# A Simple Hypothesis Test for Heteroscedasticity

Venier, Guido

1 November 2008

# A Simple Hypothesis Test for Heteroscedasticity

Guido Venier
Financial Risk Controller
guido-venier@freenet.de
http://www.myspace.com/guidovenierresearch

***Abstract:*** *The scope of this paper is the presentation of a simple hypothesis test that enables to discern heteroscedastic data from homoscedastic i.i.d. gaussian white noise. The main feature will be a test statistic that's easy applicable and serves well in committing such a test. The power of the statistic will be underlined by examples where it is applied to stock market data and time series from deterministic diffusion a chaotic time series process. It will turn out that in those cases the statistic rejects with a high degree of confidence the random walk hypothesis and is therefore highly reliable. Furthermore it will be discussed, that the test in most cases also may serve as a test for independence and heteroscedasticity in general. This will be exemplified by independent and equally distributed random numbers.*

## 1. Introduction

The history of this paper is basically, that it was originally part of [1] and was used to show that DAX and Model time series obey similar degrees of heteroscedasticity. However for the sake of economy it was left out of [1] since hetero-scedasticity could also be easily demonstrated graphically. Nevertheless the used test procedure still has got a charm with respect to its simplicity and should therefore be presented in the following. It also contributes another measure in addition to the approaches shown in [2] and can be used complementary.

## 2. Heteroscedasticity

Heteroscedasticity is a common feature observed in certain time series e.g. financial time series like interest rates or stock returns. It happens to occur when a lot of large changes follow abruptly a series of moderate changes.

*Definition 4.2.1(Heteroscedasticity)*
Define the m-sample variance estimator at sample point k of a sample of N realizations of a variable $x_1,x_2.....,x_N$ as:

$$\hat{\sigma}^2_{m,k} = \frac{1}{m-1}\sum_{i=k}^{m+k}\left(x_i - \hat{\mu}_{m,k}\right)^2$$

$$\text{where } \hat{\mu}_{m,k} = \frac{1}{m}\sum_{i=k}^{m+k} x_i$$

$$\text{with } 1<k<N \text{ and } n+k<N\ \forall\ k$$

is defined as the m-sample mean estimator at sample point k. And define the sample variance estimator by:

$$\hat{\sigma}^2_{N} = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \hat{\mu}_{N}\right)^2$$

$$\text{where } \hat{\mu}_{N} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

is defined as the sample mean estimator. In this context heteroscedasticity means that for N samples there exists significant many values k so that the m-sample estimators of non overlapping sample buckets differ significantly from the sample variance.

To have a clear discern between homoscedastcity and heteroscedasticity of course one needs a hypothesis test. By generating a test statistic, it can be defined what it means that there exist significantly many sample estimators that are distinct from the sample variance.

## 3. The Test Statistic

The hypothesis of heteroscedasticity will be tested against the Null Hypothesis of a homoscedastic Gaussian random process. To test for heteroscedasticity in time bucket τ the following test statistic is suggested:

$$\mathrm{T}(m,\alpha) = \frac{m\sum_{k=1}^{\mathrm{int}(N/m)} I_{k,m}(\alpha)}{N}$$

Where $I_k$ is an indicator function for the non overlapping buckets k = 1,2,3,…N/m indicating:

$$I_{k,m}(\alpha)\left\{\begin{array}{l} =1 \quad if \quad \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} < \dfrac{1}{m-1}\mathrm{X}_{\alpha/2}(m) \\ or \quad if \quad \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} > \dfrac{1}{m-1}\mathrm{X}_{1-\alpha/2}(m) \\ =0 \quad if \quad \dfrac{1}{m-1}\mathrm{X}_{\alpha/2}(m) \\ < \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} < \dfrac{1}{m-1}\mathrm{X}_{1-\alpha/2}(m) \end{array}\right\}$$

Where $X_\alpha(m)$ is the $\alpha$-Quantile of the $\chi^2$ distribution with m degrees of freedom.

The test is motivated by the fact that one can show under the assumption of a homoscedastic random process that:

$$\frac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} \cong \frac{1}{1-m}\chi^2(m)$$

where $\cong$ means is distributed as. I.e. under the assumption of a homoscedastcity gaussian random process the ratio of the sample variance and a m-sample variance should behave like a $\chi^2$ distributed random variable with m degrees of freedom multiplied by 1/(m-1).

Thus the Indicator function always indicates on the test level $\alpha$ if the m-sample variance differs significantly from its expected value at either the upside or the downside that should be measured best by the sample variance. Finally we conclude that:

$$\mathrm{T} \cong \frac{1}{n}B(n,p) : n = \mathrm{int}(N/m),\ p = \alpha$$

I.e. T follows a binominal distribution multiplied by 1/n with parameters n=int(N/m), which is the number of time buckets yielded by the choice of time bucket length m, and $\alpha$ is the chosen test level. To commit a statistical test regarding T, one has to check whether the observed quantity of as significant indicated variance changes exceeds a certain 1-$\beta$-quantile of the binominal distribution multiplied by 1/N or not:

$$T > \frac{1}{n}\mathrm{B}_{\beta-1}\left(\mathrm{int}(N/m),\alpha\right) \textit{ reject null}$$
$$T < \frac{1}{n}\mathrm{B}_{\beta-1}\left(\mathrm{int}(N/m),\alpha\right) \textit{ not reject null}$$

The value $\beta$ will then be the level of confidence of the test.

# 4. Examples

Table 1 and 2 show the results of the test statistic T for the DAX and Model time series of [1]. Test parameters chosen where $\alpha$=$\beta$= 0.01. The null hypothesis of a homoscedastic i.i.d Gaussian random process is rejected for every time bucket significantly at a confidence level of one percent.

| TimeBuckets | Lower-T | Upper-T | Sum-T | P-Value | Reject |
|---|---|---|---|---|---|
| 50 | 10,00% | 75,00% | 85,00% | 2,50% | TRUE |
| 60 | 15,15% | 75,76% | 90,91% | 3,03% | TRUE |
| 70 | 14,29% | 82,14% | 96,43% | 3,57% | TRUE |
| 80 | 12,00% | 72,00% | 84,00% | 4,00% | TRUE |
| 90 | 18,18% | 72,73% | 90,91% | 4,55% | TRUE |
| 100 | 15,00% | 80,00% | 95,00% | 5,00% | TRUE |
| 110 | 16,67% | 77,78% | 94,44% | 5,56% | TRUE |
| 120 | 25,00% | 75,00% | 100,00% | 6,25% | TRUE |
| 130 | 20,00% | 73,33% | 93,33% | 6,67% | TRUE |
| 140 | 21,43% | 78,57% | 100,00% | 7,14% | TRUE |
| 150 | 23,08% | 69,23% | 92,31% | 7,69% | TRUE |
| 160 | 16,67% | 66,67% | 83,33% | 8,33% | TRUE |
| 170 | 18,18% | 63,64% | 81,82% | 9,09% | TRUE |
| 180 | 27,27% | 63,64% | 90,91% | 9,09% | TRUE |
| 190 | 20,00% | 70,00% | 90,00% | 10,00% | TRUE |
| 200 | 20,00% | 70,00% | 90,00% | 10,00% | TRUE |
| 220 | 22,22% | 66,67% | 88,89% | 11,11% | TRUE |
| 250 | 25,00% | 62,50% | 87,50% | 12,50% | TRUE |

Table 1

| TimeBuckets | Lower-T | Upper-T | Sum-T | P-Value | Reject |
|---|---|---|---|---|---|
| 50 | 17,31% | 48,08% | 65,38% | 1,92% | TRUE |
| 60 | 16,28% | 44,19% | 60,47% | 2,33% | TRUE |
| 70 | 18,92% | 45,95% | 64,86% | 2,70% | TRUE |
| 80 | 18,75% | 50,00% | 68,75% | 3,13% | TRUE |
| 90 | 21,43% | 46,43% | 67,86% | 3,57% | TRUE |
| 100 | 23,08% | 50,00% | 73,08% | 3,85% | TRUE |
| 110 | 17,39% | 47,83% | 65,22% | 4,35% | TRUE |
| 120 | 23,81% | 47,62% | 71,43% | 4,76% | TRUE |
| 130 | 25,00% | 55,00% | 80,00% | 5,00% | TRUE |
| 140 | 22,22% | 50,00% | 72,22% | 5,56% | TRUE |
| 150 | 23,53% | 47,06% | 70,59% | 5,88% | TRUE |
| 160 | 25,00% | 50,00% | 75,00% | 6,25% | TRUE |
| 170 | 26,67% | 53,33% | 80,00% | 6,67% | TRUE |
| 180 | 28,57% | 50,00% | 78,57% | 7,14% | TRUE |
| 190 | 15,38% | 38,46% | 53,85% | 7,69% | TRUE |
| 200 | 30,77% | 46,15% | 76,92% | 7,69% | TRUE |
| 220 | 27,27% | 45,45% | 72,73% | 9,09% | TRUE |
| 250 | 30,00% | 50,00% | 80,00% | 10,00% | TRUE |

Table 2

**Table 1, 2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points for the model time series and DAX time series respectively. Upper-T, Lower-T percentage of samples for which the m-sample bucket variance differs significantly from its expected value indicated by I at either the upside or the downside respectively. Sum-T total percentage of m-sample bucket variance being significantly different from its expected value. P-Value is the 1-$\beta$ Quantil of the Binomial Distribution B(n,p) Multiplied by 1/N with $\beta$ = 0,01**

From the proof of general limit theorem in [3], one could conjecture, that the intermediate distribution of squares of i.i.d. random variables is a $\chi^2$ distributed random variable until also these sums of i.i.d. random variables converge to a normal distribution.

Therefore the test should also serve as a test for independence and a homoscedastcity in general. Finally table 3 shows the test results independent and equally on the interval [0,1] distributed random numbers. Please note that the test statistic only rejects the homoscedastcity and independence hypothesis for the smallest time bucket.

| TimeBuckets | UpperP | LowerP | LowerP+UpperP | P-Value | Reject |
|---|---|---|---|---|---|
| 20 | 0,00% | 4,00% | 4,00% | 1,00% | TRUE |
| 30 | 0,00% | 0,00% | 0,00% | 1,52% | FALSE |
| 40 | 0,00% | 0,00% | 0,00% | 2,00% | FALSE |
| 50 | 0,00% | 0,00% | 0,00% | 2,50% | FALSE |
| 60 | 0,00% | 0,00% | 0,00% | 3,03% | FALSE |
| 70 | 0,00% | 0,00% | 0,00% | 3,57% | FALSE |
| 80 | 0,00% | 0,00% | 0,00% | 4,00% | FALSE |
| 90 | 0,00% | 0,00% | 0,00% | 4,55% | FALSE |
| 100 | 0,00% | 0,00% | 0,00% | 5,00% | FALSE |
| 110 | 0,00% | 0,00% | 0,00% | 5,56% | FALSE |
| 120 | 0,00% | 0,00% | 0,00% | 6,25% | FALSE |
| 130 | 0,00% | 0,00% | 0,00% | 6,67% | FALSE |
| 140 | 0,00% | 0,00% | 0,00% | 7,14% | FALSE |
| 150 | 0,00% | 0,00% | 0,00% | 7,69% | FALSE |
| 160 | 0,00% | 0,00% | 0,00% | 8,33% | FALSE |
| 170 | 0,00% | 0,00% | 0,00% | 9,09% | FALSE |
| 180 | 0,00% | 0,00% | 0,00% | 9,09% | FALSE |
| 190 | 0,00% | 0,00% | 0,00% | 10,00% | FALSE |
| 200 | 0,00% | 0,00% | 0,00% | 10,00% | FALSE |
| 210 | 0,00% | 0,00% | 0,00% | 11,11% | FALSE |
| 220 | 0,00% | 0,00% | 0,00% | 11,11% | FALSE |
| 230 | 0,00% | 0,00% | 0,00% | 12,50% | FALSE |
| 240 | 0,00% | 0,00% | 0,00% | 12,50% | FALSE |
| 250 | 0,00% | 0,00% | 0,00% | 12,50% | FALSE |

**Table 1, 2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points of an i.i.d random variable equally distributed on the interval [0,1].**

## 5. Summary and Conclusions

The presented method is straight forward and shows, when applied, significant results indicating its power. It can be used in addition to the methods of [2] but should also work stand alone. Furthermore the test seems to have power in general to verify homoscedastcity and independence at the same time.

## 6 References

[1] Guido Venier "A New Model for Stock Price Movements" JAES "Journal of Applied Economic Sciences" http://www.jaes.uv.ro. Volume III Issue2(4) Fall2008

[2] Andrew W. Lo & A. Craig MacKinlay (1987) "Stock Market Prices do not follow Random Walks: evidence from a simple specification Test" Department of Finance, Wharton School, University of Pennsylvania, Philadelphia, PA 19104 USA

[3] Heinz Bauer: Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie. 4. Auflage. DeGruyter, Berlin 1991