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# Imperfect Competition in the International Energy Market: A Computerized Nash-Cournot Model

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This paper describes the conceptual structure, properties, and solution approach of a computerized model of the international energy market. The model treats energy producers as players in a multistage, noncooperative game. The goal of each player (or cartel of players) is assumed to be maximization of discounted profit subject to technical, political, and resource constraints. The model calculates that collection of intertemporal extraction and price paths from which a player can unilaterally deviate only at a loss—the open-loop, Nash equilibrium. The model integrates the theory of exhaustible resources due to Hotelling and the theory of oligopoly due to Nash and Cournot. Although useful as a teaching device to illustrate theoretical results, its main function is to facilitate analysis of real-world resource problems. The model is flexible, allowing the user to specify not only cost, demand, and reserve information but also assumptions about who belongs to what coalition. Two shortcomings deserve note. The strategies of players are restricted to time-dated (open-loop) paths. Also, lags cannot be accommodated in the current version. The restriction of the strategy space significantly increases tractability and will permit the incorporation of lags and other complications in the future. The model was built under government contract and is in the public domain.

THIS PAPER describes the conceptual structure, properties and solution approach of a computerized model of the international energy market. The model was built in 1979 at the request of the Department of Energy. Since views about parameters of the world oil market differ among potential users, the model was designed to be as flexible as possible. Simulations under markedly different input assumptions have so far been conducted at the initiative of the Department of Energy, the U.S.-Saudi Joint Commission on Economic Cooperation, Stanford University's Energy Modeling Forum, and the Shell Oil Company. Details on these simulations are reported elsewhere.[1]

*Subject classification:* 131 dynamic, imperfect competition, Hotelling, exhaustible resource; 366 open-loop, multi-stage, simulation; 473 optimization, foresight, energy prices.

[1] See Blankenship and Gaskins [1981], Salant et al. [1979, 1981], Stitt et al. [1980], and EMF VI Report [1981].



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The model described below is the natural outgrowth of the economics literature on exhaustible resources, which dates back to Hotelling [1931]. Prior to 1975, contributions to that voluminous literature assumed either perfect competition or a single monopoly, despite historical examples of intermediate market structures.[2] In 1975, the first attempt to integrate the theory of imperfect competition and exhaustible resources was completed. Salant [1975, 1976] showed how the polar cases of perfect competition and pure monopoly—as well as a continuum of intermediate market structures—could be studied within a common framework by using a dynamic analogue of the Cournot [1838] equilibrium concept (formally an open-loop, Nash noncooperative, nonzero-sum differential game).[3] The same solution concept has been applied in other dynamic settings by Prescott [1973], Flaherty [1980], and Spence [1981]. In the resource context, the Nash-Cournot solution was subsequently explored by Khalatbari [1976], Lewis and Schmalensee [1978, 1979, 1980], Loury [1980], Pakravan [1976, 1980], and Ulph and Folie [1978, 1980]. Meanwhile, a conceptually distinct approach utilizing a dynamic (open-loop) analogue of the Stackelberg [1934] solution concept was initiated by Cremer and Weitzman [1976], with subsequent contributions by Gilbert [1976], Gilbert and Goldman [1978], Hoel [1978], Marshalla [1978], and Salant [1979]. The two solution concepts can result in the same equilibrium price path—as was noted by Gilbert [1978] and explored by Ulph and Folie [1979]. However, the Stackelberg solution often displays an undesirable characteristic known as "dynamic inconsistency." This prop-

[2] For an interesting discussion of the life and death of the British coal cartel, see Levy [1909].

[3] For preliminary treatments, see Intrilligator [1971] (p. 387) or Starr and Ho [1969]. For any game in extensive form, three Nash solutions may be distinguished: the closed-loop, the open-loop, and the feedback (or subgame perfect) solutions. The latter two solutions are contained within the *set* of closed-loop solutions. In the closed-loop solution, each player's time-dated sequence of decision rules (functions of the vector of state variables known by him at each date) is optimal given the sequences of decision rules of the other players. That is, the closed-loop solution is simply the noncooperative solution defined by John Nash 30 years ago. Since such solutions tend to be plentiful, attention has recently focussed on how to restrict them usefully. In some cases, there is one (or more) solution where each player's decision at any stage is independent of previous moves. In this case, the decision rules taken as given are simple—constant functions. In this so-called open-loop solution (which is contained in the set of closed-loop solutions) each player takes as given the time-dated sequence of decisions as distinct from decision rules. The open-loop solution is used here. However, each of the foregoing solutions may admit cases where the behavior taken as given is not credible. A player may find that although a particular sequence of decision rules is optimal *given* the sequence of rules of the other players, the given sequence is implausible; for the player can see that if one departed from a particular sequence, the other players would be put in situations where their given rules would not be optimal. Closed-loop solutions where the behavior taken as given is always credible are called feedback in the optimal-control literature and subgame perfect in the game theory literature. For an illustration of the feedback solution, see Levhari and Mirman [1980].

erty—which has been investigated by Kydland [1975], Kydland and Prescott [1977], Simaan and Cruz [1973], and Ulph [1980]—is undesirable in part because it causes the forecasts for a given year to depend on the arbitrary choice of the initial date.[4] The Nash-Cournot solution does not suffer from this weakness but, of course, is vulnerable to other criticisms.[5] Alternative solutions do exist and have been explored in other contexts.[6] Promising research by Maskin, Newberry, Reinganum and Stokey [1981], and Ulph and Folie [1980] among others may lead in the future to the application of the perfect equilibrium concept to models of the international energy market.

The model described below utilizes the open-loop Nash-Cournot approach. The model is flexible enough to handle various market structures, extraction technologies, and demand functions. It can therefore be used in the classroom to illustrate most results in the theoretical literature on exhaustible resources which have been developed over the last half century.[7] Furthermore, since the computer can calculate equilibria even when inputs reflect real world complexities (for example, simultaneous consideration of technological uncertainty about the backstop, depletion effects, capacity constraints, and a market structure where competitive extractors, the OPEC cartel, and large extractors like Mexico coexist),

[4] If the Stackelberg model is solved from time zero onward and then re-solved from $t$ onward (using the reserves remaining at $t$ in the former run as the initial conditions of the latter run), the two price paths after $t$ might differ. The reason is that the Stackelberg decisions announced in the first run as planned from $t$ onward are in part chosen to influence fringe behavior *prior* to $t$; since the second game *begins* at date $t$, this consideration is no longer relevant and the optimal choice at $t$ may differ. If the Stackelberg leader's plan is time-inconsistent and the leader cannot precommit to the announced strategy, the fringe would be foolish to take it as given. In this case, the open-loop Stackelberg solution is not an equilibrium in a meaningful sense. Even if the time-inconsistent plan *is* believed by the fringe, the predicted price at time $t$ depends on the arbitrary starting date of the model. In the case above, it would depend on whether the initial date was zero or $t$. To illustrate, the forecast for 1980 of the Cremer-Weitzman [1976] oil model may depend on the particular year these authors chose to begin the Stackelberg player's problem.

[5] However, its weaknesses pale compared with those of the Stackelberg "open-loop" solution. Besides time-inconsistency discussed in the previous footnote, there are issues raised by the unexplained asymmetry of the Stackelberg setup. Although bothersome in any situation unless induced by an explicit informational structure, the problem is worse when the model is used for government policy evaluation. For then one must not only assume that the leader is smarter than the followers, *but that the U.S. policy-maker is smarter than the leader.* Finally, the Stackelberg leader's profit-maximization problem is fraught with technical difficulties. According to Cremer, the objective function of the leader may not be concave in his extraction vector even if all cost functions are convex. Furthermore, as shown in Salant [1979], the switching of followers from zero to positive extraction leads to nondifferentiable points on the leader's objective function.

[6] See, for example, Levhari and Mirman [1980].

[7] Solow [1974], Weinstein and Zeckhauser [1975], and Stiglitz [1979] provide excellent expository surveys of the results of this literature. For a more exhaustive survey, see Peterson and Fisher [1977].

the model has a useful role to play in policy analysis.

Questions like the following can at last be addressed quantitatively within a single model:

—If OPEC split apart in a specified way or, alternatively, gained control of other sources of supply, what effect would such changes in market structure have on the present and future price of oil?

—If the United States succeeded in cutting its demand for oil by a specified amount, how large would the benefits be and how would they be distributed?

—If additional research and development (R&D) could provide *earlier* resolution of the existing uncertainty about the costs of future technologies, how much would the additional R&D be worth and who would benefit from it?

In addition, the model can be used to examine the response of domestic extractors (and stockpilers) to phased decontrol of oil prices, the imposition of import fees, and other policies of current relevance.

The paper is organized as follows. In Section 1, the Nash-Cournot solution is defined by specifying the intertemporal maximization problems which are simultaneously solved in the equilibrium. Section 2 outlines how the solution is computed. Section 3 describes existing features of the model which have proved useful in applications. Section 4 illustrates the performance of the model by reporting one set of simulations. One weakness of the model in its current form is the omission of lags on either side of the market. Section 5 indicates how lags can be incorporated without altering the structure of the model substantially and concludes the paper. Discussion of existence and uniqueness of equilibrium and the convergence of algorithms to the equilibrium is relegated to the appendices. A more extensive description of both theoretical and computational aspects of the model (and computer listings and documentation) can be found in Salant et al. [1981].

## 1. THE NASH-COURNOT SOLUTION

Most world oil models are based on input assumptions about demand, costs, and market structure, and generate as output a set of extraction paths and a price path expected to prevail over time. The ultimate test of such models is the accuracy of their predictions. But even before the future unfolds, a model should be scrutinized to make sure that its assumptions are realistic and that its forecasts pass the following elementary tests.

*Feasibility:*

1. The forecast of the amount produced by any given extractor in any

given period must not exceed the assumed capacity limits of that producer at that date.

2. The forecast of the cumulative amount produced over time by a particular extractor must not exceed the assumed reserve limits of that producer.

*Consistency:*

3. The total forecasted supply of the various extractors at each date must equal the demand which the forecasted price would induce.

*Profit Maximization:*

4. No decision-maker should conjecture that discounted profits can be increased by behaving differently than the way forecasted.

A model whose forecasts fail any of these tests is open to serious criticism. The first three of these tests are unambiguous. The fourth, however, is more difficult to apply because there is an inherent ambiguity about the set of profit opportunities which firms believe they face.

The evaluation of profit opportunities by a firm ceases to be straightforward the moment the firm recognizes that its opportunities are affected by the decisions of other firms as well as by market conditions in the future. The debate about how a noncompetitive firm forecasts the concurrent responses of rivals—who in turn are making similar forecasts about it—is responsible for the multiplicity of "plausible solutions" (among them various Nash and Stackelberg solutions) known as noncooperative game theory. There is also debate about how firms forecast the future. Such forecasts affect behavior since firms must decide when to extract their limited reserves. Pindyck [1978], Hnyilicza and Pindyck [1976], Gately et al. [1976] and others assume that competitive extractors have no foresight whatsoever. Accordingly, they would produce the same amount today regardless of whether what awaited them tomorrow was either expropriation or a doubling of price. In contrast, the agents in this model are assumed to have some foresight. Special cases of this assumption include rational expectations and perfect foresight; but subjective expectations which differ across maximizing agents are also permitted to a limited extent.[8]

The forecasts of the Nash-Cournot model of the world oil market pass each of the tests discussed above—the fourth in a manner which will now be made precise. The output of the model consists of a set of paths over time—one price path ($P^t$) and as many production paths ($Q_i^t$) as there are oil-producing units or "plants." Each plant is either competitive or under the control of a particular "monopolistic" Nash player. As an

[8] See pp. 266–268.

illustration, the table below represents the output from a case where there are nine plants.

Production at Plants of Each Player

| *Year* | *Competitive Fringe* | | | *Nash Player* 1 | | *Nash Player* 2 | | | *Nash Player* 3 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $Q_1^1$ | $Q_2^1$ | $Q_3^1$ | $Q_4^1$ | $Q_5^1$ | $Q_6^1$ | $Q_7^1$ | $Q_8^1$ | $Q_9^1$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ | | | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ | | | $\vdots$ | | $\vdots$ |
| $T$ | $Q_1^T$ | $Q_2^T$ | $Q_3^T$ | $Q_4^T$ | $Q_5^T$ | $Q_6^T$ | $Q_7^T$ | $Q_8^T$ | $Q_9^T$ |
| Cumulative Production (column sums) | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ | $U_7$ | $U_8$ | $U_9$ |

The braces group together the extraction paths of (i) the competitive plants[9] (1–3) in the table, (ii) the multiple plants (4 and 5) of the first large player, (iii) the multiple plants (6–8) of the second large player, and (iv) the single plant (9) of the third large player. Below the column of annual extraction levels for each plant is the cumulative extraction ($U_i$) for that plant over the entire horizon.

Such grouping defines the industrial organization of the world oil market and *must be specified by the user.* The user must also specify the discount factor ($\beta$), any resource constraints on the cumulative production of the $i$th plant caused by the finiteness of its resource base ($R_i$), and—for each period—the extraction cost function ($C_i^t(Q_i^t)$) of each plant and any flow constraint ($Q_i^t$ max) which is imposed by technical or political considerations.

The paths computed as output pass the fourth test in the following manner. Competitive plant $i$ makes larger discounted profits—evaluated along the forecasted path of prices—by extracting in the way forecasted,

[9] Consider an equilibrium with competitive and Nash plants. Suppose each competitive plant were deleted from the model and were replaced by $n$ independent Nash players, each owning $1/n$th of its oil and having a scaled cost of extraction function. The modified model would no longer have a mixture of competitive and Nash plants; *everyone* would be a Nash player and would be treated symmetrically. For large $n$ (infinitesimal reserve sizes for each of the $n$ players), this model's equilibrium would converge to the original equilibrium with the competitive plants. Price paths would converge, and the aggregate produced by the $n$ infinitesimal Nash players would approach the production at each competitive plant which they replaced. For a discussion of this limit theorem, see Salant [1976]. It is important to note that the Nash-Cournot solution concept is completely symmetric despite the *apparent* differentiation between competitive and Nash plants, and to think of competitive plants simply as artifacts which have been introduced to summarize the aggregate behavior of many infinitesimal Nash players.

rather than in any other (feasible) fashion. That is,

$$P1 \qquad \{Q_i^t\} \text{maximizes } \textstyle\sum_{t=1}^{T} \beta^{t-1}\{\bar{P}^t Q_i^t - C_i^t(Q_i^t)\}$$

subject to

feasibility conditions: $0 \leq Q_i^t \leq Q_i^t\text{max}, \quad t = 1, T$

$$\textstyle\sum_{t=1}^{T} Q_i^t = U_i \leq R_i,$$

where a bar over a forecasted variable indicates that it is taken as given by the particular agent.

Furthermore, large Nash player $j$, owner of the set of plants $\Omega_j$, makes larger joint discounted profits—evaluated assuming plants not under the player's control produce as forecasted (at least in aggregate)—by extracting at the $i$th plant in the way forecasted, rather than in any other (feasible) fashion.[10] That is,

$$\{P^t\}\{Q_i^t\} \quad \text{maximize } \textstyle\sum_{t=1}^{T} \sum_{i\in\Omega_j} \beta^{t-1}\{P^t Q_i^t - C_i^t(Q_i^t)\}$$

$P2$ subject to
feasibility conditions:

$$0 \leq Q_i^t \leq Q_i^t\text{max}, \quad i \in \Omega_j, \quad t = 1, T$$

$$\textstyle\sum_{t=1}^{T} Q_i^t = U_i \leq R_i, \quad i \in \Omega_j$$

and the market-clearing
equations determining
price

$$\textstyle\sum_{i\in\Omega_j} Q_i^t = D^t(P^t) - \sum_{i\notin\Omega_j} \bar{Q}_i^t, \quad t = 1, T.$$

Hence, neither a competitive plant nor a large player seeking profits would have any incentive to disturb the forecasted solution by altering any of the variables under its control. *For this reason, the Nash-Cournot solution is regarded as an equilibrium.*

## 2. DETERMINATION OF EQUILIBRIUM

Having described the nature of the Nash-Cournot solution, we now explain how it is determined by the model. Briefly, the model locates the equilibrium by arbitrarily picking an initial set of multipliers, working forward through the Kuhn-Tucker conditions associated with each intertemporal maximization problem until the end of the horizon (while computing market-clearing prices) and then checking the transversality conditions for each plant. If any transversality condition is violated, the

[10] For an early discussion of cartels as multi-plant monopolists producing nonstorables, see Patinkin [1947].

initial set of multipliers is judiciously revised and the procedure repeated until convergence is achieved. Restrictions on the shapes of curves suffice to ensure that what this procedure locates are global maxima.[11] In the remainder of the section, this solution approach is described in greater detail.

We begin by describing the entire set of equations which must be simultaneously solved in equilibrium. These consist of the Kuhn-Tucker conditions associated with each agent's dynamic maximization problem as well as equations indicating that what is taken as given by each player is in fact optimal for the others. For purposes of exposition, we begin by considering a case where each of two players ($n = 2$) owns a single plant. Player 1 is assumed to be competitive, while player 2 is large. Generalizations will be discussed later.

The following notation will be used:

$D_t(P^t)$ = demand curve in year $t$;
$R_i$ = total reserves of plant $i (i = 1, \cdots n)$;
$C_i^t$ = $C_i^t(Q_i^t, t)$ = cost function for plant $i$ in year $t$;
$Q_i^t\text{max}$ = capacity constraint for plant $i$ in year $t$;
$T$ = length of finite horizon;
$r$ = the real rate of interest;
$\beta$ = the real discount factor $(1/(1 + r))$;
$\lambda_i$ = the multiplier assigned to the resource constraint of the $i$th plant $(R_i - \sum_{t=1}^T Q_i^t \geq 0)$;
$\alpha_i^t$ = the multiplier assigned to the capacity constraint of the $i$th plant in period $t (Q_i^t\text{max} - Q_i^t \geq 0)$;
$\gamma_i^t$ = the multiplier assigned to the market-clearing constraint viewed by the $i$th player in period $t (D(P_i^t) - \sum_{j \neq i} \bar{Q}_j^t - Q_i^t = 0)$:
$U_i$ = cumulative extraction at plant $i (\sum_{t=1}^T Q_i^t)$ over entire horizon
$\hat{n}$ = the number of possible states of nature under uncertainty
$\hat{t}$ = the date after which uncertainty is resolved
$\pi_{is}$ = the subjective evaluation by player $i$ that state $s$ will occur.

To solve the competitor's problem outlined above ($P1$), we form the Lagrangean:

$$L(Q_1^1, \cdots Q_1^T; \alpha_1^1, \cdots \alpha_1^T; \lambda_1) = \sum_{t=1}^T \beta^{t-1}\{\bar{P}^t Q_1^t - C_1^t(Q_1^t)\} + \alpha_1^t\{Q_1^t\text{max} - Q_1^t\} + \lambda_1\{R_1 - \sum_{t=1}^T Q_1^t\}.$$

[11] In the absence of such restrictions, the forecasted strategies would still simultaneously satisfy the first-order conditions of each player but might nonetheless be suboptimal. Indeed, as Roberts and Sonnenschein [1977] have indicated, there are cases where no pure strategies exist which are simultaneously globally optimal for the respective players. In such cases, no pure-strategy Nash equilibrium exists. Such cases never arise if the demand curve is linear or concave.

Then, the Kuhn-Tucker theorem indicates that the competitor's optimal program must satisfy each of the following conditions:

1. (a) $Q_1{}^t \geq 0$
   (b) $L_{Q_1{}^t} \leq 0$, where $L_{Q_1{}^t} = \beta^{t-1}\{\bar{P}^t - C_1{}^t(Q_1{}^t)\} - \alpha_1{}^t - \lambda_1$
   (c) $Q_1{}^t \cdot L_{Q_1{}^t} = 0$ for $t = 1, \cdots T$.
2. (a) $\alpha_1{}^t \geq 0$
   (b) $L_{\alpha_1{}^t} \geq 0$, where $L_{\alpha_1{}^t} = Q_1\text{max} - Q_1{}^t$
   (c) $\alpha_1{}^t \cdot L_{\alpha_1{}^t} = 0$ for $t = 1, \cdots T$.
3. (a) $\lambda_1 \geq 0$
   (b) $L_{\lambda_1} \geq 0$, where $L_{\lambda_1} = R_1 - \sum_{t=1}^{T} Q_1{}^t$.
   (c) $\lambda_1 \cdot L_{\lambda_1} = 0$.

It is assumed that each plant's cost function is strictly convex. Then since the competitor's objective function in $P1$ is strictly concave and the constraint set is convex, only one solution $(Q_1{}^1, \cdots, Q_1{}^T)$ to these conditions will exist for a given price sequence $\{\bar{P}^t\}$.

To solve the problems of a large, single-plant Nash player—a special case of $P2$—we form the Lagrangean[12]:

$$L(Q_2{}^1, \cdots Q_2{}^T; \alpha_2{}^1, \cdots \alpha_2{}^T; \lambda_2; P^1, \cdots P^T; \gamma_2{}^1, \cdots \gamma_2{}^T)$$
$$= \sum_{t=1}^{T} (\beta^{t-1}\{P^t Q_2{}^t - C_2{}^t(Q_2{}^t)\} + \alpha_2{}^t\{Q_2{}^t\text{max} - Q_2{}^t\}$$
$$+ \gamma_2{}^t\{D_t(P^t) - \bar{Q}_1{}^t - Q_2{}^t\}) + \lambda_2\{R_2 - \sum_{t=1}^{T} Q_2{}^t\}.$$

Then, the Kuhn-Tucker theorem indicates that the optimal program for a Nash player must satisfy the following conditions:

1. (a) $Q_2{}^t \geq 0$
   (b) $L_{Q_2{}^t} \leq 0$, where $L_{Q_2{}^t} = \beta^{t-1}\{P^t - C_2'^t 2(Q_2{}^t)\} - \gamma_2{}^t - \lambda_2 - \alpha_2{}^t$,
   (c) $Q_2{}^t \cdot L_{Q_2{}^t} = 0$ for $t = 1, \cdots T$.
2. (a) $\alpha_2{}^t \geq 0$
   (b) $L_{\alpha_2{}^t} \geq 0$, where $L_{\alpha_2{}^t} = Q_2{}^t\text{max} - Q_2{}^t$
   (c) $\alpha_2{}^t \cdot L_{\alpha_2{}^t} = 0$ for $t = 1, \cdots T$
3. (a) $\lambda_2 \geq 0$
   (b) $L_{\lambda_2} \geq 0$, where $L_{\lambda_2} = R_2 - \sum_{t=1}^{T} Q_2{}^t$
   (c) $\lambda_2 \cdot L_{\lambda_2} = 0$
4. (a) $P^t \geq 0$
   (b) $L_{P^t} \leq 0$, where $L_{P^t} = \gamma_2{}^t D_t'(P^t) + \beta^{t-1} Q_2{}^t$
   (c) $P^t \cdot L_{P^t} = 0$ for $t = 1, \cdots T$

[12] Each Nash player believes that the player controls the price. Hence, we would be more precise in stating the $i$th Nash player's problem if we denoted the player's instruments as $\{P_i{}^1, \cdots P_i{}^T\}$ subject to the constraint that $D_t(P_i{}^t) = \sum_{j \neq i} \bar{Q}_j{}^t + Q_i{}^t$. However, in equilibrium the right-hand side of this constraint is identical for each player. Hence each Nash player will select the same price in equilibrium and no confusion will arise if we drop the subscript on $\{P_i{}^t\}$.

5. $L_{\gamma_2^t} = 0$, where[13] $L_{\gamma_2^t} = D_t(P^t) - \bar{Q}_1^{\,t} - Q_2^{\,t}$ for $t = 1, \cdots T$.

These conditions will have a single solution $\{Q_2^{\,1}, \cdots, Q_2^{\,T}; P_2^{\,1}, \cdots, P_2^{\,T}\}$ for a given sequence $\{\bar{Q}_1^{\,t}\}$ if the demand curve is suitably curved.[14]

Finally, we must adjoin to these two sets of first-order conditions the requirement that what is taken as given by one player must be optimal for the others: $P^t = \bar{P}^t$ and $Q_1^{\,t} = \bar{Q}_1^{\,t}$ for $t = 1, \cdots T$. We can use this pair of equations to eliminate barred variables where they appear in (1)–(3) and (1)–(5), replacing them with unbarred variables. An equilibrium is any simultaneous solution to the foregoing conditions. These conditions define $\lambda_1$ and $\lambda_2$, as well as $Q_1^{\,t}$, $Q_2^{\,t}$, $P^t$, $\alpha_1^{\,t}$, $\alpha_2^{\,t}$, and $\gamma_2^{\,t}$ for $t = 1, \cdots T$. Conditions (1)–(3) and (1)–(5) can be solved in the following manner.

*Step 0:* Provisionally drop the transversality condition (3) at each plant and assign to each an arbitrary (positive) multiplier, $\lambda_i$.

*Step 1:* $t = 1$.

(a) Provisionally drop the requirement that the market clears (5) and pick a (positive) trial price, $P^t$.

(b) Use conditions (1) and (2) for each plant to compute production at each plant contingent on trial price $P^t$.

(c) Now consider (5). If the demand forthcoming at $P^t$ exceeds the aggregate production of all the plants calculated in Step 1b, use the sign of the excess demand to adjust $P^t$ (as discussed below) and return to Step 1b. Otherwise $t = t + 1$. If $t > T$, go to Step 2; if not, go to Step 1a.

*Step 2:* For each plant, calculate whether the cumulative sales along the production path $(\sum_{t=1}^{T} Q_i^{\,t})$ exceed the reserves $(R_i)$ available to that plant. If the transversality condition for any plant is violated, use the resulting deviation to revise the multipliers and return to Step 1. Otherwise, stop and report the equilibrium values for prices, production, and $\{\lambda_i\}$.

We now discuss Steps 1 and 2 in greater detail.

We first consider precisely how in Step 1b the competitor's conditions

[13] Condition (5) differs from the others because $\gamma_2^{\,t}$ is the multiplier for an equality constraint (market-clearing) rather than an inequality constraint. To derive (5) from the conditions for inequality constraints, replace the original equality of market-clearing by the two inequalities—that excess demand be nonnegative *and* nonpositive. For these two constraints to be simultaneously satisfied, excess demand must be zero (market-clearing). Further, the two multipliers associated with the two constraints always appear together with one subtracted from the other. Defining their difference as $\gamma_2^{\,t}$, we obtain (5).

[14] Substituting the "inverse demand curve" for price in the large player's objective function, we see that the Kuhn-Tucker conditions have a unique solution if $D_t^{-1}(\bar{Q}_1^{\,t} + Q_2^{\,t}) \cdot Q_2^{\,t} - C_2^{\,t}(Q_2^{\,t})$ is strictly concave in $Q_2^{\,t}$. If this sufficient condition is violated anywhere care must be exercised to determine if each plant is globally maximizing in the solution selected by the computer.

are manipulated in any period to determine $Q_i^t$ (and $\alpha_i^t$), given $\lambda_1$, $\lambda_2$ *and* $P^t$. Set $\alpha_1^t = 0$, satisfying (2a) and (2c). Set $L_{Q_1^t} = 0$, satisfying (1b) and (1c). Solve this equation for $Q_1^t$. Three situations may arise:

a. $0 \leq Q_1^t \leq Q_1^t\text{max}$
b. $Q_1^t < 0$
c. $Q_1^t > Q_1^t\text{max}$.

In the first case, (1a) and (2b) hold so that *each* part of conditions (1) and (2) is satisfied.

In the second case, (1a) would be violated. To satisfy (1a) and (1c), set $Q_1^t = 0$. Since $L_{Q_1^t}$ is decreasing in $Q_1^t$, (1b) will hold with strict inequality; furthermore, (2b) will also hold. Hence, (1) and (2) will be completely satisfied.

Finally, in the third case (2b) would be violated. To satisfy (2b) as well as (2c) and (1a) set $Q_1^t = Q_1^t\text{max}$. (1c) implies $L_{Q_1^t} = 0$. Thus (1b) is satisfied. Solve $L_{Q_1^t} = 0$ for $\alpha_1^t$. Since $L_{Q_1^t}$ is increasing in $Q_1^t$ and $\alpha_1^t$, the $\alpha_1^t$ determined will be strictly positive, satisfying (2a). Hence (1) and (2) will be completely satisfied for the competitor.

The conditions of the Nash player are manipulated in parallel fashion to determine in Step 1b $Q_2^t$ (and $\alpha_2^t$, $\gamma_2^t$) given $\lambda_1$, $\lambda_2$ and $P^t > 0$. From (4c), $L_{P^t} = 0$ which implies that $\gamma_2^t = \beta^{t-1}Q_2^t/-D_t'(P^t)$. Rewrite $L_{Q_2^t} = \beta^{t-1}\{P^t + Q_2^t/D_t'(P_t) - C_2^{\prime t}(Q_2^t)\} - \alpha_2^t - \lambda_2$. We can now solve for $Q_2^t$, $\alpha_2^t$ using (1) and (2) exactly as we solved for $Q_1^t$, $\alpha_1^t$ for the competitor. We can then use (4) to determine $\gamma_2^t$.

We now consider how the trial price is revised in Step 1c. If the demand forthcoming at $P^t$ does not match the sum of the supplies calculated in Step 1b for each plant, (5) will be violated. In that case, $P^t$ must be altered and the calculation of production in period $t$ repeated.

The following algorithm is used to locate the market-clearing price in each period. We begin with an arbitrarily specified price and a wide initial interval around it. Given this price, excess demand is calculated. The calculation determines how the interval and trial price are revised. In particular, if the excess demand is positive (negative), the price is increased (decreased) so that it lies midway between the previous trial price and the previous upper (lower) limit. The previous trial price then becomes the new lower (upper) limit while the upper (lower) limit remains unchanged. The procedure is then repeated using the new trial price and the new interval until convergence occurs. Once the market-clearing price is determined, the time period is advanced and a trial price for the new period is picked as a basis for the new supply calculations of Step 1b.

After the calculations for period $T$ are completed, the transition to Step 2 occurs. At this point, the cumulative sales along the production path of each plant are calculated and compared to the reserves owned by that plant. If the reserves utilized by any plant with a positive multiplier exceed (fall short of) the reserves available, the transversality condition (3) is violated and the set of multipliers assigned to the various plants must be revised. The entire process is then repeated beginning at $t = 1$.

Appendix 1 ensures under weak conditions that a set of multipliers does exist which satisfies (3) for each plant. Indeed, several such fixed points might exist. Appendix 2 establishes, for the case of linear demand, that a unique set of multipliers exists and that a simple gradient algorithm can locate it. It should be noted, however, that the algorithm currently used in the computer code to locate the equilibrium $\{\lambda_i\}$ is heuristic. There is no guarantee that this algorithm will converge. Although convergent algorithms are available, the team responsible for implementing the model on the computer judged the heuristic to be faster and, for the intended applications, preferable. In all simulations thus far, it has converged. A description of it is contained in Salant et al. [1981].

The procedure outlined above can be applied without alteration if there are $m$ competitive plants and $n$ Nash plants, each owned by a separate player. Given $P^t$, the determination of production at *each* of the $m + n$ plants still requires only the analysis of a single Kuhn-Tucker condition. If each marginal cost curve is nonlinear, $Q_i^t$ must be determined by some form of iterative procedure (e.g. Newton's method). If the marginal cost functions are linear, however, each condition can be solved explicitly for $Q_i^t$, even when the demand function is nonlinear. In the current version of the model the assumption of linear marginal cost curves has been adopted to speed computation.

A minor complication arises when some Nash players own more than one plant. In Step 1b—given $P^t$—production is calculated in the usual way for competitive plants and plants owned by single-plant Nash players. But if a player owns several plants, the solution to $P2$ indicates that the player should produce at each plant such that the marginal cost plus the user cost $(\beta^{1-t}\lambda_i)$ at each plant is equated to the common marginal revenue the player would obtain from selling another unit. This common marginal revenue equals the price per unit *minus* the losses on the *sum* of the inframarginal units produced by all the plants under the player's control. The losses occur because selling an additional unit from one of the player's plants depresses the price slightly.

To calculate production at each plant owned by a particular multiplant Nash player—given $P^t$—the following approach could be adopted.[15] First, a trial value for the common marginal revenue is selected. Production at

[15] In fact, the computer program uses a variation of this approach which will always converge. For details, see Chapter 4 of Salant et al. [1981].

each plant is then calculated and aggregated. If the Nash player sold this aggregate production at price $P^t$, the extra revenue from additional production could easily be computed. Should this value differ from the trial marginal revenue assumed initially, the trial value must be revised and the step repeated until convergence occurs. Once convergence is achieved and production at each plant is computed, the remaining steps of the solution procedure can be executed without modification.

## 3. FEATURES OF THE MODEL

Within this simple structure of plants and players, various features have been introduced to facilitate application of the model.

### Depletion Effects

In reality, the oil controlled by a given player is not of uniform quality and accessibility, therefore, all oil cannot be extracted at the same average cost. Depletion effects can be incorporated in the model by treating oil that can be extracted at a particular marginal cost as the reserves of a separate plant (whose marginal cost curve perhaps has a negligible upward slope); this plant is then assigned either to the competitive sector or to one of the large players.

In this approach, a player is not assumed to be *constrained* to extract from the pools assigned to the player in a particular sequence. Nonetheless, it is optimal, in the absence of flow constraints or increasing marginal costs, to extract from the cheapest pool first.[16] Hence, if the player were constrained to extract from the cheapest source first, the imposition of the constraint would not affect the player's optimal strategy. Nature itself sometimes imposes this constraint by making the oil which is deepest in the ground the most difficult to extract.

### Perfect Substitutes in Use

Oil produced from liquefied coal can be made virtually indistinguishable from ordinary petroleum. Consequently, consumers would attempt to choose one to the exclusion of the other if their prices differed; but the resulting excess supply of the more expensive fuel would depress its price until the two fuels were equally expensive. In such a case, a barrel of liquefied coal and a barrel of petroleum would sell at the same price, and a liquefied coal plant could be treated as if it were an oil plant.

[16] Intuitively, it is sensible to extract from the more expensive pool first only if its "user cost" is cheap enough to more than compensate for the higher extraction cost of the pool. But in that case it is *always* optimal to extract from the more expensive pool instead of the cheaper pool and the cheaper pool must be assigned a zero user cost. But then it would be impossible to assign a strictly smaller (nonnegative) "user cost" to the more costly pool.

The only difficulty with this approach is the specification of the marginal cost curve of such a plant. In reality, it depends not only on the technology of liquefaction but also on the price of hard coal (an imperfect substitute for oil). The path of coal prices must therefore either be specified exogenously or determined within the model. Each of these approaches has been pursued separately. In the model described in the text,[17] an exogenous path of coal prices is used outside the model to construct the marginal-cost curve for the backstop technology.

Like liquefied coal, natural gas is treated as a perfect substitute for oil. Heat can be produced for industrial use from natural gas or oil. It is assumed that $M$ units of oil can replace 1 unit of gas in the production of any given amount of heat, and that this constant marginal rate of technical substitution drives the relative factor costs into the same ratio. Hence, oil and gas prices are linearly related:

$$M = (P_{\text{gas}} + C_P^{\text{gas}})/(P_{\text{oil}} + C_P^{\text{oil}}).$$

It should be noted that factor costs to industrial users include not only the price paid to extractors ($P$) but the additional costs (refining, transport, etc.) of getting the input to the factory where the production of heat occurs. These latter costs are denoted as $C_P^{\text{gas}}$ and $C_P^{\text{oil}}$.

### Imperfect Substitutes in Use

The prices of energy sources which are imperfect substitutes for oil (coal, nuclear power, etc.) are not rigidly linked to the oil price and must be forecasted simultaneously. To analyze such cases, the model of the text has been generalized. The generalization utilizes the same equilibrium concept, the same plant-player distinction, and the same solution approach. Its output, however, consists of a price path for each imperfect substitute. The imperfect-substitutes model is described in more detail in Appendix 3.

### "Backstop" Marginal Costs

In the single-product model and most other oil models, the marginal cost of producing liquefied coal must be specified exogenously. This is methodologically sound only if liquefaction costs do not depend on oil prices. *In reality*, however, the cost of liquefying coal depends on the price of coal. And the price of coal is affected by the price of oil, an imperfect substitute. It is commonplace but erroneous to ignore the fact that backstop marginal costs *depend* indirectly on oil prices. When the oil market tightens and oil prices rise, the prices of substitutes like coal also rise, and coal liquefaction becomes more costly than it was formerly.

[17] See Appendix 3.

This methodological error can be avoided in the multiproduct model since the price of coal is determined endogenously. The multiproduct model requires as input a specification of the liquefaction technology for transforming coal into a perfect substitute for oil. Since the model determines the price path for coal endogenously it also determines the path of the backstop marginal cost curve.

### Uncertainty

The model can forecast the consequence of two types of uncertainty: (1) uncertainty that, with constant probability per period, an oil plant will be nationalized without compensation or an asset alternative to oil will be confiscated; and (2) uncertainty, to be resolved at a known date, about which of several outcomes will subsequently occur.

The first type is equivalent to a change in the interest rate under certainty. The extractor whose plant is in danger of expropriation acts exactly as one would in the absence of uncertainty if the interest rate were higher, while an extractor whose alternative assets may be confiscated acts as one would under certainty if the interest rate were lower.[18]

The second type of uncertainty applies to any situation where the date of resolution of the uncertainty is known but the outcome is not. Suppose there is uncertainty about which of $\hat{n}$ paths of a particular exogenous parameter will obtain after some date ($y$), and that the uncertainty will be resolved on a known prior date ($\hat{t} \leq y$). The model would then forecast a price path and extraction paths for each plant through date $\hat{t}$ and $\hat{n}$ sets of such paths from date $\hat{t}+1$ onward—a contingent forecast for each of the $\hat{n}$ events which might be realized. Consider the forecast over the entire horizon given a particular state $s$ is realized at the end of period $\hat{t}$. Then the forecast (for any $s$) will pass the two feasibility and the consistency tests outlined on pp. 255–256. Moreover, no decision-maker will conjecture that *expected* discounted profits can be increased by behaving differently than in the contingent ways forecasted. For example, competitive plant $i$ faced with price path $\{\bar{P}^t\}$ through $\hat{t}$ and a distribution of $\hat{n}$ possible price paths $\{\bar{P}_s^t\}$ thereafter will plan to extract $\{Q_i^t\}$ through

[18] If the time until expropriation is exponentially distributed, it has been shown by Long [1975] that the additional risk is equivalent to an interest rate increase. For example, if the risk of field expropriation is 10% per year and the real return on safe financial assets is 4%, oil extractors will act exactly as they would in the absence of field-expropriation risk if the real rate of interest were 16% (1.04/0.9 − 1). Indeed, Mead [1979] has argued that the historic 4-fold increase in oil prices is *entirely* attributable to the removal of nationalization risk from a *competitive* world oil industry rather than from exertion of market power by OPEC. Alternatively, if there is no risk of field expropriation, but instead there is a risk that the extractor's financial assets may be confiscated, the additional risk is equivalent to an interest rate decrease. For example, if OPEC can earn $r\%$ abroad on an asset if it is not confiscated, and nothing otherwise; its expected return is only $[(1-\alpha)(1+r)-1]$, where $\alpha$ is the probability per year that the asset will be confiscated.

$\hat{t}$ and $\{Q_{is}^t\}$ thereafter, depending on which of the $\hat{n}$ states is realized, where

$$\{Q_i^t\},\ \{Q_{is}^t\}\text{maximize } \sum_{t=1}^{\hat{t}} \beta^{t-1}\{\bar{P}^t Q_i^t - C_i^t(Q_i^t)\} + \sum_{s=1}^{\hat{n}} \pi_{is} \sum_{t=\hat{t}+1}^{T} \beta^{t-1}\{\bar{P}_s^t Q_{is}^t - C_{is}^t(Q_{is}^t)\}$$

$P1'$ subject to
feasibility conditions

$$0 \le Q_i^t \le Q_i^t\text{max}, \quad t = 1, \hat{t}$$

$$0 \le Q_{is}^t \le Q_{is}^t\text{max}, \quad t = \hat{t}+1, T, \quad s = 1, \hat{n}$$

$$\sum_{t=1}^{\hat{t}} Q_i^t + \sum_{t=\hat{t}+1}^{T} Q_{is}^t = U_{is} \le R_i, \quad s = 1, \hat{n}$$

where $\pi_{is}$ is the evaluation of competitive player $i$ that state $s$ will occur. A similar modification can be made to $P2$, the maximization problem for a typical Nash player. The model solves these uncertainty problems within the structure outlined above.[19]

One application of this uncertainty feature has been the evaluation of R&D programs. The cost of a future substitute is uncertain. At the existing intensity of R&D, it is assumed this uncertainty will be resolved at a known future date. A more intense R&D effort is assumed to shorten the period of uncertainty by a determinate amount without affecting the various possible outcomes or their odds of occurring. Under these assumptions, the model has been used to quantify the expected benefits from intensified R&D programs and to analyze the distribution of these expected benefits among market participants.[20]

It is important to note that players may disagree about their subjective

[19] Solutions to such maximization problems involve a single extraction path for each plant through date $\hat{t}$ and then $\hat{n}$ extraction paths thereafter, i.e., one for each state which might occur at date $\hat{t}$. Cumulative extraction along the two phases must equal $R$. Associated with the extraction path from date zero to $\hat{t}$ is a multiplier ($\lambda$); associated with each extraction path from date $\hat{t}+1$ onward is a state-dependent multiplier ($\lambda_s$). If the program is optimal, the multiplier associated with the first phase capitalized to $\hat{t}$ must equal the weighted average of the $\hat{n}$ state-dependent multipliers of the second phase. The weights are subjective probabilities and may differ among players. The optimality condition ensures that no possibilities exist for a player to arbitrage between the phase of the program prior to the realization and the phase subsequent to the realization. The equilibrium is located in the following way. As usual, each plant is assigned an initial multiplier. The implied extraction paths are calculated through date $\hat{t}$. Using the remaining stocks at each plant and following the approach on p. 261, the model calculates from date $\hat{t}+1$ the equilibrium multipliers for each plant in each of the $\hat{n}$ possible states. Finally, a test is conducted at each plant to see if the initial multiplier capitalized to $\hat{t}$ equals the expected multiplier in the second phase. If not, the assignment of initial multipliers is judiciously revised and the entire procedure is repeated until convergence occurs. For further details, see Salant et al. [1981].

[20] See the Appendix to Chapter 2 of Salant et al. [1981].

evaluations about which state will occur. This accounts for subscript $i$ in $\pi_{is}$: it is the evaluation *by the ith player* that state $s$ will occur. It need not accord with either the judgments of other players or the objective probability (if such an entity exists). In the R&D application, for example, one player may feel the low cost technology is likely to be implemented while another assigns little weight to this event.

The point is important because models of this type are often mistakenly criticized as assuming "perfect" foresight. Confusion has arisen because it is often assumed *for convenience* that all agents assign the same weight to a given outcome ($\pi_{is}$ is independent of $i$). If the common subjective distribution matches the objective distribution, expectations are said to be "rational." If the common subjective distribution assigns probability one to a particular state and it occurs, the event is said to be "fully anticipated"—otherwise it is said to be "unanticipated." But these are special cases of a more general formulation. The model can also forecast the equilibrium when people evaluate future events differently. It is a weakness of the model, however, that these possibly different expectations are not *revised* as market experience accumulates.

### Unanticipated Changes

Suppose, instead, that agents were "confident" of the paths of the exogenous variables over time, but at time $x$ it became clear that some future segment of one of the exogenous paths would be different than originally anticipated. Such situations of surprise can be analyzed within the uncertainty provision of the model. Every player is assumed to attach 100% of the probability weight to the path originally anticipated and 0% to the path which ultimately results.

## 4. EXAMPLE

The purpose of this section is to indicate by example how the model may be applied to the international energy market. Extensive discussion of other experiments is reported elsewhere.[21]

The experiments summarized here were conducted several years ago (prior to the upheaval in Iran) to evaluate the potential impact on price of U.S. energy policies aimed at reducing domestic demand by 1 million barrels per day (roughly 1% of world consumption). The assumption of unconstrained backstop supply at prices in excess of $30, then regarded as the conventional wisdom of oil market "experts," severely depressed the forecasted prices. For each case run, a comparable experiment was conducted with the demand curve in each period shifted to the left by the amount of 0.365 billion barrels per day. The experiments were run

[21] See footnote 1.

TABLE I

RESOURCE COSTS

| Stock Marginal Cost[a] | Resources[b] | Plant | |
|---|---|---|---|
| 8.0 | 30.0 | US | Oil |
| 10.0 | 35.0 | US | Oil |
| 15.0 | 95.0 | US | Oil |
| 20.0 | 95.0 | US | Oil |
| 25.0 | 90.0 | US | Oil |
| 1.5 | 179.0 | US | Gas |
| 2.0 | 302.0 | US | Gas |
| 3.0 | 249.0 | US | Gas |
| 4.0 | 235.0 | US | Gas |
| 5.0 | 61.0 | US | Gas |
| 4.0 | 93.0 | ROW | Oil |
| 5.0 | 93.0 | ROW | Oil |
| 10.0 | 218.0 | ROW | Oil |
| 15.0 | 209.0 | ROW | Oil |
| 20.0 | 332.0 | ROW | Oil |
| 25.0 | 326.0 | ROW | Oil |
| 0.8 | 214.0 | ROW | Gas |
| 1.0 | 281.0 | ROW | Gas |
| 2.0 | 342.0 | ROW | Gas |
| 3.0 | 354.0 | ROW | Gas |
| 4.0 | 265.0 | ROW | Gas |
| 5.0 | 61.0 | ROW | Gas |
| 1.0 | 600.0 | OPEC | Oil |
| 5.0 | 178.0 | OPEC | Oil |
| 10.0 | 119.0 | OPEC | Oil |
| 15.0 | 62.0 | OPEC | Oil |
| 0.1 | 1736.0 | OPEC | Gas |
| 1.0 | 515.0 | OPEC | Gas |
| 2.0 | 345.0 | OPEC | Gas |
| 3.0 | 179.0 | OPEC | Gas |
| 4.0 | 60.0 | MEX | Oil |
| 5.0 | 40.0 | MEX | Oil |
| 10.0 | 35.0 | MEX | Oil |
| 15.0 | 30.0 | MEX | Oil |
| 20.0 | 35.0 | MEX | Oil |
| 0.5 | 96.0 | MEX | Gas |
| 1.0 | 65.0 | MEX | Gas |
| 2.0 | 56.0 | MEX | Gas |
| 3.0 | 48.0 | MEX | Gas |
| 4.0 | 39.0 | MEX | Gas |

[a] For oil, in 1975 dollars per barrel; for gas, in 1975 dollars per thousand cubic feet.

[b] For oil, in billions of barrels; for gas, in trillions of cubic feet.

using both linear and nonlinear (constant elasticity) demand curves. All prices and costs are in 1975 dollars. To convert to 1980 dollars, multiply by 1.45.

The input data are summarized below:

1. *Market Structure:* Mexico and OPEC are large, multiplant Nash players; the United States and the rest-of-world (ROW) are competitive
2. *Real Interest Rate:* 5%
3. *Backstop Cost and Availability:* Always available at \$30
4. *Flow Constraints:* None
5. *Demand Curve:* Linear: pivots out around the fixed vertical intercept of $13(1 + 1/\eta)$ over time, where $\eta$ is the point elasticity at the \$13 price. The demand curve for $t = 0$ passes through the point $P = 13$ (1975 dollars per barrel), $Q = 32.8$ (billion barrels per year)

$$Q = (1.0274)^t\{32.8 + 13M_0 - M_0P_t\}, \quad t = 1, 2, \cdots 8,$$
$$(1.0411)^{t-8}\{40.717 + 16.138M_0 - 1.2414M_0P_t\}, \quad t = 9, 10 \cdots$$

where $M_0 = 2.523\eta$ and $\eta = 0.3$.
6. *Resource Costs:* See Table I

The real price of oil in 1978 forecasted under these assumptions was \$12.09 (in 1975 dollars). This translates to approximately \$17.50 in 1980 dollars. Conservation was expected to affect the world oil price little because (1) marginal cost curves were assumed to be elastic, (2) no flow constraints were imposed, and (3) the amount conserved was minor relative to the size of world consumption. On the other hand, a leftward shift in demand for an exhaustible resource does have one depressing effect on price which is not present if the good is inexhaustible. A cut in demand lowers the value of oil in the ground ($\lambda_i$) and hence induces more rapid pumping at any price. In the simulations conducted, conservation of 1 million barrels per day lowered the predicted price in that year insignificantly—to \$12.04. The effect of conservation under alternative market structures is summarized below:

| | Predicted Real Price of Oil in 1978 (in 1975 dollars) | |
|---|---|---|
| | Unshifted demand | Demand shifted leftward 0.365 billion barrels per year |
| Perfect competition | 7.82 | 7.78 |
| Reference case | 12.09 | 12.04 |
| OPEC controls all but backstop | 30.00 | 29.87 |
| OPEC controls even the backstop[22] | 30.49 | 30.24 |

[22] When OPEC is assigned all plants except the backstop, the price begins at \$30 and *remains* there. When OPEC owns even the backstop, the price begins slightly higher but then rises over time.

It is evident that the predicted 1978 price was extremely sensitive to market structure, but regardless of structure it was relatively insensitive to the conservation effort.

The demand curve used in the preceding experiments was linear. To evaluate the importance of this assumption, the experiment was repeated using the following constant-elasticity specification:

5′. *Demand Curve:* Constant elasticity: The demand curve for $t = 0$ passes through the point $P = 13$, $Q = 32.8$ and shifts out over time at the same rate as the linear curve. Hence, if $\eta$ is set the same, the linear and constant elasticity curves will be tangent at $P = 13$ on each period.

$$Q = (1.0274)^t\{32.8\}\{13^\eta P^{-\eta}\}, \quad t = 1, 2, \cdots 8$$

$$(1.0411)^{t-8}\{40.717\}\{13^\eta P^{-\eta}\}, \quad t = 9, 10.$$

Using this specification ($\eta = 0.3$) the real price of oil forecasted for 1978 was \$12.80 under the market structure described in assumption 1. (The change in functional form turns out to matter little when an unlimited backstop is available at a relatively low price.) A leftward shift in demand of 1 million barrels a day at all prices cut the predicted 1978 price to \$12.78. Hence, the forecasted price remained insensitive to the conservation effort when the functional form of the demand curve was changed to constant elasticity.

## 5. EXTENSIONS

Since conceptions of the world oil market differ, construction of a model useful to everyone is impossible. Inevitably, features of the market that someone feels "essential" will be neglected. The model here is no exception. In particular, it does not yet take proper account of:

—Lags in supply
—Lags in demand
—Transportation and distribution costs.

However, each of these features can be handled within the existing framework without a major alteration in the computer program. Lags in supply can be handled by making investment in capacity expansion endogenous. This involves little more than the introduction of one more multiplier to represent the discounted future value of having a larger capacity from some point onward, as discussed in Switzer and Salant [1979].[23] If appropriate separability is assumed, lags in demand can be handled by running the model in reverse—beginning at a terminal date

[23] Indeed, this extension may be implemented in the near future.

with an arbitrary assignment of multipliers on resource constraints and moving backward to 1978. Inclusion of transportation costs involves the introduction of a wedge to raise above the export price the price of a good which, in equilibrium, is imported. Such modifications present no conceptual problems but limited resources and a desire to keep the computer program manageable dictated that work on them be deferred.

As a final note, it should be remembered that although the model was constructed to examine cartel behavior in the international energy market, this is only one application. Whenever one is thinking within the Hotelling framework about an exhaustible resource problem, consideration should be given to applying the model described in this paper to the problem at hand.

## APPENDIX 1: EXISTENCE OF EQUILIBRIUM

It can be shown (under weak assumptions) that each assignment of multipliers to the $n$ plants determines (through first-order conditions like those on pp. 260–261) a market price and extraction rate[24] at each plant in each time period. Consequently, each assignment of multipliers determines cumulative extraction ($U_i$) at each plant over the specified time horizon ($T$).

However, if a player's extraction policy is optimal, it must also satisfy transversality conditions for each plant:

$$\lambda_i \geq 0, \quad R_i - U_i(\lambda_1, \cdots \lambda_n) \geq 0, \quad \text{and} \quad \lambda_i\{R_i - U_i\} = 0, \quad i = 1, n.$$

That is, either a plant uses all of its reserves or its resource constraint has a zero shadow price. We now prove that a set of multipliers satisfying these transversality conditions exists.[25]

Consider $n$-tuples $(\lambda_1, \cdots \lambda_n)$ such that $P^c \geq \lambda_i \geq 0$, where $P^c$ is the largest vertical intercept of the $T$ demand curves (that is, the lowest price which induces zero demand in every period).[26] Clearly, the set of all such $n$-tuples is closed, bounded, and convex.

Next consider a transformation which takes a trial assignment $(\lambda_1, \cdots \lambda_n)$ and revises it to $(\lambda_1', \cdots \lambda_n')$ based on the excess of $U_i$ over $R_i$:

$$\lambda_i' = \min\{P^c, \max(0, \lambda_i + \beta[U_i(\lambda_1, \cdots \lambda_n) - R_i])\}, \quad i = 1, n \quad \text{for any} \quad \beta > 0.$$

In words,

1. If the old $\lambda_i$ resulted in overexhaustion, set the new $\lambda_i' = \lambda_i + \beta(U_i - R_i)$ or $P^c$, whichever is smaller.
2. If the old $\lambda_i$ resulted in underexhaustion, set the new $\lambda_i' = \lambda_i + \beta(U_i - R_i)$ or 0, whichever is larger.

[24] The proof does not apply to cases where competitive plants have constant marginal extraction costs because this creates an indeterminacy in supply at such plants. Some existence results for these cases can be found in Salant ([1976], p. 1090) and Lewis and Schmalensee ([1979], Section 2.3). The proof here *does* apply, however, when the marginal cost curves of competitive plants have slopes which are arbitrarily close to zero.

[25] See Appendix 2 for a discussion of uniqueness.

[26] Such an intercept is assumed to exist.

Denote the transformation by which each new $\lambda_i'$ is computed as $T_i(\lambda_1, \cdots \lambda_n)$. That is, $\lambda_i' = T_i(\lambda_1, \cdots \lambda_n)$.

It can be verified that $T_i$ is a continuous function whose image lies in the interval $[0, P^c]$. Since $T$ is a continuous transformation from a non-empty compact convex set into itself, Brouwer's fixed point theorem ensures that there exists a point $(\lambda_1^*, \cdots \lambda_n^*)$ which remains unchanged under the transformation. That is

$$\lambda_i^* = T_i(\lambda_1^*, \cdots \lambda_n^*), \quad i = 1, n$$

or,

$$\lambda_i^* = \min\{P^C, \max(0, \lambda_i^* + \beta[U_i(\lambda_1^*, \cdots \lambda_n^*) - R_i])\}, \quad i = 1, n.$$

Analysis of this equation reveals three ways it might be satisfied:

a. $\lambda_i^* = 0$ and $U_i(\lambda_1^*, \lambda_2^*, \cdots, 0, \cdots, \lambda_n^*) - R_i \leq 0$,
b. $\lambda_i^* = P^C$ and $U_i(\lambda_1^*, \lambda_2^*, \cdots, P^C, \cdots, \lambda_n^*) - R_i \geq 0$, or
c. $0 < \lambda_i^* < P^C$ and $U_i(\lambda_1^*, \lambda_2^*, \cdots, \lambda_i, \cdots, \lambda_n^*) - R_i = 0$ for $i = 1, n$.

In words, each component of the fixed point must have one of the following characteristics:

a. $\lambda_i^* = 0$ and yet underexhaustion still occurs.
b. $\lambda_i^* = P^C$ and yet overexhaustion still occurs.
c. $\lambda_i^* \in (0, P^C)$ and cumulative extraction matches the initial stock.

Although each component of the fixed point must satisfy one of these three conditions, other considerations imply that it will never satisfy the second condition. For, if $\lambda_i^* = P^C$ the user cost for the $i$th plant would be so high that no extraction would occur in any period (condition (1) on p. 260 would imply $Q_i^t = 0$). Hence cumulative extraction would be zero and could not exceed the plant's reserves (assumed without loss of generality to be strictly positive).

Therefore, there exists an $n$-tuple of Lagrange multipliers which is unchanged under the transformation, $T$. For this to be the case, a zero multiplier must be assigned to plants with reserves to spare ($R_i \geq U_i$) and a strictly positive multiplier to plants which exhaust their reserves ($R_i = U_i$). Such a fixed point, therefore, would satisfy the transversality conditions for each plant.

Since all other first-order conditions for optimality hold by construction, satisfaction of the transversality conditions is sufficient to establish the existence of equilibrium *provided* all agents have strictly concave objective functions.[27] In that case the foregoing proof establishes the existence of a Nash-Cournot equilibrium (and, *a fortiori*, of monopolistic and competitive equilibrium). The method of proof used here seems more satisfactory than the equation counts of Goldsmith [1974] and more general than the proofs of Salant [1976] and Lewis and Schmalensee [1979].[28]

[27] If they do not, the computer may forecast strategies which satisfy all the necessary conditions for optimality but which are suboptimal (see footnote 11).

[28] The proof in this appendix was originally stated in Salant [1977]. It turns out that Lewis and Schmalensee independently approached the existence question in the same way but omitted the proof from their 1979 paper because they were unable to prove to their satisfaction that $U_i(\lambda_1, \cdots, \lambda_n)$ is continuous.

## APPENDIX 2: UNIQUENESS OF EQUILIBRIUM AND CONVERGENCE OF ALGORITHMS

The fixed-point theorem of Appendix 1 neither establishes the uniqueness of the Nash equilibrium nor demonstrates that algorithms exist which will converge to it. Preliminary research suggests the merits of an indirect approach to these questions.

Given the exogenous data of the problem, suppose that a function can be constructed which attains a constrained local maximum at each Nash equilibrium. Then, a demonstration that this function is strictly concave and the constraint set is convex would imply that only one local maximum existed, and hence only one Nash equilibrium. Furthermore, it is well known that under these concavity assumptions algorithms exist which will converge to the constrained global optimum and hence to the unique Nash equilibrium.[29]

We illustrate this approach for the case of linear demand. Readers interested in extending the results to other cases will find helpful the related work of Spence [1976]. Suppose there are $m$ large players and $n$ plants, indexed by integers 1, 2, $\cdots$ $n$. Let $\Omega_j$ be the set of integers of plants controlled by large player $j$. Let $\Omega_0$ be the set of integers of plants assigned to the competitive fringe. Suppose the inverse demand curve at $t$ is $P^t = F^t - A^tQ^t$, where $A^t$, $F^t > 0$. Then ignoring corners for ease of exposition, the first-order conditions which define the Nash equilibrium are:

1. For competitive plants:

$$\beta^{t-1}\{F^t - A^t \textstyle\sum_{k=1}^{n} Q_k{}^t - C_2'{}^t(Q_i{}^t)\} - \lambda_i = 0, \qquad i \in \Omega_0,\ t = 1, \cdots T$$

2. For plants owned by Nash players:

$$\beta^{t-1}\{F^t - A^t \textstyle\sum_{k=1}^{n} Q_k{}^t - A^t \sum_{k\in\Omega_j} Q_k{}^t - C_2'{}^t(Q_i{}^t)\} - \lambda_i = 0,$$

$$i \in \Omega_j,\ j = 1, \cdots m,\ t = 1, \cdots T,\ i = 1, \cdots n$$

and $R_i - \sum_{t=1}^{T} Q_i{}^t = 0$. The price in each period is simply $P^t = F^t - A^t \sum_{k=1}^{n} Q_k{}^t$. Sale of an additional unit depresses the price by $A^t$. As the first-order conditions indicate, the competitor evaluates the additional revenue and costs from selling an additional unit without regard to the losses on the inframarginal units which the reduction of the price by $A^t$ will induce. The large Nash players take this additional factor into account through the additional term

$$-A^t \textstyle\sum_{k\in\Omega_j} Q_k{}^t.$$

Now consider the following artificial problem:

$$\text{maximize}_{\{Q_i{}^t\}} \textstyle\sum_{t=1}^{T} \beta^{t-1} M^t(Q_1{}^t, \cdots Q_n{}^t)$$

$$\text{subject to } Q_i{}^t \geq 0$$

$$Q_i{}^t\text{max} - Q_i{}^t \geq 0, \quad i = 1, \cdots n,\ t = 1, \cdots T,$$

[29] See Arrow et al. [1958] and Zangwill [1969]. It should be noted that the algorithm currently used in the model is heuristic and has not been proved to converge even for linear demand. This algorithm was selected because it was felt to be more satisfactory over a wide class of demand specifications.

and $$R_i - \sum_{t=1}^{T} Q_i^t \geq 0, \quad i = 1, \cdots n$$

where we define $M^t$ as follows:

$$M^t = F^t \sum_{k=1}^{n} Q_k^t - (A^t/2)\,[(\sum_{k=1}^{n} Q_k^t)^2 + \sum_{j=1}^{m} (\sum_{k\in\Omega_j} Q_i^t)^2] - \sum_{k=1}^{n} C_k^t(Q_k^t).$$

The objective function is strictly concave since $M^t$ is the sum of a finite number of concave and strictly concave functions (assuming strict convexity of the cost functions).[30] The constraint set is linear and hence convex. Finally, it should be noted that the sensitivity of $M^t$ to an increase in $Q_i^t$ depends on the industrial position of the $i$th plant. In particular:

1. For competitive plants:

$$\partial M^t/\partial Q_i^t = F^t - A^t \sum_{k=1}^{n} Q_k^t - C_2'^t(Q_i^t). \quad i \in \Omega_0,\ t = 1, \cdots T.$$

2. For plants owned by Nash players:

$$\partial M^t/\partial Q_i^t = F^t - A^t \sum_{k=1}^{n} Q_k^t - A^t \sum_{k\in\Omega_j} Q_i^t - C_2'^t(Q_i^t).$$
$$i \in \Omega_j,\ j = 1, \cdots m,\ t = 1, \cdots T.$$

It should be evident that the necessary conditions for a maximum to this constrained optimization problem are identical to the conditions defining the Nash equilibrium. Because the artificial function is strictly concave over a convex set, the Nash equilibrium is unique and convergent algorithms can be devised to locate it.

For nonlinear demand curves, it might turn out that there are multiple Nash equilibria. If so, it will be impossible to construct a strictly concave function which attains a local maximum at each Nash equilibrium.

## APPENDIX 3: THE IMPERFECT-SUBSTITUTES MODEL

Nuclear energy and (unliquefied) coal are imperfect substitutes for oil. Hence, the prices of these alternative fuels are not rigidly linked to the price of oil although the magnitudes of their supplies certainly affect the oil price. To analyze such cases, the "single-product" model has been generalized. A "multiproduct" model has been built which utilizes the same equilibrium concept, the same plant-player distinction, and the same solution approach. The output of the multiproduct Nash-Cournot model includes, in addition to the output previously described, a price path for *each* substitute for oil which is considered. A system of demand curves, each a function of all the energy prices, replaces the single demand curve of the smaller model. The forecasts of the multiproduct model pass each of the elementary tests of pp. 255–256 in the following sense.

A competitive plant extracting fuel type $K$ makes larger discounted profits (evaluated along the forecasted path of prices for fuel type $K$) by extracting in the way forecasted rather than in any other feasible fashion. A competitor's optimization problem is similar to $P1$ (with $\bar{P}^t$ replaced by $\bar{P}_K^t$) and need not be repeated.

[30] The square of a sum is convex, and its negative is concave since it is a convex function (the square function) of an increasing convex function (the sum function).

Nash play $j$ may own sets of plants producing different fuel types. Let $\Omega_{j,K}$ denote the set of plants controlled by player $j$ which produce fuel type $K$, for $K = 1, 3$. Each Nash player makes larger joint discounted profits (evaluated assuming plants not under the player's control produce as forecasted) by extracting at plant $i$ in the way forecasted rather than in any other (feasible) fashion. That is, price and extraction paths are chosen to

$$\text{maximize}_{\{P_K{}^t\},\{Q_i{}^t\}} \sum_{K=1}^{3} \sum_{t=1}^{T} \sum_{i\in\Omega_{j,K}} \beta^{t-1}\{P_K{}^t Q_i{}^t - C_i{}^t(Q_i{}^t)\}$$

$P3$ subject to
feasibility conditions

$$0 \le Q_i{}^t \le Q_i{}^t\text{max} \quad \text{for } i \in \Omega_{j,K},\ t = 1, T \quad \text{and} \quad K = 1, 3$$

$$\sum_{t=1}^{T} Q_i{}^t = U_i \le R_i \quad \text{for } i \in \Omega_{j,K} \quad \text{and} \quad K = 1, 3$$

and the market-clearing
equations implicitly determining
the three prices:

$$\sum_{i\in\Omega_{j,K}} Q_i{}^t = D_K{}^t(P_1{}^t, P_2{}^t, P_3{}^t) - \sum_{i\in\bar{\Omega}_{j,K}} \bar{Q}_i{}^t \quad \text{for } t = 1, T \quad \text{and} \quad K = 1, 3$$

where $\bar{\Omega}_{j,K}$ is the set of plants producing fuel type $K$ which are not controlled by player $j$.

In a multiproduct world, the Nash player must take careful account of *interfuel* substitution by users. Since the Nash player accepts as given production by rivals, the player believes an increase in its own production at a plant producing one fuel type will disturb the prices of all three fuel types. This belief influences profit-maximizing strategy.

The "multiproduct" model has not yet been fully implemented on the computer. In particular, multiplant players can only own combinations of gas and oil plants. They cannot also own coal plants. Furthermore, since depletion effects are handled in the manner previously described, there is no place in the model for a large Nash player (as distinguished from a set of competitors) which owns different grades of coal. This shortcoming will soon be eliminated.

On the positive side, however, the larger model does calculate endogenous price paths for oil, coal and nuclear energy.[31] And it does have a competitive liquefaction process which operates when oil becomes sufficiently expensive relative to coal. Only if this "backstop" process is operating (and is unconstrained) do the prices of coal and oil become rigidly linked.

The technology for converting coal to oil is assumed to be linear (up to the constraint) and can change from period to period. Thus, while in the smaller model a marginal cost curve for the "backstop" must be specified exogenously, in the larger model what is specified exogenously is the amount of coal required to produce (through liquefaction) another barrel of oil in each period.[32] The larger model will then *calculate* among other things the coal price path (and, by implication, the path of the marginal cost curve for producing liquefied coal).

Since the larger and smaller models use the same approach, they will generate

[31] For details on the imperfect-substitutes model see Salant et al. [1981].

[32] See pp. 265–266.

identical equilibria under identical conditions. To verify this, specify a set of inputs and calculate the equilibrium in the larger model. Fit the smaller model with the same set of oil and gas plants, the same market structure, interest rate, etc. Use the path of coal prices determined in the larger model to calculate the path of the marginal cost curve of the backstop, which is required as an exogenous input in the smaller model. Substitute the price paths for coal and nuclear energy determined in the larger model into the demand curve for oil in the larger model to obtain the (nonstationary) demand curve for oil in each period as a function only of its price. The smaller model requires such a sequence of demand curves as an exogenous input. Now run the smaller model using these exogenous inputs to obtain the oil-price path and the extraction paths for each oil and gas plant. These paths will be identical to the corresponding paths previously calculated on the larger model. In short, the two models are compatible.

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