i with values $v_i \leq \tilde{r}$, as they contribute nothing to the virtual surplus (in any period). Therefore, buyers with values less than the reserve price $\tilde{r}$ will bid up to their true values, dropping out of the auction before the reserve price is met.

Of the remaining buyers (those with values greater than $\tilde{r}$), the first to exit the auction is buyer m. She will drop out of the auction when the price reaches

$$\begin{aligned}\widetilde{\beta}_{m,n}(v_m,\ldots,v_n) &= v_m - \delta\mathbb{E}\left[\Pi_n(\bar{v},\ldots,\bar{v},v_m,v_{m+1},\ldots,v_n) - \Pi_{n-1}(\bar{v},\ldots,\bar{v},v_{m+1},\ldots,v_n)\right]\\ &= v_m - \delta^{m-1}\int_{\tilde{r}}^{v_m}\lambda^{m-1}(v')\,dv'.\end{aligned}$$

Notice that this price is strictly greater than the reserve price, as $v_m > \tilde{r}$ and $\lambda(v') \in [0,1]$ for all $v' \in [0,\bar{v}]$. The $m-1$ remaining buyers observe this exit, and revise their anticipated exit prices. As expected, these exit prices are monotonic in their values, leading to the agents sequentially dropping out of the auction in increasing order of their values. Thus, the auction will end when the price reaches

$$\widetilde{\beta}_{2,n}(v_2,\ldots,v_n) = v_{i_2} - \sum_{j=2}^{m}\delta^{j-1}\int_{v_{j+1}}^{v_j}\lambda^{j-1}(v')\,dv',$$

where $v_{m+1} := \tilde{r}$. Buyer 1, the buyer with the highest value present on the market, will win the current-period object. In addition, the price she pays in the auction is exactly equal to the dynamic virtual pivot mechanism transfer described in Equation (7). Thus, if each buyer bids according to the functions described above, the sequential ascending auction leads to outcomes identical to those of the dynamic virtual pivot mechanism. Moreover, as shown in Proposition 5, these

[17]This fact follows from a simple alteration to the argument found in Appendix B.

bidding strategies, coupled with the appropriate off-equilibrium beliefs, form a perfect Bayesian equilibrium.

8. CONCLUSION

In this paper, we examine a private-values, single-unit demand environment where buyers and objects arrive at random times. We show that the efficient allocation policy may be implemented by using a dynamic variant of the classic Vickrey-Clarke-Groves mechanism. Moreover, by generalizing the static Myerson (1981) payoff- and revenue-equivalence results to our setting, we are able to derive a revenue-maximizing direct mechanism. This mechanism succeeds in maximizing a seller's profits by applying an efficient dynamic pivot mechanism to buyers' virtual values.

We also examine indirect mechanisms in this setting, finding that a sequence of ascending auctions serves as a natural and intuitive institution that corresponds to the direct mechanisms described above. This open auction format allows each buyer to learn her competitors' values, and hence determine her own marginal contribution to the social welfare. When each buyer exits each auction at the price such that she is indifferent between winning the object and obtaining her marginal contribution, we obtain a decentralized price discovery mechanism that yields outcomes identical to those of the centralized direct mechanisms.

These results set the stage for several additional avenues of inquiry. For instance, suppose that objects need not be allocated in the period of their arrival, but can instead be placed in inventory and allocated in future periods. While some properties of our solution are maintained (for instance, the efficient policy continues to allocate objects to higher-ranked buyers before moving onto competitors with lower values, and a dynamic VCG mechanism will continue to be efficient), the indirect implementation results do not follow immediately. For example, in the case of storable objects, the sale of an object to a particular agent imposes an additional externality on her competitors, as the number of objects available in the future decreases. This reduces the incentives for buyers to truthfully reveal their private information, as this information may be of great strategic value to competitors.

Another natural extension of our model is the generalization to the case in which agents may demand multiple units. This, however, introduces additional intertemporal tradeoffs in any auction mechanism, as expected future payoffs in individual valuations are no longer identical functions of individual values when buyers have differential demands. An alternative line of research relaxes the assumption that buyer entries and exits are exogenous. It is unclear what outcomes will arise when buyers may instead condition their entry on market conditions. We leave these questions, however, for future work.

Appendix A. Proofs

Proof of Lemma 1. We first show the necessity of the three conditions for incentive compatibility and individual rationality. So, suppose that the mechanism $\mathcal{M}$ is both incentive compatible and individually rational. Fix any $t \in \mathbb{N}_0$, any $i \in \mathcal{I}_t$, and arbitrary $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$. Then incentive compatibility implies that, for all $v_i \in \mathbf{V}$,

$$\widehat{U}_i(v_i, \alpha_t, k_t) = \max_{v_i' \in \mathbf{V}} \left\{ q_i(v_i', \alpha_t, k_t) v_i - m_i(v_i', \alpha_t, k_t) \right\}.$$

Thus, $\widehat{U}_i(v_i, \cdot)$ is an affine maximizer, and is hence a convex function of v_i. Moreover, for all $v_i, v_i' \in \mathbf{V}$, incentive compatibility is equivalent to

$$\begin{aligned}
\widehat{U}_i(v_i', \alpha_t, k_t) &\geq q_i(v_i, \alpha_t, k_t) v_i' - m_i(v_i, \alpha_t, k_t) \\
&= q_i(v_i, \alpha_t, k_t) v_i - m_i(v_i, \alpha_t, k_t) + q_i(v_i, \alpha_t, k_t)(v_i' - v_i) \\
&= \widehat{U}_i(v_i, \alpha_t, k_t) + q_i(v_i, \alpha_t, k_t)(v_i' - v_i).
\end{aligned}$$

Thus, $q_i(v_i, \cdot)$ is a subderivative of $\widehat{U}_i(v_i, \cdot)$ at v_i. Since $\widehat{U}_i(v_i, \cdot)$ is convex in v_i, it is absolutely continuous and hence differentiable almost everywhere, implying that at every point of differentiability,

$$\frac{\partial}{\partial v_i} \widehat{U}_i(v_i, \alpha_t, k_t) = q_i(v_i, \alpha_t, k_t).$$

Since $\widehat{U}_i(v_i, \cdot)$ is convex, this implies that q_i must be nondecreasing in v_i.

Moreover, every absolutely continuous function is equal to the definite integral of its derivative, implying that

$$\widehat{U}_i(v_i, \alpha_t, k_t) = \widehat{U}_i(0, \alpha_t, k_t) + \int_0^{v_i} q_i(v_i', \alpha_t, k_t)\, dv_i'$$

for all $v_i \in \mathbf{V}$. Finally, since q_i is nondecreasing in v_i, the requirement of individual rationality is then satisfied for all v_i only if

$$\widehat{U}_i(0, \alpha_t, k_t) \geq 0.$$

Hence, the three conditions are necessary conditions for $\mathcal{M}$ to be incentive compatible and individually rational.

We now show the sufficiency of the three conditions for incentive compatibility and individual rationality. Suppose that $\mathcal{M}$ satisfies the three conditions, and fix any $t \in \mathbb{N}_0$, any $i \in \mathcal{I}_t$, and arbitrary $(\alpha_t, k_t) \in \{0,1\}^{\mathcal{I}} \times \mathcal{K}$. Note first that q_i is nondecreasing in v_i and $\widehat{U}_i(0, \alpha_t, k_t) \geq 0$ immediately imply that $\widehat{U}_i(v_i, \alpha_t, k_t) \geq 0$ for all $v_i \in \mathbf{V}$, and so $\mathcal{M}$ is individually rational.

Now, for any $v_i, v_i' \in \mathbf{V}$, the second condition implies that

$$\widehat{U}_i(v_i', \alpha_t, k_t) = \widehat{U}_i(v_i, \alpha_t, k_t) + \int_{v_i}^{v_i'} q_i(v_i'', \alpha_t, k_t)\, dv_i''.$$

If $v_i < v_i'$, then q_i nondecreasing implies that $q_i(v_i, \alpha_t, k_t) \leq q_i(v_i', \alpha_t, k_t)$. Thus,

$$\widehat{U}_i(v_i', \alpha_t, k_t) \geq \widehat{U}_i(v_i, \alpha_t, k_t) + q_i(v_i, \alpha_t, k_t)(v_i' - v_i).$$

Similarly, if $v_i > v_i'$, then q_i nondecreasing implies that $q_i(v_i, \alpha_t, k_t) \geq q_i(v_i', \alpha_t, k_t)$. Therefore,

$$\begin{aligned}\widehat{U}_i(v_i', \alpha_t, k_t) &= \widehat{U}_i(v_i, \alpha_t, k_t) - \int_{v_i'}^{v_i} q_i(v_i'', \alpha_t, k_t)\, dv_i'' \\ &\geq \widehat{U}_i(v_i, \alpha_t, k_t) - q_i(v_i, \alpha_t, k_t)(v_i - v_i') = \widehat{U}_i(v_i, \alpha_t, k_t) + q_i(v_i, \alpha_t, k_t)(v_i' - v_i).\end{aligned}$$

However, this inequality is, as shown above, equivalent to incentive compatibility. Since $v_i, v_i' \in \mathbf{V}$ were chosen arbitrarily, this implies that $\mathcal{M}$ is incentive compatible. □

Proof of Lemma 2. Note first that any two allocation rule that satisfy the conditions of the lemma yield the same expected payoff to the social planner. To see this, note that the only variation permitted is in the allocation of objects to agents with zero valuation and in the breaking of ties. Given the allocations to all other agents, choosing to allocate to agents with value zero yields neither an increase nor decrease in the realized surplus. Moreover, although the second condition regarding the allocation to agents in $\mathcal{A}_\sim$ allows for various mixtures over this set of agents, the outcome of these mixtures is always the same: exactly $k_t - |\mathcal{A}_+|$ of these agents receive an object. Different choices among these outcomes does not affect future payoffs, as the arrival process of agents and objects is orthogonal to these allocative decisions, and all agents depart the system at the same exogenously given rate $1 - \gamma$.

With this in mind, let $\widehat{\mathbf{x}}$ denote a deterministic allocation rule that allocates an object to the highest-ranking agents (including those with value equal to zero), where ties are broken arbitrarily (but without randomization). Fix any policy $\mathbf{x}_0$ that yields the planner a strictly higher payoff than the policy $\widehat{\mathbf{x}}$, and define

$$\mathcal{Z}^0 := \left\{(h_t, \alpha_t, k_t) \in \mathcal{H} \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} : \mathbf{x}_0(z) \neq \widehat{\mathbf{x}}(z), \mathbf{x}_0(z') = \widehat{\mathbf{x}}(z') \text{ for all } z' \to z\right\}.$$

Thus, $\mathcal{Z}^0$ is the set of all histories and arrivals z such that $\mathbf{x}_0$ and $\widehat{\mathbf{x}}$ disagree at z, but agree on all of z's prefixes; that is, $\mathcal{Z}^0$ is the set of "first" or "earliest" disagreements between $\mathbf{x}_0$ and $\widehat{\mathbf{x}}$. Since $\mathbf{x}_0$ does strictly better than $\widehat{\mathbf{x}}$, this set must have nonzero measure (with respect to the measure induced by the arrival processes), as otherwise the two policies would agree almost everywhere (and hence yield identical payoffs).

For each $z \in \mathcal{Z}^0$, note that the policy $\mathbf{x}_0$ induces a probability distribution over outcomes, where an outcome is an assignment of objects to agents. Denote by $\Sigma^0(z)$ the set of outcomes induced by $\mathbf{x}_0$ at history z. Thus, an outcome $\sigma \in \Sigma^0(z)$ is associated with a subset of buyers present that receive an object. Let $a_j(\sigma)$ denote the j-th highest-valued agent that receives an object under $\mathbf{x}_0$ in outcome σ. Similarly, let $b_j(z)$ denote the j-th highest-valued agent overall.

Define for each $z \in \mathcal{Z}^0$ and for each $\sigma \in \Sigma^0(z)$, we define the "continuation policy" $\mathbf{x}_1^\sigma(z)$ to be the allocation rule that allocates to the highest-ranking agents present at time z, and is equal to $\mathbf{x}_0$ at all successors of z except that it allocates to agent $a_j(\sigma)$ whenever $\mathbf{x}_0$ allocates to agent $b_j(z)$. Thus, $\mathbf{x}_1^\sigma(z)$ is the same as $\mathbf{x}_0$ except that it "swaps" the allocation decisions of $a_j(\sigma)$ and $b_j(z)$. Since $v_{a_j(\sigma)} \leq v_{b_j(z)}$ (with a strict inequality for at least one j), the expected payoff to the planner under $\mathbf{x}_1^\sigma$ is greater than that of $\mathbf{x}_0$ along this branch of the mechanism tree. To see why this is true, consider any $v > v'$ and $t < t'$. Since $\delta < 1$, we have

$$(\delta^t v + \delta^{t'} v') - (\delta^t v' + \delta^{t'} v) = (\delta^t - \delta^{t'})(v - v') > 0.$$

Thus, even if agents do not depart the market, the planner's payoff along this path is increases.

Thus, define the allocation policy $\mathbf{x}_1$ to be equal to $\mathbf{x}_0$ at all histories that are not successors to histories in $\mathcal{Z}^0$. Furthermore, for each $z \in \mathcal{Z}^0$, we define $\mathbf{x}_1(z)$ to be the stochastic policy that chooses $\mathbf{x}_1^\sigma(z)$ with the probability that $\mathbf{x}_0$ leads to outcome σ. Since this leads to an increase in the planner's payoff over $\mathbf{x}_0$ along every successor history to those in $\mathcal{Z}^0$, and this set has positive measure, it must be the case that $\mathbf{x}_1$ yields a strictly greater payoff that $\mathbf{x}_0$.

If $\mathbf{x_1}$ yields the planner a payoff less than or equal to that of $\widehat{\mathbf{x}}$, transitivity of the planner's payoffs leads to a contradiction, implying that there does not exist a policy $\mathbf{x_0}$ such that $\mathbf{x_0}$ does strictly better than $\widehat{\mathbf{x}}$, and hence that $\widehat{\mathbf{x}}$ is optimal.

On the other hand, if $\mathbf{x}_1$ yields a payoff greater than that of $\widehat{\mathbf{x}}$, we define the set

$$\mathcal{Z}^1 := \left\{(h_t, \alpha_t, k_t) \in \mathcal{H} \times \{0,1\}^{\mathcal{I}} \times \mathcal{K} : \mathbf{x}_1(z) \neq \widehat{\mathbf{x}}(z), \mathbf{x}_1(z') = \widehat{\mathbf{x}}(z') \text{ for all } z' \to z\right\}$$

to be the set of $\mathbf{x}_1$'s "first disagreements" with $\widehat{\mathbf{x}}$. We may repeat the procedure above to then define a new policy $\mathbf{x}_2$ that agrees with $\widehat{\mathbf{x}}$ at every $z \in \mathcal{Z}^1$, but does strictly better than either $\mathbf{x}_1$. Notice that, if $\mathbf{x}_1$ does better than $\widehat{\mathbf{x}}$, then we have arrived at a contradiction.

Proceeding in this manner, we construct a sequence of policies $\{\mathbf{x}_s\}_{s=0}^\infty$ with associated expected payoffs $\{W_s\}_{t=0}^\infty$ such that $W_s < W_{s+1}$ for all $s \in \mathbb{N}_0$. Note, however, that for all $s \in \mathbb{N}_0$, $\mathbf{x}_s$ agrees with $\widehat{\mathbf{x}}$ on *at least* all histories of length s. Since δ^s approaches zero as s becomes increasingly large, this implies that

$$\lim_{s\to\infty} W_s = \widehat{W},$$

where $\widehat{W}$ is the planner's expected payoff from following policy $\widehat{\mathbf{x}}$. Moreover, since $\{W_s\}$ is an increasing sequence, this implies that

$$\widehat{W} \geq W_s \text{ for all } s \in \mathbb{N}_0,$$

a contradiction. It must therefore be the case that there does not exist a policy $\mathbf{x}_0$ that yields the planner a strictly higher payoff than $\widehat{\mathbf{x}}$. Therefore, we may conclude that $\widehat{\mathbf{x}}$ is, in fact, a socially optimal policy. ☐

Proof of Proposition 1. As already established, the discounted expected probability of receiving an object under the efficient allocation rule $\hat{q}_i(v_i, \alpha_t, k_t)$ is nondecreasing in v_i. Given the result of Lemma 1, this implies that the VCG mechanism is incentive compatible. To see that it is individually rational, note that

$$\widehat{U}_i(v_i, \alpha_t, k_t) = \hat{q}_i(v_i, \alpha_t, k_t)v_i - \hat{m}_i(v_i, \alpha_t, k_t) = \mathbb{E}\left[w_i(\alpha_t, k_t, \mathbf{v}_t)\right],$$

where the expectation is taken with respect the values of agents $j \in \mathcal{A}_t \setminus \{i\}$. However, since the social planner always has available to her the option of ignoring i's presence on the market (which imposes no externalities on the other agents), i's marginal contribution to the social welfare must be nonnegative. Thus, $\widehat{U}_i(v_i, \alpha_t, k_t) \geq 0$, implying that $\widehat{\mathcal{M}}$ is individually rational.

In order to show that truth-telling is a dominant strategy for all agents, fix an arbitrary agent $i \in \mathcal{I}_t$ for arbitrary $t \in \mathbb{N}_0$, and suppose that i knows the reported values $\mathbf{v}_{-i} \in \times_{j \neq i} \mathbf{V}$ of all agents other than i. Then, by reporting a value v_i' upon her arrival in state (α_t, k_t), agent i's payoff under the VCG mechanism is

$$\mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} \hat{x}_{i,s}(\alpha_s, k_s, (v_i', \mathbf{v}_{-i}))\right](v_i - v_i') + w_i(\alpha_t, k_t, (v_i', \mathbf{v}_{-i}))$$
$$= \mathbb{E}\left[\sum_{s=t}^{\infty} \sum_{j \in \mathcal{I}} \delta^{s-t} \hat{x}_{j,s}(\alpha_s, k_s, (v_i', \mathbf{v}_{-i})) v_j\right] - W_{-i}(\alpha_t^{-i}, k_t, \mathbf{v}_{-i}).$$

Since $\widehat{\mathbf{x}}$ is the efficient policy, the first term above is maximized by setting $v_i' = v_i$. Moreover, the second term does not depend on v_i'. Hence, i's expected payoff is maximized by truthful reporting of her value, regardless of the reports of the other agents or the state upon i's arrival; that is, truth-telling is a dominant strategy. □

Proof of Lemma 3. Fix an arbitrary period $t \in \mathbb{N}_0$, and let α_t and k_t indicate the set of agents and objects present on the market, respectively. Consider an agent $i \in \mathcal{A}_t$ with value v_i, and suppose that $n_t - m - 1$ buyers have dropped out of the period-t auction, revealing values $\mathbf{y}_t^{>m}$, where $n_t := |\mathcal{A}_t|$ is the number of agents present, and $m \in \{1, \ldots, n_t - 1\}$. We wish to show first that $v_i > v_j > y_t^{m+1}$ implies that

$$\widehat{\beta}_{m,n_t}^t(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := v_i - w^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$$
$$> v_j - w^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}) =: \widehat{\beta}_{m,n_t}^t(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}).$$

Notice that

$$\begin{aligned}
&w^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}) - w^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) \\
&= \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_j, \mathbf{y}_t^{>m})\Big] - \delta \mathbb{E}\Big[W_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_j, \mathbf{y}_t^{>m})\Big] \\
&- \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_i, \mathbf{y}_t^{>m})\Big] + \delta \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_i, \mathbf{y}_t^{>m})\Big] \\
&= \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_j, \mathbf{y}_t^{>m})\Big] - \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_i, \mathbf{y}_t^{>m})\Big],
\end{aligned}$$

since removing either i or j in the following period, conditional on their being the m-th highest-ranked agent, does not differentially affect the order of anticipated future allocations to any other agents. In particular, since the the efficient allocation rule $\widehat{\mathbf{x}}$ makes assignments based solely on the ranking of valuations, it will choose the same assignments in future periods when i or j have been removed from the market.

Moreover, by naïvely treating buyer j as though her true value were v_i, we can provide a bound on the difference above. In particular, we have

$$\delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_j, \mathbf{y}_t^{>m})\Big] - \delta \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_i, \mathbf{y}_t^{>m})\Big]$$
$$\geq \mathbb{E}\left[\sum_{s=t+1}^{\infty} \delta^{s-t} \hat{x}_{i,s}(\alpha_s, k_s, \mathbf{v}_s) | \mathbf{v}_t = (\bar{\mathbf{v}}^{n_t - m}, v_i, \mathbf{y}_t^{>m})\right](v_j - v_i).$$

Thus, if $v_i > v_j$, then

$$\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) - \widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) \geq (v_i - v_j)\left(1 - \mathbb{E}\left[\sum_{s=t+1}^{\infty} \delta^{s-t}\hat{x}_{i,s}(\alpha_s, k_s, \mathbf{v}_s) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, \mathbf{y}_t^{>m})\right]\right) > 0$$

since the discounted expected probability of receiving an object in the future is bounded above by $\delta < 1$. Thus, $\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$ is strictly increasing in v_i.

Additionally, note that if $v_i > v_j = y_t^{m+1}$, then

$$\begin{aligned}
w^{t+1}&(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) - w^{t+1}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1}) \\
&= \delta\Big(\mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{m+1}, v_j, \mathbf{y}_t^{>m+1})\Big] \\
&\qquad - \mathbb{E}\Big[W(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_j, \mathbf{y}_t^{>m+1})\Big]\Big) \\
&\quad - \delta\Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_j, \mathbf{y}_t^{>m+1})\Big] \\
&\qquad - \mathbb{E}\Big[W_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{m+1}, v_j, \mathbf{y}_t^{>m+1})\Big]\Big).
\end{aligned}$$

However, the second difference above may be rewritten as

$$\begin{aligned}
\mathbb{E}&\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_j, \mathbf{y}_t^{>m+1})\Big] \\
&- \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_i, \mathbf{y}_t^{>m+1})\Big] \\
&+ \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_i, \mathbf{y}_t^{>m+1})\Big] \\
&\qquad - \mathbb{E}\Big[W_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^{m+1}, v_j, \mathbf{y}_t^{>m+1})\Big].
\end{aligned}$$

Thus,

$$w^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) - w^{t+1}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1})$$

is the sum of three differences. The first is the expected gain in social welfare when increasing i's value from v_i to $\bar{v}$. The second is the expected gain in social welfare (when i is not on the market) from increasing j's value from v_j to v_i. Finally, the third difference is the expected loss in social welfare (when j is not present) from decreasing i's value from $\bar{v}$ to v_i. However, since $v_j < v_i$, the presence or absence of j from the market has no influence on when the efficient policy allocates to i, regardless of whether i's value is v_i or $\bar{v}$. Therefore, the gain from the first difference equals the loss from the third difference, implying that

$$\begin{aligned}
w^{t+1}&(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) - w^{t+1}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1}) \\
&= \delta\Big(\mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_j, \mathbf{y}_t^{>m+1})\Big] \\
&\qquad - \mathbb{E}\Big[W_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1}) | \mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_i, \mathbf{y}_t^{>m+1})\Big]\Big).
\end{aligned}$$

Moreover, by (again) naïvely treating buyer j as though her true value were v_i, we can provide a bound on the difference above, which may be used to show that

$$\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1}) - \widehat{\beta}^t_{m+1,n_t}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) > 0.$$

Thus, the exit of the buyer with rank $(m+1)$ does not induce the immediate exit of any buyer with a higher value. Therefore, since $\widehat{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1})$ is strictly increasing in v_i, the price at which this exit occurs fully reveals the value of the $(m+1)$-th highest-ranked buyer.

Since m was arbitrarily chosen, this implies that the drop-out points of buyers bidding according to the strategy described by Equation (3) are fully revealing of the buyers' values. $\square$

Proof of Proposition 2. Fix an arbitrary period $t \in \mathbb{N}_0$, and let k_t denote the number of objects present, and $n_t := |\mathcal{A}_t|$ denote the number of agents present. As discussed above, the bidding strategies $\widehat{\beta}^t_{m,n_t}$ are strictly increasing; therefore, the multi-unit uniform-price ascending auction ends allocates the k_t objects to the group of buyers with the k_t highest values. Recall that if $k_t \geq n_t$, the auction ends immediately, and all buyers present receive an object for free. Similarly, in the dynamic pivot mechanism, each buyer i receives an object, and makes a payment $\hat{p}^F_{i,t}$ given by

$$\hat{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = v_i - w^F_i(\alpha_t, k_t, \mathbf{v}_t),$$

where w^F_i is the agent's marginal contribution to the social welfare.[18] However, since there are sufficient objects present for each agent to receive one, i does not impose any externalities on the remaining agents; thus,

$$w^F_i(\alpha_t, k_t, \mathbf{v}_t) = w_i(\alpha_t, k_t, \mathbf{v}_t) = v_i,$$

implying that $\hat{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = 0$. In this case, then, the allocation and payments of the auction mechanism and the dynamic pivot mechanism are the same.

Suppose instead that $k_t < n_t$; that is, there are more agents present than objects. Denote by i_m the bidder with the m-th highest value. Then each agent who receives an object pays the price at which buyer i_{k_t+1} drops out of the auction, which is given by

$$\widehat{\beta}^t_{k_t+1,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}) = v_{i_{k_t+1}} - w^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}).$$

In the dynamic pivot mechanism, on the other hand, each agent i who receives an object pays a price

$$\begin{aligned}
\hat{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) &= v_i - w^F_i(\alpha_t, k_t, \mathbf{v}_t) \\
&= v_i - \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)v_j\right] + \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}\setminus\{i\}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s^{-i}, k_s, \mathbf{v}_s)v_j\right] \\
&= v_i - \left(\sum_{m=1}^{k_t} v_m + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)v_j\right]\right) \\
&\quad + \left(\sum_{m=1}^{k_t} v_m + (v_{i_{k_t+1}} - v_i) + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\hat{x}_{j,s}(\alpha_s^{-i,-i_{k_t+1}}, k_s, \mathbf{v}_s)v_j\right]\right).
\end{aligned}$$

[18]Note that since i is receiving an object, her total and flow marginal contributions are equal.

Rearranging and rewriting this expression yields

$$\begin{aligned}\hat{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) &= v_{i_{k_t+1}} - \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}}\delta^{s-t}\hat{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)v_j\right] \\ &\qquad + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}}\delta^{s-t}\hat{x}_{j,s}(\alpha_s^{-i,-i_{k_t+1}}, k_s, \mathbf{v}_s)v_j\right] \\ &= v_{i_{k_t+1}} - w^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}),\end{aligned}$$

where the second equality follows from the fact that $w^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}})$ is defined to be the expected future marginal contribution of the agent with the $(k_t + 1)$-th highest value, conditional on agents with higher values (which includes i) receiving an object today.

Thus, following the bidding strategies $\widehat{\beta}^t_{m,n_t}$ leads to period-t prices and allocations identical to those of the dynamic pivot mechanism. Since the period t was arbitrary, as was the state (α_t, k_t), this equivalence holds after each history. Thus, the two mechanisms are outcome equivalent. □

Proof of Proposition 3. We prove this proposition by making use of the one-shot deviation principle. Consider any period with $n_t := |\mathcal{A}_t|$ buyers on the market and k_t objects present. Suppose that all bidders other than player i are using the conjectured equilibrium strategies. We must show that bidder i has no profitable one-shot deviations from the collection of cutoff points $\{\widehat{\beta}^t_{m,n_t}\}$. More specifically, we must show that i does not wish to exit the auction earlier than prescribed, nor does she wish to remain active later than specified.

Once again labeling agents such that buyer i_1 has the highest value and buyer i_{n_t} has the lowest, note that if $v_i < v_{i_{k_t}}$, bidding according to $\{\widehat{\beta}^t_{m,n_t}\}$ implies that i does not win an object in the current period. Therefore, exiting earlier than specified does not affect i's current-period returns. Moreover, since the bidding strategies are memoryless, neither future behavior by i's competitors nor i's future payoffs will be affected by an early exit. Suppose, on the other hand, that i has one of the k_t highest values; that is, that $v_i \geq v_{i_{k_t}}$. As established by Proposition 2, i receives an object, paying a price such that her payoff is exactly equal to her marginal contribution to the social welfare. Deviating to an early exit, however, leads to agent i_{k_t+1} winning an object instead of buyer i. Moreover, i's expected payoff is then $w^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}})$, which we defined as i's future expected marginal contribution. This is a profitable one-shot deviation for i if, and only if,

$$w^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}) \geq v_i - \widehat{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}).$$

Rearranging this inequality yields

$$\widehat{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}) \geq v_i - w^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}) = \widehat{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}),$$

where the equality comes from the definition of $\widehat{\beta}^t_{k_t,n_t}$ in Equation (3). Since $v_i > v_{i_{k_t+1}}$, this contradicts the conclusion of Lemma 3. Thus, i does not wish to exit the auction early.

Alternately, if $v_i \geq v_{i_{k_t}}$, then planning to remain active in the auction *longer* than specified does not change i's payoffs, as i will win an object regardless. If, on the other hand, $v_i < v_{i_{k_t}}$, then delaying exit from the period-t auction can affect i's payoffs. Since bids in future periods do not depend on information revealed in the current period, this only occurs if i remains in the auction long enough to win an object. If i wins, she pays a price equal to the exit point of $i_{v_{k_t}}$, whereas if she exits, she receives as her continuation payoff her marginal contribution to the social welfare. So, suppose that $i = i_m$ for some $m > k_t$. Then a deviation to remaining active in the auction is profitable if, and only if,

$$v_m - \widehat{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t}}, \ldots, v_{i_{m-1}}, v_{m+1}, \ldots, v_{i_{n_t}}) \geq w^{t+1}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}).$$

Rearranging this inequality yields

$$\begin{aligned} \widehat{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t}}, \ldots, v_{i_{m-1}}, v_{m+1}, \ldots, v_{i_{n_t}}) &\leq v_m - w^{t+1}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}) \\ &= \widehat{\beta}^t_{m-1,n_t}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}), \end{aligned}$$

where the equality comes from the definition of $\widehat{\beta}^t_{m-1,n_t}$ in Equation (3). As above, the fact that $v_m < v_{i_{k_t}}$ contradicts the conclusion of Lemma 3. Therefore, i does not desire to remain active in the auction long enough to receive an object.

Thus, we have shown that no player has any incentive to deviate from the prescribed strategies when on the equilibrium path. In particular, using the bidding strategies $\widehat{\beta}^t_{m,n_t}$ is sequentially rational given players' beliefs along the equilibrium path. Recall, however, that we have specified off-equilibrium beliefs such that buyers "ignore" their past observations when they observe a deviation from equilibrium play, updating their beliefs to place full probability on the valuation that rationalizes the deviation; they believe that the deviating agent is *currently* being truthful with regards to the strategies $\widehat{\beta}^t_{m,n_t}$. The argument above then implies that continuing to bid according to the specified strategies remains sequentially rational with respect to these updated beliefs. Thus, bidding according to the cutoffs in Equation (3) is optimal along the entire game tree: this strategy profile forms a perfect Bayesian equilibrium of the sequential ascending auction mechanism. □

Proof of Proposition 4. As discussed above, $\tilde{q}_i(v_i, \alpha_t, k_t)$, the discounted expected probability of receiving an object under the revenue-maximizing allocation policy, is nondecreasing in v_i, implying that the virtual VCG mechanism is incentive compatible. To see that it is individually rational, note that the expected utility from participating in the mechanism of an agent i with value $v_i = 0$ is

$$\widehat{U}_i(0, \alpha_t, k_t) = -\tilde{m}_i(0, \alpha_t, k_t) = \mathbb{E}\left[\widetilde{w}_i(\alpha_t, k_t, \mathbf{v}_t) | v_i = 0\right],$$

where the expectation is taken with respect to the values of agents $j \in \mathcal{A}_t \setminus \{i\}$. However, since

$$\varphi_i(0) = -\frac{1}{f_i(0)} < 0,$$

the revenue-maximizing allocation rule *never* allocates an object to agent i. Therefore, the optimal policy yields exactly the same outcome whether or not i is present, implying that $\widetilde{w}_i = 0$ regardless of the realizations of other buyers' values. Thus, by Lemma 1, $\widehat{\mathcal{M}}$ is individually rational.

In order to show that truth-telling is a dominant strategy for all agents, fix an arbitrary agent $i \in \mathcal{I}_t$ for arbitrary $t \in \mathbb{N}_0$, and suppose that all agents other than i report values $\mathbf{v}_{-i} \in \times_{j \neq i} \mathbf{V}$, which need not be truthful. Then, by reporting a value v'_i upon her arrival in state (α_t, k_t), agent

i's payoff under the virtual VCG mechanism is

$$\mathbb{E}\left[\sum_{s=t}^{\infty} \tilde{x}_{i,s}(\alpha_s, k_s, (v_i', \mathbf{v}_{-i}))\right](v_i - v_i') + \widetilde{w}_i(\alpha_t, k_t, (v_i', \mathbf{v}_{-i}))$$
$$= \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s, k_s, (v_i', \mathbf{v}_{-i}))\left(\varphi_i^{-1}(\varphi_j(v_j)) - \tilde{r}_i\right)\right] - \Pi_{-i}^{i}(\alpha_t^{-i}, k_t, \mathbf{v}_{-i}).$$

Since $\widetilde{\mathbf{x}}$ is an efficient policy for maximizing the above sum of "transformed" virtual values, the first term above is maximized by setting $v_i' = v_i$. Moreover, the second term does not depend on v_i'. Hence, i's expected payoff is maximized by truthful reporting of her value, regardless of the reports of the other agents or the state upon i's arrival; that is, truth-telling is a dominant strategy in the virtual Vickrey-Clarke-Groves mechanism. □

Proof of Proposition 5. The proof of this proposition parallels the developments of Section 5. In particular, we will first show that bids are monotone and fully revealing, as in Lemma 3. Then, we will show that following the postulated bidding strategies leads to an identical outcome as the dynamic virtual pivot mechanism in a manner similar to the proof of Proposition 2. Finally, we will, analogously to Proposition 3, show that these strategies do, in fact, form a perfect Bayesian equilibrium of the sequential auction mechanism.

Claim. *The bid functions $\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$ are increasing in v_i for all $m = 1, \ldots, n_t - 1$. Moreover, if $v_i > y_t^{m+1}$, then*

$$\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) > \widetilde{\beta}^t_{m+1,n_t}(\alpha_t, k_t, \mathbf{y}_t^{>m}).$$

Proof. Fix an arbitrary period $t \in \mathbb{N}_0$, and let α_t and k_t indicate the set of agents and objects present on the market, respectively. Consider an agent $i \in \mathcal{A}_t$ with value v_i, and suppose that $n_t - m - 1$ buyers have dropped out of the period-t auction, revealing values $\mathbf{y}_t^{>m}$, where $n_t := |\mathcal{A}_t|$ is the number of agents present, and $m \in \{1, \ldots, n_t - 1\}$. We wish to show first that $v_i > v_j > y_t^{m+1}$ implies that

$$\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m}) := v_i - \widetilde{w}^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$$
$$> v_j - \widetilde{w}^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}) =: \widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}).$$

Notice that

$$\widetilde{w}^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m}) - \widetilde{w}^{t+1}(\alpha_t, k_t, v_i, \mathbf{y}_t^{>m})$$
$$= \delta\mathbb{E}\Big[\Pi(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_j, \mathbf{y}_t^{>m})\Big] - \delta\mathbb{E}\Big[\Pi_{-j}(\alpha_{t+1}^{-j}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_j, \mathbf{y}_t^{>m})\Big]$$
$$- \delta\mathbb{E}\Big[\Pi(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_i, \mathbf{y}_t^{>m})\Big] + \delta\mathbb{E}\Big[\Pi_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_i, \mathbf{y}_t^{>m})\Big]$$
$$= \delta\mathbb{E}\Big[\Pi(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_j, \mathbf{y}_t^{>m})\Big] - \delta\mathbb{E}\Big[\Pi(\alpha_{t+1}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m}, v_i, \mathbf{y}_t^{>m})\Big],$$

since removing either i or j in the following period, conditional on their being the m-th highest-ranked agent, does not differentially affect the order of anticipated future allocations to any other agents. In particular, since the the revenue-maximizing allocation rule $\widetilde{\mathbf{x}}$ makes assignments based solely on the ranking of valuations, it will choose the same assignments in future periods when i or j have been removed from the market.

Moreover, by naïvely treating buyer j as though her true value were v_i, we can provide a bound on the difference above. In particular, we have

$$\begin{aligned}\delta\mathbb{E}\Big[\Pi(\alpha_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m},v_j,\mathbf{y}_t^{>m})\Big] \\ &\geq \delta\mathbb{E}\Big[\Pi(\alpha_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m},v_i,\mathbf{y}_t^{>m})\Big] \\ &\quad + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\delta^{s-t}\tilde{x}_{i,s}(\alpha_s,k_s,\mathbf{v}_s)|\mathbf{v}_t = (\bar{\mathbf{v}}^{n_t-m},v_i,\mathbf{y}_t^{>m})\right](v_j - v_i).\end{aligned}$$

Thus, if $v_i > v_j$, then

$$\begin{aligned}\widetilde{\beta}^t_{m,n_t}(\alpha_t,k_t,v_i,\mathbf{y}_t^{>m}) - \widetilde{\beta}^t_{m,n_t}(\alpha_t,k_t,v_i,\mathbf{y}_t^{>m}) \\ &\geq (v_i - v_j)\left(1 - \mathbb{E}\left[\sum_{s=t+1}^{\infty}\delta^{s-t}\tilde{x}_{i,s}(\alpha_s,k_s,\mathbf{v}_s)|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,\mathbf{y}_t^{>m})\right]\right) > 0\end{aligned}$$

since the discounted expected probability of receiving an object in the future is bounded above by $\delta < 1$. Thus, $\widetilde{\beta}^t_{m,n_t}(\alpha_t,k_t,v_i,\mathbf{y}_t^{>m})$ is strictly increasing in v_i.

Additionally, note that if $v_i > v_j = y_t^{m+1}$, then

$$\begin{aligned}\widetilde{w}^{t+1}(\alpha_t,k_t,v_j,\mathbf{y}_t^{>m+1}) - \widetilde{w}^{t+1}(\alpha_t,k_t,v_i,v_j,\mathbf{y}_t^{>m+1}) \\ &= \delta\Big(\mathbb{E}\Big[\Pi(\alpha_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m+1},v_j,\mathbf{y}_t^{>m+1})\Big] \\ &\qquad - \mathbb{E}\Big[\Pi(\alpha_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,v_j,\mathbf{y}_t^{>m+1})\Big]\Big) \\ &\quad - \delta\Big(\mathbb{E}\Big[\Pi_{-i}(\alpha^{-i}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,v_j,\mathbf{y}_t^{>m+1})\Big] \\ &\qquad - \mathbb{E}\Big[\Pi_{-j}(\alpha^{-j}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m+1},v_j,\mathbf{y}_t^{>m+1})\Big]\Big).\end{aligned}$$

However, the second difference above may be rewritten as

$$\begin{aligned}\mathbb{E}\Big[\Pi_{-i}(\alpha^{-i}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,v_j,\mathbf{y}_t^{>m+1})\Big] \\ &- \mathbb{E}\Big[\Pi_{-i}(\alpha^{-i}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,v_i,\mathbf{y}_t^{>m+1})\Big] \\ &+ \mathbb{E}\Big[\Pi_{-i}(\alpha^{-i}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m},v_i,v_i,\mathbf{y}_t^{>m+1})\Big] \\ &\qquad - \mathbb{E}\Big[\Pi_{-j}(\alpha^{-j}_{t+1},k_{t+1},\mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^{m+1},v_j,\mathbf{y}_t^{>m+1})\Big].\end{aligned}$$

Thus,

$$\widetilde{w}^{t+1}(\alpha_t,k_t,v_j,\mathbf{y}_t^{>m+1}) - \widetilde{w}^{t+1}(\alpha_t,k_t,v_i,v_j,\mathbf{y}_t^{>m+1})$$

is the sum of three differences. The first is the expected gain in virtual surplus when increasing i's value from v_i to $\bar{v}$. The second is the expected gain in virtual surplus (when i is not on the market) from increasing j's value from v_j to v_i. Finally, the third difference is the expected loss in virtual surplus (when j is not present) from decreasing i's value from $\bar{v}$ to v_i. However, since $v_j < v_i$, the presence or absence of j from the market has no influence on when the optimal (revenue-maximizing) policy allocates to i, regardless of whether i's value is v_i or $\bar{v}$. Therefore, the gain from the first difference equals the loss from the third difference, implying that

$$\begin{aligned}\widetilde{w}^{t+1}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) - \widetilde{w}^{t+1}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1})\\ = \delta\Big(\mathbb{E}\Big[\Pi_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_j, \mathbf{y}_t^{>m+1})\Big]\\ - \mathbb{E}\Big[\Pi_{-i}(\alpha_{t+1}^{-i}, k_{t+1}, \mathbf{v}_{t+1})|\mathbf{v}_t = (\bar{\mathbf{v}}^m, v_i, v_i, \mathbf{y}_t^{>m+1})\Big]\Big).\end{aligned}$$

Moreover, by (again) naïvely treating buyer j as though her true value were v_i, we can provide a bound on the difference above, which may be used to show that

$$\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1}) - \widetilde{\beta}^t_{m+1,n_t}(\alpha_t, k_t, v_j, \mathbf{y}_t^{>m+1}) > 0.$$

Thus, the exit of the buyer with rank $(m+1)$ does not induce the immediate exit of any buyer with a higher value. Therefore, since $\widetilde{\beta}^t_{m,n_t}(\alpha_t, k_t, v_i, v_j, \mathbf{y}_t^{>m+1})$ is strictly increasing in v_i, the price at which this exit occurs fully reveals the value of the $(m+1)$-th highest-ranked buyer.

Since m was arbitrarily chosen, this implies that the drop-out points of buyers bidding according to the strategy described by Equation (6) are fully revealing of the buyers' values. □

CLAIM. *Following the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ in every period t in the sequential ascending auction mechanism is outcome equivalent to the dynamic virtual pivot mechanism.*

Proof. Fix an arbitrary period $t \in \mathbb{N}_0$, and let k_t denote the number of objects present, and $n_t := |\mathcal{A}_t|$ denote the number of agents present. As shown above, the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ are strictly increasing; therefore, the multi-unit uniform-price ascending auction ends allocates the k_t objects to the group of buyers with the k_t highest values greater than the reserve.[19] Recall that if $k_t \geq n_t$, the auction ends immediately upon the price reaching the reserve value $\tilde{r}$, and all buyers present receive an object at that price. Similarly, in the dynamic virtual pivot mechanism, each buyer i with $v_i > \tilde{r}$ receives an object, and makes a payment $\tilde{p}^F_{i,t}$ given by

$$\tilde{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = v_i - \widetilde{w}^F_i(\alpha_t, k_t, \mathbf{v}_t),$$

where $\widetilde{w}^F_i$ is the agent's marginal contribution to the virtual surplus.[20] However, since there are sufficient objects present for each agent with a non-negative virtual value to receive one, i does not impose any externalities on the remaining agents; thus,

$$\widetilde{w}^F_i(\alpha_t, k_t, \mathbf{v}_t) = \widetilde{w}_i(\alpha_t, k_t, \mathbf{v}_t) = v_i - \tilde{r},$$

implying that $\tilde{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) = \tilde{r}$. In this case, then, the allocation and payments of the auction mechanism and the dynamic pivot mechanism are the same.

Suppose instead that $k_t < n_t$; that is, there are more agents present than objects. Denote by i_m the bidder with the m-th highest value. Then each agent who receives an object pays the greater

[19] Recall that buyers with values less than $\tilde{r}$ bid up to their true value, as they are never allocated an object, and so their future expected contribution to the virtual surplus is zero.

[20] Note that since i is receiving an object, her total and flow marginal contributions are equal.

of the reserve price $\tilde{r}$ and the price at which buyer i_{k_t+1} drops out of the auction, which is given by

$$\widehat{\beta}^t_{k_t+1,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}) = v_{i_{k_t+1}} - w^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}).$$

If $v_{i_{k_t+1}} < \tilde{r}$, then the situation is identical to the previous case. Therefore, assume that $v_{i_{k_t+1}} \geq \tilde{r}$.

In the dynamic virtual pivot mechanism, on the other hand, each agent i who receives an object pays a price

$$\begin{aligned}
\tilde{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) &= v_i - w^F_i(\alpha_t, k_t, \mathbf{v}_t) \\
&= v_i - \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right] \\
&\quad + \mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{j\in\mathcal{I}\setminus\{i\}} \delta^{s-t}\tilde{x}_{j,s}(\alpha^{-i}_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right] \\
&= v_i - \left(\sum_{m=1}^{k_t}(v_m - \tilde{r}) + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right]\right) \\
&\quad + \left(\sum_{m=1}^{k_t}(v_m - \tilde{r}) + (v_{i_{k_t+1}} - v_i) + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha^{-i,-i_{k_t+1}}_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right]\right).
\end{aligned}$$

Rearranging the above expression allows us to rewrite it as

$$\begin{aligned}
\tilde{p}^F_{i,t}(\alpha_t, k_t, \mathbf{v}_t) &= v_{i_{k_t+1}} - \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right] \\
&\qquad + \mathbb{E}\left[\sum_{s=t+1}^{\infty}\sum_{j\in\mathcal{I}} \delta^{s-t}\tilde{x}_{j,s}(\alpha^{-i,-i_{k_t+1}}_s, k_s, \mathbf{v}_s)(v_j - \tilde{r})\right] \\
&= v_{i_{k_t+1}} - \widetilde{w}^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}),
\end{aligned}$$

where the second equality follows from the fact that $w^{t+1}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}})$ is defined to be the expected future marginal contribution to the virtual surplus of the agent with the (k_t+1)-th highest value, conditional on agents with higher values (which includes i) receiving an object today. Therefore, following the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ leads to period-t prices and allocations identical to those of the dynamic pivot mechanism. Since the period t was arbitrary, as was the state (α_t, k_t), this equivalence holds after each history. Thus, the two mechanisms are outcome equivalent. □

Finally, it remains to be seen that the bidding strategies in Equation (6) do, in fact, form an equilibrium. As in the case of the sequential ascending auction with no reserve, the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ are strictly increasing. Behavior along the equilibrium path is therefore perfectly separating, implying that Bayesian updating fully determines beliefs. In order to determine optimality *off* the equilibrium path, we again suppose that, after a deviation, buyers ignore their past observations and the history of the mechanism, and instead believe that the deviating agent is *currently* truthfully revealing her value in accordance with the bidding strategies $\widetilde{\beta}^t_{m,n_t}$.

CLAIM. *Suppose that in each period, buyers bid according to the cutoff strategies given in Equation (6). This strategy profile, combined with the system of beliefs described above, forms a perfect Bayesian equilibrium of the sequential ascending auction mechanism with reserve price $\tilde{r}$.*

Proof. We prove this claim by making use of the one-shot deviation principle. Consider any period with $n_t := |\mathcal{A}_t|$ buyers on the market and k_t objects present. Suppose that all bidders other than player i are using the conjectured equilibrium strategies. We must show that bidder i has no profitable one-shot deviations from the collection of cutoff points $\{\widetilde{\beta}^t_{m,n_t}\}$. More specifically, we must show that i does not wish to exit the auction earlier than prescribed, nor does she wish to remain active later than specified.

Once again labeling agents such that buyer i_1 has the highest value and buyer i_{n_t} has the lowest, note that if $v_i < \max\{v_{i_{k_t}}, \tilde{r}\}$, bidding according to $\{\widetilde{\beta}^t_{m,n_t}\}$ implies that i does not win an object in the current period. Therefore, exiting earlier than specified does not affect i's current-period returns. Moreover, since the bidding strategies are memoryless, neither future behavior by i's competitors nor i's future payoffs will be affected by an early exit. Suppose, on the other hand, that $v_i > \tilde{r}$ and that i has one of the k_t highest values; that is, that $v_i \geq \max\{v_{i_{k_t}}, \tilde{r}\}$. As established by Proposition 2, i receives an object, paying a price such that her payoff is exactly equal to her marginal contribution to the virtual surplus. Deviating to an early exit, however, leads either to agent i_{k_t+1} winning an object (if $v_{i_{k_t+1}} \geq \tilde{r}$) instead of buyer i, or to an object being discarded. Moreover, i's expected payoff is then $\widetilde{w}^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}})$, which we defined as i's future expected marginal contribution to the virtual surplus. This is a profitable one-shot deviation for i if, and only if,

$$\widetilde{w}^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}) \geq v_i - \widetilde{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}).$$

Rearranging this inequality yields

$$\begin{aligned}
\widetilde{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t+1}}, \ldots, v_{i_{n_t}}) &\geq v_i - \widetilde{w}^{t+1}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}) \\
&= \widetilde{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_i, v_{i_{k_t+2}}, \ldots, v_{i_{n_t}}),
\end{aligned}$$

where the equality comes from the definition of $\widetilde{\beta}^t_{k_t,n_t}$ in Equation (6). Since $v_i > v_{i_{k_t+1}}$, this contradicts the conclusion of the first claim above. Thus, i does not wish to exit the auction early.

Alternately, if $v_i \geq \max\{v_{i_{k_t}}, \tilde{r}\}$, then planning to remain active in the auction *longer* than specified does not change i's payoffs, as i will win an object regardless. If, on the other hand, $v_i < \max\{v_{i_{k_t}}, \tilde{r}\}$, then delaying exit from the period-t auction can affect i's payoffs. Since bids in future periods do not depend on information revealed in the current period, this only occurs if i remains in the auction long enough to win an object. If i wins, she pays a price equal to the larger of $\tilde{r}$ and the exit point of $i_{v_{k_t}}$, whereas if she exits, she receives as her continuation payoff her marginal contribution to the virtual surplus. So, suppose that $i = i_m$ for some $m > k_t$. Then a deviation to remaining active in the auction is profitable if, and only if,

$$v_m - \widetilde{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t}}, \ldots, v_{i_{m-1}}, v_{m+1}, \ldots, v_{i_{n_t}}) \geq \widetilde{w}^{t+1}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}).$$

Rearranging this inequality yields

$$\begin{aligned}\widetilde{\beta}^t_{k_t,n_t}(\alpha_t, k_t, v_{i_{k_t}}, \ldots, v_{i_{m-1}}, v_{m+1}, \ldots, v_{i_{n_t}}) &\leq v_m - \widetilde{w}^{t+1}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}) \\ &= \widetilde{\beta}^t_{m-1,n_t}(\alpha_t, k_t, v_m, \ldots, v_{i_{n_t}}),\end{aligned}$$

where the equality comes from the definition of $\widetilde{\beta}^t_{m-1,n_t}$ in Equation (6). As above, the fact that $v_m < v_{i_{k_t}}$ contradicts the conclusion of the claim above regarding the monotonicity of bids. Therefore, i does not desire to remain active in the auction long enough to receive an object.

Thus, we have shown that no player has any incentive to deviate from the prescribed strategies when on the equilibrium path. In particular, using the bidding strategies $\widetilde{\beta}^t_{m,n_t}$ is sequentially rational given players' beliefs along the equilibrium path. Recall, however, that we have specified off-equilibrium beliefs such that buyers "ignore" their past observations when they observe a deviation from equilibrium play, updating their beliefs to place full probability on the valuation that rationalizes the deviation; they believe that the deviating agent is *currently* being truthful with regards to the strategies $\widetilde{\beta}^t_{m,n_t}$. The argument above then implies that continuing to bid according to the specified strategies remains sequentially rational with respect to these updated beliefs. Thus, bidding according to the cutoffs in Equation (6) is optimal along the entire game tree: this strategy profile forms a perfect Bayesian equilibrium of the sequential ascending auction mechanism. □

Thus, bidding in each period according to the strategy described in Equation (6) forms a perfect Bayesian equilibrium of the sequential ascending auction with reserve price $\tilde{r}$; moreover, this equilibrium is outcome equivalent to the dynamic virtual pivot mechanism. □

APPENDIX B. AN ILLUSTRATIVE EXAMPLE: SUPPORTING RESULTS

In this appendix, we formally state and prove two results relating to the example in Section 3. The first characterizes the social planner's payoff function from following the efficient policy $\widehat{\mathbf{x}}$ in a setting where there is exactly one object available in each period and at most one new entrant arrives in each period (with probability $\rho \in [0,1]$). Before we can prove the result, however, we require the following technical lemma. Denote by $\mathbf{C}(\mathbf{V}^n)$ the set of continuous real-valued functions on $\mathbf{V}^n$. In addition, for any $k < n$, let $\mathbf{C}_k(\mathbf{V}^n) \subseteq \mathbf{C}(\mathbf{V}^n)$ denote the subset of such functions that do *not* depend on their first k arguments. We endow $\mathbf{C}(\mathbf{V}^n)$ with the sup-metric d_∞, where

$$d_\infty(f,g) := \sup\{|f(x) - g(x)| : x \in \mathbf{V}^n\} \text{ for all } f,g \in \mathbf{C}(\mathbf{V}^n).$$

This implies that $\mathbf{C}(\mathbf{V}^n)$ is a complete metric space.

LEMMA B.1 ($\mathbf{C}_k(\mathbf{V}^n)$ is closed)**.**
For any $k \leq n$, $\mathbf{C}_k(\mathbf{V}^n)$ is a closed subset of $\mathbf{C}(\mathbf{V}^n)$.

Proof. Fix any convergent sequence $\{f_m\}_{m=1}^\infty$ in $\mathbf{C}(\mathbf{V}^n)$ such that $f_m \in \mathbf{C}_k(\mathbf{V}^n)$ for all $m \in \mathbb{N}$, and let $f^* \in \mathbf{C}(\mathbf{V}^n)$ denote the (uniform) limit of this sequence. Suppose, however, that $f^* \notin \mathbf{C}_k(\mathbf{V}^n)$. Then there exist $x, y \in \mathbf{V}^n$ such that $x_i = y_i$ for $i = k+1, k+2, \ldots, n$, but $f^*(x) \neq f^*(y)$. Let

$$\epsilon := |f^*(x) - f^*(y)| > 0.$$

Since f_m converges to f^*, there exists $M_x \in \mathbb{N}$ such that $|f_m(x) - f^*(x)| < \frac{\epsilon}{2}$ for all $m > M_x$. Similarly, there exists $M_y \in \mathbb{N}$ such that $|f_m(y) - f^*(y)| < \frac{\epsilon}{2}$ for all $m > M_y$. Therefore, for any $m > \max\{M_x, M_y\}$,

$$\begin{aligned}\epsilon = |f^*(x) - f^*(y)| &\leq |f^*(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f^*(y)| \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

a contradiction. The first inequality above follows from the triangle inequality, and the second is due to the fact that $f_m \in \mathbf{C}_k(\mathbf{V}^n)$ implies $f_m(x) = f_m(y)$. Therefore, we must have $f^*(x) = f^*(y)$; that is, there are no $x, y \in \mathbf{V}^n$ such that x and y agree on their last $n-k$ arguments but $f^*(x) \neq f^*(y)$. Thus, we may conclude that $f^* \in \mathbf{C}_k(\mathbf{V}^n)$. □

Define W_0 to be the expected value to the social planner when there are no buyers present on the market. Then

$$W_0 = \delta\left[(1-\rho)W_0 + \rho\int_0^{\bar{v}} (v' + W_0)\, dF(v')\right] = \frac{\delta\rho}{1-\delta}\int_0^{\bar{v}} v'\, dF(v').$$

We now denote by $W_n(v_1, \ldots, v_n)$ the expected value to the social planner when there are n buyers with values $v_1 > \cdots > v_n$.

LEMMA B.2 (Characterization of W_n)**.**
The planner's expected payoff from implementing the efficient policy $\widehat{\mathbf{x}}$ when there are n buyers present with

values $v_1 > \cdots > v_n$ is given by

$$W_n(v_1, \ldots, v_n) = W_0 + \sum_{j=1}^{n} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv',$$

where $\lambda(v') := \frac{1-\delta(1-F(v'))}{1-\delta\rho(1-F(v'))}$.

Proof. We begin by showing that W_1 has the desired form and then proceed inductively. Note that W_1 is a fixed point of the operator $\widehat{T}_1 : \mathbf{C}(\mathbf{V}) \to \mathbf{C}(\mathbf{V})$ defined by

$$[\widehat{T}(g)](x) := x + \delta \left[(1-\rho) W_0 + \rho \int_0^{\bar{v}} g(y)\, dF(y) \right].$$

This operator is clearly a self-map from $\mathbf{C}(\mathbf{V})$ into itself. Furthermore, it is straightforward to see that $\widehat{T}_1$ is a contraction. Fix any $g, g' \in \mathbf{C}(\mathbf{V})$ such that $g' > g$. Then

$$[\widehat{T}_1(g' - g)](x) = \delta\rho \left[\int_0^{\bar{v}} (g'(y) - g(y))\, dF(y) \right] > 0.$$

Furthermore, for any $g \in \mathbf{C}(\mathbf{V})$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\widehat{T}_1(g + \alpha)](x) = [\widehat{T}_1(g)](x) + \delta\rho\alpha.$$

Since $\delta\rho < 1$, we may apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem.[21] These imply that $\widehat{T}_1$ has a unique fixed point W_1 such that

$$W_1(v_1) = v_1 + \delta \left[(1-\rho) W_0 + \rho \int_0^{\bar{v}} W_1(v')\, dF(v') \right].$$

Differentiating this expression with respect to v_1 yields

$$W_1'(v_1) = 1.$$

Finally, note that $W_1(0) = W_0$, since a buyer with value zero adds nothing to the social welfare. Since the continuity of F implies the continuity of W_1', we may apply the Fundamental Theorem of Calculus, yielding

$$W_1(v_1) = W_0 + v_1 = W_0 + \delta^0 \int_0^{v_1} \lambda^0(v')\, dv'.$$

Now consider $W_n(v_1, \ldots, v_n)$ for arbitrary $n > 1$, and suppose that W_{n-1} takes the desired form.[22] W_n is defined to be a fixed point of the operator $\widehat{T}_n : \mathbf{C}(\mathbf{V}^n) \to \mathbf{C}(\mathbf{V}^n)$ given by

$$\begin{aligned}[\widehat{T}_n(g)](x_1, \ldots, x_n) &:= x_1 + \delta(1-\rho) W_{n-1}(x_{-1}) \\ &\quad + \delta\rho \left[\sum_{j=0}^{n-2} \int_{x_{n-j+1}}^{x_{n-j}} g(x_2, \ldots, x_{n-j}, y, x_{n-j+1}, \ldots, x_n)\, dF(y) + \int_{x_2}^{\bar{v}} g(y, x_{-1})\, dF(y) \right].\end{aligned}$$

Note that for any $g, g' \in \mathbf{C}(\mathbf{V}^n)$, $[\widehat{T}_n(g' - g)](x)$ is given by

$$\delta\rho \left[\sum_{j=0}^{n-2} \int_{x_{n-j+1}}^{x_{n-j}} [g' - g](x_2, \ldots, x_{n-j}, y, x_{n-j+1}, \ldots, x_n)\, dF(y) + \int_{x_2}^{\bar{v}} [g' - g](y, x_{-1})\, dF(y) \right].$$

[21]See Section C.6 of Ok (2007) for precise statements of the results we are applying.

[22]Notice that this implies that all of the cross-derivatives of W_{n-1} are identically zero.

If $g' > g$, then this expression is strictly positive. Furthermore, for any $g \in \mathbf{C}(\mathbf{V}^n)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\widehat{T}_n(g+\alpha)](x) = [\widehat{T}_n(g)](x) + \delta q \alpha.$$

Since $\delta q < 1$, Blackwell's monotonicity and discounting conditions are satisfied. Thus, Blackwell's Contraction Lemma and the Banach Fixed Point Theorem imply that $\widehat{T}_n$ has a unique fixed point W_n such that

$$W_n(v_1, \dots, v_n) = x_1 + \delta(1-\rho)W_{n-1}(v_{-1}) \\ + \delta\rho\left[\sum_{j=0}^{n-2}\int_{v_{n-j+1}}^{v_{n-j}} W_n(v_2, \dots, v_{n-j}, v', v_{n-j+1}, \dots, v_n))\, dF(v') + \int_{v_2}^{\bar{v}} W_n(v', v_{-1})\, dF(v')\right].$$

Differentiating this expression implicitly with respect to v_1 yields

$$W_n^{(1)}(v_1, \dots, v_n) = 1 = \delta^0 \lambda^0(v_1).$$

Note that this expression is independent of n and of v_j for $j \neq 1$, implying that $W_n^{(1,j)}$ is identically zero for all $j \neq 1$.

Similarly, implicit differentiation with respect to v_2 yields

$$W_n^{(2)}(v_1, \dots, v_n) = \delta(1-\rho)W_{n-1}^{(1)}(v_{-1}) \\ + \delta\rho\left[\sum_{j=0}^{n-2}\int_{v_{n-j+1}}^{v_{n-j}} W_n^{(1)}(v_2, \dots, v', \dots, v_n)\, dF(v') + \int_{v_2}^{\bar{v}} W_n^{(2)}(v', v_{-1})\, dF(v')\right].$$

Since $W_n^{(1,j)}$ is identically zero,

$$\sum_{j=0}^{n-2}\int_{v_{n-j+1}}^{v_{n-j}} W_n^{(1)}(v_2, \dots, v', \dots, v_n)\, dF(v') = F(v_2).$$

Furthermore, $W_n^{(2,1)} = W_n^{(1,2)} = 0$ implies that

$$\int_{v_2}^{\bar{v}} W_n^{(2)}(v', v_{-1})\, dF(v') = (1 - F(v_2))W_n^{(2)}(v_1, v_{-1})$$

Thus, making use of the inductive hypothesis that $W_{n-1}^{(1)}(v_{-1}) = 1$, we may conclude that

$$W_n^{(2)}(v_1, \dots, v_n) = \delta(1-\rho) + \delta\rho F(v_2) + \delta\rho(1 - F(v_2))W_n^{(2)}(v_1, \dots, v_n) = \delta\lambda(v_2).$$

Once again, note that this expression is independent of n and of v_j for any $j \neq 2$, implying that $W_n^{(2,j)}$ is identically zero for all $j \neq 2$.

Proceeding inductively, consider the derivative of W_n with respect to its k-th argument, where $k \leq n$. We have

$$W_n^{(k)}(v_1, \dots, v_n) = \delta(1-\rho)W_{n-1}^{(k-1)}(v_{-1}) + \delta\rho\left[\sum_{j=0}^{n-k}\int_{v_{n-j+1}}^{v_{n-j}} W_n^{(k-1)}(v_2, \dots, v', \dots, v_n)\, dF(v') \right. \\ \left. + \sum_{j=n-k+1}^{n-1}\int_{v_{n-j+1}}^{v_{n-j}} W_n^{(k)}(v_2, \dots, v', \dots, v_n)\, dF(v') + \int_{v_2}^{\bar{v}} W_n^{(k)}(v', v_{-1})\right].$$

Applying the same simplifications as above, along with our inductive hypothesis that $W_{n-1}^{(k-1)}(v_{-1}) = \delta^{k-2}\lambda^{k-2}(v_k)$, we have

$$\begin{aligned} W_n^{(k)}(v_1,\dots,v_n) &= \delta(1-\rho)W_{n-1}^{(k-1)}(v_{-1}) + \delta\rho F(v_k)W_{n-1}^{(k-1)}(v_{-1}) + \delta\rho(1-F(v_k))W_n^{(k)}(v_1,\dots,v_n) \\ &= \delta^{k-1}\lambda^{k-1}(v_k). \end{aligned}$$

Finally, note that $W_n(0,\dots,0) = W_0$ since, as with a single buyer with value zero, "null" agents provide no social benefit. By induction on n, we may then conclude that

$$W_n(v_1,\dots,v_n) = W_0 + \sum_{j=1}^{n} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv'$$

for all $n \in \mathbb{N}$ and all $(v_1,\dots,v_n) \in \mathbf{V}^n$. □

The second result we wish to show in this appendix is a characterization of the discounted expectation of buyer 2's future marginal contribution to the social welfare. In order to do so, we make use of the following lemma.

LEMMA B.3.
Suppose that, prior to the potential entry of a new agent, there are n buyers present with values $v_1 > \cdots > v_n$, and that the planner is using the efficient allocation rule $\widehat{\mathbf{x}}$. Then the planner's expected payoff is given by

$$\frac{\rho}{1-\delta}\int_0^{\bar{v}} v'\, dF(v') + \sum_{j=1}^{n}\delta^{j-1}\int_0^{v_j}\lambda^j(v')\, dv'.$$

Proof. Recall that a new entrant arrives with probability $\rho \in [0,1]$, and that there is always a single object available to be allocated. Therefore, applying Lemma B.2, the planner's expected payoff when *entering* a period with n buyers whose values are $v_1 > \cdots > v_n$ is, letting $v_0 := \bar{v}$ and $v_{n+1} := 0$,

$$\begin{aligned} &(1-\rho)W_n(v_1,\dots,v_n) + \rho\sum_{k=0}^{n}\int_{v_{k+1}}^{v_k} W_{n+1}(v_1,\dots,v',\dots,v_n)\, dF(v') \\ &\quad = W_0 + (1-\rho)\sum_{j=1}^{n}\delta^{j-1}\int_0^{v_j}\lambda^{j-1}(z)\, dz \\ &\qquad + \rho\sum_{k=0}^{n}\int_{v_{k+1}}^{v_k}\left(\sum_{j=1}^{k}\delta^{j-1}\int_0^{v_j}\lambda^{j-1}(z)\, dz + \delta^k\int_0^{v'}\lambda^k(z)\, dz + \sum_{j=k+1}^{n}\delta^j\int_0^{v_j}\lambda^j(z)\, dz\right) dF(v') \\ &\quad = W_0 + (1-\rho)\sum_{j=1}^{n}\delta^{j-1}\int_0^{v_j}\lambda^{j-1}(z)\, dz \\ &\qquad + \rho\sum_{j=1}^{n}\left[\delta^{j-1}F(v_j)\int_0^{v_j}\lambda^{j-1}(z)\, dz + \delta^j(1-F(v_j))\int_0^{v_j}\lambda^j(z)\, dz\right] \\ &\qquad + \rho\sum_{k=0}^{n}\left[\delta^k\int_0^{v_k}(F(v_k)-F(z))\lambda^k(z)\, dz - \delta^k\int_0^{v_{k+1}}(F(v_{k+1})-F(z))\lambda^k(z)\, dz\right], \end{aligned}$$

where the final equality follows from interchanging the orders of summation and integration. Combining terms and rewriting this expression yields

$$W_0 + \rho \int_0^{\bar{v}} (1 - F(z))\, dz + \sum_{j=1}^{n} \int_0^{v_j} \Big((1 - \rho(1 - F(z)))\delta^{j-1}\lambda^{j-1}(z) + \rho(1 - F(z))\delta^j \lambda^j(z) \Big)\, dz.$$

Note, however, that

$$\begin{aligned}
(1 - \rho(1 - F(z)))&\delta^{j-1}\lambda^{j-1}(z) + \rho(1 - F(z))\delta^j \lambda^j(z) \\
&= \delta^{j-1}\lambda^{j-1}(z)\left[(1 - \rho(1 - F(z))) + \delta\rho(1 - F(z))\lambda(z)\right] \\
&= \delta^{j-1}\lambda^{j-1}(z)\left[(1 - \delta\rho(1 - F(z))\lambda(z) + \delta\rho(1 - F(z))\lambda(z)\right] = \delta^{j-1}\lambda^j(z).
\end{aligned}$$

Therefore, the planner's expected payoff is given by

$$\frac{\rho}{1-\delta} \int_0^{\bar{v}} v'\, dF(v') + \sum_{j=1}^{n} \delta^{j-1} \int_0^{v_j} \lambda^j(v')\, dv'. \qquad \square$$

With this result in hand, it is a matter of straightforward subtraction in order to determine that the discounted expectation of buyer 2's future marginal contribution to the social welfare is given by

$$\sum_{j=1}^{n} \delta^{j-1} \int_0^{v_j} \lambda^{j-1}(v')\, dv',$$

as i_2 is the highest-ranking buyer *entering* the next period.

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