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Abstract

This brief paper derives the value of headway, i.e. the time interval between departures, for a scheduled service. It presents a consistent framework in which users have scheduling costs, time costs and planning costs. The model represents both users who arrive at the station to choose just the next departure and users who plan for a specific departure. Planning for a specific departure is costly but becomes more attractive at longer headways. Simple expressions for the user cost result. In particular, the marginal cost of headway is large at short headways and smaller at long headways. The difference in marginal costs is the value of time multiplied by half the headway.

1 Introduction

This paper presents an economic analysis of the value of headway for a scheduled service such as a bus route, a train or an air connection. Such an analysis seems to be lacking in the literature. The problem derives from the (rather obvious) observation that users of scheduled services cannot choose their departure time freely, they are constrained to the departure times of the service, where the headway is the time between departures. It then becomes important to know how the size of the headway affects user costs. Insights into this may inform analyses of demand as well as welfare economic analyses of changes in the supply of scheduled services. There seems to be no analysis in the literature of this issue.

For a frequent service we may imagine users do not plan to use a specific departure, they just arrive in order to catch the next departure. From their perspective, the time to the next departure is random. This observation has led many to include just the average waiting time, half the headway, into the user cost. In the present case that accounts for scheduling considerations, they are seen to incur costs of waiting as well as costs due to the uncertainty about the time at which they will arrive at their destination.

For a less frequent service we may imagine that users plan which departure they want to use. In this case, the choice of service is based on scheduling considerations. The choice between planning and not planning we can imagine is governed by a cost of planning, which may include the effort involved in consulting the time table, the timing of the trip to the station as well as a planned wait at the station.

We consider a scheduled service that runs with a time interval between departures of h minutes. In other words, h is the headway. We are not concerned with the travel time on the service and take this to be simply zero in order to simplify the analysis. Hence a user who boards the service at time t also arrives at time t. If the travel time is known and not random, the cost of travel time may just be added to the cost expression obtained in this papers.

The present analysis is based on scheduling preferences (Vickrey, 1969; Small, 1982; Arnott et al., 1993), where the user cost is described in terms of travel time and a scheduling cost given as a function of the time of arrival at the destination relative to a preferred arrival time. We consider a continuum of users, each of whom has a preferred arrival time (PAT), such that the PATs are uniformly distributed over time. Consider a user who has scheduling cost given by the function D and a preferred arrival time (PAT) t^*, such that the scheduling cost of actually arriving at time t is $D(t - t^*)$. The simplest case of D arises when $D(s) = \gamma s^+ + \beta s^-$. In general we assume that D is convex with a minimum at $D(0) = 0$ and that $-D'(0_-) < D'(0_+)$. It is customary also to allow for a convex or linear time cost $A(\cdot)$ with $A(0) = 0$ and $A' > 0$ that is a function of travel time such that the total time cost becomes $A + D$. In the linear case we define $A(t) = \alpha t$ and get the $(\alpha, \beta, \gamma)-$framework that has been used in many papers. A user who arrives at the station at time t_1 and catches a service at time t_2 then incurs a total cost of $A(t_2 - t_1) + D(t_2)$.

The layout of the paper is the following. Section 2 treats the case of long headways where the users plan which departure to use. Section 3 treats the case of a frequent service where users do not bother to consult the time table but merely appear at the station to catch the next departure. Section 4 integrates these two cases through the concept of a planning cost, such that users will plan for a specific departure if the benefit of planning exceeds the cost. Section 5 discusses how the model may be applied in practice.

2 A service with long headway

We first consider the case of a service with long headways, where a user plans which departure he wants to use. He does not wait and so his cost is given only in terms of the scheduling cost function D. The time cost A is not relevant. Instead of thinking of users being uniformly distributed over time, we may take the perspective of a single user and consider arrivals to be uniformly distributed over time. We may take a user with PAT=0 as representative of all users.

Consider the time t defined by

$$D(t-h) = D(t). \tag{1}$$

This equation defines a unique t with $t > 0$ since D is convex and has minimum at 0. In the case of a linear D, we have $t = h\beta/(\gamma+\beta)$. This is illustrated in figure 2

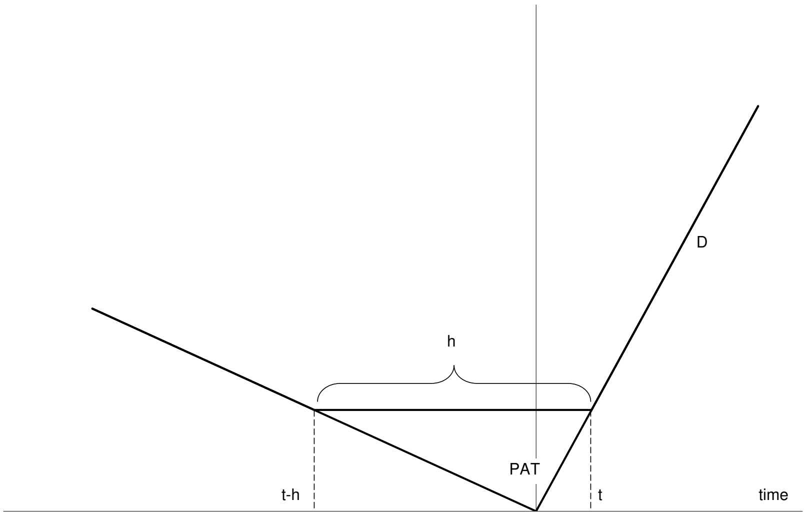

Figure 1: Optimal interval around the PAT

The equation (1) states that the user is indifferent between arriving t minutes late and $-(t-h)$ minutes early. Anything in between is preferred. Since the travel time on the service is normalised to zero, he will choose a departure in the interval $[t-h, t]$ defined by (1) and be at most t minutes late. With departures considered to be uniformly distributed over time, the expected scheduling cost is then

$$C_p(h) = \frac{1}{h}\int_{t-h}^{t} D(s)\,ds,$$

where we use the subscript p to denote that this cost applies to a planning user. In the case of a linear scheduling cost D, we find that $C_p(h) = h\frac{\gamma\beta}{2(\gamma+\beta)}$.

Since the user with PAT=0 is representative, we have that $C_p(h)$ is the average scheduling cost for all users.

We may use the convexity of D to find a bound for $C_p(h)$, namely

$$C_p(h) \leq \frac{1}{h}\left(\frac{D(t)\,t}{2} - \frac{D(t-h)(t-h)}{2}\right) = \frac{D(t)}{2}, \tag{2}$$

with equality when D is linear. This bound will be useful below.

The marginal cost of headway can be found by differentiating the cost with respect to h. Note first that the maximal time a planning user will be late, t, is a function of h. We may then find how t changes by differentiating (2) with respect to h.

$$D'(t)\,t' = D'(t-h)\left(t'-1\right)$$

such that

$$t' = \frac{-D'(t-h)}{D'(t) - D'(t-h)}.$$

This positive by the assumptions on D, such that increasing the headway will increase the maximal lateness of a planning user. In the case of a linear D this becomes $t' = \beta/(\gamma+\beta)$.

Differentiating $C_p(h)$ we find that[1]

$$\begin{aligned} C_p'(h) &= \frac{D(h)}{h}t' - \frac{D(t-h)}{h}\left(t'-1\right) - \frac{C_p(h)}{h} \\ &= \frac{D(t) - C_p(h)}{h}. \end{aligned}$$

In the case of a linear D we have $C_p'(h) = \frac{\gamma\beta}{2(\gamma+\beta)}$, which is constant as a function of h.

Use the bound on $C_p(h)$ in (2) to find that

$$C_p'(h) \geq \frac{D(t)}{2h} > 0,$$

such that the marginal cost of headway is strictly positive for general convex D.

It may be of interest how the marginal cost of headway depends on h. Differentiate again to find that

$$\begin{aligned} C_p''(h) &= \frac{D'(t)t' - C_p'(h)}{h} - \frac{C_p'(h)}{h} \\ &= -\frac{1}{h}\frac{D'(t)D'(t-h)}{D'(t) - D'(t-h)} - \frac{2C_p'(h)}{h}. \end{aligned}$$

[1] Use the rule that

$$\frac{d}{dx}\int_{g(x)}^{f(x)} h(x,y)dy = h(x,f(x))f'(x) - h(x,g(x)g'(x) + \int_{g(x)}^{f(x)} \frac{d}{dx}h(x,y)dy.$$

The first term here is positive since $D'(t-h) < 0$ while the second term is negative. I have not been able to be more definite about the sign of C''_p, so it seems that the marginal cost of headway may decrease or increase as headway increases. If the marginal cost of headway is low, then it must increase, but it can possibly decrease when the marginal cost of headway is high.

3 A service with short headway

We now suppose that service is so frequent that a user will not consult the time table but appear at the station in order to catch the next departure, not knowing exactly when that will be. We may again take a user with PAT=0 as representative. If he arrives at the station at, say, time $t-h$, where t will be found below, then the next departure time is uniformly distributed over the interval $[t-h, t]$. His cost if the next departure occurs at time s is

$$A(s-t+h) + D(s)$$

and the expected cost is

$$C_u(t) = \frac{1}{h}\int_{t-h}^{t} A(s-t+h) + D(s)\, ds,$$

where the subscript u denotes that the cost applies to an unplanning user. The optimal arrival time t is found by setting the marginal expected cost equal to zero.[2]

$$\frac{\partial C_u(t)}{\partial t} = \frac{1}{h}(A(h) + D(t)) - \frac{1}{h}(A(0) + D(t-h)) - \frac{1}{h}\int_{t-h}^{t} A'(s-t+h)\, ds = 0.$$

Assume that A is linear, $A(s) = \alpha s$. Then the equation reduces to

$$\frac{1}{h}(\alpha h + D(t)) - \frac{1}{h}(D(t-h)) - \alpha\frac{1}{h}\int_{t-h}^{t} ds = 0,$$

such that the first order condition becomes

$$D(t-h) = D(t),$$

which is exactly the same as in the planned arrival case. Taking t to be optimally chosen, we may derive the expected cost for an unplanning user C_u using the expression for the average cost C_p for a planning user in (2). The expected cost then becomes

$$\begin{aligned} C_u(h) &= \frac{\alpha}{h}\left[s^2/2 - (t-h)s\right]_{t-h}^{t} + C_p(h) \\ &= \frac{\alpha h}{2} + C_p(h). \end{aligned}$$

[2]Using the same rule of differentiation as above.

We may differentiate this expression with respect to h to find the marginal cost of headway for an unplanning user as

$$C_u'(h) = \frac{\alpha}{2} + C_p'(h).$$

This is exactly $\alpha/2$ larger in the unplanned case than in the planned case. The term $\alpha/2$ corresponds to the average waiting time of an unplanning user. We see that the marginal cost comprises an additional term, namely the marginal average scheduling cost of a planning user.

4 To plan or not to plan

We have established the scheduling cost as a function of the headway for two situations. In one the users are assumed to plan for a specific departure while they are not planning in the other. Otherwise the situations are completely identical. To complete the story, we therefore need to explain why some users plan and other do not. At the same time we want the model to have the property that planning is more worthwhile at longer headways.

Assume that a user has a planning cost of $\zeta > 0$. If he plans for a specific departure, he will incur a total cost of $C_p(h) + \zeta$. If he does not plan, then his cost is $\alpha h/2 + C_p(h)$. Choosing the minimum cost option, he will then plan if $\zeta < \alpha h/2$. Assume that planning costs are distributed in the population with some cumulative distribution function Φ with bounded support. This will result in an interval of headways such that more and more users will decide to plan as the headway increases. The average user cost at headway h is

$$C(h) = C_p(h) + \left(1 - \Phi\left(\frac{\alpha h}{2}\right)\right)\frac{\alpha h}{2} + \int_0^{\frac{\alpha h}{2}} \zeta\phi(\zeta)\,d\zeta. \tag{3}$$

The average marginal cost of headway then becomes

$$\begin{aligned} C'(h) &= C_p'(h) + \left(1 - \Phi\left(\frac{\alpha h}{2}\right)\right)\frac{\alpha}{2} - \phi\left(\frac{\alpha h}{2}\right)\frac{\alpha^2 h}{4} + \frac{\alpha^2 h}{4}\phi\left(\frac{\alpha h}{2}\right) \\ &= C_p'(h) + \left(1 - \Phi\left(\frac{\alpha h}{2}\right)\right)\frac{\alpha}{2}, \end{aligned}$$

which is always positive. Differentiate again to find that

$$C''(h) = C_p''(h) - \phi\left(\frac{\alpha h}{2}\right)\frac{\alpha^2}{4}$$

such that the cost is concave in headway when scheduling costs are linear. We may suppose that ϕ has support on some finite interval I with $0 < \min I$, such that nobody will plan for short headways while everybody will plan for long headways. Then for short headways we have $C'(h) = C_p'(h) + \alpha/2$ while for long headways we have a lower marginal cost of $C'(h) = C_p'(h)$.

5 Application

For the application of the model we need to know first the scheduling preferences. We assume linear scheduling costs expressed by (α, β, γ) and that these have been estimated. A number of studies have estimated these parameters, e.g. Bates et al. (2001) and Small (1982), see the review in Fosgerau et al. (2008). The parameter α may be thought to equal the value of travel time. If the marginal cost of waiting time is thought to be higher, the value of α may be increased correspondingly, while keeping the values of β and γ fixed.

It is harder to presume that we know the distribution of planning costs in the population. The planning cost makes no difference if we desire to compare two services where either all users plan or all users do not plan. It is however necessary to know the cost distribution in order to account for the planning cost in the case when two service schedules are compared where at least one of them involve both planning and unplanning users.

Fortunately, it is possible to form an opinion about the share of users who will choose a specific departure at different headways. For example, one may observe when passengers arrive at train platforms and which departure they get on. If users arrive at a constant rate then we may think they are not planning. If on the other hand they all arrive close to the next departure, then we may think they are all planning. In between we will have some users arriving at random and some users arriving close to the next departure. If we can estimate the share of users who plan as a function of headway, and if we know α, then we can identify $\Phi\left(\frac{\alpha h}{2}\right)$.

We hence assume that Φ is known. In particular we know the support of Φ, that is, we know the maximum headway at which no users plan $h_{\min}$ and also the minimum headway at which all users plan $h_{\max}$. This defines the interval over which the planning costs are distributed. Introduce for brevity of notation a function to truncate the headway at $h_{\min}$ and $h_{\max}$ by $\Lambda(h) = (x \vee h_{\min}) \wedge h_{\max}$ (where $\vee$ is the maximum operator and $\wedge$ is the minimum operator) and let also $\Delta = h_{\max} - h_{\min}$.

We could assume for simplicity that the distribution of planning costs in (3) is uniform, such that

$$\Phi\left(\frac{\alpha h}{2}\right) = \frac{\Lambda(h) - h_{\min}}{\Delta}$$

or

$$\Phi(\zeta) = \frac{2\zeta/\alpha - h_{\min}}{\Delta}, \phi(\zeta) = \frac{2}{\alpha\Delta}$$

for values of ζ in the appropriate interval. Recall that with linear scheduling cost we have $C_p(h) = C'_p h$ and $C'_p = \frac{\gamma\beta}{\gamma+\beta}$. We can then write the user cost function as

$$
\begin{aligned}
C(h) &= C_p(h) + \left(1 - \Phi\left(\frac{\alpha h}{2}\right)\right)\frac{\alpha h}{2} + \int_{\frac{\alpha h_{min}}{2}}^{\frac{\alpha \Lambda(h)}{2}} \zeta \phi(\zeta)\, d\zeta \\
&= C_p' h + \left(\frac{h_{max} - \Lambda(h)}{\Delta}\right)\frac{\alpha h}{2} + \frac{1}{\alpha \Delta}\left(\frac{\alpha^2 \Lambda^2(h)}{4} - \frac{\alpha^2 h_{min}^2}{4}\right) \\
&= C_p' h + \frac{h\alpha}{2}\frac{h_{max} - \Lambda(h)}{\Delta} + \frac{\alpha}{4}\frac{\Lambda^2(h) - h_{min}^2}{\Delta}.
\end{aligned}
$$

This expression simplifies when h is outside the interval where users change from not planning to planning.

$$
\begin{aligned}
h &< h_{min} : C(h) = C_p' h + \frac{\alpha h}{2}, C'(h) = C_p' + \frac{\alpha}{2} \\
h &> h_{max} : C(h) = C_p' h + \alpha \frac{h_{max} + h_{min}}{2}, C'(h) = C_p'
\end{aligned}
$$

Note here that that the last term in $C(h)$ when $h > h_{max}$ is $E(\zeta) = \alpha(h_{max} + h_{min})/4$.

Substituting numerical values for the scheduling parameters yields an expression for the cost associated with headway. Figure 2 uses $(\alpha, \beta, \gamma) = (1, 0.5, 2)$, $h_{min} = 5$ and $h_{max} = 15$. We see that the cost curve is steep with slope $C'(p) + \alpha/2$ up to the point h_{min} where some users begin to plan. The curve is dashed in the interval $[h_{min}, h_{max}]$ where more and more users switch to planning. It is drawn here as a straight line but the shape depends on the distribution of planning costs in the population. Thereafter the curve becomes again linear with the smaller slope of $C'(p)$. The first and last line segments have been extended with light dashed segments to indicate that the cost curve is bounded above by these lines. The intersection of the extension of the last line segment with the y-axis corresponds to the average planning cost when all users plan.

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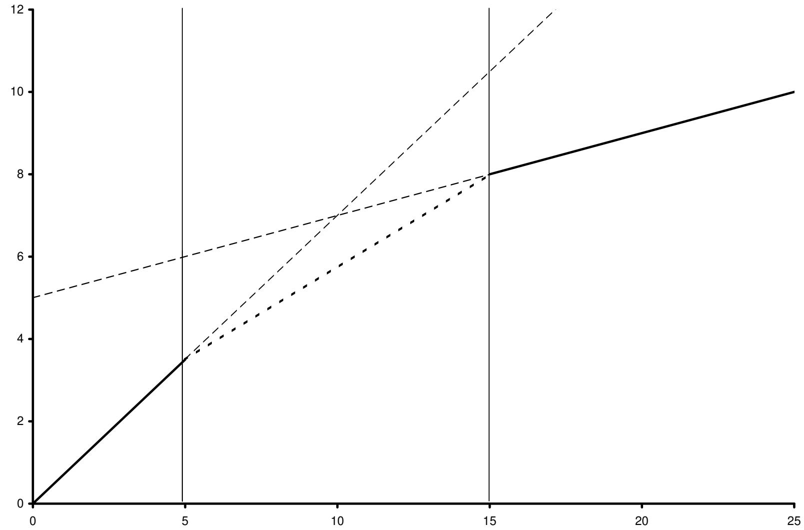

Figure 2: Illustration of the cost function as a function of headway

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