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Optimal Induced Innovation and Growth with Congestion of a Limited Natural Resource

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Abstract

In a simple Neoclassical Growth Model with endogenous technical change, I expand on the hypothesis of Induced Innovation including a production externality from a fixed input, called 'land', which represents the carrying capacity of the earth's atmosphere. Land is assumed to be congested by the use of labor and capital in production. A market economy where land is free will fail to reach a steady state, and may end up in either of three possible cases: (i) a catastrophe driven by overaccumulation; (ii) a state in which Induced Innovation stops capital deepening but not environmental decline; (iii) a path of perpetual decumulation of capital resembling an industrial counterrevolution. A planned economy, instead, will assign a shadow-price to land, thus setting in motion the Induced Innovation engine and fostering land-augmenting technological progress which will reduce environmental stress. The unique equilibrium if this economy is found to be locally asymptotically stable in the numerical analysis for substitution elasticities smaller than 1. The corresponding direction of technical change is characterized by constant shares of all inputs, a positive growth rate of labor- and land-augmenting technologies, and by a rate of growth of capital-augmentation equal to zero.

Keywords: Induced Innovation, Climate Change, Technological Change, Functional Distribution of Income.

JEL Classification System: O30, Q55.

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1 Introduction

In recent years, the global level of attention on climate change has risen considerably, spreading from the scientific community to policy-making and the general public. The Intergovernmental Panel on Climate Change (IPCC) has published to date 4 Assessment Reports, the most recent of which after the plenary meeting held in November 2007, summarizing the agreement reached by the nations of the world on the key findings and the uncertainties about the issues at stakes. Quoting from the IPCC Synthesis Report 2007:¹

Warming of the climate system is unequivocal, as is now evident from observations of increases in global average air and ocean temperatures, widespread melting of snow and ice and rising global average sea level (p. 30).

There is also a widespread consensus that greenhouse gases (GHGs) are to be ascribed among the causes of global warming: the concentration of CO₂ (carbon dioxide), CH₄ (methane) and N₂O (nitrous oxide) in the atmosphere increased as a result of human activities, mostly because of fossil fuel use and agriculture. GHGs accumulate in the atmosphere for very long time, and concentration of such gases lead to warming of land and oceans, as data on global average surface temperature and sea level display without any doubt for the more recent periods. In the words of the IPCC Synthesis Report,

There is *very high confidence* that the global average net effect of human activities since 1750 has been one of warming (*ibid.*, p. 37).

To complete this brief snapshot of the facts, IPCC's best estimate of the temperature change over the coming century is between 1.8 and 4.0°C. Other than melting of permanent ice packs and consequent sea rise, coastal erosions, floods, climate change has important economic effects, too, as the nations participating to the 1997 Kyoto Conference recognized in agreeing to put forward a system of economic incentives to limit the emissions of GHG. Among such effects we can include: (i) higher crop productivity at mid- to high latitudes for local mean temperature increase, and decrease in crop productivity at tropical regions for the same reason; (ii) increasing exposure of industries, settlements and societies in coastal and river flood plains, areas whose economies are linked with climate-sensitive resources, regions prone to extreme weather events, especially where rapid urbanization is occurring; (iii) effects in health status because of malnutrition, deaths due to extreme weather events, increased frequency of cardio-respiratory diseases due to higher concentrations of ground-level ozone in urban areas related to climate change, and the altered spatial distribution of some infectious diseases. (IPCC [18], p. 48).

Global warming is likely to display its economic consequences in the long-run, due to geophysical time constants such as the half-life of atmospheric carbon dioxide, so that climate change is naturally incorporated in growth models.² On the other hand, it is universally

¹Available for download at <http://www.ipcc.ch/ipccreports/index.htm>.

²See Foley [12], also for a discussion of why the usual time-horizon of growth model is to some extent inappropriate to assess the economic impact of global environmental change.

recognized that technological change the key driving force of economic growth. Stylized facts of post-war capitalist development, such as constancy of input shares and of capital productivity coupled with rising labor productivity, don't support neutrality of technological progress on factor augmentation, pointing instead toward explicit consideration of the bias, or direction of technical change. The direction of technological progress has fundamental implications on the functional distribution of income, as shown by several authors (Kennedy [19], Drandakis and Phelps [11], Nordhaus [28] are early examples. More recently, the interest on the bias of technical change has flourished again: see above all Acemoglu [2], [1]). In a world facing fundamental challenges from global warming, three important questions to address are: i) what can be the role of economic incentives on patterns of technical change directed at reducing GHG emission or environmental stress in general? ii) Will this technological progress be biased? iii) If so, what are the implied patterns of income distribution of such technological change?

If we adopt the standard, old-fashioned Neoclassical view of exogenous technical change, in a closed economy the impact of emission permits or distortionary taxes on technology will be that of textbook-substitution from GHG emitting inputs to other inputs to production according to their relative prices. Conversely, a New Endogenous Growth approach will emphasize the role of profit-appropriation in non-competitive markets for more environmentally friendly, idea-based intermediate products in bolstering the endogenous reduction of GHG emissions from the producers of the final goods, and the role of taxes/subsidies in fostering this process.

A different standpoint to analyze the phenomenon of interest, related the second one, is rooted in Marx's view of capitalist social relations as a powerful source of *induced* technical progress arising from the capitalists' incentive to develop and adopt cost-reducing methods of production. These ideas, also appearing in Hicks [17], have been developed into formal models mostly in the 1960s starting with the thought-provoking contribution by Kennedy [19], followed by a number of articles by some of the most prominent scholars of the past century. Despite some obvious similarities, tracing back to the parallel between Marx's and Schumpeter's ideas on technological progress, a key difference between the Induced Innovation view and the New Growth view of technical change is that income distribution is the ultimate focus of the former whereas the latter has primarily to do with economic efficiency.³ Also, a wave of strong criticism to the Induced Innovation approach, sometimes by its same proponents, determined its decadence after the 1970s, while New Growth Theory still enjoys the favor of the profession. Nevertheless, the Induced Innovation literature has proven to be able to explain in an economically appealing way some key facts of capitalist development, and this, together with the unavoidable degree of arbitrariness in picking models one likes,⁴ is the reason why I choose to follow this approach instead of the more recent one.

This paper develops a model that is Neoclassical in spirit but that incorporates Induced Innovation to provide an answer to the three questions above. The production function of the stylized economy I study, which is a simple extension of the one in Nordhaus [28], features a fixed input of production, which I will call 'land' and represents the carrying capacity of the earth's atmosphere. Consistently with the agreement reached by the nations participating to the IPCC, atmosphere capacity is assumed to be congested by 'human activities', namely by the use of labor and capital in production. The other distinctive

³This is of course a bird-fly distinction, but it will suffice for the purposes of this paper.

⁴As a matter of fact, I'm interested in developing a model of *optimal* growth, and the types of growth analyzed by New Growth theorists are clearly not optimal in the sense of Pareto. This is another justification for my choice.

feature of the model is Kennedy’s [19] ‘Innovation Possibility Frontier’ (IPF henceforth), a function describing the trade-off between different types of factor-augmentation for given growth possibilities of the economy.⁵ The trade-off between factor-augmenting technological changes includes land-augmenting technologies.

Optimal growth pursued by an omniscient, benevolent planning authority requires (shadow) pricing of every input including land, thus setting in motion the Induced Innovation engine of technical change on it. The equilibrium path of technical change is characterized by constant input shares, a positive growth rate of both labor and land augmentation and a zero growth rate of capital-augmenting technical progress. In a market economy where land is free, instead, the induced technical change mechanism is prevented to operate by failure to price the fixed resource. In the unpriced land case, if any, land augmentation will be too small, and the economy will fail to reach its steady state progressively reducing its production possibilities. This process can be characterized by either one of three scenarios, according to the actual shape of the IPF: (i) never-ending capital deepening, which in turn will produce increasing congestion on land; (ii) steady capital accumulation but decreasing land-efficiency; (iii) industrial regress taking place through progressive capital decumulation. Therefore, in all the three contexts failure to price land is a cause of concern for a market economy.

Before moving forward to illustrate the model and its properties, one objection has to be raised and disposed of. The roots of the induced innovation concept point toward considering cost-based technical change a feature peculiar of capitalist, and not of socially planned economies. In other words, if it makes perfect sense to consider profit-seeking capitalists to innovate to economize on market input costs, the extension of this line of reasoning to a social planner appears cumbersome, in what innovation in this context would be induced by social, and not actual market prices. Although in different institutional settings, however, both social and market prices reflect social scarcity of objects with economic relevance, if we adopt a Neoclassical point of view. If one agrees with this sentence, it makes no difference whether or not prices have a monetary content. Induced innovation is therefore the power of social scarcity of inputs to production to generate technical change. It is under this interpretation that the narrative of planned as opposed to market economy is followed in this paper.

2 The Model

2.1 Basic Assumptions and Definitions

In a simple one-sector economy, production of output requires labor, L , capital, K and a natural resource, which we call ‘land’, representing the carrying capacity of the earth’s atmosphere. The production possibilities of this economy are bounded by the following function:

$$Y = h(\theta)F(AL, BK) \tag{1}$$

⁵It must be said that it was Carl Christian von Weiszacker to first develop the IPF, which he called ‘technical progress function’. That work however was never published, and this is why the paper by Kennedy is the one usually referred to in the literature.

where $\theta \equiv T/F(AL, BK)$, T being a parameter which summarizes technical change on land. The total amount of land available in the economy is normalized to 1, and we assume $h' > 0, h'' < 0$. The term $h(\theta)$ in the production function is meant to capture the ‘effects of human activities’ on climate change. The idea is that the way in which capital and labor are utilized for production purposes (described in a stylized way by F) congest the atmosphere capacity. On the other hand, F is the typical linearly homogeneous neoclassical production function, with A, B being positive parameters denoting respectively labor- and capital-augmenting technical change. It is easy to check that the production function (1) displays constant returns to scale in effective inputs. Defining $y \equiv Y/L, x \equiv \frac{BK}{AL}$, the intensive form of (1) is:

$$y = h(\theta)Af(x) = h\left(\frac{\phi}{Af(x)}\right)Af(x) \quad (2)$$

where $\phi \equiv T/L$. Also, if $k \equiv K/L$, we have $x = Bk/A$. The standard regularity (Inada) conditions on f are assumed to be satisfied. Population grows exponentially at the exogenous rate n . There is no public sector: aggregate demand will equal consumption plus investment. Denoting by δ the ‘radioactive’, exogenous depreciation rate, the accumulation equation is $\dot{K} = sY - \delta K$, where s is the propensity to save. From what above, accumulation of capital per worker follows the law of motion:

$$\dot{k} = sh(\theta)Af(x) - (\delta + n)k \quad (3)$$

Productive factors are paid their marginal product. We distinguish the case of a market economy not pricing land from the case of a planned economy in which the land externality is accounted for. If land is not priced, a market economy will consider it as a mere externality. Hence, the land share in output would be zero, and the production share in output will equal 1. The capital share in the market economy is $\frac{f'(x)}{f(x)}x$, and the market labor share is given by $\frac{f(x)-f'(x)x}{f(x)}x \equiv \omega(x)$. Also, the elasticity of substitution between capital and labor in f is defined as:

$$\sigma \equiv -\frac{f'(x)[f(x) - xf'(x)]}{xf(x)f''(x)} \quad (4)$$

Conversely, denote the land share in the planned economy as $\frac{\partial Y/\partial T}{Y}T = \frac{h'(\theta)}{h(\theta)}\theta \equiv \lambda(\theta)$. In this case, the capital share will be $[xf'(x)/f(x)][h(\theta) - \theta h'(\theta)]/h(\theta) = (1 - \omega(x))(1 - \lambda(\theta))$, and the labor share will equal $\omega(x)(1 - \lambda(\theta))$. Hence, the production share in the planned economy will be $1 - \lambda(\theta)$. It is worth observing that what I call ‘land share’ is nothing else than what is traditionally referred to as ‘mitigation’ in recent climate change literature, as for instance in Nordhaus [30]. Symmetrically, we define:

$$\eta \equiv -\frac{h'(\theta)[h(\theta) - \theta h'(\theta)]}{\theta h(\theta)h''(\theta)} \quad (5)$$

Finally, we extend the traditional framework of Induced Innovation by assuming that at each moment in time the growth rates of labor-, land- and capital-augmenting technical change are related by a three-dimensional version of Kennedy’s [1964] IPF. Denoting $\frac{\dot{A}}{A} \equiv \alpha, \frac{\dot{T}}{T} \equiv \tau, \frac{\dot{B}}{B} \equiv \beta, (\alpha, \tau, \beta) \in \Upsilon \subset \mathbb{R}^3$, the IPF written in explicit form is :

$$\beta = g(\alpha, \tau), \quad \text{with } \nabla g < 0, \quad D^2g \text{ negative definite} \quad (6)$$

and we assume, following Drandakis and Phelps [11], that there exist $0 < \bar{\alpha} < \infty, 0 < \bar{\tau} < \infty$ such that $g(\bar{\alpha}, \bar{\tau}) = -\infty$, so that the frontier is allowed to cross the axes and take values

below zero for finite values of its arguments.

The Induced Innovation theorists of the '60s and '70s utilized a functional specification of the kind $\beta = g(\alpha)$, thus imposing zero land-augmenting technical change. The inclusion of τ in the domain of the IPF, together with the form of the production function (1), are the building block of the analysis carried in this paper.

2.2 Optimal Direction of Technical Change

Let us start with the problem of allocating the given growth possibilities of the economy into different factor-augmenting technologies, as in the models by Kennedy [19], Drandakis and Phelps [11], Samuelson [32]. The analysis, which extends the framework by Nordhaus [28] is notationally simpler than that including the choice of optimal rate of technical change, and thus leads to an easier understanding of the implications arising from the assumptions made on technology for the patterns of technical change and income distribution arising when the congestion effects of human activities on atmosphere are optimally accounted for.

2.2.1 The Planner's Problem

Consider an omniscient, benevolent social planner willing to maximize the present discounted value of consumption per capita over an infinite horizon.⁶ The frontier (6) describes the trade-off in allocating the given growth rate of technological progress in different factor-augmenting improvements. The problem faced by the planning authority is:

$$\begin{aligned} \text{Choose } s, \alpha, \tau \quad & \text{to maximize} \quad V(0) = \int_0^\infty e^{-\rho t} \left[(1-s)h(\theta)Af\left(\frac{Bk}{A}\right) \right] dt \\ \text{subject to} \quad & \dot{k} = sh(\theta)Af(x) - (\delta + n)k \\ & \dot{B} = g(\alpha, \tau)B \\ & \dot{A} = \alpha A \\ & \dot{T} = \tau T \end{aligned} \quad (7)$$

The associated Hamiltonian is:

$$\begin{aligned} \mathcal{H} = e^{-\rho t} & \left\{ (1-s)h\left(\frac{\phi}{Af\left(\frac{Bk}{A}\right)}\right) Af\left(\frac{Bk}{A}\right) + p_1 \left[sh\left(\frac{\phi}{Af\left(\frac{Bk}{A}\right)}\right) Af\left(\frac{Bk}{A}\right) - (\delta + n)k \right] \right\} \\ & + e^{-\rho t} \{ p_2 e^{\alpha_{ss}t} g(\alpha, \tau)B + p_3 \alpha A + p_4 \tau T \} \end{aligned} \quad (8)$$

where α_{ss} denotes the steady state value for the growth rate of labor augmentation.⁷ Also, the initial conditions

$$A(0) = A_0, \quad B(0) = B_0, \quad T(0) = T_0, \quad k(0) = k_0 \quad \text{given} \quad (9)$$

⁶Ramsey [31] argued that a planning authority shouldn't discount the streams of consumption in its maximization program. If this is the case, neither the Maximum Principle, which relies on convergence of the integral (7), nor the Dynamic Programming approach, which requires discounting as a sufficient condition, are the correct techniques for solving this problem. Brock [6], McKenzie [24] developed a different method to solve undiscounted growth problems, known as the 'Value-Loss Approach'. See also Khan and Mitra [20] for a geometric interpretation in a multi-sector framework.

⁷Observe that the adjoint variable for B is assumed to be $p_2 e^{\alpha_{ss}t}$, so that the paper compares directly with the analysis in Nordhaus [28].

must be fulfilled, together with non-negativity of the shadow-prices $p_i(t) \geq 0 \forall t, \forall i = 1, \dots, 4$, and the transversality conditions, which we state in terms of the shadow-prices given that our problem has a free-end point:⁸

$$\lim_{t \rightarrow \infty} e^{-\rho t} p_1(t) = \lim_{t \rightarrow \infty} e^{-(\rho - \alpha_{ss})t} p_2(t) = \lim_{t \rightarrow \infty} e^{-\rho t} p_3(t) = \lim_{t \rightarrow \infty} e^{-\rho t} p_4(t) = 0 \quad (10)$$

from which $\rho > \alpha_{ss}$ must hold for any positive value of $p_2(t)$. The first order necessary conditions for an ordinary maximum of the Hamiltonian are:

$$\frac{\partial \mathcal{H}}{\partial s} = (p_1 - 1)Ah(\theta)f(x) = 0 \quad (11)$$

$$\frac{\partial \mathcal{H}}{\partial \alpha} = p_2 e^{\alpha_{ss} t} Bg_\alpha + p_3 A = 0 \quad (12)$$

$$\frac{\partial \mathcal{H}}{\partial \tau} = p_2 e^{\alpha_{ss} t} Bg_\tau + p_4 T = 0 \quad (13)$$

and are also sufficient because of concavity of g with respect of both α, τ and the fact that $\partial^2 \mathcal{H} / \partial s^2 = 0$. Also, recall that even if in principle B is allowed to grow exponentially, we are not maximizing over B but on what makes it grow, and our assumptions on the production function and the IPF are enough to ensure concavity. On the other hand, the necessary conditions for optimality are existence of continuous function $p_i(t), i = 1, \dots, 4$ such that, denoting $\gamma \equiv 1 - s(1 - p_1)$:

$$\rho p_1 - \dot{p}_1 = \gamma B f'(x)[h(\theta) - \theta h'(\theta)] - (\delta + n)p_1 \quad (14)$$

$$(\rho - \alpha_{ss})p_2 - \dot{p}_2 = \gamma e^{-\alpha_{ss} t} k f'(x)[h(\theta) - \theta h'(\theta)] - g(\alpha, \tau)p_2 \quad (15)$$

$$\rho p_3 - \dot{p}_3 = \gamma h(\theta)[h(\theta) - \theta h'(\theta)][f(x) - f'(x)x] + \alpha p_3 \quad (16)$$

$$\rho p_4 - \dot{p}_4 = \gamma \frac{h'(\theta)}{L} + \tau p_4 \quad (17)$$

Since the constraint set is convex, for f, h, g being strictly concave and the other state variables being described by linear functions, the above equations together with the transversality conditions are also sufficient to characterize the optimal path.

2.2.2 Steady State in the Planned Economy

Equations (9)-(17) describe a system of necessary and sufficient conditions for optimality of the program (7) whose long-run solution we are interested in. One way to find such solution, followed by Nordhaus [28], is to note that at a steady state all shadow-prices must be constant. Using the resulting equilibrium values of the adjoint variables, we are able to solve for the long-run quantities we are interested in, that is the effective capital-labor ratio x , the effective land θ , and the growth rates of labor-augmenting and land-augmenting technical change. Setting all shadow-prices constant, and noting that at an equilibrium $p_1 = 1 = \gamma$ from (11), we obtain:

$$Bf'(x) = \frac{(\rho + \delta + n)}{h(\theta) - \theta h'(\theta)} = \frac{(\rho + \delta + n)}{[1 - \lambda(\theta)]h(\theta)} \quad (18)$$

⁸See Sethi and Thompson [33], p.75 for a taxonomy of terminal conditions for a broad class of models.

$$p_2 = \frac{A[h(\theta) - \theta h'(\theta)]x f'(x)}{B[\rho - \alpha_{ss} - g(\alpha, \tau)]} e^{-\alpha_{ss} t} = \frac{A[1 - \lambda(\theta)]h(\theta)[1 - \omega(x)]f(x)}{B[\rho - \alpha_{ss} - g(\alpha, \tau)]} e^{-\alpha_{ss} t} \quad (19)$$

$$p_3 = \frac{[h(\theta) - \theta h'(\theta)][f(x) - f'(x)x]}{\rho - \alpha} = \frac{[1 - \lambda(\theta)]h(\theta)\omega(x)f(x)}{\rho - \alpha} \quad (20)$$

$$p_4 = \frac{h'(\theta)}{(\rho - \tau)L} \quad (21)$$

It is first useful to derive an equation of motion for θ . Logarithmic differentiation of $\frac{\dot{\phi}}{A f(x)}$ yields:

$$\begin{aligned} \frac{\dot{\theta}}{\theta} &= \left(\frac{\dot{\phi}}{\phi} - \frac{\dot{A}}{A} - \frac{x f'(x) \dot{x}}{f(x) x} \right) \\ &= \left\{ (\tau - \alpha - n) - [1 - \omega(x)] \frac{\dot{x}}{x} \right\} \end{aligned} \quad (22)$$

The next step is to find a dynamic equation for x . Because $p_1 = 1$ always along an optimal control, we can differentiate totally with respect to time (18) to derive:

$$\begin{aligned} \dot{B} f'(x) + B f''(x) \dot{x} &= \frac{\rho + \delta + n}{(1 - \lambda)^2 h^2} [\lambda' h - h'(1 - \lambda)] \dot{\theta} \\ &= B f'(x) \left[\frac{\lambda'}{1 - \lambda} - \frac{h'}{h} \right] \frac{\dot{\theta}}{\theta} \\ &= B f'(x) \left[\frac{\eta - 1}{\eta} \frac{\lambda}{\theta} - \frac{\lambda}{\theta} \right] \dot{\theta} \\ &= -B f'(x) \frac{\lambda}{\eta} \frac{\dot{\theta}}{\theta} \end{aligned}$$

Divide both sides by $B f'(x)$, multiply and divide the second addendum in the RHS of the last equation by $\frac{x f(x)}{f(x) - x f'(x)}$, and use (4) to get:

$$\frac{\dot{B}}{B} - \frac{\omega}{\sigma} \frac{\dot{x}}{x} = -\frac{\lambda}{\eta} \frac{\dot{\theta}}{\theta}$$

Substituting (22) and solving for \dot{x}/x , we obtain the law of motion for the effective capital-labor ratio:

$$\frac{\dot{x}}{x} = \frac{\eta \sigma}{\eta \omega(x) + \sigma [1 - \omega(x)] \lambda(\theta)} \left[g(\alpha, \tau) - \frac{\lambda(\theta)}{\eta} (\alpha + n - \tau) \right] \quad (23)$$

where it is understood that η, λ are functions of θ and σ, ω depend on x .

Evaluation of (15) and (16) at constant shadow-prices and substitution in (12) yields the optimal direction of labor-augmenting technical change:

$$-g_\alpha = \frac{\omega(x)}{1 - \omega(x)} \left[\frac{\rho - \alpha_{ss} - g(\alpha_{ss}, \tau_{ss})}{\rho - \alpha_{ss}} \right] \quad (24)$$

Whereas, inserting the equilibrium values (19) and (21) into (13) yields the optimal direction of land augmentation:

$$-g_\tau = \frac{\lambda(\theta)}{1 - \lambda(\theta)} \frac{1}{1 - \omega(x)} \left[\frac{\rho - \alpha_{ss} - g(\alpha_{ss}, \tau_{ss})}{\rho - \tau_{ss}} \right] \quad (25)$$

From which it is apparent that when land is not priced $g_\tau = 0$ as in the model without land.

We are now able to characterize the steady state of this model. Since in steady state

$\dot{x}/x = 0$, $\tau_{ss} = \alpha_{ss} + n$ in (22). Hence, $g(\alpha_{ss}, \tau_{ss}) = 0$. Summarizing, a long-run equilibrium of our system is:

$$\tau_{ss} = \alpha_{ss} + n \quad (26)$$

$$g(\alpha_{ss}, \tau_{ss}) = \beta_{ss} = 0 \quad (27)$$

$$-g_{\alpha,ss} = \frac{\omega(x_{ss})}{1 - \omega(x_{ss})} \quad (28)$$

$$\frac{\lambda(\theta_{ss})}{1 - \lambda(\theta_{ss})} = -g_{\tau,ss}[1 - \omega(x_{ss})] \left(\frac{\rho - \alpha_{ss} - n}{\rho - \alpha_{ss}} \right) \quad (29)$$

where the last equation is solved for the ratio $\lambda/(1 - \lambda)$ for comparative statics purposes, and requires as an additional condition that $\rho > \alpha_{ss} - n = \tau_{ss}$. This equilibrium exists and is unique for $\sigma \neq 1 \cap \eta \neq 1$, paralleling what shown by the cited authors. In fact, when either of the substitution elasticities equals one, the innovation possibility frontier is not able to pin down the ratio of factor shares, for it is only the form of the (Cobb-Douglas, in this case) production function determining the factor distribution of income. When $\sigma \neq 1$, instead, $\omega(x) \in [0, 1]$ and $\omega(x)/(1 - \omega(x)) \in [0, \infty)$. Similarly, for $\eta \neq 1$, $\lambda(\theta) \in [0, 1]$ and $\lambda(\theta)/(1 - \lambda(\theta)) \in [0, \infty)$. Observe also that, since (26) is sufficient to determine the equilibrium rate of land augmentation given (28), the role of (29) is to pin down the equilibrium value of θ .

The optimal direction of land-augmenting technical progress resulting from this model compares interestingly to that in Foley [12]. In his paper, he expands on Kennedy's result assuming that the growth rate of land-augmenting technologies is a function of the land share only. If we increase the dimensionality of IPF, instead, we see that (i) land augmentation depends negatively on the capital share and thus positively on the labor share, and (ii) the direction of land augmentation relates positively on both the rate of labor-augmenting technical progress and on population growth rate. Hence, being the direction of land-augmenting technical change derived in an optimizing framework, it will display important feedback effects from the other endogenous and exogenous variables of the model, which were ruled out by assumption in the previous treatments of the subject.

We can also compute the optimal long-run savings rate for the planned economy. Since $\frac{\dot{x}}{x} = \left(\frac{\dot{B}}{B} + \frac{\dot{k}}{k} - \frac{\dot{A}}{A} \right) = 0$, we have:

$$g(\alpha_{ss}, \tau_{ss}) + \frac{sBh(\theta)f(x_{ss})}{x_{ss}} - (\delta + n) - \alpha_{ss} = 0$$

so that, using (18):

$$s_{ss} = [1 - \omega(x_{ss})][1 - \lambda(\theta_{ss})] \left(\frac{\alpha_{ss} + \delta + n}{\rho + \delta + n} \right) \quad (30)$$

The optimal savings rate is always less than 1, for $\rho > \alpha_{ss}$ from the transversality conditions. The savings rate of the benchmark model with no land is easily obtained setting $\lambda = 0$.

2.3 The Dynamical System

The solution approach we adopted above amounts to find a long-run equilibrium of the system as values for the variables of interest that ensure constant shadow-prices of all the

state variables. This method has the disadvantage of being silent of what happens out of equilibrium.⁹

The dimensionality of the problem at hand may look too high to enable us to study the dynamical system arising from the maximization program solved by the social planner. Nonetheless, a closer look at the sufficient conditions for a maximum of (8) reveals that we can exploit (11) on the one hand, and make use of the IPF and of its relative shadow-price on the other, to end up in a fully determined 4-dimensional system in two state variables, θ, x , and two control variables, α, τ . In order to do so, we proceed as it is usually done in standard courses on growth theory in ‘eliminating’ the adjoint variables of the Hamiltonian from the picture so that we can focus on the behavior of the control variables and state variables only.¹⁰

Let us start in standard fashion by totally differentiating (12) with respect to time:

$$\dot{p}_2 e^{\alpha_{ss} t} B g_\alpha + \alpha p_2 e^{\alpha_{ss} t} B g_\alpha + p_2 e^{\alpha_{ss} t} \dot{B} g_\alpha + p_2 e^{\alpha_{ss} t} B g_{\alpha\alpha} \dot{\alpha} = -\dot{p}_3 A - p_3 \dot{A}$$

Using (12) and (15), we have, rearranging:

$$e^{\alpha_{ss} t} B g_\alpha \left[(\rho - \alpha_{ss}) p_2 - \frac{A}{B} e^{-\alpha_{ss} t} (1 - \omega(x)) f(x) (1 - \lambda(\theta) h(\theta)) \right] + p_2 e^{\alpha_{ss} t} B g_{\alpha\alpha} \dot{\alpha} = -\dot{p}_3 A$$

Making use of (16), and then of (12) again, we obtain:

$$\begin{aligned} e^{\alpha_{ss} t} B g_\alpha \left[(\rho - \alpha_{ss}) p_2 - \frac{A}{B} e^{-\alpha_{ss} t} [1 - \omega(x)] f(x) [1 - \lambda(\theta)] h(\theta) \right] + p_2 e^{\alpha_{ss} t} B g_{\alpha\alpha} \dot{\alpha} \\ = A(1 - \lambda(\theta) h(\theta) \omega(x) f(x) + (\rho - \alpha) p_2 e^{\alpha_{ss} t} B g_\alpha \end{aligned}$$

We can now harmlessly substitute (19) in the previous equation. Simplifying, we obtain:

$$\frac{1}{\rho - \alpha_{ss} - g(\alpha, \tau)} g_{\alpha\alpha} \dot{\alpha} [1 - \omega(x)] = \omega(x) + \frac{\rho - \alpha}{\rho - \alpha_{ss} - g(\alpha, \tau)} g_\alpha [1 - \omega(x)]$$

from which, finally:

$$\dot{\alpha} = \frac{1}{g_{\alpha\alpha}} \left\{ g_\alpha (\rho - \alpha) + \frac{\omega(x)}{1 - \omega(x)} [\rho - \alpha_{ss} - g(\alpha, \tau)] \right\} \quad (31)$$

Similar calculations lead to:

$$\dot{\tau} = \frac{1}{g_{\tau\tau}} \left\{ g_\tau (\rho - \tau) + \frac{\lambda(\theta)}{1 - \lambda(\theta)} \frac{1}{1 - \omega(x)} [\rho - \alpha_{ss} - g(\alpha, \tau)] \right\} \quad (32)$$

It is obvious that equations (31) and (32) alone have the same equilibrium values as (24) and (25), and yield the same long-run equilibrium we found above when considered together with (22) and (23).

Summing up, we have derived a dynamical system formed by (22), (23), (31) and (32). This system can be studied in the standard way, as we will do in the following section.

⁹This issue may ring a bell to economists following the Classical-Marxian school of thought, too. We found an equilibrium of the system at constant long-run prices. Long-run prices give us a complete description of the ‘center of gravitation’ of the system itself. However, from long-run prices we cannot infer law of motions for all the variables of interest, and therefore we are not able to study the ‘gravitation’ process for a system starting out of the long-run equilibrium. Nordhaus [28] was aware of this problem, in recognizing that ‘We have not shown that in the general case [that is, for other initial conditions] the optimal path is to go to the Harrod equilibrium’ (p.61), the ‘Harrod Equilibrium’ in this paper being equations (18), (27), (28), (30).

¹⁰See Barro and Sala-i-Martin [5], for instance, for an example of this method in the simple Neoclassical Growth Model case.

2.4 Stability Analysis

I now study the behavior of the dynamical system above in a neighborhood of its steady state. I will consider the following three cases: (i) planned economy without land; (ii) market economy with unpriced land, and (iii) planned economy with land. Although the first case is not immediately relevant for the purposes of this paper, it is of interest in itself because Nordhaus [28], who first studied the model without land, did not provide an analysis of the solution paths outside the equilibrium. This special case is also interesting because it is simple enough to be studied analytically.

2.4.1 Planned Economy without Land Constraint on Production

Suppose for a moment to be a growth theorist not concerned with the environment, wishing to study the simplest no-externality scenario arising in the special case where $h(\theta) = 1$, $\dot{\theta}/\theta = 0$ always. A reason for studying such special case can be a strong prior on atmosphere carrying capacity as a non-scarce factor, or simply the lack of conclusive evidence on environmental stress available in the early stages of research on economic growth. The dynamics of our system take place in the two-dimensional plane (x, α) . The nontrivial rest point of this economy is the Harrod equilibrium $-g_\alpha = \frac{\omega(x_{ss})}{1-\omega(x_{ss})}$, $g(\alpha_{ss}) = 0$ which ensures constancy of the effective capital-labor ratio. The Jacobian matrix evaluated at the steady state is:

$$J_{Nordhaus,ss} = \begin{pmatrix} 0 & -\frac{\sigma(x_{ss})}{1-\omega(x_{ss})}x_{ss} \\ \frac{1}{g_{\alpha\alpha}} \left(\frac{1-\sigma}{\sigma}\right) \left[\frac{\omega(x_{ss})}{(1-\omega(x_{ss}))x_{ss}}\right] & \rho - \alpha_{ss} + \frac{1}{g_{\alpha\alpha}} \left(\frac{\omega(x_{ss})}{1-\omega(x_{ss})}\right)^2 \end{pmatrix}$$

The determinant is finite and negative if and only if $\sigma \in (0, 1)$ and positive if $\sigma > 1$, for $g_{\alpha\alpha} < 0$. Thus, if $0 < \sigma < 1$ the two eigenvalues are of opposite sign, and the Harrod equilibrium is saddle-path stable. Conversely, when $\sigma > 1$, we need to look at the trace of the matrix, too. A necessary and sufficient condition for the trace to be negative is:

$$-g_{\alpha\alpha} > \frac{1}{\rho - \alpha_{ss}} \left(\frac{\omega(x_{ss})}{1 - \omega(x_{ss})}\right)^2 \quad (33)$$

If this is the case, under $\sigma > 1$ both eigenvalues are of equal sign and sum up to a negative number: the long-run equilibrium is stable. Conversely, if the inequality has the wrong sign, both eigenvalues are positive and the system is unstable. Therefore, the term in the RHS of the inequality (33) acts as a bifurcation parameter of the two-dimensional version of our system, provided that labor and capital are gross-substitutes in the production function.¹¹ The inequality (33) says that, even when $\sigma > 1$, the system may be stable or unstable according to how concave is Kennedy's IPF. It is important to point out, however, that there is very little empirical evidence supporting a substitution elasticity between capital and labor greater than 1. I will briefly survey this evidence in describing the calibration and simulation exercise I carry in the general four-dimensional case. In light of such evidence,

¹¹In defining productive inputs as gross-substitutes if the elasticity of substitution is greater than 1, i follow Acemoglu [1].

the analysis of the case $\sigma > 1$ is included only for completeness.

These new findings on the model pair interestingly with the important result that, if $\sigma < 1$, an economy that remains in the Harrod equilibrium maximizes the preference functional (8), whereas the Harrod equilibrium is not the optimal path if $\sigma > 1$.¹² Thus, our simple economy either has an optimal saddle-path stable equilibrium as in the typical Neoclassical Growth Model, or a potentially unstable equilibrium that, however, does not maximize the preference functional (8), and it is not grounded in the available evidence on capital-labor substitution.¹³ Of course, when $\sigma = 1$ an equilibrium with a role for the IPF doesn't exist in this model, so that it is needless to study its properties.

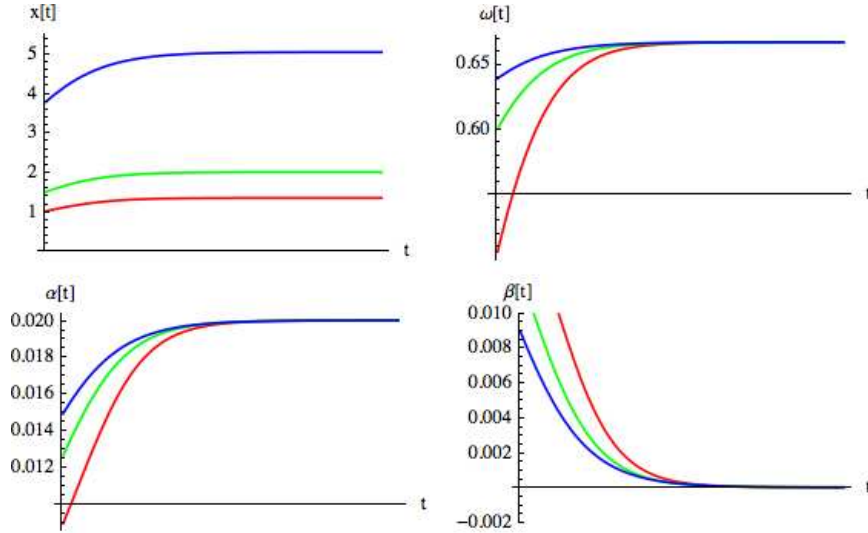


Figure 1: Simulation results for the planned economy without land over 200 periods. Elasticity of substitution: $\sigma = \{.3 \text{ (red)}, .5 \text{ (green)}, .7 \text{ (blue)}\}$.

Figure 1 displays the results of a simulation round over 200 periods of the model without land under the following calibration: $g(\alpha) = q - \frac{a}{\nu}\alpha^\nu$, $q = .02$, $\nu = 2$, and a being calibrated internally so as to solve $g(\alpha) = 0$. The discount rate is set exogenously at $\rho = .05$, equal to the depreciation rate, the population growth rate $n = .02$ and the elasticity of substitution equals $1/2$. The endogenous variable x is set as to solve $\omega(x_{ss}) = 2/3$, which is roughly the observed labor share in total output in advanced capitalist economies.

Given that we are dealing with a two-point boundary value problem with a saddle-path stable equilibrium, in order to compute the solution I specified an initial condition for the state variable x and a terminal condition for the control variable α , and then used a shooting algorithm to force the system back to its stable manifold. Standard duality arguments guarantee that the terminal condition on α is equivalent to a terminal condition on p_2 . As expected, the eigenvalues of the Jacobian matrix, evaluated numerically at several

¹²See Nordhaus [28] for a proof.

¹³Drandakis and Phelps [11] studied a dynamical system without land in the plane $(K, 1 - \omega)$, and found that the system is stable or unstable according to σ being less than or greater than one respectively. The same result holds true in the Classical scenario without a land constraint studied as a special case in Foley [12].

points along the optimal trajectories, are of opposite sign.¹⁴

2.4.2 Unpriced Land in the Market Economy

Consider a market economy (whose relative variables are denoted by the subscript M) where land is not priced. In this case, $-g_{\alpha M} = \omega/(1 - \omega)$, but $-g_{\tau M} = 0$. Our assumptions on the IPF imply that

$$-g_{\tau}^{-1}(0) < -g_{\tau}^{-1} \left[\frac{\lambda(\theta_{ss})}{1 - \lambda(\theta_{ss})} \frac{1}{1 - \omega(x_{ss})} \left(\frac{\rho - \alpha_{ss}}{\rho - \alpha_{ss} - n} \right) \right]$$

Hence, $g_{\tau}^{-1}(0) = \tau_M < \alpha_{ss} + n$, so that $g(\alpha, \tau_M) > 0$. Depending on the actual shape of the IPF, and provided that $\sigma, \eta > 0$, three cases may arise, but a steady state is never reached in any of them. Hence, there is no need for the study of the Jacobian matrix. The three possible scenarios are:

1. $\dot{x}/x > 0$ (*Foley's Catastrophe*). From (23), the effective capital-labor ratio fails to reach its steady state value, and grows forever at a strictly positive rate, as long as $\sigma, \eta > 0$ (recall that although land is not priced the elasticity of output with respect to land is never zero in this model). Failure to price land completely overthrows the ability of an elasticity of substitution smaller than one to hold back capital deepening, contrary to the typical feature of the early models of induced innovation without externalities.¹⁵ As a consequence, $\dot{\theta}/\theta$ doesn't reach its steady state either and keeps growing at a negative rate, so that land becomes increasingly congested reducing the production possibilities of the economy. Output will inevitably tend to zero, due to the destructive interplay of diminishing productivity of x in f and increasingly negative impact of θ on h (recall that $h'' < 0$).

Hence, the unpriced land scenario in this model is as hopeless as the correspondent case in the Classical model by Foley [12]. The same forces of endlessly increasing capital deepening that determine an always rising effective capital-labor ratio are at work here, and their effect is enhanced by the congestion externality on land. As a result, the market capital share rises depressing the labor share. But the higher the capital share, the worse the impact of a rising x on output per worker, as it is easily seen by differentiating y with respect to x when land congestion is taken into account:

$$\frac{\partial y}{\partial x} = Ah(\theta)f'(x) - Ah'(\theta)(1 - \omega)f(x)$$

If the effective capital-effective labor ratio keeps growing without reaching its steady state value, the inability of a market economy to price land triggers all forces driving technical progress to work in the wrong direction.

2. $\dot{x}/x = 0$ (*Environmental Decline with Steady Capital Accumulation*). The effective capital-labor ratio is constant, but θ fails to reach a steady state, shrinking forever

¹⁴The software used for simulations is *Mathematica* 6, and the code is available from the author upon request.

¹⁵This is just another way of saying that an equilibrium with induced innovation involves a constant capital-labor ratio, and this condition will not be met in this case.

toward zero. Induced technical change succeeds in holding back capital deepening, but land augmentation is not enough to overcome the congestion effect of production on land. The eventual state of zero production, however, is reached at a slower pace than in the previous case, given that there is no overaccumulation.

3. $\dot{x}/x < 0$ (*Industrial Counterrevolution - The Luddite's Dream*). The forces at work in case 1 are reversed: capital decumulates forever, and land becomes less and less congested without any need of land augmentation.¹⁶ At lower levels of output, the effective capital-labor ratio becomes more productive, but decreasing returns to effective land are also at work in lowering the impact of diminishing factor intensity on production. Capital accumulation, and output as a consequence, keeps tending to to but never reaches zero because of the Inada condition $f'(0) = \infty$ which rules out the option of not undertaking production on the basis of economic convenience.

We conclude that failure to price land is never harmless on a market economy. The prevailing scenario will eventually depend on the actual functional form of the IPF, but in any case it is hard not to be concerned about the harmful effects of unpriced land on patterns of capitalist economic growth. In particular, case 2 must not be underestimated, in what it leads to a state of the world similar to case 1 without sharing with the latter its evident catastrophic signs.

2.4.3 Planned Economy with Land: Numerical Analysis

In this section, I analyze the stability properties of the full model numerically. From a calibration standpoint, the moments of the endogenous variables to match are: i) a long-run growth rate of labor productivity roughly equal to 2%; ii) a roughly constant capital productivity; iii) a labor share of about 2/3 of output. Parameters exogenously given to the model will be the discount rate ρ , the depreciation rate δ , and the population growth rate n . The depreciation rate is assumed to be 5% per period, and the population growth rate to equal 2% per period, both in standard fashion in the macro literature. As for the discount rate, we need to assume ρ to be big enough to ensure a positive and finite share of land in output. In particular, the calibration of the discount rate must satisfy $\rho > \alpha_{ss} + n$. Hence, for these simulation rounds, I set $\rho = .05$.

The next step is to assume a functional form and calibrate the IPF. In what above, I only required the function to be decreasing and concave in both α, τ . The easiest specification one can think of is an exponential one:

$$g(\alpha, \beta) = q - \frac{a}{\nu}(\alpha + \beta)^\nu$$

Under a quadratic exponent ν , the parameters a, q can be easily calibrated internally. In fact, observation of (28) reveals that the LHS of the equation doesn't depend on the intercept q . Evaluating the equation at $(\alpha_{ss}, \tau_{ss}) = (.02, .04)$ and solving for a yields $a = 33.333$. Using this value, we can then set q so as to solve $g(\alpha_{ss}, \tau_{ss}) = 0$, which returns $q = .06$.

¹⁶An issue not considered here is that, if idle machines cannot be destroyed completely, disposal of capital in the environment may be a cause of land congestion, or more generally an environmental threat too, as are for instance idle nuclear plants.

Finally, we need to assign values to the substitution elasticities σ, η . The vast literature on estimation of the elasticity of substitution between capital and labor is far from conclusive, but most of the articles on the subject suggest that the aggregate elasticity of substitution is significantly less than 1.¹⁷ Nadiri [26], Nerlove [27] and Hamermesh [16] survey a range of early estimates of the elasticity of substitution, which are generally between 0.3 and 0.7. David and Van de Klundert [7] provide an estimate of σ to be in the neighborhood of 0.3. Using the translog production function, Griffin and Gregory [14] estimate elasticities of substitution for nine OECD economies between 0.06 and 0.52. Berndt [4], on the other hand, finds an estimate of the elasticity of substitution equal to 1, but does not control for a time trend, creating a strong bias towards 1. Using more recent data, and various different specifications, Krusell, Ohanian, Rios-Rull, and Violante [15] and Antras [3] also find estimates of the elasticity significantly less than 1. Estimates implied by the response of investment to the user cost of capital also typically yield an elasticity of substitution between capital and labor significantly less than 1 (see, e.g., Chirinko [8], Chirinko, Fazzari and Mayer [9] and [10], or Mairesse, Hall and Mulkay [22]).

Assigning a value of .5 to the elasticity of substitution σ seems a fairly appropriate average of the estimation results surveyed above. On the other hand, an estimate of the substitution elasticity between effective land and f is not only not available but also in principle problematic to obtain. To break ties, I assume $\eta = \sigma$, and as a robustness exercise I evaluate solutions for these elasticities equal to 0.3, 0.5, 0.7, which imply steady a sequence of steady-state values $\{(x_{ss,i}, \theta_{ss,i})\}_{i=1}^3$ equal to $\{(1.35, 1.9), (2, 4.5), (5.04, 33.43)\}$ respectively. The simulation round for the full model depicted in Figure 2 is obtained by taking a discrete-time approximation of the system.¹⁸

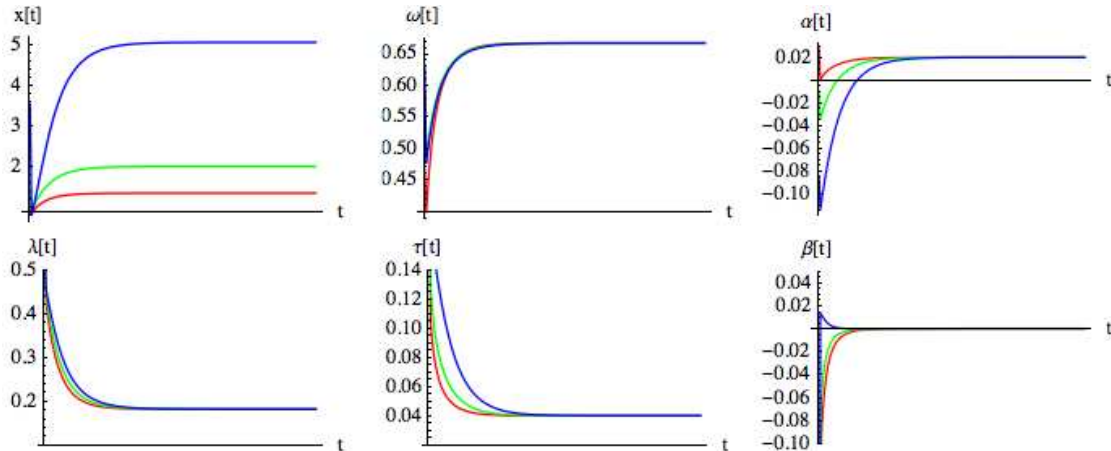


Figure 2: Simulation results for the full model: $\rho = .05$, $\delta = .05$, $n = .02$. Elasticities of substitution: $\sigma = \eta = \{0.3 \text{ (red)}, 0.5 \text{ (green)}, 0.7 \text{ (blue)}\}$.

The Jacobian matrix, evaluated numerically for the discretized system, has four distinct

¹⁷The following survey closely follows Acemoglu [2].

¹⁸As it is clear from the dynamical system, this approximation is harmless. The system is solved in *Mathematica* using the function ‘FindRoot’. Due to the two point-boundary nature of the problem, initial conditions are given for the state variables x, θ and terminal conditions for the variables α, τ . Standard duality arguments guarantee that a terminal condition on the control variables is equivalent to a terminal condition on the adjoint variables. On the other hand, as a consistency check on the optimality of the solutions, I computed equilibrium paths assigning different values to the terminal t , up to 2000 periods, and find that the solution for the first 200 periods is not sensitive to the choice of the terminal horizon.

eigenvalues inside the unit circle, and therefore the equilibrium is locally stable. The simulation round displayed in Figure 2 show that the model converges fairly quickly to its balanced growth path. Given our calibration, labor augmentation settles onto a 2% growth rate, land augmentation grows at 4%, and the labor share converges to a long-run value of 2/3 of F . The land share stabilizes around 18% of total output, which is fairly high compared to the results on mitigation appearing, for instance, in the DICE model by Nordhaus [30]. The reason of this discrepancy has to be found in the requirement on the discount rate. Typically, models of climate change assume exogenous technical change, and for simulation purposes calibrate the discount rate very low compared to macroeconomic growth models. Here, the discount rate has to be set fairly high for consistency reasons: one the one hand, the use of the Maximum Principle to solve the planner's problem requires $\rho > \alpha_{ss}$. On the other hand, we want the land share/mitigation to be positive and finite, which in turns imposes $\rho > \alpha_{ss} + n$.

2.5 Balanced Growth Path

The growth rate of the planned economy at a balanced growth path will be given by:

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{h'(\theta)}{h(\theta)}\theta \left(\frac{\dot{\phi}}{\phi} - \frac{\dot{A}}{A} - \frac{xf'(x)\dot{x}}{f(x)x} \right) + \frac{\dot{A}}{A} + \frac{xf'(x)\dot{x}}{f(x)x} \\ &= \lambda(\theta_{ss})(\tau_{ss} - n - \alpha_{ss}) + \alpha_{ss} \\ &= \alpha_{ss} \end{aligned} \tag{34}$$

On the other hand, a planned economy with no externality from land grows at the warranted rate $\alpha_{ss} + n$. Finally, a market economy in the unpriced land case will grow at the (unbalanced) rate:

$$\frac{y\dot{M}}{y} = \lambda \left(\frac{\eta + (1 - \lambda)}{\eta} \right) (\tau - \alpha - n) + \alpha + \frac{\eta\sigma}{\eta\omega(x) + \sigma[1 - \omega(x)]\lambda(\theta)} (1 - \omega)(1 - \lambda)g(\alpha, \tau)$$

As it is intuitive, there is a one-to-one correspondence between the three scenarios discussed above and $\frac{y\dot{M}}{y} \begin{cases} \geq \\ < \end{cases} \alpha_{ss}$.

2.6 Comparative Dynamics

In analyzing the dynamical properties of our solution path, we found that the long-run equilibrium of the system is unique and locally saddle-path stable. It is then of interest to study the effect of changes in the exogenous variables of the model on its steady state. Consider first an increase in the population growth rate. It is clear from (26) that the long-run growth rate of land-augmenting technical change must increase. On the other hand, differentiating the right-hand side of (29) with respect to n we can see that the ratio $\lambda/(1 - \lambda)$, and therefore the land share, will fall, as it is intuitive given that there is full employment and the whole population growth will be absorbed in production. Also, the savings rate responds positively, and in the same way, to both n and δ , given $\rho - \alpha_{ss} > 0$.

An increase in the discount rate ρ will reduce the optimal savings rate given the higher

degree of time-impatience, as it is standard in growth theory. What is perhaps surprising is that a higher discount rate determines a higher share of land in total output. In fact,

$$\frac{\partial \lambda / (1 - \lambda)}{\partial \rho} = -g_{\tau,ss} \frac{(1 - \omega)n}{(\rho - \alpha_{ss})^2} > 0$$

This result is, however, much less counterintuitive than it seems. A look at (17) reveals that when the discount rate increases, the shadow-price of land augmentation increases. The central plan office compensates for a higher time-impatience of its citizens by increasing technical change directed at reducing congestion on land. To put it differently, since an amount in ρ determines an increase in present consumption, the social planner has to compensate for the higher consumption through an increase in mitigation on land. Figure 3 plots the solution paths for $\rho = .1$. The increase in time-impatience determines a dramatic increase in the equilibrium land share. All the other parameters are calibrated as above.

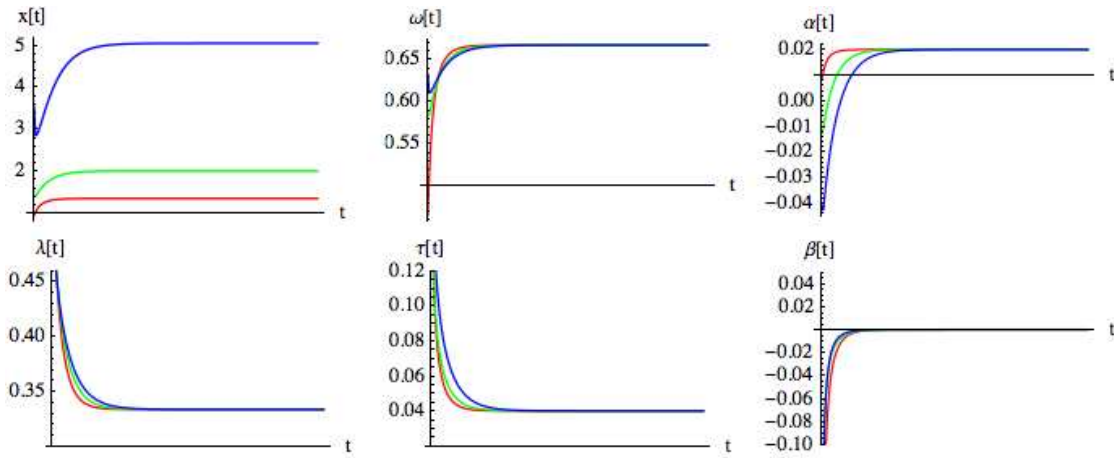


Figure 3: Simulation results for $\rho = .1$, all the other parameters as above.

On the other hand, an increase in population growth requires recalibration of the IPF, and this is due to the fact that $\beta = 0$ must hold in equilibrium. In Figure 4, I increase population growth to .4, and consequently $\tau_{ss} = .6$, and reassign values to a, q so as to ensure $g(.2, .6) = 0$. The plot for $\sigma = \eta = .7$ is pathological, but still converges eventually to a steady state.

2.7 Optimal Rate of Technical Change

In the above setup, we studied the determination of the optimal direction of technological change. For the framework to become an endogenous growth model we need to include also the planner's choice of intensity, or rate, of technical change. The conclusions about the long-run equilibrium of the system reached in the previous section are not sensitive to the inclusion of the optimal rate of technical change in the planner's problem.

Consider a social planner determining both the direction and the rate of technological change by allocating part of the labor force into the educational sector, as in Uzawa [36]. If we follow the Neoclassical tradition in imposing that labor is fully employed, the amount of labor in the educational sector will be a fraction of the total labor force: $L_e = (1 - u)L$.

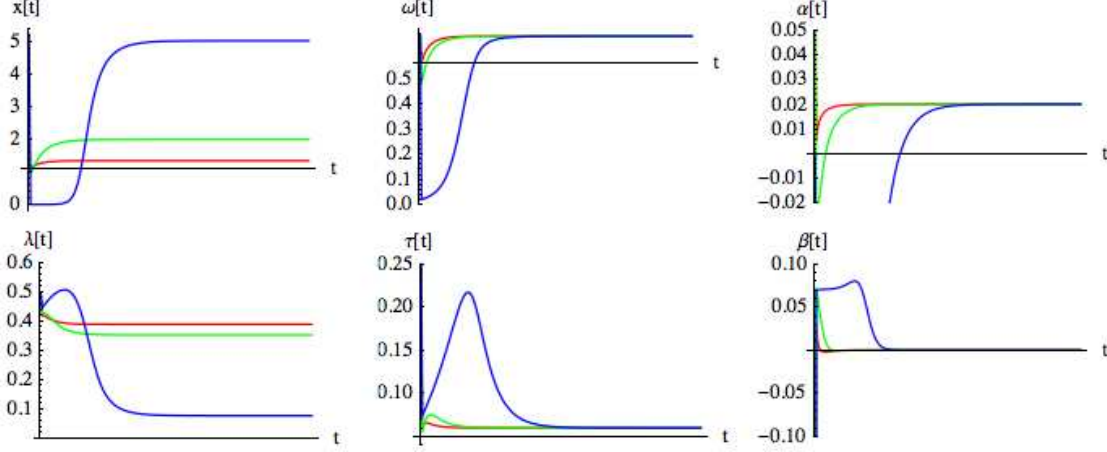


Figure 4: Simulation results for $n = .04$, $a = 25$, $q = .08$

However, workers in L_e can be employed in any of the factor-augmenting technologies. To keep things simple, I make the assumption that land augmentation is subject to the same technology as other factor-augmentations, and that the planner chooses the portion ν of workers L_e employed on production of land-augmenting technologies. Therefore, we can rewrite equations (6) as:

$$\dot{A} = \alpha\xi(1 - \nu u)A, \quad \dot{T} = \tau\xi[1 - u(1 - \nu)]T, \quad \dot{B} = g(\alpha, \tau)\xi(1 - \nu u)B \quad (35)$$

where g is exactly as above. Following Uzawa [36], we assume that the function ξ is concave enough to ensure that the present discounted value of consumption per capita converges as $t \rightarrow \infty$, that is we assume: $\xi(1) < \rho < \xi(0) + \xi'(0)$, and $\xi' > 0, \xi'' < 0$. The planning problem is to choose s, α, τ, u, ν to maximize (7) under the constraints (3), (35), (9) and (10). The Hamiltonian of the problem is:

$$\mathcal{H} = e^{-\rho t} \left\{ (1 - s)Ah \left(\frac{\phi}{Au f(x/u)} \right) u f \left(\frac{x}{u} \right) + p_1 \left[sAh \left(\frac{\phi}{Au f(x/u)} \right) u f \left(\frac{x}{u} \right) - (\delta + n) \frac{A}{B} x \right] \right\} \\ + e^{-\rho t} \left\{ p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t} g(\alpha, \tau)\phi(1 - \nu u)B + p_3 \alpha \phi(1 - \nu u)A + p_4 \tau \phi[(1 - u(1 - \nu)]T \right\} \quad (36)$$

where the conjugate variable for B is now $p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t}$ because of the specification of the technology for production of factor-augmentation. The first-order conditions for an ordinary maximum of (36) are:

$$(p_1 - 1)Ah(\theta)u f \left(\frac{x}{u} \right) = 0 \quad (37)$$

$$\frac{\partial \mathcal{H}}{\partial \alpha} = p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t} g_\alpha \xi(1 - \nu u)B + p_3 \xi(1 - \nu u)A = 0 \quad (38)$$

$$\frac{\partial \mathcal{H}}{\partial \tau} = p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t} g_\tau \xi(1 - \nu u)B + p_4 \xi[1 - u(1 - \nu)]T = 0 \quad (39)$$

$$\frac{\partial \mathcal{H}}{\partial u} = \gamma A [h(\theta) - \theta h'(\theta)] \left[f \left(\frac{x}{u} \right) - \frac{x}{u} f' \left(\frac{x}{u} \right) \right] \\ - \xi' \left\{ \nu \left[p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t} g(\alpha, \tau)B + p_3 \alpha A \right] + (1 - \nu)p_4 \tau T \right\} = 0 \quad (40)$$

$$\frac{\partial \mathcal{H}}{\partial \nu} = p_4 \xi' u \tau T - \left[p_2 e^{\alpha_{ss}\phi(1-\nu_{ss}u_{ss})t} g(\alpha, \tau)B + p_3 \alpha A \right] \xi' u = 0 \quad (41)$$

Equations (14)-(17) modify as follows:

$$\dot{p}_1 = (\rho + \delta + n)p_1 - \gamma [h(\theta) - \theta h'(\theta)] f' \left(\frac{x}{u} \right) B \quad (42)$$

$$\begin{aligned} \dot{p}_2 &= [\rho - \alpha_{ss}\xi(1 - \nu_{ss}u_{ss}) - g(\alpha, \tau)\xi(1 - \nu u)] p_2 \\ &\quad - [h(\theta) - \theta h'(\theta)] \gamma f' \left(\frac{x}{u} \right) e^{-\alpha_{ss}\xi(1 - \nu_{ss}u_{ss})t} \frac{A}{B} x \end{aligned} \quad (43)$$

$$\dot{p}_3 = [\rho - \alpha\xi(1 - \nu u)] p_3 - \gamma [h(\theta) - \theta h'(\theta)] u \left[f \left(\frac{x}{u} \right) - \frac{x}{u} f' \left(\frac{x}{u} \right) \right] \quad (44)$$

$$\dot{p}_4 = \{ \rho - \tau\xi[1 - u(1 - \nu)] \} p_4 - \gamma \frac{h'(\theta)}{L} u f \left(\frac{x}{u} \right) \quad (45)$$

while equations (18) and (28)-(30) become:

$$B_{ss} f' \left(\frac{x_{ss}}{u_{ss}} \right) = \frac{\rho + \delta + n}{[1 - \lambda(\theta)]h(\theta)} \quad (46)$$

$$g(\alpha_{ss}, \tau_{ss}) = \psi(\theta_{ss}) \{ \tau_{ss}\xi[1 - u_{ss}(1 - \nu_{ss}) - n - \alpha_{ss}\xi(1 - \nu_{ss}u_{ss})] \} = 0 \quad (47)$$

$$-g_\alpha = \frac{\omega(x)}{1 - \omega(x)} \quad (48)$$

$$-g_\tau = \frac{\lambda(\theta_{ss})}{1 - \lambda(\theta_{ss})} \frac{1}{1 - \omega(x_{ss})} \left[\frac{\rho - \alpha_{ss}\xi(1 - \nu_{ss}u_{ss})}{\rho - \tau_{ss}\xi[1 - u_{ss}(1 - \nu_{ss})]} \right] \frac{\xi[1 - u_{ss}(1 - \nu_{ss})]}{\xi(1 - \nu_{ss}u_{ss})} \quad (49)$$

$$s_{ss} = [1 - \omega(x_{ss})][1 - \lambda(\theta_{ss})] \left[\frac{\alpha_{ss}\xi(1 - \nu_{ss}u_{ss}) + \delta + n}{\rho + \delta + n} \right] \quad (50)$$

The difference with the equilibrium conditions in the previous section being only the appearance of the intensity values multiplying the factor-augmentation rates. Also, the equation determining the optimal innovation intensity is, from (40):

$$\xi'(u_{ss}) = \frac{\omega(x)}{\nu \left[\frac{(1 - \omega(x))\lambda(\theta)(n - \tau_{ss})}{\rho - [\alpha_{ss} + \lambda(\theta)(n - \tau_{ss})]\xi(1 - \nu_{ss}u_{ss})} + \frac{\omega(x)\alpha_{ss}\xi(1 - \nu_{ss}u_{ss})}{\rho - \alpha_{ss}\xi(1 - \nu_{ss}u_{ss})} \right] + (1 - \nu_{ss}) \frac{\lambda(\theta)\tau_{ss}\xi[1 - u_{ss}(1 - \nu_{ss})]}{\rho - \tau_{ss}\xi[1 - u_{ss}(1 - \nu_{ss})]}} \quad (51)$$

and from (41) we have:

$$\frac{(1 - \omega(x))\lambda(\theta)(n - \tau_{ss})}{\rho - [\alpha_{ss} + \lambda(\theta)(n - \tau_{ss})]\xi(1 - \nu_{ss}u_{ss})} + \frac{\alpha_{ss}\omega(x)}{\rho - \alpha_{ss}\xi(1 - \nu_{ss}u_{ss})} = \frac{\lambda(\theta)\tau_{ss}}{\rho - \tau_{ss}\xi[1 - u_{ss}(1 - \nu_{ss})]} \quad (52)$$

The long-run equilibrium of the model including the choice of intensity of technical change is exactly the same as the simpler case of choice of direction only, and involves positive land and labor augmentation, zero capital augmentation, and constant shares of all inputs. The conclusions we reached above extend to the market economy where land is not priced. The same properties of the resulting dynamical system can be derived by assuming u, ν to be constant over time at their equilibrium values.

3 Discussion

Economists have been dealing with environmental issues for a long time. Sophisticated models for environmental policy evaluation have been developed over the past three decades, a

recent example being the DICE-2007 studied in [30]. In such model, the issue of direction of technical change in presence of externalities from the atmosphere capacity is not addressed, as technical change is assumed to be Hicks-neutral. Hicks-neutrality of technical change is a natural outcome of models of Induced Innovation when there is no accumulating factor,¹⁹ but it is at odd with a basic stylized fact of capitalist development: increasing labor productivity coupled with rising capital/labor ratio. The present model produces a Harrod-neutral path of technological progress as a special case in the planned economy without land, as the standard model of Induced Innovation with capital accumulation. Also, zero capital augmentation is one of the steady state outcomes in the planned economy with land.

The viewpoint I took in this paper is that induced technical change can be a powerful source of economic growth, and that the early contributions on Induced Innovation have proved to be successful in reproducing some of the long-run features of capitalist economies. Hence, I chose to adopt a modification of a fairly old model of optimal technical change with Induced Innovation, first studied by Nordhaus [28]. The model presented here can be seen as a Neoclassical counterpart of the Classical framework developed recently by Foley [12], and shares with the latter the basic idea: assigning a (shadow-) price to land will induce cost-reducing technical change directed at economizing the use of land in production, thus reducing environmental stresses. Differently from Nordhaus, I introduced an externality arising from a fixed natural resource, affecting production of output. Unlike Foley's model, in which the dependence of each factor-augmenting technical change on its own share in costs is assumed, I augmented the dimensionality of Kennedy's [19] IPF to include land-augmenting technical change. Such feature of the present framework creates important equilibrium feedbacks among different kinds of factor-augmentation, including land, that were ruled out by assumption in the previous analyses of the subject. Another difference is that Foley's model is closed by a Goodwin predator-prey cycle, whereas full employment in the planned economy is imposed here. This feature of the model calls for relaxing the assumption of a fully-employed labor force in order to address the possible presence of trade-offs or trade-ins between labor market institutions and environmental policies. Finally, the scenario depicted by Foley in the unpriced land case is only one of the possible cases arising in the present model. The reason behind this different result lays in the different assumption on technology I made in this paper, which explicitly considers land congestion through human activities in agreement with the consensus reached by the nations participated in the IPCC.

The Induced Innovation approach has been sharply criticized for its lack of microfoundations, especially in [29], and this weakness, among others, was responsible for the decline of growth models based on induced technical change. At the firm level, in fact, it is not clear how innovation can be financed and priced if there are constant returns to scale and competition. The problem of reconciling economic growth with competition, however, is common to the whole early growth literature, and involves removing the assumption of constant returns to scale in the production function. Twenty years of literature on Endogenous Growth have addressed this issue, spanning from AK frameworks, to models of human capital accumulation, to R&D-based growth models. All these strands of literature feature increasing returns. Recent models with decreasing returns to scale reconciled growth and competition, using the fact that firms in regime of decreasing returns have inframarginal rents to finance R&D expenditure.²⁰ In a planned economy, the complication of financing

¹⁹See Samuelson [32] for an enlightening illustration of this point.

²⁰A very recent example of a model with decreasing returns and induced innovation, also providing detailed references to the literature, is Zamparelli [38]. The price to pay for this reconciliation between growth and

innovation is not compelling, as long as the social planner is able to choose the of the labor force working in the production of new technologies, as it happens in Uzawa [36], and in the present model with a minor modification due to the increased dimensionality of the IPF. However, because of the underlying assumption of competition, a meaningful comparison between market and planned economies in this model can be made only with an exogenous rate of technical progress as in section 2.2, and this is the reason why I didn't extend the comparisons to the endogenous growth framework of section 2.7.

Another matter of criticism is the assumed stationarity of the trade-off between factor augmentations represented by the IPF. Magat [23], and Skott [35] explored the implications of depletion of innovation possibilities on the dynamics of factor shares in a model based on that of Drandakis and Phelps [11]. Depletion of innovation possibilities can be seen as a way to capture environmental decline occurring at a faster pace than what is implied by the aggregate production function 1, as it seems to be the case in the very recent years. This extension is left for future research. On the other hand, in the market counterpart of the present model based on imperfect competition, there is no need for an explicit specification of the IPF (see Acemoglu [1]), because each factor-augmenting technical progress will be determined by its own technology. 'Innovation possibility frontiers' will be implicitly determined by the relative slopes of such technologies, and their slope will be time-varying.

The most important assumption in this paper is that human activities, in the form of use of capital and labor in production, congest a limited resource, which I called 'land'. In their controversial 1972 book commissioned by the Club of Rome, Meadows *et al.* [25] explored the consequences of the interaction between capitalist production, exponential population growth and a limited resource utilized in production such as oil. Their predictions were contradicted by history, because of technical progress in oil extraction which made more oil available for human activities. Several interesting lessons can be learned from this story. First, I believe oil to be a striking example of the power of induced innovation as a source of economic growth. Fuel-efficiency was not a problem when oil was cheap on the market. Increases in oil price are more and more fostering induced technical change to make more efficient the use of such resource, and are also setting in motion the forces of substitution between productive inputs, in what other sources of energy are being explored to reduce the world's dependence on petroleum. Land pricing is likely to produce the same kind of mechanisms of induced technical progress we have seen happening with fossil fuels, and this mechanism will cope with progressive substitution from land-congesting to more land-friendly inputs. Second, world population in 1972 was 3,860 billion. Today is over 6,600 billion. Given such spectacular growth, it makes sense to explore its implications for technical change under the type of limits to production possibilities postulated in this paper. We found that land-augmenting technical progress must grow at a rate equal to the sum of the growth rate of labor augmentation plus the rate of population growth to ensure existence of a steady state to the economy. An immediate extension of this model is to explore its properties with endogenous population growth, and it is left for future research.

A key implication of the congestion hypothesis in the unpriced market case is that labor and capital will each appropriate a portion part of the land's contribution to the productive process, so that the market will remunerate factors more than it is socially optimal. As a consequence, if labor is fully employed in the planned economy, it cannot be in the market economy not pricing land. This poses problems additional to the ones already outlined in the comparisons of the two economies regarding their innovative ability. On the other

competition, however, is that the optimal dimension of the firm tends to zero.

hand, it points toward extensions of the framework to open economies in order to study the interaction between environmental policies and movements of labor and capital across countries, with the difficulty that such interaction will occur when the country not pricing land is out of the steady state path.

A final methodological remark can be made considering that the whole argument of this paper lays on the implicitly assumed ability of an economist to aggregate across different inputs to production into categories such as ‘labor’, ‘capital’, and ‘land’. As pointed out by Samuelson [32], from a mathematical standpoint there is no difference between the *arguments* producing output according to the technology (1). Here, and usually in models of economic growth, a meaningful distinction between labor, capital, and land can be made only regarding their respective law of motions: ‘capital’ accumulates, ‘labor’ is scarce, ‘land’ is fixed.²¹ The peculiar aspects of the use of labor in production, such as the contractual features of labor relation as opposed to those of ‘land’ and ‘capital’, are not addressed in this paper, as it is assumed that each factor is paid its marginal product. Tavani [34] analyzes at a micro level the differences in enforceability of contracts for different factors in a two-input production model, showing that they are in general relevant for the direction of technical change in market economies, even when productive inputs are not distinguishable as far as their motion over time is considered. Different ways of enforcing the contracts regulating the use of inputs in production must be a feature of a more realistic model of capitalist technological progress.

4 Conclusion

In this paper, I extended an early model of induced technical change first studied by Nordhaus [28]. The extension amounts to include a production externality from a fixed resource (land) congested by the use of labor and capital in production, and to increase the dimensionality of Kennedy’s [19] IPF by adding land-augmenting technical change. In standard fashion, I was also able to study both analytically and numerically the dynamical system that results out of the infinite horizon problem (7), this way providing an account of the equilibrium and out-of-equilibrium dynamics of the model, and addressing long-standing questions remained unanswered in the framework. I showed that:

1. The planned equilibrium without land is either saddle-path stable if the elasticity of substitution between labor and capital is less than one, or stable if the substitution elasticity is greater than one *and* the IPF is sufficiently steep at the steady state, a precise meaning of the adjective ‘steep’ being given by the fulfillment of the inequality (33).
2. A market economy not pricing land always fails to reach a steady state, and may end

²¹The present model has another feature that is perhaps interesting for economists familiar with the early Induced Innovation literature. It is not hard to show that, if we assume all shadow-prices to be constant over time at their steady state values, the long-run optimal solution involves an equally positive growth rate of all factor-augmenting technologies, and thus equal shares of all factors in output. This result has been called ‘Kindleberger Paradox’ by Samuelson [32], and it is typical of models of Induced Innovation when there is no accumulating factor. Here, it arises because constant shadow prices will not reflect the aforementioned differences between productive inputs.

up in either one of three worrying scenarios: (i) a catastrophe led by overaccumulation of capital: (ii) a slower environmental decline where capital deepening is held back by Induced technical change but land congestion is not; (iii) a path of industrial regress where capital progressively decumulates.

3. A planned economy with land has an equilibrium, unique when $\sigma, \eta \neq 1$, in which the shares of all inputs are constant, the rates of labor and land augmentation are positive, and the rate of capital-augmenting technical change is zero. I showed that, under the calibration proposed, the equilibrium path is locally asymptotically stable.

These findings lose some of their importance if we consider that a proper account of endogenous growth in a market economy requires removing the hypothesis of competition that underlies the comparisons made in this paper. Influential models of directed technical change in non-competitive markets have been developed recently by Acemoglu [1], [2], although environmental externalities are absent in those frameworks. A different type of exercise might be to study the role of land congestion in the natural market counterpart of the growth model analyzed in section 2.7, that of human capital due to Lucas [21]. I also don't address intergenerational equity issues (see for instance Greiner and Semmler [13]), nor the role played by uncertainty in climate change (Weitzman [37]). All the above directions in which the Induced Innovation framework can be extended to include externalities from the atmosphere capacity appear to be fruitful areas for further investigation.

References

- [1] Acemoglu, Daron [2002]. 'Directed Technical Change'. *Review of Economic Studies*, Vol. 69: 781-810.
- [2] Acemoglu, Daron [2003]. 'Labor and capital-augmenting Technical Change'. *Journal of the European Economic Association*, Vol. 1: 1-37.
- [3] Antras, Pol, [2001]. 'Is the U.S. Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution'. Mimeo, MIT.
- [4] Berndt, Ernst, [1976]. 'Reconciling Alternative Estimates of the Elasticity of Substitution' *Review of Economics and Statistics* 58, No. 1, pp. 59-68.
- [5] Barro, R. and Xavier Sala-i-Martin [2005]. *Economic Growth*, 2nd Edition. Cambridge, MA: The MIT Press.
- [6] Brock, William, A. [1970] 'On Existence of Weakly Maximal Programmes in a Multi-Sector Economy'. *Review of Economic Studies*, Vol. 37: 275-280.
- [7] David, Paul, and Th. Van de Klundert, [1965]. 'Biased Efficiency Growth and Capital-Labor Substitution in the U.S., 1899-1960,' *American Economic Review*, 55, 357-393.
- [8] Chirinko, Robert S., [1993]. 'Business Fixed Investment: a Critical Survey of Modeling Strategies, Empirical Results and Policy Implications'. *Journal of Economic Literature*, XXXI, 1875-1911.

- [9] Chirinko, Robert S., Steven M. Fezari and Andrew P. Mayer, [1999]. ‘How Responsive Is Business Capital Formation to Its User Cost?’ *Journal of Public Economics*, 75, 53-80.
- [10] Chirinko, Robert S., Steven M. Fezari and Andrew P. Mayer, [2001] ‘That Elusive Elasticity: A Long-Panel Approach to Estimating the Price Sensitivity of Business Capital’. Federal Reserve Bank of St. Louis, working paper.
- [11] Drandakis, E.M, and Edmund Phelps [1965]. ‘A Model of Induced Invention, Growth and Distribution’. *The Economic Journal*, Vol. 76, No. 304: 823-840.
- [12] Foley, Duncan K. [2003]. ‘Endogenous Technical Change with Externalities in a Classical Growth Model’. *Journal of Economic Behavior and Organization*, Vol. 52: 167-189.
- [13] Greiner, Alfred, and Willi Semmler [2008]. *The Global Environment, Natural Resources, and Economic Growth*. Oxford, UK: Oxford University Press.
- [14] Griffin, James M. and Paul R. Gregory, [1976] ‘An Intercountry Translog Model of Energy Substitution Responses’, *American Economic Review* 66, 845-857.
- [15] Krusell, Per; Lee Ohanian and Victor Rios-Rull and Giovanni Violante, ‘Capital Skill Complementary and Inequality’, *Econometrica*, 68 No. 5 (2000), 1029-1053.
- [16] Hamermesh, David S. [1993]. *Labor Demand*. Princeton, NJ: Princeton University Press.
- [17] Hicks, John [1932]. *The Theory of Wages*. London, UK. McMillan.
- [18] Intergovernmental Panel on Climate Change [2007]. *Climate Change 2007: Synthesis Report*. Available for download at <http://www.ipcc.ch/ipccreports/index.htm>.
- [19] Kennedy, Charles [1964] ‘Induced Bias in Innovation and the Theory of Distribution. *Economic Journal*, Vol. 74: 541-47.
- [20] Khan, M. Ali, and Tapan Mitra [2005]
- [21] Lucas, Robert E., Jr. [1988] ‘On the Mechanics of Economic Development’. *Journal of Monetary Economics*, Vol. 22: 3-42.
- [22] Mairesse, Jacques, Bronwyn H. Hall and Benoit Mulkey, [1999]. ‘Firm-Level Investment in France and the United States: An Exploration over What We Have Returned in Twenty Years’ *Annales d’Economie et Statistiques* 55, 27-67.
- [23] Magat, Wesley [1979] ‘Technological Change with Depletion of Innovation Possibilities: Implications for the Dynamics of Factor Shares’. *Economic Journal*, 89: 614-623.
- [24] McKenzie, Lionel [1976] ‘Turnpike Theory’. *Econometrica*, Vol. 44, No. 5: 841-865.
- [25] Meadows, D.H, Dennis L. Meadows, Jorgen Randers and William W. Behrens III [1972]. *The Limits to Growth ...*
- [26] Nadiri, M. I. [1970]. ‘Some Approaches to Theory and Measurement of Total Factor Productivity: a Survey’ *Journal of Economic Literature*, VIII 8, pp. 1117-77.

- [27] Nerlove, Mark, [1967] ‘Recent Empirical Studies of the CES and Related Production Functions’ in M. Brown (editor) *The Theory and Empirical Analysis of Production*, New York
- [28] Nordhaus, William [1967]. ‘The Optimal Rate and Direction of Technical Change’, in Shell, Karl, ed. *Essays on the Theory of Optimal Economic Growth*. Cambridge, MA: The M.I.T. Press.
- [29] Nordhaus, William [1973] ‘Some Skeptical Thoughts on the Theory of Induced Innovation’. *Quarterly Journal of Economics*, Vol. 87, No. 2: 208-219.
- [30] Nordhaus, William [2007] *The Challenge of Global Warming: Economic Models and Environmental Policy*, mimeo: Yale University.
- [31] Ramsey, F. P. ‘A Mathematical Theory of Savings’, *Economic Journal*, Vol. 38 (1928), 543-559.
- [32] Samuelson, Paul [1965]: ‘A Theory of Induced Invention along Kennedy-Weizsacker Lines’. *Review of Economics and Statistics*, Vol. 47 n. 4: 343-356.
- [33] Sethi, Suresh P., and Gerald L. Thompson [2000]. *Optimal Control Theory*. New York, NY. Springer.
- [34] Tavani, Daniele [2008]: ‘The Role of Technology in Factor-Discipline and the Bias of Technical Change: a Microeconomic Analysis’, mimeo: The New School for Social Research.
- [35] Skott, Peter [1981]. ‘Technological advance with depletion of innovation possibilities: A comment and some extensions’. *Economic Journal*, 91: 977-87.
- [36] Uzawa, Hirofumi [1965] ‘Optimum Technical Change in an Aggregative Model of Economic Growth’. *International Economic Review*, Vol. 6: 18-31.
- [37] Weitzman, Martin [2008]: ‘On Modeling and Interpreting the Economics of Catastrophic Climate Change’, mimeo: Harvard University.
- [38] Zamparelli, Luca [2007]: ‘Direction and Intensity of Technical Change: a Micro Model’, mimeo: The New School for Social Research.