The value of useless information

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31 March 2009

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February 1, 2009

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Abstract

There are many situations in which individuals do not expect to find out which outcome will occur. The standard vNM Expected Utility model is not necessarily appropriate for these cases, since it does not distinguish between lotteries for which the outcomes are observed by the agent and lotteries for which they are not. This paper provides an axiomatic model which makes this distinction, and which admits preferences for observing the outcome as well as preferences for remaining in doubt. This framework can accommodate behavioral patterns that are inconsistent with the vNM model, and that have motivated the development of models that differ significantly from the standard vNM framework. In particular, this framework accommodates self-handicapping, in which an agent chooses to impair his own performance. It also admits a status quo bias, without having recourse to framing effects. Several other examples are provided. In one example, voters prefer to remain ignorant, and as the importance of the relevant issues increases, their incentive to acquire information *decreases*.

Keywords: Value of information, uncertainty, recursive utility, doubt, unobserved outcomes, unresolved lotteries.

*I would like to thank Jing Li, Antonio Penta, Philipp Kircher, Michela Tincani, Wojciech Olszewski, Deniz Selman, David Dillenberger, Eleanor Harvill, Karl Schlag and especially my advisors Alvaro Sandroni and Andrew Postlewaite for invaluable comments and suggestions.

1 Introduction

Models of decision-making under uncertainty usually assume that the agents expect to observe the resolution of uncertainty ex-post. However, there are many situations in which individuals never find out which outcome occurs. In addition to preferring some outcomes to others, individuals may not be indifferent between remaining in doubt and observing the resolution of uncertainty. For instance, many people do not want to know whether the goods they buy have been made by children. Consider also the classical example of genetic diseases. As Pinker (2007) discusses, "the children of parents with Huntington's disease [HD] usually refuse to take the test that would tell them whether they carry the gene for it". HD is a neurodegenerative disease with severe physical and cognitive symptoms. It reduces life expectancy significantly, and there is currently no known cure. A person can take a predictive test to determine whether he himself will develop HD. A prenatal test can also be done to determine whether his unborn child will have the disease as well.[1] In an experimental study, Adam et al. (1993) find low demand for prenatal testing for HD. This is supported by a number of other studies as well, and Simpson et al. (2002) find that the demand for prenatal testing is significantly lower than the demand for predictive tests. That is, individuals who are willing to know their own HD status are unwilling to find out their unborn child's status. The prenatal test is done at a stage in which parents can still terminate the pregnancy, hence observing the result is an important decision. As for parents who do not consider pregnancy termination to be an option, the information could still impact the way they decide to raise their child. For example, if they know that their child will develop HD, then they might choose to prepare him psychologically for the difficult choices he himself would one day have to make. On the other hand, if they know that he will not develop HD, then they would have no such considerations.

The parents' preferences to avoid the test may seem puzzling; "given the technical feasibility of prenatal testing in HD, and the severity of the disorder, it might be expected that prenatal diagnosis would be frequently requested" (Simpson (2002)). It may appear particularly puzzling that a person who prefers to know now rather than later his own HD status also chooses not to find out whether his unborn child will develop the disease.[2] But note that the average age of onset for HD is high enough that the subjects who do not see the result of the prenatal test may *never* find out whether their children are affected. That is, while choosing the predictive test mostly reveals a preference for temporal resolution, choosing (or refusing) the prenatal test mainly reveals a preference for observing an outcome (or remaining in doubt). It is precisely this type of preference that is the focus of this paper.[3]

The standard von Neumann-Morgenstern (vNM) expected utility model cannot accommodate

[1] An affected individual has a 50% chance of passing the disease to each child. The average age of onsets varies between ages 35 and 55. See Tyler et al. (1990) for details.

[2] The prenatal test is not costless, as the procedure does involve a small chance of miscarriage. However, this cost appears small, compared to the severity of the disease.

[3] In particular, this paper does not consider other factors that are present in the HD example, such as parents' concern that their child will be treated differently if it is known that he has HD, as discussed in Simpson (2002).

preferences for knowing which outcome occurs or preferences for remaining in doubt, since it does not make a distinction between lotteries for which the final outcomes are observed and lotteries for which they are not. Redefining the outcome space to include whether the prize is observed does not resolve the issue, as is shown in the appendix. The argument makes use of the notion that observability should not in itself affect the value of a price. It appears plausible that if an agent expects an outcome z to occur with probability 1, then his utility would be the same whether he observes it or not, for he is certain that it occurs.[4] Hence, his utility is simply u_z, as opposed to u_{z_o} (observed) or u_{z_u} (unobserved). Since the outcome z has the same value to the agent whether it is labeled as 'z, observed' or 'z, unobserved', there is no degree of freedom in the standard vNM model for expanding the outcome space to include the observability of z. In addition, it would be difficult to interpret the meaning of receiving the prize 'z, unobserved', since the agent cannot know he has received the prize without observing it. The observability of an outcome is fundamentally connected to the *uncertainty* of receiving the prize, and not just to the value of the prize.

This paper provides an axiomatic model that accommodates preferences for remaining in doubt or observing the resolution of uncertainty. The agent's primitive preferences are taken over general lotteries that lead either to outcomes that he observes or to lotteries that never resolve (denoted unresolved lotteries), from his frame of reference, in the sense that he never observes which outcome occurs.[5]

This framework extends the standard vNM model, and for that reason makes similar assumptions. In particular, a version of the independence axiom is taken to hold. The standard vNM independence axiom is taken over lotteries that lead only to final outcomes, without specifying whether the agent observes the resolution of these lotteries. In this framework, the independence axiom is taken over more general lotteries which lead to either observed outcomes or to unresolved lotteries. The justification for assuming the independence axiom in this richer space is that both observed outcomes and unresolved lotteries are final prizes that the agent receives, the only difference being that one prize is an outcome and the other is a lottery. It is also assumed that the agent is indifferent between observing a specific outcome and receiving an unresolved lottery that places probability 1 on that same outcome, since he is certain of the outcome's occurrence. The observation in itself has no effect on the value of the outcome in this model. This property restricts the agent's allowable preferences over unresolved lotteries, as is demonstrated in section 2.

The central result of this paper is a representation theorem that separates the agent's risk atti-

[4]The term observation is defined as learning what the outcome is. For example, observing child labor is taken to mean that the agent learns that child labor occurs. It does not mean that he sees images of child labor taking place, which could in itself be a difficult experience.

[5]Throughout this paper, probabilities are taken to be objective. With subjective probabilities, there are cases in which it may seem more natural to interpret the preferences as state-dependent. For a person who does not know whether he is talented, for instance, it is unclear whether talent is better viewed as a state of the world or a consequence.

tude over lotteries whose outcomes he observes from his risk attitude over unresolved lotteries. These two attitudes are distinct, and need not coincide. Henceforth, the term 'caution' is used instead of 'risk-aversion' for unresolved lotteries, since the agent is not taking any risks per se if he does not observe the outcome. That is, there is no 'risk' that the agent will obtain the worse outcome rather than the better outcome for an unresolved lottery, since he observes *neither* outcome. His final prize is the unresolved lottery itself, not the outcome that ensues without his knowledge. There is no formal justification for having his valuation of these unresolved lotteries be dictated by his risk-attitude. For that reason, his caution and his risk-aversion need not be identical, and his caution must be elicited directly from his preferences over unresolved lotteries. The difference between the agent's risk-aversion and his caution induce his doubt-attitude. An agent who is always more risk-averse than he is cautious is demonstrated to be doubt-prone, while an agent who is relatively more cautious is doubt-averse. These terms are defined formally in section 2, and the exact relation between risk-aversion, caution and doubt-attitude is characterized in theorem 6.

Since this model is an extension of the standard vNM framework, the assumptions made are closely related to the vNM axioms. But note that the distinction between whether an agent expects to observe the final outcome or not is also ignored in alternative models, such as models of non-expected utility and cumulative prospect theory. These frameworks therefore do not take into account the agent's doubt-attitude. However, it is possible to extend different classes of models to make the distinction between resolved and unresolved lotteries, and to obtain a corresponding representation theorem. Section 4 provides a method for extending alternative models to incorporate unresolved lotteries. A new axiom is presented, since these alternative models typically do not assume the vNM independence axiom.

The model presented here can accommodate seemingly unrelated behavioral patterns that are inconsistent with the standard vNM model, and that have motivated frameworks that are significantly different. Two important examples are self-handicapping and the status quo bias.

Consider first self-handicapping, in which individuals choose to reduce their chances of succeeding at a task. As discussed in Benabou and Tirole (2002), people may "choose to remain ignorant about their own abilities, and [...] they sometimes deliberately impair their own performance or choose overambitious tasks in which they are sure to fail (self-handicapping)." This behavior has been studied extensively, and seems difficult to reconcile with the standard EU theory.[6] For that reason, models that study self-handicapping make a substantial departure from the standard vNM assumptions. A number of models follow Akerlof and Dickens' (1982) approach of endowing the agents with manipulable beliefs or selective memory. Alternatively, Carillo and Mariotti (2000) consider a model of temporal-inconsistency, in which a game is played between the selves, and Benabou and Tirole (2002) use both manipulable beliefs and time-inconsistent

[6] Berglass and Jones (1978) conduct an experiment in which they find that males take performance-inhibiting drugs, and argue that they do so precisely because it interferes with their performance.

agents.[7]

The frameworks mentioned above capture a notion of self-deception, which involves either a hard-wired form of selective memory (or perhaps a rule of thumb), or some form of conflict between distinct selves. These models are typically not axiomatized. In contrast, this paper simply extends the vNM framework, and so the agents cannot manipulate their beliefs (in fact, all probabilities are objective), and do not have access to any other means for deceiving themselves. Yet it can still accommodate the decision to self-handicap, as is shown in section 3. Intuitively, a doubt-prone agent prefers doing worse in a task if this allows him to avoid information concerning his own ability. This is essentially a formalization of the colloquial 'fear of failure'; an agent makes less effort so as to obtain a coarser signal.

This model can also accommodate a status quo bias in some circumstances. The status quo bias refers to a well-known tendency individuals have to prefer their current endowment or decision to other alternatives. This phenomenon is often seen as a behavioral anomaly that cannot be explained using the vNM model. On the other hand, it can be accommodated using loss aversion, which refers to the agent being more averse to avoiding a loss than to making a gain (Kahneman, Knetch and Thaler (1991)). The status quo bias is therefore an immediate consequence of the agent taking the status quo to be the reference point for gains versus losses. The vNM model does not allow an agent to evaluate a bundle differently based on whether it is a gain or a loss, and hence cannot accommodate a status quo bias. Arguably, this is an important systematic violation of the vNM model, and is one of the reasons cited by Kahneman, Knetch and Thaler (1991) for suggesting "a revised version of preference theory that would assign a special role to the status quo".

However, in some settings, the model presented here also admits a status quo bias, even without having recourse to the notion of reference point, gains or losses.[8] In the cases where the choices also have an informational component on the agent's ability to perform a task well, a doubt-prone agent has incentive to choose the bundle that is less informative. This leads to a status quo bias when it is reasonable to assume that holding to the status quo, or inaction, is a less informative indicator of the agent's ability than other actions.

In addition, since this model does not make use of the reference point notion, there is no arbitrariness in defining what constitutes a gain and what constitutes a loss. The bias of a doubt-prone agent is always towards the least-informative signal of his ability. In fact, in instances

[7]See also Compte and Postlewaite (2004), who focus on the positive welfare implications of having a degree of selective memory (assuming such technology exists) in the case where performance depends on emotions. Benabou (2008) and Benabou and Tirole (2006a, 2006b) explore further implications of belief manipulation, particularly in political economy settings, in which multiple equilibria emerge. Brunnermeier and Parker (2005) treat a general-equilibrium model in which beliefs are essentially choice variables in the first period; an agent manipulates his beliefs about the future to maximize his felicity, which depends on future utility flow. Caplin and Leahy (2001) present an axiomatic model where agents have 'anticipatory feelings' prior to resolution of uncertainty, which may lead to time inconsistency. Koszegi (2006) considers an application of Caplin and Leahy (2001).

[8]There are, however, examples of the status quo bias for which this model does not seem to provide as natural an explanation as loss-aversion does.

where the status quo provides the most informative signal, the bias would be *against* the status quo. For example, an individual could have incentive to change hobbies frequently rather than obtaining a sharp signal of his ability in one particular field.

The framework presented here admits other instances of seemingly paradoxical behavior. In one example, an individual pays a firm to invest for him, even though he does not expect that firm to have superior expertise. In other words, the agent's utility not only depends on the outcome, but also on who makes the decision. This result is not due to a cost of effort, but rather to the amount of information acquired by the decision maker. This framework can also be used in a political economy setting, as there are many government decisions that are never observed by voters. As shown in section 3, voters may have strong incentives to remain ignorant over these issues, even if information is free. This is in line with the well-known observation that there has been a consistently high level of political ignorance amongst voters in the US (see Bartels (1996) for details). Surprisingly, this model suggests that if voters care more about policies that they may never observe, then they have *less* incentive to acquire information. Finally, this framework can also be adapted to provide an alternative theoretical foundation for anticipated regret.[9] However, this discussion is outside the scope of this paper, and is deferred to future research.

The approach used in this paper is related to, but distinct from, the recursive expected Utility (REU) framework introduced by Kreps and Porteus (1978), and extended by Epstein and Zin (1989), Segal (1990) and Grant, Kajii and Polak (1998, 2000).[10] These earlier contribution address the issue of temporal resolution, in which an agent has a preference for knowing now versus knowing later. While the REU framework treats the issue of the timing of the resolution, this paper treats the case of *no* resolution. It may appear that simply adding a 'never' stage to the REU space would yield an equivalent representation, but in fact this is not the case. This distinction is formally discussed in section 4 of the paper. Fundamentally, an agent's preferences in a dynamic setting are allowed to differ from period to period, which is the reason why a person is not indifferent to the timing of resolution. If unresolved lotteries are introduced, then they must also be allowed to differ from period to period. For instance, an agent may not have the same preferences over unresolved lotteries after ten years as he does initially. These different preferences cannot be represented by a set of 'never' stages in an REU framework, because each 'never' stage cannot lead to any later stages. In other words, each unresolved lottery is itself a terminal node, and cannot lead to any subsequent nodes. In addition to the formal differences between the two frameworks, there are also interpretational ones. The REU model captures a notion of 'anxiety' (wanting to know sooner or later) which is distinct from the notion of

[9]See Loomes and Sugden (1982) for a theoretical model of anticipated regret, and Zeelenberg (1999) for a review.

[10]See also Dillenberger (2008). Selden's (1978) framework is closely related to the Recursive EU model.

doubt-proneness (not wanting to know at all) addressed here.
This model does not assume the independence axiom for preferences over unresolved lotteries, for reasons discussed in section 2. Instead, it is argued that the agent should satisfy an 'information scrambling consistency' property, which is itself satisfied by a rank-dependent utility (RDU) representation. The agent's doubt-attitude, risk-aversion and cautiousness restrict the allowable weighting function over the probabilities, and under some conditions, this weighting function must be linear.

This paper is structured as follows. Section 2 introduces the model and derives the representation theorem. Doubt-proneness and doubt-aversion are then defined, and implications of the doubt-attitude of agents on the representation are discussed. Section 3 presents applications of this model. Section 4 relaxes the main independence axiom of the framework, and introduces an axiom that allows different classes of models to incorporate outcomes that are never observed. Section 5 concludes. All proofs are in the appendix.

2 Model

2.1 General Structure and Representation Theorem Template

This section derives a template for a representation theorem, which is then made precise in the following subsections. The following objects are used:

- $\mathcal{Z} = [\underline{z}, \bar{z}] \subset \Re$ is the outcome space.
- $\mathfrak{L}_0$ is the set of simple probability measures on $\mathcal{Z}$. For $f = (z_1, p_1; z_2, p_2; ...; z_m, p_m) \in \mathfrak{L}_0$, z_i occurs with probability p_i. The notation $f(z_i)$ is also used to mean the probability p_i (in lottery f) that z_i occurs.
- $\mathfrak{L}_1$ is the set of simple lotteries over $\mathcal{Z} \cup \mathfrak{L}_0$. For $X \in \mathfrak{L}_1$, the notation $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N)$ is used. Here, z_i occurs with probability q_i^I, and lottery f_j occurs with probability q_j^N. Note that $\sum_{i=1}^{n} q_i^I + \sum_{i=1}^{m} q_i^N = 1$. The reason for using this notation, rather than the simpler enumeration $q_1, q_2, ..., q_n$ is explained below.
- $\succeq$ denotes the agent's preferences over $\mathfrak{L}_1$. $\succ$, $\sim$ are defined in the usual manner.

For any $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N)$, the agent expects to observe the outcome of the first-stage lottery. He knows, for instance, that with probability q_i^I, outcome z_i occurs, and furthermore he knows that he will observe it. Similarly, he knows that with probability q_i^N, lottery f_i occurs. However, although he does observe that he is now faced with lottery f_i, he does *not* observe the outcome of f_i. Lottery f_i is referred to as an 'unresolved'

lottery. The q_i^I's, q_i^N's are used to distinguish between the probabilities that lead to prizes where he is fully informed of the outcome (since he directly observes which z occurs), and the probabilities that lead to prizes where he is *not* informed (since he only observes the ensuing lottery).[11] The superscript I in q_i^I stands for 'Informed', and N in q_i^N for 'Not informed'.
Denote the degenerate one-stage lottery that leads to $z_i \in \mathcal{Z}$ with certainty $\delta_{z_i} = (z_i, 1) \in \mathfrak{L}_0$. The degenerate lottery that leads to $f_i \in \mathfrak{L}_0$ with certainty is denoted $\delta_{f_i} = (f_i, 1) \in \mathfrak{L}_1$. Note that all lotteries of form $X = f$, where $f \in \mathfrak{L}_0$, are purely resolved (or 'informed') lotteries, in the sense that the agent expects to observe whatever outcome occurs. Similarly, all lotteries of form $X = \delta_f$, where $f \in \mathfrak{L}_0$, are purely unresolved lotteries. With slight abuse, the notation $f \succeq f'$ (or $\delta_f \succeq \delta_{f'}$) is used, where $f, f' \in \mathfrak{L}_0$. In addition, $f \succeq \delta_f$ (or $\delta_f \succeq f$) indicates that the agent prefers (not) to observe the outcome of lottery f than to remain in doubt.

Assumptions are now made to allow the agent's preferences $\succeq$ to be represented by functions $u : \mathcal{Z} \to \Re$, and an $H : \mathfrak{L}_0 \to \mathcal{Z}$ in the following way: for $X, Y \in \mathfrak{L}_1$, $X \succ Y$ if and only if $W(X) > W(Y)$, where W is of the form:

$$W(X) = \sum_{i=1}^{n} q_i^I u(z_i) + \sum_{i=1}^{m} q_i^N u\left(H(f_{z_i})\right)$$

This is essentially a standard vNM EU representation, where receiving lottery f_{z_i} as a prize has the same value to the agent as receiving the outcome $H(f_{z_i}) \in \mathcal{Z}$. The conditions for obtaining this representation are presented in this subsection, and the next subsections consider assumptions that further qualify H.

Axiom **A.1** is assumed throughout:

AXIOM A.1 (Certainty): Take any $z_i \in \mathcal{Z}$, and let $X = \delta_{z_i} = (z_i, 1)$ and $X' = (\delta_{z_i}, 1)$. Then $X \sim X'$.

The certainty axiom **A.1** concerns the case in which an agent is certain that an outcome z_i occurs. In that case, it makes no difference whether he is presented with a resolved lottery that leads to z_i for sure or an unresolved lottery that leads to z_i for sure. He is indifferent between the two lotteries. Hence axiom **A.1** does not allow the agent to have a preference for being informed of something that he already knows for sure.
The following three axioms are standard.

[11] Note that it would be straightforward to extend the model to allowing for subsequent resolved lotteries. However, it would make the notation more cumbersome. For 3 periods, for instance, the preferences would be taken over $\mathfrak{L}_2$, where $\mathfrak{L}_2$ is the set of simple lotteries over $\mathcal{Z} \cup \mathfrak{L}_1$. In this case, the second-stage lottery could also lead either to an outcome that he observes, or to a lottery whose outcome he does not observe. For more periods, the notation would make use of recursion, i.e. $\mathfrak{L}_t$ is the set of simple lotteries over $\mathcal{Z} \cup \mathfrak{L}_{t-1}$.

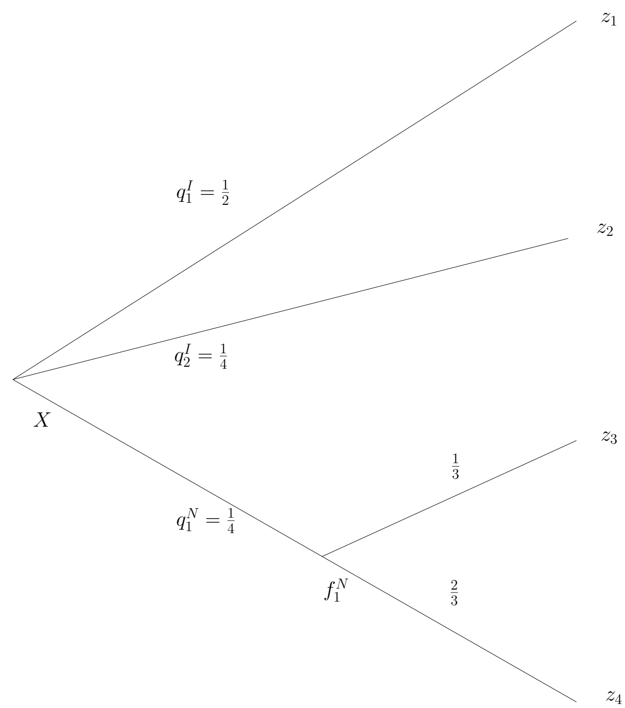

Figure 1: Lottery $X = (z_1, \frac{1}{2}; z_2, \frac{1}{4}; f_1, \frac{1}{4})$,where $f_1 = (z_3, \frac{1}{3}; z_4, \frac{2}{3})$

AXIOM A.2 (Weak Order): $\succeq$ is complete and transitive.

AXIOM A.3 (Continuity): $\succeq$ is continuous in the weak convergence topology. That is, for each $X \in \mathfrak{L}_\mathbf{1}$, the sets $\{X' \in \mathfrak{L}_\mathbf{1} : X' \succeq X\}$ and $\{X' \in \mathfrak{L}_\mathbf{1} : X \succeq X'\}$ are both closed in the weak convergence topology.

AXIOM A.4 (Independence): For all $X, Y, Z \in \mathfrak{L}_\mathbf{1}$ and $\alpha \in (0, 1]$, $X \succ Y$ implies $\alpha X + (1 - \alpha)Z \succ \alpha Y + (1 - \alpha)Z$.

Focusing on axiom **A.4**, it is noteworthy that the agent's preferences $\succeq$ are on a bigger space than in the standard framework. The independence axiom in the standard vNM model is taken on preferences over lotteries over outcomes, since all lotteries lead to outcomes that are eventually observed. In this paper, the agent's prize is not always an outcome z_i, and can instead be an unresolved lottery f_i. However, by assumption **A.4**, there is no axiomatic difference between receiving an outcome z_i as a prize and obtaining an unresolved lottery f_i as a prize. Under this approach, the rationale for using the independence axiom in the standard model holds in this case as well. Since this section aims to depart as little as possible from the vNM Expected Utility model, the independence axiom **A.4** is assumed throughout. This assumption is relaxed in section 4 and replaced with a weaker axiom.

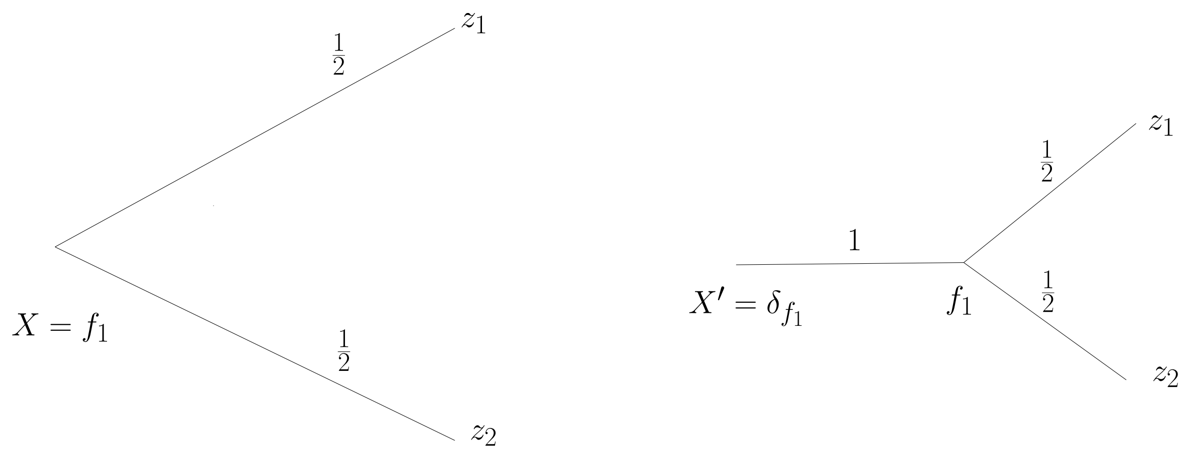

Figure 2: Lotteries $X = f_1 = (z_1, \frac{1}{2}; z_2, \frac{1}{2})$, $X' = \delta_{f_1}$ with the same reduction.

Note that the axiom of reduction, under which only the ex-ante probability of reaching each outcome matters, is *not* taken to hold in this setting.[12] Under reduction, the sequential aspect of the lottery does not affect the agent's preferences, which is arguably the case if the delay between the lotteries is insignificant. But if an agent receives the lottery f_i as a prize, then from his frame of reference the uncertainty never resolves. The delay before observing the final outcome is not short or insignificant, as it is in fact infinite.

If the reduction axiom were to hold, it would immediately imply that the agent is always indifferent between receiving a resolved and an unresolved lottery. To illustrate this point, consider the two lotteries $X = (z_1, \frac{1}{2}; z_2, \frac{1}{2}) = f_1$ and $X' = \delta_{f_1}$ (see figure 2). Note that in both lotteries X and X', there is a $\frac{1}{2}$ probability of reaching z_1, and a $\frac{1}{2}$ of reaching z_2. However, for lottery X, the agent observes the final outcome, while for lottery X' he does not. If he were to be indifferent between X and X', then he would also be indifferent between observing and not observing the outcome. The reduction axiom essentially removes the distinction between lotteries whose outcomes are observed and the ones whose outcomes are not, and therefore does not allow the agent to judge them differently.[13]

The following lemma paves the way for the general representation template that follows.

Lemma 1 (**Informed certainty equivalent**). *Suppose axioms **A.1** through **A.3** hold. There exists an H: $\mathfrak{L}_0 \to \mathcal{Z}$ such that for all $f \in \mathfrak{L}_0$, $\delta_{H(f)} \sim \delta_f$.*

[12] Formally, reduction holds if, for all $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N)$, $X' = (z'_1, {q'}_1^I; z'_2, {q'}_2^I; ...; z'_n, {q'}_{n'}^I;\ f'_1, {q'}_1^N; f'_2, {q'}_2^N; ...; f'_{m'}, {q'}_{m'}^N) \in \mathfrak{L}_1$ such that $q^I(z) + \sum q^N(z) f(z) = {q'}^I(z) + \sum {q'}^N(z) f'(z)\ \forall z$, $X \sim X'$.

[13] See Grant,Kajii and Polak (1998) for a similar discussion in the case of early and late resolution of uncertainty. See Segal (1990) for a discussion of the related notion of time-neutrality.

For any lottery f that the agent knows he will not observe, there exists an informed certainty equivalent $H(f)$: the agent is indifferent between his prize being an unresolved lottery f and obtaining an outcome $H(f)$.[14] One interpretation is that if he does not expect the uncertainty to resolve, then it is as though the outcome $H(f)$ occurs. Since it is not necessarily the case that this aggregation is identical to his attitude towards risk (i.e. his marginal utility) for the informed lotteries, he may not be indifferent between remaining in doubt and observing the resolution of uncertainty. The theorem below follows naturally from the existence of H and from the assumptions made so far.

Representation Theorem. *Suppose axioms **A.1** through **A.4** hold. Then there exist a continuous and bounded function $u : \mathcal{Z} \to \Re$, and an $H : \mathfrak{L}_0 \to \mathcal{Z}$ such that for all $X, Y \in \mathfrak{L}_1$,*

$$X \succ Y \text{ if and only if } W(X) > W(Y)$$

where W is defined to be: for all $X = (z_1, q_1^I; ...; z_n, q_n^I;\ f_1, q_1^N; ...; f_m, q_m^N)$,

$$W(X) = \sum_{i=1}^{n} q_i^I u(z_i) + \sum_{i=1}^{m} q_i^N u\left(H(f_{z_i})\right)$$

Moreover u is unique up to positive affine transformation. If $H(f)$ has more than one element, then any element can be chosen arbitrarily.

Under this representation, preferences over the resolved part of lotteries are of the standard EU form, with utility function u. Take a lottery $X \in \mathfrak{L}_1$, in which the agent obtains outcome z_i with probability q_i^I. In this case, $u(z_i)$ enters his $W(X)$ functional linearly, weighted by q_i^I. As for an unresolved lottery f_j that he obtains with probability q_j^N, it has an informed certainty equivalent $H(f_i)$. Hence $u(H(f_i))$ also enters his functional linearly, weighted by q_j^N. In that sense, the representation is an EU representation, where obtaining an unresolved lottery f_j as a prize is equivalent to obtaining a final outcome $H(f_j)$. The task now is to find a suitable representation of H.

2.2 Representations of H

The discussion that follows considers axioms on the unresolved lotteries, that is, only lotteries of the form $X = \delta_f$. As there is a natural isomorphism between these lotteries and one-stage lotteries, the preference relation $\succeq_N$ is defined in this way, for convenience: $\delta_f \succeq \delta_{f'}$ implies $f \succeq_N f'$ (and similarly for $\sim_N$, $\succ_N$).

[14] $H(f)$ is not necessarily unique, but the agent must be indifferent between the possible outcomes. That is, if $H(f) = z$ and $H(f) = z'$ can both occur, then $\delta_z \sim \delta_{z'} \sim \delta_f$. Hence either outcome can be chosen arbitrarily in the representation that follows.

Since this model is an extension of the standard vNM framework, it might seem that the preferences over the unresolved lotteries should also have an Expected Utility form. The only additional axiom required for this representation is the independence axiom over $\succeq_N$. However, this does not admit preferences which appear natural, as will be shown. A weaker axiom is then assumed, and it is demonstrated that under certain restrictions over risk-aversion and doubt-attitude, the stronger independence axiom must in fact hold.

As a useful first step, the EU representation is first obtained. Since reduction has not been assumed, the independence axiom over the uninformed preference relation $\succeq_N$ is not implied by the independence axiom **A.4**. It must therefore be explicitly assumed, although it is later argued that this axiom is not adequate for this setting.

AXIOM H.1 (Independence for $\succeq_N$): For all $f, f', f'' \in \mathfrak{L}_o$ and $\alpha \in (0,1]$, $f \succ_N f'$ implies $\alpha f + (1-\alpha)f'' \succ_N \alpha f' + (1-\alpha)f''$.

All the axioms required for an EU representation of $\succeq_N$ now hold.

Theorem 2 (**EU Representation for Purely Unresolved Lotteries**). *Suppose axioms **A.1-A.4** and axiom **H.1** hold. Then there exists a continuous and bounded function $v : \mathcal{Z} \to \Re$ such that for any $f, f' \in \mathfrak{L}_o$,*

$$f \succ_N f' \texttt{ if and only if } \sum_{z \in \mathcal{Z}} v(z)f(z) > \sum_{z \in \mathcal{Z}} v(z)f'(z)$$

Moreover, v is unique up to positive affine transformation. Furthermore, the following holds for H (where Ev denotes the expectation of v):

$$H(f) = v^{-1}(Ev) = v^{-1}\left(\sum_{z \in \mathcal{Z}} v(z)f(z)\right)$$

Note that v is the utility function associated with unresolved lotteries, and u remains the utility function associated with the general lotteries (and final outcomes).[15] In this special case, the preferences over $\succeq$, represented by $W(X)$ (defined in the representation theorem), are essentially reduced to a two-stage Kreps-Porteus REU form, with a different interpretation. Instead of u being associated with an 'earlier' stage and v with a 'later' stage, in this representation u is associated with the lotteries that are resolved and v with the lotteries that are unresolved.[16] However, and perhaps surprisingly, extending an REU model with two stages or more to allow

[15] It is also case that $u(z) > u(z') \Leftrightarrow v(z) > v(z')$.

[16] If v is a positive affine transformation of u, then this collapses to a standard *EU* representation.

for unresolved lotteries is *not* equivalent to adding a 'never' stage. This discussion is deferred to section 4.

Limitations of the independence axiom

In the Recursive EU setting with delay in resolution, it could be argued that the agent has a different risk-attitude in the second stage than in the first stage. This in turn drives his preference for acquiring information sooner or later, and determines his 'anxiety' factor. But this argument faces a greater challenge in the context of this model. The agent never observes the second stage, and hence is not taking any risks, in the usual sense of the term. Instead, one could focus on the interpretation that $v(z)$ represents the weight of each outcome z, and that the agent's attitude towards doubt is induced by the difference in his relative weighting of the outcomes, when the uncertainty does not resolve.

The function v, therefore, contains different notions which cannot be disentangled. It incorporates the agent's valuation of each outcome as well as a notion of caution. In addition, v fully captures the way he forms his perception of the unresolved lotteries, since $v^{-1}(Ev)$ is his informed certainty equivalent. The relation between v and u, in turn, determines his attitude towards doubt.

To illustrate this point, consider again the case of the agent who has had a bad performance (t_b), a mediocre one (t_m), or a good one (t_g). There are three lotteries over outcomes: $f = (t_b, \frac{1}{3}; t_m, \frac{1}{3}; t_g, \frac{1}{3})$, $f' = (t_b, \frac{1}{2}; t_g, \frac{1}{2})$ and $\delta_m = (1, t_m)$.[17] Assume that if he expects to observe the outcome, a risk-averse agent has a preference for being certain his performance was mediocre rather than having the lottery f, and might prefer the less risk lottery f to lottery f': $\delta_m \succ f \succ \delta_{f'}$. Furthermore, suppose that $f \succ_N f' \succ_N \delta_m$. For instance, the agent might prefer to remain in doubt and obtain f' rather than obtaining δ_m and being certain of a mediocre performance, because of the way he forms his perception if he does not see the outcome. Since he is risk-averse when he expects to observe the outcome, then perhaps he is also cautious when he does not expect to observe the outcome, and prefers f to f'. f is better for a cautious agent, and has the benefit, for a doubt-prone agent, of also being similarly uninformative.

The plausibility of these preferences depends on the interaction between the notions of risk, caution and doubt-attitude. He is cautious and prefers lottery f to f', and he also prefers to stay in doubt rather than knowing that he is mediocre. Note, however, that these preferences violate independence. In fact, they violate the stronger axiom of betweenness, and so do not fall in the Dekel (1986) class of preferences.[18]

This example highlights the possible conflicting attitudes that are merged together in the function v. In particular, an agent can be optimistic about his perception of the unobserved outcome

[17] Alternatively, consider a donor to a charity, who does not know whether his donation is being put to the best possible use.

[18] Note that $f = \frac{2}{3}f' + \frac{1}{3}\delta_m$. Hence this is a violation of independence (and betweenness) since the following does not hold: $f' \succ_N \frac{2}{3}f' + \frac{1}{3}\delta_m \succ_N \delta_m$. More specifically, this violates quasi-convexity.

and still be cautious. The number of different notions merged together suggests that a more flexible representation should be allowed for the preferences over unresolved lotteries, even while choosing to stay within the standard framework for the general lotteries.

In this example, $f' \succ_N \delta_m$ does not necessarily imply that $f' \succ af' + (1-a)\delta_m \succ_N \delta_m$ for all $a \in (0,1)$. Now let $\tilde{f}' = (t_{\tilde{b}}, \frac{1}{2}; t_{\tilde{g}}, \frac{1}{2})$, where $t_{\tilde{b}}, t_{\tilde{g}}$ are such that $f' \sim_N \tilde{f}$ and $t_b < t_{\tilde{b}} < t_{\tilde{g}} < t_g$.[19] With the independence axiom **H.1** over unresolved lotteries, it would follow that $af' + (1-a)\delta_m \sim_N a\tilde{f}' + (1-a)\delta_m$ for all $a \in (0,1)$. In other words, there is no difference in the agent's risk-aversion (caution), in the standard sense of the term. But using the same reasoning as in the example above, this model should allow a strict preference, since the agent may have a preference for being more or less informed. That is, since the interval $[t_{\tilde{b}}, t_{\tilde{g}}]$ is smaller than the interval $[t_b, t_g]$, the lottery $a\tilde{f}' + (1-a)\delta_m$ is less 'scrambled' than the lottery $af' + (1-a)\delta_m$. Hence, this model should allow the agent to have this type of preference:

1. $af' + (1-a)\delta_m \succ_N a\tilde{f}' + (1-a)\delta_m$ for some $a \in (0,1)$

   or

2. $a\tilde{f}' + (1-a)\delta_m \succ_N af' + (1-a)\delta_m$ for some $a \in (0,1)$

It may also be the case that for some $a \in (0,1)$, the agent has a preference for more scrambled information (case 1) and for some $a' \in (0,1)$, the agent has a preference for less scrambled information. Suppose, for now, that there exists some $a \in (0,1)$ such that case 1 holds. Let $\hat{f} = (t_{\hat{b}}, \frac{1}{2}; t_{\hat{g}}, \frac{1}{2})$, where $t_{\hat{b}}, t_{\hat{g}}$ are such that $\hat{f} \sim_N f$. If $t_{\hat{b}} < t_b < t_g < t_{\hat{g}}$, then it should also be the case that $a\hat{f}' + (1-a)\delta_m \succ_N af' + (1-a)\delta_m$. If, instead $t_b < t_{\hat{b}} < t_{\hat{g}} < t_g$, then it should instead be that $af' + (1-a)\delta_m \succ_N a\hat{f}' + (1-a)\delta_m$. That is, the optimistic (pessimistic) agent with a preference for more (less) scrambled information prefers a lottery with a larger (smaller) distance between the good and the bad outcome.

This property is generalized in the next part of the discussion. It is then shown that this property is satisfied by rank-dependent utility (henceforth RDU). Following this, doubt-attitude is defined, and the relation between doubt-attitude, risk aversion and caution is characterized in theorem 6.

Rank-dependent utility

Although this section considers RDU axioms for the preference relation associated with unresolved lotteries, note that for the general preference relation, $\succeq$, the independence axiom **A.4** still holds. For that reason, the overall representation will consist of a combination of the EU and the RDU frameworks. The representation theorem template presented earlier still holds,

[19] If monotonicity holds in this example, then with stochastic dominance, either $t_b < t_{\tilde{b}} < t_{\tilde{g}} < t_g$ or $t_{\tilde{b}} < t_b < t_g < t_{\tilde{g}}$ must hold.

but the H function will no longer have the form $v^{-1}(Ev)$. Note that if the independence axiom **A.4** were to be relaxed as well, it would *not* be equivalent to relaxing the independence axiom in each stage of the Recursive EU model. This is discussed in more detail in section 4.
Hereafter it is assumed, for simplicity, that higher outcomes are strictly preferred to lower outcomes, i.e. $z \succ_N z' \Leftrightarrow z > z'$.[20] The following notation is used: for lottery $f = (z_1, p_1; z_2, p_2; ...; z_m, p_m) \in \mathfrak{L}_{\mathbf{o}}$, the $z_i's$ are rank-ordered; i.e. $z_m \succ_N ... \succ_N z_1$. In addition, p_i^* denotes the probability of reaching outcome z_i or an outcome that is weakly preferred to z_i. That is, $p_i^* = \sum_{j=i}^m p_j$. Note that for the least-preferred outcome z_1, $p_1^* = 1$. Probabilities p_i^* are referred to here as 'decumulative' probabilities. Following Abdellaoui (2002), the rank-dependent utility form is defined in this manner:

Definition (RDU) Rank-dependent utility (RDU) holds if there exists a strictly increasing continuous probability weighting function $w : [0,1] \to [0,1]$ with $w(0) = 0$ and $w(1) = 1$ and a strictly increasing utility function $v : \mathcal{Z} \to \Re$ such that for all $f, f' \in \mathfrak{L}_{\mathbf{o}}$,

$$f \succ_N f' \text{ if and only if } V_{RDU}(f) > V_{RDU}(f')$$

where V_{RDU} is defined to be: for all $f = (z_1, p_1; z_2, p_2; ...; z_m, p_m)$,

$$V_{RDU}(f) = v(z_1) + \sum_{i=2}^{m} [v(z_i) - v(z_{i-1})] w(p_i^*)$$

Moreover, v is unique up to positive affine transformation.

If RDU holds, then the function H is represented as follows, as shown in the appendix:

$$H(f) = v^{-1}\left(V_{RDU}(f)\right)$$

Note that if the weighting function w is linear, then V_{RDU} reduces to the standard EU form.[21] The standard motivation for rank-dependent utility is to separate the notion of diminishing marginal utility from that of probabilistic risk aversion, which expected utility does not do. The aim here is different; in fact the standard EU form still holds for the general setting. Instead, this model separates the notion of caution (which remains identical to diminishing marginal unresolved utility) from his optimism (or pessimism) in the way he forms his perception of the unobserved outcome. Specifically, as discussed in the previous subsection, an optimistic agent prefers to have more scrambled information. He prefers to know less, so as to form a more

[20]It follows from the certainty axiom **A.1** that if the the higher outcomes are preferred to the lower outcomes, $\delta_z \succ \delta_{z'} \Leftrightarrow z > z'$.

[21]This is not the most common form of RDU. Given the rank-ordering above, the typical form would be $V_{RDU} = \sum_{i=1}^{n-1} [w(p_i^*) - w(p_{i+1})^*] v(z_i) + w(p_n) v(z_n^*)$. It is easy to check that the two representations are identical.

reassuring perception of the outcome. A pessimistic agent, on the other hand, prefers sharper information, since knowing less would lead him to form a more negative perception. This property is summarized below:

Definition (ISC) $\succeq_N$ satisfies information scrambling consistency (ISC) if:
let $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$, $f' = (z_1, p_1; ...z'_i; p_i; z'_{i+1}, p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and *case 1*: $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$ (*case 2*: $(z_i, z_{i+1}) \subset (z'_i, z'_{i+1})$). If, for some $a \in (0, 1)$ and some $z \in (z'_i, z'_{i+1})$:

$$af + (1-a)\delta_z \succeq_N af' + (1-a)\delta_z$$

then it must also be that:

$$a\tilde{f} + (1-a)\delta_{\tilde{z}} \succeq_N a\tilde{f}' + (1-a)\delta_{\tilde{z}}$$

for any $\tilde{f} = (\tilde{z}_1, p_1; ...\tilde{z}_i; p_i; \tilde{z}_{i+1}, p_{i+1}; ...; \tilde{z}_n, p_n)$, $\tilde{f}' = (\tilde{z}_1, p_1; ...\tilde{z}'_i; p_i; \tilde{z}'_{i+1}, p_{i+1}; ...; \tilde{z}_n, p_n)$ and $\tilde{z}$ such that $\tilde{z} \in (\tilde{z}'_i, \tilde{z}'_{i+1}) \subset (\tilde{z}_i, \tilde{z}_{i+1})$ (*case 2*: $\tilde{z} \in (\tilde{z}_i, \tilde{z}_{i+1}) \subset (\tilde{z}'_i, \tilde{z}'_{i+1})$).

A preference for more scrambled information corresponds to case 1, i.e. preferring $af + (1-a)\delta_z \succ af' + (1-a)\delta_z$ when $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$. Similarly, a preference for less scrambled information corresponds to case 2.[22] Note that the ISC property allows an agent to prefer more scrambled information for some $a \in (0, 1)$ and less scrambled information for another $a' \in (0, 1)$. In other words, an agent's pessimism or optimism may depend on how likely he believes an outcome to occur.[23]. This property is satisfied by an RDU representation:

Theorem 3. *Suppose that RDU holds for $\succeq_N$. Then $\succeq_N$ satisfies ISC.*

In the discussion above, optimism (pessimism) have been associated with a preference for more (less) scrambled information, as described by the two cases in the ISC property.[24] But note that the notions of optimism (pessimism) have a different meaning in the RDU setting, as they are associated with concavity (convexity) of the weighting function w (see Wakker (1994)). In fact, if the weighting function is concave (convex), then an agent always prefers more (less) scrambled information, and so the two notions of optimism (pessimism)coincide.

[22]This a separate notion from risk: for a risk-averse agent in the standard EU setting, $f \sim_N f'$ implies that $af + (1-a)\delta_z \succ_N af' + (1-a)\delta_z$, for any $a \in (0, 1)$ and any $z \in \mathcal{Z}$.

[23]Note also that this notion of pessimism (optimism) is separate from risk. In a standard EU setting, $f \sim_N f'$ would always $af + (1-a)\delta_z \succ af' + (1-a)\delta_z$, for any $a \in (0, 1)$ and any $z \in \mathcal{Z}$.

[24]Only the cases $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$ and $(z_i, z_{i+1}) \subset (z'_i, z'_{i+1})$ are considered. For any other case, $f \sim_N f'$ would violate stochastic dominance, which will not be allowed in this model.

Theorem 4. *Suppose that $\succeq_N$ satisfies RDU, and let w be the associated weighting function. Then w is concave (convex) if and only if:*

for any $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$, $f' = (z_1, p_1; ...z'_i; p_i; z'_{i+1}, p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_{\mathfrak{o}}$ such that $f \sim_N f'$, and $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$, and for all $a \in (0,1)$ and $z \in (z_i, z_{i+1})$, the following must hold:

$$af + (1-a)\delta_z \succeq_N af' + (1-a)\delta_z$$

$$(convex\ w: \quad af' + (1-a)\delta_z \succeq_N af + (1-a)\delta_z)$$

There is therefore no formal difference between an agent who always prefers more (less) scrambled information, as defined above, and an optimist (pessimist), in the usual RDU sense of the term. The axiomatic foundation of the RDU representation is now briefly discussed, in the context of this model. Suppose that

$$f_\alpha = (z_1, p_1; ...; \alpha, p_i; ...; z_m, p_m) \succeq_N (z'_1, p_1; ...; \beta, p_i; ...; z'_m, p_m) = f'_\beta$$
$$f'_\kappa = (z'_1, p_1; ...; \kappa, p_i; ...; z'_m, p_m) \succeq_N (z_1, p_1; ...; \gamma, p_i; ...; z_m, p_m) = f_\gamma$$

where $\alpha, \beta, \gamma, \kappa \in \mathcal{Z}$.

Comparing lotteries f_α and f_γ, the only difference is in whether α or γ is reached with probability p_i. Since all the other outcomes are the same in both lotteries and are reached with the same probabilities, the difference is in the value of outcome α compared to the value of outcome γ (and similarly for f'_β,f'_κ and β, κ). In the comparison of $f_\alpha \succeq_N f'_\beta$ and $f'_\kappa \succeq_N f_\gamma$, all the probabilities of reaching the (rank-preserved) outcomes are the same. For that reason, it is assumed in this model that the switch in preference is due to a difference in the value of outcomes α and β relative to γ and κ, and not in the way the probabilities are aggregated. It is precisely this property that RDU provides: if $f_\alpha \succeq_N f'_\beta$ and $f'_\kappa \succeq_N, f_\gamma$, and if $\succeq_N$ is of the RDU form, then $v(\alpha) - v(\beta) \geq v(\gamma) - v(\kappa)$. Note that this does not depend on the choice of $z's$ and $p's$, and so the following axiom, adapted from Wakker (1994), must hold:

AXIOM H.1RA (Wakker tradeoff consistency for $\succeq_N$): Let $f_\alpha = (z_1, p_1; ...; \alpha, p_i; ...; z_m, p_m)$, $f_\gamma = (z_1, p_1; ...; \gamma, p_i; ...; z_m, p_m)$, $f'_\beta = (z'_1, p_1; ...; \beta, p_i; ...; z'_m, p_m)$ and $f'_\kappa = (z'_1, p_1; ...; \kappa, p_i; ...; z'_m, p_m)$. If:

$$f_\alpha \succeq_N f'_\beta$$
$$f'_\kappa \succeq_N f_\gamma$$

then for any lotteries $g_\alpha = (\hat{z}_1, \hat{p}_1; ...; \alpha, \hat{p}_i; ...; \hat{z}_{\hat{m}}, \hat{p}_{\hat{m}})$, $g_\gamma = (\hat{z}_1, \hat{p}_1; ...; \gamma, \hat{p}_i; ...; \hat{z}_{\hat{m}}, \hat{p}_{\hat{m}})$, $g'_\beta = (\hat{z}'_1, \hat{p}_1; ...; \beta, \hat{p}_i; ...; \hat{z}'_{\hat{m}}, \hat{p}_{\hat{m}})$, $g'_\kappa = (\hat{z}'_1, \hat{p}_1; ...; \kappa, p_i; ...; \hat{z}'_{\hat{m}}, \hat{p}_{\hat{m}})$ such that $g_\gamma \succeq_N g'_\kappa$, it must be that $g_\alpha \succeq_N g'_\beta$.

Under this axiom, only the values of α,β,γ and κ are relevant to the ordering of the agent's preferences when all the probabilities of reaching all other outcomes are the same across the four lotteries. In fact, as shown in Wakker (1994), this axiom is sufficient, along with stochastic dominance and continuity, for the RDU representation to hold.

Theorem 5 (**RDU Representation for Purely Unresolved Lotteries**). *Suppose axioms* ***A.1-A.4****, and* ***H.1R*** *hold. In addition, suppose that $\succeq_N$ satisfies stochastic dominance. Then: RDU holds for $\succeq_N$. Furthermore, $H(f) = v^{-1}\left(V_{RDU}(f)\right)$.*

Consider again the notions of optimism and pessimism in this context. An agent who always prefers more scrambled information is referred to as an optimist, and has a concave weighting function w. Extensive research has been done on the shape that seems to hold, empirically, on w in the usual RDU setting.[25] As this a different setting, assumptions over the shape of w are not made. In particular, while it is common to assume that w is S-shaped (concave on the initial interval and convex beyond), an empirical discussion of w for the unresolved lotteries is outside the scope of this paper. Instead, it is shown that the induced preferences to remain in doubt or not to remain in doubt have strong implications on the weighting function w. In particular, under certain conditions described below, the weighting function is constrained, and under strong enough restrictions it must be linear.

Implications of doubt-aversion and doubt-proneness

Doubt-aversion and doubt-proneness are defined in the following way:

Definition (Doubt-attitude)

- An agent is *doubt-prone* if: (i) there exists no $f \in \mathfrak{L}_o$ such that $f \succ \delta_f$ and (ii) there exists some f such that $\delta_f \succ f$.
- An agent is *doubt-averse* if (i) there exists no $f \in \mathfrak{L}_o$ such that $\delta_f \succ f$ and (ii) there exists some f such that $f \succ \delta_f$.
- For two agents A and $\tilde{A}$ with associated preference relations $\succeq$ and $\tilde{\succeq}$, agent A is *at least as* doubt-prone as agent $\tilde{A}$ if, for all $f \in \mathfrak{L}_o$, (i) $\delta_f \tilde{\succ} f \Longrightarrow \delta_f \succ f$, and (ii) $f \succ \delta_f \Longrightarrow f \tilde{\succ} \delta_f$.

In other words, an agent who (weakly, and strictly for one lottery) prefers not to observe than to observe the outcome of a lottery is doubt-prone, and an agent who always prefers to observe the outcome is doubt-averse. No strong stance is taken in this section concerning whether attention should be restricted mostly to doubt-proneness or to doubt-aversion, or indeed, to

[25]See Karni and Safra (1990), and Prelec (1998) for an axiomatic treatment of w.

doubt-proneness in some range and doubt-aversion in another. The result below connects the assumptions on doubt-proneness to properties of the probability weighting function $w(p)$; a similar result hold for doubt-aversion, and is provided in the appendix.

Theorem 6. *Suppose that axioms **A.1** through **A.4** and the RDU axioms hold, and let u and v be the utility functions associated with the resolved and unresolved lotteries, respectively, and w be the decision weight associated with the unresolved lotteries. In addition, suppose that u, v are both differentiable. Then:*
(i) If there exists a $p \in (0,1)$ such that $p < w(p)$, then there exists an $f \in \mathfrak{L}_o$ such that $\delta_f \succ f$. Similarly, if there exists $p' \in (0,1)$ such that $p' > w(p')$, then there exists an $f' \in \mathfrak{L}_o$ such that $f' \succ \delta'_f$.
(ii) If $\succeq$ exhibits doubt-proneness, then $p \leq w(p)$ for all $p \in (0,1)$. Moreover, if v exhibits stronger diminishing marginal utility than u, then $\succeq_N$ violates quasi-convexity. (that is, there exists some $f', f'' \in \mathfrak{L}_o$, and $\alpha \in (0,1)$ such that $f' \succ f''$ and $\alpha f' + (1-\alpha)f'' \succ_N f'$).

The differentiability assumption, though common, may seem bothersome as it is not taken over the primitives. Alternatively, an assumption could be made over the primitives that guarantees (for instance) strict concavity of u and v, which would in fact be sufficient for the result.[26] Given the results above, an assumption or deduction over the attitude towards doubt has testable implications over the attitude towards the aggregation of probabilities, and vice-versa. In addition, these implications can be disentangled from the attitude towards diminishing marginal utility. Since it is not necessary that w satisfies the same empirical properties as for the typical case considered under rank-dependent utility, an experimental study would be useful for a better sense of the shape of w.
If, in addition to doubt-proneness, mean-preserving risk-aversion (in the standard sense) of $\succeq_N$ is assumed, then the RDU representation collapses to the recursive EU representation:

Corollary. *Suppose that axioms **A.1** through **A.4** and the RDU axioms hold, and let u and v be the utility functions associated with the resolved and unresolved lotteries, respectively, and w be the decision weight associated with the unresolved lotteries. In addition, suppose that u, v are both differentiable. Then:*
If $\succeq$ displays doubt-proneness and $\succeq_N$ displays mean-preserving risk-aversion, then V_{RDU} must be of the EU form. That is, $w(p) = p$ for all $p \in \mathfrak{L}_o$. It also follows that both u and v are concave, and that $u = \lambda \circ v$ for some continuous, concave, and increasing λ.

This result further shows that attitude toward risk and attitude towards doubt constrain the probability weighting function, and can in fact completely characterize it.[27]

[26] For a discussion of the differentiability assumption, see Chew, Karni and Safra (1987).
[27] This last corollary is similar to a result in Grant, Kajii and Polak (2000) but with a notion of doubt-proneness that is weaker than the preference for late-resolution that would be required in the framework they use; the

Returning to the task example, note that if the assumption of mean-preserving risk aversion is to be maintained, then it cannot be that the agent is doubt-prone everywhere, as this would imply by the last result that the unresolved lotteries satisfy the vNM axioms. However, this is not consistent with these preferences' violation of independence. Hence $\tilde{f} \succ \delta_{\tilde{f}}$ for some $\tilde{f} \in \mathfrak{L}_{\mathfrak{o}}$. Since the agent prefers δ_f to f, for $f = (t_b, \frac{1}{3}; t_m, \frac{1}{3}; t_g, \frac{1}{3})$, he is therefore doubt-prone in some region and doubt-averse in others. If instead the assumption of mean-preserving risk-aversion is discarded, then it is possible for him to be doubt-prone everywhere. Note that this entails that quasi-convexity is violated, which corresponds precisely to the violation discussed in motivating the use of this framework. Finally, in the typical case of a regressive S-shaped w function, it must be that the agent is doubt-prone for some lotteries and doubt-averse for others, by theorem 6.

3 Applications

Two applications are considered in this section. In the first, an agent's utility depends directly on his ability, since it is related to his self-image. He may never fully observe his ability, but his success at performing tasks provides him with an imperfect signal. How well he performs a task also depends on his effort. Performing a task better provides him with a reward, and so in the standard EU setting, he would always put in as much effort as he can if effort is costless. In this setting, however, there is a tradeoff between obtaining a better reward by putting in more effort and obtaining a coarser signal of ability by putting in less effort. Under some conditions, the agent has an incentive to self-handicap, as is shown below. This setup also accommodates other well known behavioral patterns. Under one version of this setup, an agent has an incentive to remain with the status quo. In another version of this setup, a risk-neutral agent prefers less risky bonds with a lower expected return to more risky stocks with a higher expected return. This agent is also willing to pay a firm to invest for him, even if he knows that the firm does not have superior expertise.

In another application, voters all have the same preferences, but they do not know who the better candidate is. However, they can acquire this information at no cost. It is shown that there are equilibria in which they choose to remain ignorant, and the wrong candidate is as likely to win as the right candidate.[28]

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difference in assumptions is due to the difference in settings. It is also of note that under Grant, Kajii and Polak (2000)'s restriction, there is no need to assume differentiability, as it is in fact implied.

[28]In some cases, 'disappointment' may seem an appropriate notion in the circumstances described below. A person's fear of failure may stem from not wanting to be disappointed by what he finds out about himself, or not wanting to be disappointed by the outcome. This terms is not used in this paper to avoid confusion, as it has a distinct meaning in other settings. Disappointment aversion is typically used in discussions of the Allais Paradox, as a possible explanation for the common ratios effect (see Gul (1991) for a theoretical model).

3.1 Preservation of self-image

A general setup is first introduced, and the implications of the results are then analyzed in different contexts. The agent is assumed to place direct value on his ability (or talent), independently of the effect it has on his monetary reward. Arguably, individuals care about their self-image, and would rather think of themselves as talented than untalented. Their success at achieving their goals, given how much effort they put in, provides them with imperfect signals of their talent.

Suppose then that the agent is endowed with talent $t \in [\underline{t}, \overline{t}] \in \Re$. He does not know what his talent is, but his prior probability of having talent t is $p(t)$. The agent chooses effort $e \in [\underline{e}, \overline{e}] \in \Re$, to obtain a reward $m \in [\underline{m}, \overline{m}] \in \Re$. Although the agent may never observe his talent, he does observe m. The reward depends on his talent, the effort he puts in, and an intrinsic uncertainty. Let $p(m|e,t)$ denote his probability of receiving reward m given his effort e and his talent t. Since he does not know what his talent is ex-ante, his prior probability of receiving m given effort e is $p(m|e) = \sum_{t \in [\underline{t}, \overline{t}]} p(m|e,t)p(t)$. Assume that the expected reward is higher if he puts in more effort for any given talent, and it is higher if he is more talented at any given effort level: $Em(e,t) > Em(e,t') \Leftrightarrow t > t'$, and $Em(e,t) > Em(e',t) \Leftrightarrow e > e'$.[29]

The agent's value function W depends on both his reward m and on his intrinsic talent t. Assume that his utility for m is linear; more precisely, his expected utility over m is $Em(e)$. In addition, it is linearly separable from his utility over t. He is weakly risk-averse over t (for both resolved and unresolved lotteries) as well as doubt-prone.[30] As in the theory section, let u be his resolved utility, and let v be his unresolved utility.

If the agent expects to observe both his talent t and his reward m, then his value function is:

$$W(e) = Em(e) + Eu(t)$$

Since effort is costless, it is immediate that he should put in the highest level of effort, $e = \overline{e}$. But now suppose that he does *not* necessarily observe his talent ex-post. In this case, when he receives his monetary reward, he simply updates his probability on his talent, given m and his chosen effort level e. His value function is therefore:

$$W(e) = Em(e) + \sum_m p(m|e) u \circ v^{-1}(Ev(t|m,e))$$

Depending on the functional form, the agent might not put in effort $e = \overline{e}$. His effort level also depends on his incentive to obtain the least information concerning his talent, since he is

[29] All the probability distributions in this section have finite support.

[30] Note that by the corollary of theorem 6, the weighting function here is linear, w(p) = p. Note also that the agent being doubt-prone and risk-averse in the unresolved lotteries also implies that he is risk-averse in the resolved lotteries, by the same corollary.

doubt-prone. In other words, he takes into account what the combination of his effort and the reward he obtains allow him to deduce about his talent. Suppose that there is an effort level e_o (the 'ostrich' effort) that is entirely uninformative, i.e. $p(t|m, e_o) = p(t)$ for all $t \in [\underline{t}, \overline{t}]$ and for all $m \in [\underline{m}, \overline{m}]$. Note that e_o provides the agent with the highest expected utility of talent. That is, define

$$C(e) \equiv u \circ v^{-1}(Ev(t)) - \sum_m p(m|e) u \circ v^{-1}(Ev(t|m, e))$$

As shown in the appendix, it is always the case that $C(e) \geq 0$ for a doubt-prone agent, with $C(e_o) = 0$. Redefining the value function to be $\tilde{W}(e) = W(e) - u \circ v^{-1}(Ev(t))$, the agent maximizes

$$\tilde{W}(e) = Em(e) - C(e)$$

Hence $C(e)$ is effectively the 'shadow' cost of effort due to acquiring information that he would rather ignore. The optimal effort level depends on the importance of the expected reward $Em(e)$ relative to the agent's disutility of acquiring information concerning his talent, as is captured by $C(e)$. As an illustration, a simple example is provided.

Numerical Example

Let $\underline{e} = \underline{t} = 0$, $\overline{e} = \overline{t} = 1$, $p(t = 0) = \frac{1}{2}$ and $p(t = 1) = \frac{1}{2}$. The agent's reward m only takes value \$0 and \$100. The probability of obtaining reward $m = \$100$ given e and t are:

$$p(m = \$100|t = 1, e) = e$$

$$p(m = \$100|t = 0, e) = 0$$

and $p(m = \$0|t, e) = 1 - p(m = \$100|t, e)$. The utility functions are $u = a\sqrt{t}$ for some $a > 0$, and $v = t$.

Note that in this example, the completely uninformative effort e_o is equal to 0. At effort $e = 0$, he is sure to obtain \$0, and his posterior on his talent is the same as his prior. As he puts in more effort, he obtains a sharper signal of his talent. If he puts in maximum effort $e = 1$, then he will fully deduce his talent ex-post: if he obtains \$100 then he knows he has talent $t = 1$, and if he obtains \$0 then he knows he has talent $t = 0$. His value function is now:

$$\tilde{W}(e) = 50 - C(e)$$

where $C(e) = \frac{a}{2}(\sqrt{2} - e - \sqrt{2 - 3e + e^2})$.

The optimal level of effort e^* is in the full range $[0, 1]$, depending on a. More precisely, for interior solutions, e^* is the smaller root of the equation $e^2 - 3e + \frac{2d-9}{d-4} = 0$, where $d = \left(\frac{200}{a} + 2\right)^2$. As a increases, the monetary reward m becomes less significant, and e^* decreases. As a decreases, the utility of talent becomes less significant, and the effort level increases (see appendix for details).

Self-handicapping

The setup presented here can be applied to several different contexts, the most immediate of which is self-handicapping. There is strong anecdotal evidence that people are sometimes restrained by a 'fear of failure', and will not put in as much effort as they could. Berglas and Jones (1978) find in an experiment that individuals deliberately impede their own chances of success, and attribute this behavior to people's desire to protect the image of the self.[31] The amount of optimal self-handicapping depends on the doubt-attitude of the agent, and how good of a signal he expects to obtain. As discussed above, choosing a higher effort level leads to a tradeoff between the improved reward $Em(e)$ and the incurred cost $C(e)$ of learning more about one's actual talent. This model also confirms Berglas and Jones' intuition that those who are more likely to self-handicap are not the most successful or the least successful, but rather those who are uncertain about their own competence. Akerlof and Dickens' (1982) observation that people will remain ignorant so as to protect their ego is also in agreement with the implications of this framework.

Status quo bias

The endowment effect and status quo bias are analyzed by Kahneman, Knetch and Thaler (1991), and are explained using framing effects and loss aversion. The agent's preference for avoiding a loss is taken to be stronger than his preference for making a gain, and the reference point for what constitutes a gain or a loss is assumed to be the status quo. However, Samuelson and Zeckhauser (1988) do not view the status quo bias to be solely a consequence of loss-aversion: "Our results show the presence of status quo bias even when there are no explicit gain/loss framing effects.... Thus, we conclude that status quo bias is a general experimental finding – consistent with, but not solely prompted by, loss aversion." The framework discussed here can be applied to some settings in which a status quo bias is present.

Suppose that e now represents a choice over different bundles rather than effort. In addition, suppose that acquiring a bundle also carries information concerning prizes that the individual may never observe. In this case, rather than representing a cost of effort, $C(e)$ represents the cost of deviating from the bundle over which one has the most bias. Since $C(e)$ is smallest when $e = e_o$ (the ostrich effort), the bias here is towards what is least informative. This result is therefore consistent with the status quo bias when inaction (keeping the same bundle) is less informative than taking action. Note, however, that when keeping the status quo bundle is more informative than obtaining other bundles, then a doubt-prone agent would be biased *against* the status quo.

The key difference between the model presented here and the standard vNM model is that

[31] See Benabou and Tirole (2002) for an explanation that uses manipulable beliefs.

this model allows for an asymmetry in the value of acquiring a bundle compared to losing that bundle. The bundle itself does not change value based on whether the agent is endowed with it or not, and in that sense there is no framing effect. Instead, acquiring a new bundle *in itself* has different informational implications than selling it. In the case where the unobserved prize is the agent's ability, then acquiring a new bundle may provide him with more information on his ability than keeping the one he currently has.

Bonds, stocks and paternity

Consider the case in which e represents an investment decision rather than effort. A higher e represents a more risky investment, but in expectation it leads to a higher monetary reward. As before, t corresponds to a notion of talent. A more talented individual makes a wiser investment choice and therefore obtains a higher expected monetary reward, given the chosen risk level. For instance, $\underline{e}$ might be a portfolio consisting solely of bonds, while $\overline{e}$ consists solely of higher-risk stocks. Assume also that $e_o = \underline{e}$. In other words, the riskless option is also least informative concerning the agent's potential as an investor.

In this setting, although the agent is risk-neutral in money, his chosen bundle e^* may still consist of more bonds than it would if the reward were purely monetary, as there is a bias towards $\underline{e}$.[32]

In addition, suppose that a firm exists which offers to invest the agent's money in his place. Even if the agent puts the same prior on his ability in investing as he does on the firm's, he still agrees to pay. Since the optimal level of risk in this case is $\overline{e}$, he is willing to pay up to $Em(\overline{e}) - Em(e^*) + C(e^*)$. In fact, even if the firm were to choose the suboptimal level e^*, he would be willing to pay up to $C(e^*)$.

In the standard EU model, the agent's choice would only depend on the monetary reward he expects to obtain. In contrast, the framework presented here allows the agent's choice to depend on the decision making process as well as on the reward he expects to receive. That is, the agent bases his choice on the *manner* in which he expects to obtain the monetary reward.

3.2 Political Ignorance

The high degree of political ignorance of voters has been thoroughly researched, particularly in the US (see Bartels (1996)). Given the length of electoral campaigns in American politics, the amount of media coverage and the accessibility of informational sources, it seems that the cost of acquiring information should not be prohibitive for voters. Note that there are political issues whose resolution the voters may never observe. For instance, the voters may choose not to observe the amount of foreign aid given, the degree of nepotism, or the government stance on interrogation methods. For those issues, a doubt-prone agent may have incentive to ignore information even if information is free. In other words, making information more accessible would not necessarily have a strong impact on the individual's informativeness on these issues. Since

[32] Of course, no claim is made concerning the empirical significance of this effect.

voters affect the election result as a group, each individual's decision to acquire information has an externality on other voters and on *their* decision to acquire information. This section discusses a very simple example in which voters' information acquisition plays a dominant role on the other voters' decision to acquire information. Although voting is sincere, there is a strategic aspect to the decision to acquire information.

Consider an economy in which N citizens care about issue $\gamma \in [0, 1]$, which is determined by a politician that they vote for. They can choose never to observe what the politician does. Suppose that there are two candidates, A and B. One of the two will choose policy $\gamma = 0$ if elected, and the other will choose $\gamma = 1$. The voters do not know which one is which, and place probability $\frac{1}{2}$ that A will choose $\gamma = 0$, and $\frac{1}{2}$ that A will choose $\gamma = 1$ (and similarly for B). However, they can acquire that information at no cost, if they choose to do so. Let p_i be the ex-post probability that the ith agent places on the winner being the candidate who implements $\gamma = 1$, where $i \in \{1, .., N\}$. The timing is as follows:

1) Each voter decides whether or not to observe where candidates A and B stand. A voter cannot force another voter to acquire information.

2) Each voter votes sincerely, i.e. he votes for the candidate on whom he places a higher probability of implementing policy γ that he prefers. If he is indifferent or if he places equal probability on either candidate implementing his preferred policy, then he tosses a fair coin and votes accordingly.

3) The candidate who obtains the majority wins the election. In case of a tie, a coin toss determines the winner. The winner then implements the policy he prefers, and there is no possibility of reelection.

Now suppose that every voter prefers γ to be higher. In addition, every voter is also strictly doubt-prone. Let his value function be W_I if he acquires information and W_N if he does not. Even though every voter prefers the candidate who implements $\gamma = 1$, and even though information is free, there is still an equilibrium in which no one acquires information, and the candidate who implements $\gamma = 0$ wins with probability $\frac{1}{2}$. This equilibrium is Pareto-dominated (in expectation) by the other equilibria, in which at least a strict majority of agents acquires information, and the candidate who implements $\gamma = 1$ wins with probability 1. This is briefly shown below.

1) Equilibrium in which no voter is informed:

If no other voter is informed, then voter i does not acquire information either. Since $p_i \in (0, 1)$ if no one else is informed, it follows that $W_I < W_N$ (on his own he cannot force $p_i \in \{0, 1\}$).

Unless agent i is certain that either the right candidate or the wrong candidate always wins the election, i.e. that $p_i = 1$ or that $p_i = 0$, he does not acquire information.
Note that there is no equilibrium in which a minority of voters acquires information, since each voter in the minority has incentive to deviate. Note also that the difference between W_I and W_N for a given $p_i \in (0, 1)$ is higher if the difference between the agent's utility of $\gamma = 1$ and $\gamma = 0$ is larger.

2) Equilibrium in which at least a strict majority is informed:

If at least a strict majority is informed, then the right candidate wins with probability 1. Hence $p_i = 1$ for each agent i, and so he is indifferent, since $W_I = W_N$. Note, however, that this equilibrium does not survive if each voter i places an arbitrarily small probability $\delta > 0$ that each of the other voters does not acquire information.

The externality of information plays an excessive role in this simple example, however it may still have an impact in a more realistic model. In particular, this example suggests that as the difference between the agent's utility of the good policy and his utility of the bad policy increases, a doubt-prone agent has *less* incentive to acquire information.

4 Extensions

In this section, a general methodology for extending other models is first presented. The relation between this model and Kreps-Porteus is then discussed, and it is shown, using the general methodology introduced here, that the models are formally distinct, even if independence axioms are to hold at every stage. This last result may appear counterintuitive, since it may appear that a 'never' stage is formally equivalent to a 'much later' stage, but with a different interpretation. The reasons for the distinction between the two models is also discussed.

4.1 General Methodology

The vNM EU model has been extended in this paper to allow for the distinction between lotteries that lead to observed outcomes and lotteries that never resolve, from the agent's viewpoint. A general methodology for extending other models to make this distinction as well is now provided. These models do not need to satisfy the general independence axiom **A.4**. A new axiom is introduced instead. This axiom is weak enough to accommodate a broad class of continuous preferences, including a strict preference for randomization.
Suppose that an agent is indifferent between receiving an outcome $\tilde{z}$ as a final prize and an

unresolved lottery f. It is now assumed that the agent is also indifferent between receiving unresolved lottery f with some probability q and prize $\tilde{z}$ with probability q. In other words, the agent's valuation, or perception, of unresolved lottery f is assumed to be independent of the probability with which he received it, or on the probability of receiving any other prize. The value placed on unresolved lottery f and the value placed on outcome $\tilde{z}$ are always the same.

AXIOM E.1 (Unresolved lottery equivalent): For all $f \in \mathfrak{L}_0$ such that $\delta_f \sim \delta_{H(f)}$, and for all $X, \tilde{X} \in \mathfrak{L}_1$ such that $X = (z_1, q_1^I; ...; z_n, q_n^I;\ f, q; f_2, q_2^N; ...; f_m, q_m^N)$ and $\tilde{X} = (z_1, q_1^I; ...; z_n, q_n^I; H(f), q;\ f_2, q_2^N; ...; f_m, q_m^N)$, the following holds: $X \sim \tilde{X}$.

Recall that $H(f) \in \mathcal{Z}$ is well-defined for all $f \in \mathfrak{L}_0$ (by lemma 1), and that this does *not* require the general independence axiom **A.4**. Axiom **E.1** has not been explicitly assumed in the main model because it is trivially implied.

Lemma 2. *Suppose axioms **A.1** through **A.4** hold. Then axiom **E.1** holds.*

However, without the independence axiom **A.4**, it is no longer the case that **E.1** necessarily holds. If it is explicitly assumed, however, then any lottery $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N) \in \mathfrak{L}_1$ can be replaced with a lottery $\hat{X} = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ H(f_1), q_1^N; H(f_2), q_2^N; ...; H(f_m), q_m^N) \in \mathfrak{L}_0$. Note that $X \sim \hat{X}$, by a repeated application of axiom **E.1**. This property essentially reduces two-stage lotteries to one-stage lotteries. It therefore allows a straightforward extension of different types of frameworks, so as to distinguish between resolved and unresolved lotteries. To emphasize this point, suppose that a 'simple model' is loosely defined as follows:

Definition (Simple Model) A simple model $\langle \hat{\succeq}, W, \mathcal{T} \rangle$ consists of :

- A preference relation $\hat{\succeq}$ over one-stage lotteries in $\mathfrak{L}_0$.
- A representation $W : \mathfrak{L}_0 \to \Re$ for which $x \hat{\succeq} x' \Leftrightarrow W(x) \geq W(x')$ for all $x, x' \in \mathfrak{L}_0$.
- A set of axioms $\mathcal{T}$ that allow $\hat{\succeq}$ to be closed in the weak convergence topology, and that are sufficient for representation W to hold.

Then, any simple model can be expanded to accommodate the distinction between resolved and unresolved lotteries, in the following way. Take a simple model $\langle \hat{\succeq}, W, \mathcal{T} \rangle$. Since it is usually implicitly assumed that the agent will observe the outcome of a lottery, suppose that for all $x, x' \in \mathfrak{L}_0$, $x \hat{\succeq} x' \Leftrightarrow x \succeq x'$. That is, the set of axioms $\mathcal{T}$ is taken to hold for all resolved lotteries. If in addition, axioms **A.1** through **A.3** and axiom **E.1** hold, then $\succeq$ is represented as follows: for any $X, X' \in \mathfrak{L}_1$, $X \succ X' \Leftrightarrow W(\hat{X}) > W(\hat{X}')$.[33] As for a representation of H, note that the set of axioms for unresolved lotteries considered in the paper can also be replaced by a second simple model $\langle \hat{\succeq}_N, W_N, \mathcal{T}_N \rangle$.

Conditions for obtaining doubt-neutrality (indifference between observing and not observing the outcome) for preferences that satisfy **A.1** through **A.3** are now provided. This simple result demonstrates that assuming doubt-neutrality has strong implication on the agent's allowable preferences, independently of the independence axiom **A.4**. Recall that for lotteries $f, f' \in \mathfrak{L}_o$, the notation $f \succ f'$ denotes a comparison between lotteries that the agent expects to observe; while $\delta_f \succ \delta_{f'}$ denotes a comparison between the same lotteries, but they remain unresolved.

Doubt-neutrality result. *Suppose axioms **A.1** through **A.3** hold. Then the following three conditions are equivalent:*

(i) $f \sim \delta_f$ *for all* $f \in \mathfrak{L}_o$

(ii) $f \succ f' \quad \Rightarrow \quad \delta_f \succ \delta_{f'}$ *for all* $f, f' \in \mathfrak{L}_o$

(iii) $\delta_f \succ \delta_{f'} \quad \Rightarrow \quad f \succ f'$ *for all* $f, f' \in \mathfrak{L}_o$

In words, suppose that an agent has a choice between observing and not observing the outcome of a lottery. Then he is always indifferent, for this type of choice, if and only if the order between any lotteries $f, f' \in \mathfrak{L}_o$ is always strictly preserved. That is, if he strictly prefers f to f' when he expects to observe the outcome, then he also strictly prefers f to f' if he does not expect to see the outcome. Arguably, condition (i) is often violated, even in models that depart significantly from the standard vNM model. Consider, for instance, the following variant of Machina's (1989) mother example. Suppose that a donor to a charity has no strict preference over which worthwhile cause receives the benefit from his donation, but he prefers that it be decided randomly, for reasons of fairness. He may still prefer not to observe which cause receives it, and to remain in doubt (and perhaps this encourages him to donate to an umbrella organization rather a more targeted one). It must therefore be the case that there are some lotteries f, f' over the recipients which he ranks differently based on whether he observes the outcome.

Finally, note that a number of models have a dynamic component. The Kreps-Porteus framework, for instance, allows for preferences for temporal (sequential) resolution. The method

[33] Where, as before, for $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N)$,
$\hat{X} = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ H(f_1), q_1^N; H(f_2), q_2^N; ...; H(f_m), q_m^N) \in \mathfrak{L}_o$, and similarly for X' and $\hat{X}'$.

presented in this section for extending models can also be used at every stage to admit preferences over unresolved lotteries. It may appear that this procedure is equivalent to adding one last 'unresolved stage' to the sequence, but in fact it is not, as is now shown.

4.2 Relation to Kreps-Porteus

Suppose now, for simplicity, that there are 2 stages of resolution (early and late) in a Kreps-Porteus (KP) setup. In addition, suppose that all the KP axioms hold so that an expected utility representation holds at both stage. The early and late stages have associated utility functional denoted by u_e and u_l, respectively. Define the expectation operator in the usual way. The objects used are:

- Let $\mathfrak{R}_1$ be the set of simple lotteries over $\mathfrak{L}_\mathbf{0}$ (recall that $\mathfrak{L}_\mathbf{0}$ is the set of simple lotteries over the outcome space $\mathcal{Z}$).
- Denote typical element $\mathcal{X} = (f^I_{1,e}, q^I_{1,e}; ....; f^I_{n,e}, q^I_{n,e}) \in \mathfrak{R}_1$, where $f^I_{i,e} \in \mathfrak{L}_\mathbf{0}$.
- The KP agent has preferences $\succeq^K$ over lotteries $\mathcal{X} \in \mathfrak{R}_1$.

His preferences can be represented as follows:

For $\mathcal{X}, \mathcal{Y} \in \mathfrak{R}_1$, $\mathcal{X} \succ^K \mathcal{Y}$ if $\mathbf{W}(\mathcal{X}) > \mathbf{W}(\mathcal{Y})$, where $\mathbf{W}$ is of the following form:

$$\mathbf{W}(\mathcal{X}) = \sum q^I_{i,e} u_e \left(u_l^{-1}\left(Eu_l(z|f^I_{i,e})\right)\right)$$

Using the methodology from the previous section to allow KP to incorporate this model, suppose now that at every stage, there is a possibility of reaching unresolved lotteries. The objects used are now:

- Let $\hat{\mathfrak{R}}_1$ be the set of simple lotteries over $\mathfrak{L}_\mathbf{1} \cup \mathfrak{L}_\mathbf{0}$.
- Denote typical element $\hat{\mathcal{X}} = (\hat{f}^I_{1,e}, q^I_{1,e}; ...; \hat{f}^I_{n,e}, q^I_{n,e};\ f^N_{1,e}, q^N_{1,e}; ...; f^N_{m,e}, q^N_{m,e}) \in \hat{\mathfrak{R}}_1$, where $\hat{f}^I_{i,e} \in \mathfrak{L}_\mathbf{1}$, and $f^N_{j,e} \in \mathfrak{L}_\mathbf{0}$.
- Elements $\hat{f}^I_{i,e} \in \mathfrak{L}_\mathbf{1}$ lead to late stage lotteries over prizes $z_{i,l} \in \mathcal{Z}$ that are observed and lotteries $f^N_{j,l} \in \mathfrak{L}_\mathbf{0}$ that are unresolved, with the following notation:
  $\hat{f}^I_{i,e} = (z_1, q^I_{1,l}; ...; z_{n,l}, q^I_{n,l};\ f^N_{1,l}, q^N_{1,l}; ...; f^N_{m,l}, q^N_{m,l})$
- The KP agent has preferences $\hat{\succeq}^K$ over lotteries $\hat{\mathcal{X}} \in \hat{\mathfrak{R}}_1$.

Suppose now that an independence axiom for unresolved lottery holds at every stage. That is, define $\succeq_{N,e}$ and $\succeq_{N,l}$ in the natural way, and let an independence axiom hold for each of these preferences. In this case, there are unresolved utility functions v_e, v_l associated with $\succeq_{N,e}$ and

$\succeq_{N,l}$, respectively. Note that v_e and v_l need not be the same, since $\succeq_{N,e}$ and $\succeq_{N,l}$ are separate. Hence, there are four utility functions in this setting: u_e and u_l, which are associated with early and late general lotteries, as well as v_e and v_l which are associated with early and late unresolved lotteries. It is immediate, therefore, that having a KP model that accommodates unresolved lotteries is formally distinct from simply adding a 'never' stage, as this can only account for one additional utility function. The reason for this distinction is that the agent's perception of the unresolved lotteries need not be the same in the early stage as it is in the second stage.

Note that there is another, and perhaps more fundamental, difference between temporal resolution and lack of resolution. While the early stage leads to the eventual occurrence of the late stage, there is no notion of sequence for unresolved lotteries. That is, the first unresolved lottery cannot lead to a second lottery; each unresolved lottery is a final prize, and hence a terminal node. For that reason, while the KP representation will have terms such as $u_e(u_l^{-1}(\cdot)$, there cannot be an equivalent unresolved term, $v_e(v_l^{-1}(\cdot)$. To make this point less abstract, consider the representation for the agent's preference in this context:

For $\hat{\mathcal{X}}, \hat{\mathcal{Y}} \in \hat{\mathfrak{R}}_1$, $\hat{\mathcal{X}} \hat{\succ}^K \hat{\mathcal{Y}}$ if $\hat{\mathbf{W}}(\hat{\mathcal{X}}) > \hat{\mathbf{W}}(\hat{\mathcal{Y}})$, where $\hat{\mathbf{W}}$ is of the following form:

$$\begin{aligned}\hat{\mathbf{W}}(\hat{\mathcal{X}}) = \sum q^I_{i,e} u_e \left(u_l^{-1} \left(E u_l(z|f^I_{i,e}) + \sum q^N_{i,l} u_l \left(v_l^{-1} \left(E v_l(z|f^N_{i,l}) \right) \right) \right) \right) \\ + \sum q^N_{i,e} u_e \left(v_e^{-1} \left(E v_e(z|f^N_{i,e}) \right) \right)\end{aligned}$$

Note that in this representation, both utility functions v_e and v_l are terminal, in the sense that the expectations are over outcomes, and not over any further lotteries. Suppose now that the agent is indifferent between early and late resolution of uncertainty. In other words, suppose that the functions $u_e(\cdot)$ and $u_l(\cdot)$ are identical (up to positive affine transformation), and denoted by $u(\cdot)$. Then reduction does hold for resolved lotteries, in the sense that only the probability of reaching an observed outcome z matters, and not the sequence. Let this probability of reaching observed outcome z be simply $q^I(z)$. Then the representation of the agent's preferences are now reduced to:

$$\begin{aligned}\hat{\mathbf{W}}(\hat{\mathcal{X}}) = \sum q^I(z) u(z) + \sum q^I_{i,e}(z) \left(\sum q^N_{i,l} u \left(v_l^{-1} \left(E v_l(z|f^N_{i,l}) \right) \right) \right) \\ + \sum q^N_{i,e} u \left(v_e^{-1} \left(E v_e(z|f^N_{i,e}) \right) \right)\end{aligned}$$

While the notation is cumbersome, this representation demonstrates that each unresolved lottery is essentially a final prize, and its value depends on whether it is obtained early or late. An agent's preferences over unresolved lotteries are allowed to vary in time, even when he has neutral preferences over the timing of resolution of uncertainty.

5 Closing remarks

This paper provides a representation theorem for preferences over lotteries whose outcome may never be observed. The agent's perception of the unobserved outcome, relative to his risk-aversion, induces his attitude towards doubt. This relation is captured by his resolved utility function u, his unresolved utility function v and his unresolved decision weighting function w. The model presented here is an extension of the vNM framework, and it does not entail a significant axiomatic departure. However, it can accommodate behavioral patterns that are inconsistent with expected utility, and that have motivated a wide array of different frameworks. For instance, doubt-prone individuals have an incentive to self-handicap, and this incentive is higher if they are less certain about their competence. Note that this model does not allow agents to be delusional, since they are unable to mislead themselves into having false beliefs. Doubt-prone individuals are also more likely to choose the status quo bundle, if making a decision is more informative than inaction. In addition, an agent who is risk-neutral may still favor less risky investments, and would pay a firm to invest for him, even if it does not have superior expertise. The agent's attempt to preserve his self-image implies that his utility depends not only on the outcome that results, but also on the action taken. In a political economy context, doubt-proneness encourages political ignorance. When individuals derive more utility from the policies that they are not required to observe, they have *less* incentive to acquire information. Moreover, agents have a greater disutility from acquiring information if they are more ignorant ex-ante.

Finally, note that experiments that address the impact of anticipated regret frequently allow for foregone outcomes that agents do not observe (see Zeelenberg (1999)). These experiments would be useful in determining plausible degrees of doubt-proneness, although this is outside the scope of this paper.

Appendix

The appendix is structured as follows. Part 1 explains why the standard EU model is inappropriate when the agent does not expect to observe the resolution of uncertainty. Part 2 provides an example of the 'preservation of self-image' application. All the proofs are in part 3.

A.1 Limitations of the standard EU model

This example illustrates the problem with using the standard vNM EU model when there are outcomes that the agent never expects to observe. Consider the simple case of an agent who has performed a task and does not know how well he has done. There are no future decisions that depend on his performance. For example, as a simple adaptation of Savage's omelet, suppose that the agent does not know whether he has fed his guests a good omelet or a bad one. With probability p_t, he has done well ($\bar{t}$), and with probability $(1 - p_t)$ he has done badly ($\underline{t}$). He prefers having done well to having done badly, although this will have no future repercussions. Given the choice between remaining forever in doubt (D) and perfectly resolving the uncertainty, (ND), it might appear that he compares:

$$U_D = p_t u(\bar{t}) + (1 - p_t) u(\underline{t})$$

to

$$U_{ND} = p_t u(\bar{t}) + (1 - p_t) u(\underline{t})$$

and that since $U_D = U_{ND}$, he is indifferent. But U_D is *not* necessarily the right function to use if he chooses to remain in doubt, because from his frame of reference the final outcome will not be $\bar{t}$ or $\underline{t}$. That is, he does not expect to 'obtain' ex-post utility $u(\bar{t})$ or $u(\underline{t})$ because he does not expect to observe either $\bar{t}$ or $\underline{t}$. As it is not clear what his perception of the consequence is if he does not expect the uncertainty to be resolved (from his viewpoint), his expected utility is undetermined. In its current form, the standard EU model does not offer a method for evaluating this choice. Using U_D effectively ignores that the relevant frame of reference is the agent's, not the modeler's.[34]

Redefining the outcome space to include the observation itself does not eliminate the problem. Suppose that the outcome space is taken to be $Z = \{\bar{t}_D, \underline{t}_D, \bar{t}_{ND}, \underline{t}_{ND}\}$ where $\bar{t}_D$ represents the outcome that he did well but doubts it, $\bar{t}_{ND}$ that he did well and does not doubt it, and so

[34] This issue is not resolved by starting with preferences over lotteries as primitives. In the standard framework, the agent has primitive preferences over lotteries over outcomes, and he is not allowed to choose between lotteries whose resolution he observes and lotteries whose resolution he does not observe. He is therefore not given the option to express those preferences.

forth. He therefore compares the following:

$$U_D = p_t u(\bar{t}_D) + (1 - p_t)u(\underline{t}_D)$$

to

$$U_{ND} = p_t u(\bar{t}_{ND}) + (1 - p_t)u(\underline{t}_{ND})$$

It is difficult to interpret the meaning of the consequence 'did well, but doubts it' from his frame of reference, since it is not clear what it means to be in doubt if he knows that he has done well. In addition, his preferences over $\bar{t}_D$ and $\underline{t}_D$ are completely pinned down. Consider the two extremes, $p_t = 1$ and $p_t = 0$. When $p_t = 1$, there is no intrinsic difference between U_D and U_{ND}, since he knows that he has done well. Hence, $u(\bar{t}_D) = u(\bar{t}_{ND})$. Similarly, when $p_t = 0$, he knows he has done badly, and so $u(\underline{t}_D) = u(\underline{t}_{ND})$. It then follows that $U_D = U_{ND}$ for any $p_t \in [0, 1]$. This definition of the outcome space is essentially the same as simply $Z = \{\bar{t}, \underline{t}\}$. His indifference between remaining in doubt and not remaining in doubt is a consequence of following this approach, it is not implicit from the standard EU model.
Redefining the outcome space so that his utility is constant if he remains in doubt is even more problematic. Suppose that $Z = \{\bar{t}_{ND}, \underline{t}_{ND}, D\}$, letting $\bar{t}_{ND}$ be the outcome 'talented and he does not remain in doubt (he observes the outcome)', $\underline{T}_{ND}$ be the outcome 'untalented and he observes it', and letting D mean that he does not observe the outcome, hence remaining in doubt. He now compares:

$$U_D = u(D)$$

to

$$U_{ND} = p_t u(\bar{t}_{ND}) + (1 - p_t)u(\underline{t}_{ND})$$

However, in the limit $p_t \to 1$, U_D should approach U_{ND}, which only occurs if $u(D) = u(\bar{t}_{ND})$. But in that case, as $p_t \to 0$, U_D does not approach U_{ND}, and so there is an unavoidable discontinuity.

A.2 Applications

Numerical Example (Preservation of Self-image)

The following is a more general version of the numerical example provided in the main body of the paper. Suppose he puts in effort $e \in [0, 1]$, and obtains reward $m \in [0, 100]$. He also has an unobserved talent $t \in [0, 1]$. The agent is doubt-prone and risk-averse for both resolved and unresolved lotteries on talent. Specifically, $u = at^{1/2}$ for some $a > 0$, and v = t. His expected utility of money is linearly separable from his utility of talent, and is equal to his expected reward Em. He therefore maximizes:

$$\tilde{W}(e) = Em(e) - C(e)$$

where $C(e) \equiv u \circ v^{-1}(Ev(t)) - \sum_m p(m|e) u \circ v^{-1}(Ev(t|m,e))$

The agent's prior is q that talent $t = 0$, and $1-q$ that talent $t = 1$. He can put in level $e \in [\underline{e}, \overline{e}]$. Given that he has talent $t = 1$ or $t = 0$ and puts in effort e, his respective probabilities of obtaining monetary reward $m = 100$ are $p(100|t = 1, e) = e$ and $p(100|t = 0, e) = be$, for $b \in [0, 1)$.

Note that the ostrich effort e_0 in this example is $e = 0$, since he is certain to obtain $m = 0$, independently of his talent. It follows from the probabilities given above that:

$$p(\$0|1, e) = 1 - e$$

$$p(\$0|0, e) = 1 - be$$

$$p(100|e) = e(q + b(1 - q)$$

$$p(\$0|e) = 1 - e(q + b(1 - q))$$

$$p(1|100, e) = \frac{q}{q + b(1 - q)}$$

Solving:

$$W(e) = 100 * p(100|e) + a\left(p(0|e)p(\bar{t})p(0|\bar{t}, e)\right)^{1/2} + a\left(p(100|e)p(\bar{t})p(100|\bar{t}, e)\right)^{1/2}$$
$$= e(100\beta + a(\beta q)^{1/2}) + aq^{1/2}\left(1 - e(1 + \beta) + \beta e^2\right)^{1/2}$$

where $\beta = q + b(1 - q)$. Let $\gamma = 100\beta + a(\beta q)^{1/2}$, and $D = \frac{4\gamma^2}{a^2 q}$. Then, from the first order conditions, we obtain:

$$e^2(\beta C - 4\beta^2) + e(4\beta - C)(1 + \beta) + C - (1 + \beta)^2 = 0$$

The example in the text corresponds to the case $b = 0$, $q = 1/2$, and so $\beta = 1/2$, $\gamma = 50 + \frac{a}{2}$, and $d = 2D = \left(\frac{200}{a} + 2\right)^2$.

A.3 Proofs

Lemma 1 (Informed certainty equivalent). *Proof.* Define $\succeq_N$ in the same way as in the text, i.e. $\delta_f \succeq \delta_{f'} \Leftrightarrow f \succeq_N f'$ (and similarly for $\sim_N$, $\succ_N$). Note that $\succeq_N$ inherits continuity, and so there exists a function $H : \mathfrak{L}_o \to \mathcal{Z}$ such that $\delta_{H(f)} \sim_N f$ for all $f \in \mathfrak{L}_o$. By the certainty axiom **A.3**, it follows that $\delta_{H(f)} \sim \delta_{\delta_{H(f)}}$. Hence $\delta_{H(f)} \sim \delta_f$.

Representation Theorem. *Proof.* Let $X = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ f_1, q_1^N; f_2, q_2^N; ...; f_m, q_m^N)$. By lemma 1, $\delta_f \sim \delta_{H(f)}$ for any $f \in \mathfrak{L}_o$. Hence, by a well-known implication of the independence axiom **A.4**, $X \sim \tilde{X}$, where $\tilde{X} = (z_1, q_1^I; z_2, q_2^I; ...; z_n, q_n^I;\ H(f_1), q_1^N; H(f_2), q_2^N; ...; H(f_m), q_m^N)$, and so $X \sim \tilde{X}$. Defining $\tilde{Y}$ similarly, $Y \sim \tilde{Y}$. By transitivity, $X \succ Y \Rightarrow \tilde{X} \succ \tilde{Y}$. Note that all lotteries $\tilde{X}$ and $\tilde{Y}$ are one-stage lotteries, with final outcomes as prizes. Define the preference relation $\succ_I$ in the following way: $X \succ Y \Rightarrow \tilde{X} \succ_I \tilde{Y}$. All the EU axioms hold on $\succ_I$, and so $\tilde{X} \succ \tilde{Y}$ if and only if $W(\tilde{X}) > W(\tilde{Y})$, where

$$W(\tilde{X}) = \sum_{i=1}^{n} q_i^I u(z_i) + \sum_{i=1}^{m} q_i^N u\left(H(f_{z_i})\right)$$

and W is unique up to positive affine transformation. But since $X \succ Y \Rightarrow \tilde{X} \succ \tilde{Y}$, it follows that $X \succ Y$ if and only if $W(\tilde{X}) > W(\tilde{Y})$, which completes the proof.

Theorem 2. *Proof.* Since all the axioms required for an EU representation of $\succeq_N$ hold, it is immediate that $\succeq_N$ can be represented by an expected utility function v. For any $f \in \mathfrak{L}_o$, $\delta_H(f) \sim_N f$, since $\delta_H(f) \sim f$ (by definition of H), and $\delta_{\delta_H f} \sim \delta_H(f)$ (by the certainty axiom **A.1**. Hence $v(H(f)) = \sum_{z \in \mathcal{Z}} v(z)f(z)$. It follows that $H(f) = v^{-1}\left(\sum_{z \in \mathcal{Z}} v(z)f(z)\right)$. If there exists more than one $v^{-1}(\cdot)$, any can be chosen arbitrarily: suppose $v(\tilde{z}) = v(\tilde{z}') = \sum_{z \in \mathcal{Z}} v(z)f(z)$. Then by the certainty axiom **A.1**, $\delta_{\tilde{z}} \sim \delta_{\delta_{\tilde{z}}}$ and $\delta_{\tilde{z}'} \sim \delta_{\delta_{\tilde{z}'}}$, hence $\delta_{\tilde{z}} \sim \delta_{\tilde{z}'}$, from which it follows that $u(\tilde{z}) = u(\tilde{z'})$. Since they have the same value, either $\tilde{z}$ or $\tilde{z}'$ can be used in the representation. Note also that it follows from the certainty axiom **A.1** (and transitivity) that $\delta_z \succ \delta_{z'} \Leftrightarrow \delta_{\delta_z} \succ \delta_{\delta_{z'}}$ for all $z, z' \in \mathfrak{L}_o$. Hence $u(z) > u(z') \Leftrightarrow v(z) > v(z')$ for all $z, z' \in \mathfrak{L}_o$.

Theorem 3. *Proof.* Case 1 is shown below, and case 2 can be proven in a similar way (by changing all the signs). Suppose RDU holds for $\succeq_N$.
There are two cases two consider:

(a) f, f' have more than 2 elements:
Let $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$, $f' = (z_1, p_1; ...z_i'; p_i; z_{i+1}', p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and $(z_i', z_{i+1}') \subset (z_i, z_{i+1})$. Suppose that, for some $a \in (0,1)$ and some $z \in (z_i', z_{i+1}')$,

$$af + (1-a)\delta_z \succeq_N af' + (1-a)\delta_z$$

Since RDU holds:

$$f \sim_N f' \Rightarrow V_{RDU}(f) = V_{RDU}(f')$$

$$\Rightarrow v(z_1)+\sum_{j=2}^{i-1} w(p_j^*)[v(z_j)-v(z_{j-1})]+w(p_i^*)[v(z_i)-v(z_{i-1})]+w(p_{i+1}^*)[v(z_{i+1})-v(z_i)]$$

$$+w(p_{i+2}^*)[v(z_{i+2})-v(z_{i+1})]+\sum_{j=i+3}^{n} w(p_j^*)[v(z_j)-v(z_{j-1})] \quad =$$

$$v(z_1)+\sum_{j=2}^{i-1} w(p_j^*)[v(z_j)-v(z_{j-1})]+w(p_i^*)[v(z_i')-v(z_{i-1})]+w(p_{i+1}^*)[v(z_{i+1}')-v(z_i')]$$

$$+w(p_{i+2}^*)[v(z_{i+2})-v(z_{i+1}')]+\sum_{j=i+3}^{n} w(p_j^*)[v(z_j)-v(z_{j-1})]$$

$$\Rightarrow \frac{w(p_{i+1}^*)-w(p_{i+2}^*)}{w(p_i^*)-w(p_{i+1}^*)}=\frac{v(z_i')-v(z_i)}{v(z_{i+1})-v(z_{i+1}')} \tag{1}$$

Note that $af+(1-a)\delta_z=(z_1,ap_1;...z_i;ap_i;z,1-a;z_{i+1},ap_{i+1};...;z_n,ap_n)$, where the ranking of z is due to $z\in(z_i',z_{i+1}')\subset(z_i,z_{i+1})$. Similarly, $af'+(1-a)\delta_z=(z_1,ap_1;...z_i';ap_i;z,1-a;z_{i+1}',ap_{i+1};...;z_n,ap_n)$. Using the condition

$$af+(1-a)\delta_z \succeq_N af'+(1-a)\delta_z$$

it follows that

$$\Rightarrow v(z_1)+\sum_{j=2}^{i-1} w(ap_j^*+1-a)[v(z_j)-v(z_{j-1})]+w(ap_i^*+1-a)[v(z_i)-v(z_{i-1})]$$

$$+w(ap_{i+1}^*+1-a)[v(z)-v(z_i)]+w(ap_{i+1}^*)[v(z_{i+1})-v(z)]$$

$$+w(ap_{i+2}^*)[v(z_{i+2})-v(z_{i+1})]+\sum_{j=i+3}^{n} w(ap_j^*)[v(z_j)-v(z_{j-1})] \quad \geq$$

$$v(z_1)+\sum_{j=2}^{i-1} w(ap_j^*+1-a)[v(z_j)-v(z_{j-1})]+w(ap_i^*+1-a)[v(z_i')-v(z_{i-1})]$$

$$+w(ap_{i+1}^*+1-a)[v(z)-v(z_i')]+w(ap_{i+1}^*)[v(z_{i+1}')-v(z)]$$

$$+w(ap_{i+2}^*)[v(z_{i+2})-v(z_{i+1}')]+\sum_{j=i+3}^{n} w(ap_j^*)[v(z_j)-v(z_{j-1})]$$

$$\Rightarrow \frac{w(ap_{i+1}^*)-w(ap_{i+2}^*)}{w(ap_i^*+1-a)-w(ap_{i+1}^*+1-a)}\geq\frac{v(z_i')-v(z_i)}{v(z_{i+1})-v(z_{i+1}')} \tag{2}$$

Combining (1) and (2), we obtain:

$$\frac{w(ap_{i+1}^*)-w(ap_{i+2}^*)}{w(ap_i^*+1-a)-w(ap_{i+1}^*+1-a)}\geq\frac{w(p_{i+1}^*)-w(p_{i+2}^*)}{w(p_i^*)-w(p_{i+1}^*)} \tag{3}$$

Note that this does not depend on the utility function v, but only on the weighting function

w. Take any $\tilde{f} = (\tilde{z}_1, p_1; ...\tilde{z}_i; p_i; \tilde{z}_{i+1}, p_{i+1}; ...; \tilde{z}_n, p_n)$, $\tilde{f}' = (\tilde{z}_1, p_1; ...\tilde{z}'_i; p_i; \tilde{z}'_{i+1}, p_{i+1}; ...; \tilde{z}_n, p_n)$ and $\tilde{z}$ such that $\tilde{z} \in (\tilde{z}'_i, \tilde{z}'_{i+1}) \subset (\tilde{z}_i, \tilde{z}_{i+1})$. It must be that $a\tilde{f} + (1-a)\delta_{\tilde{z}} \succeq_N a\tilde{f}' + (1-a)\delta_{\tilde{z}}$. Suppose not, i.e. suppose that $a\tilde{f}' + (1-a)\delta_{\tilde{z}} \succ_N a\tilde{f} + (1-a)\delta_{\tilde{z}}$. Then, redoing a similar calculation to the one above, we obtain:

$$\frac{w(ap^*_{i+1}) - w(ap^*_{i+2})}{w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a)} < \frac{w(p^*_{i+1}) - w(p^*_{i+2})}{w(p^*_i) - w(p^*_{i+1})} \tag{4}$$

which contradicts (3). Hence ISC holds for this case.

(b) f, f' have exactly 2 elements:
Let $f = (z_1, 1-p; z_2, p)$, $f' = (z'_1, 1-p; z'_2, p) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and $(z'_1, z'_2) \subset (z_1, z_2)$. Suppose that, for some $a \in (0,1)$ and some $z \in (z'_1, z'_2)$. If $\succeq_N$ satisfies RDU, then:

$$f \sim_N f' \Rightarrow v(z_1) + w(p)[v(z_2) - v(z_1)] = v(z'_1) + w(p)[v(z'_2) - v(z'_1)]$$

$$\Rightarrow w(p) = \frac{v(z'_1) - v(z_1)}{[v(z'_1) - v(z_1)] + [v(z_2) - v(z'_2)]}$$

$$\Rightarrow \frac{w(p)}{1 - w(p)} = \frac{v(z'_1) - v(z_1)}{v(z_2) - v(z'_2)} \tag{5}$$

Since $af + (1-a)\delta_z = ((z_1, a(1-p); z, 1-a; z_2, ap)$ and $af' + (1-a)\delta_z = ((z'_1, a(1-p); z, 1-a; z'_2, ap)$, the condition $af + (1-a)\delta_z \succ_N af' + (1-a)\delta_z$ implies (using a similar calculation to the one used for obtaining (3)) that

$$\Rightarrow \frac{w(ap)}{1 - w(ap + 1 - a)} \geq \frac{v(z'_1) - v(z_1)}{v(z_2) - v(z'_2)} \tag{6}$$

and combining (4) and (5), it follows that

$$\Rightarrow \frac{w(ap)}{1 - w(ap + 1 - a)} \geq \frac{w(p)}{1 - w(p)} \tag{7}$$

As before, this does not depend on the $v's$, but only on the weighting function w. Take any $\tilde{f} = (\tilde{z}_1, 1-p; \tilde{z}_2, p)$, $\tilde{f}' = (\tilde{z}'_1, p_1; \tilde{z}'_2, p_2)$ and $\tilde{z}$ such that $\tilde{z} \in (\tilde{z}'_1, \tilde{z}'_2) \subset (\tilde{z}_1, \tilde{z}_2)$. It must be that $a\tilde{f} + (1-a)\delta_{\tilde{z}} \succeq_N a\tilde{f}' + (1-a)\delta_{\tilde{z}}$. Suppose not, i.e. suppose that $a\tilde{f}' + (1-a)\delta_{\tilde{z}} \succ_N a\tilde{f} + (1-a)\delta_{\tilde{z}}$. Then, redoing a similar calculation to the one above, we obtain:

$$\Rightarrow \frac{w(ap)}{1 - w(ap + 1 - a)} < \frac{w(p)}{1 - w(p)} \tag{8}$$

which contradicts (7). Hence ISC holds for this case as well, which completes the proof.

The following lemma is used in the proof of theorem 4:

Lemma 2t. *Let $w : [0,1] \to [0,1]$. Take any $p, q, p', p' \in [\underline{p}, \overline{p}] \subseteq [0,1]$ such that $p > p' > q'$, $q > q'$. Then if w is concave on $[\underline{p}, \overline{p}]$:*

$$\frac{w(p) - w(q)}{p - q} \leq \frac{w(p') - w(q')}{p' - q'}$$

if w is convex on $[\underline{p}, \overline{p}]$:

$$\frac{w(p) - w(q)}{p - q} \geq \frac{w(p') - w(q')}{p' - q'}$$

Proof. The proof is only shown for a concave function w. We make use of the following well-known result that a function f is concave if and only if for any $\tilde{p} > \tilde{q} > \tilde{r}$,

$$\frac{f(\tilde{p}) - f(\tilde{q})}{\tilde{p} - \tilde{q}} \leq \frac{f(\tilde{p}) - f(\tilde{r})}{\tilde{p} - \tilde{r}} \leq \frac{f(\tilde{q}) - f(\tilde{r})}{\tilde{q} - \tilde{r}} \tag{9}$$

We now directly prove the claim for each of the three possible cases:

(i) $p > q > p' > q'$
Using (9) twice,

$$\frac{w(p) - w(q)}{p - q} \leq \frac{w(q) - w(p')}{q - p'} \leq \frac{w(p') - w(q')}{p' - q'}$$

(ii) $p > p' > q > q'$
Using (9) twice,

$$\frac{w(p) - w(q)}{p - q} \leq \frac{w(p') - w(q)}{p' - q} \leq \frac{w(p') - w(q')}{p' - q'}$$

(iii) $p > p' = q > q'$
In this case, the result follows immediately from (9):

$$\frac{w(p) - w(q)}{p - q} \leq \frac{w(q) - w(q')}{q - q'} = \frac{w(p') - w(q')}{p' - q'}$$

which completes the proof.

Theorem 4. *Proof.* Suppose that $\succeq_N$ satisfies RDU. We first show (A) that the weighting function w is concave implies that for any $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$,
$f' = (z_1, p_1; ...z'_i; p_i; z'_{i+1}, p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$, and for all $a \in (0,1)$ and $z \in (z_i, z_{i+1})$,

$$af + (1-a)\delta_z \succeq_N af' + (1-a)\delta_z$$

We then prove the converse (B).

Proof of (A) Suppose that the weighting function w is concave. We proceed by contradiction. There are two cases to consider:

(a) f, f' have more than two elements: Let $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$, $f' = (z_1, p_1; ...z'_i; p_i; z'_{i+1}, p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$. Suppose there exists some $a \in (0,1)$ and some $z \in (z_i, z_{i+1})$ such that $af' + (1-a)\delta_z \succ_N af + (1-a)\delta_z$. Using the derivation of theorem 3, it follows that

$$\frac{w(ap^*_{i+1}) - w(ap^*_{i+2})}{w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a)} < \frac{w(p^*_{i+1}) - w(p^*_{i+2})}{w(p^*_i) - w(p^*_{i+1})} \tag{10}$$

We now show:

(I) $w(ap^*_{i+1}) - w(ap^*_{i+2}) \geq a\left(w(p^*_{i+1}) - w(p^*_{i+2})\right)$
Note that $p^*_{i+1} > p^*_{i+2} > ap^*_{i+2}$, since $a \in (0,1)$, and using the definition of p^*. It is immediate that $ap^*_{i+1} > ap^*_{i+2}$. It follows, therefore, from lemma 2t, that:

$$\frac{w(p^*_{i+1}) - w(p^*_{i+2})}{p^*_{i+1} - p^*_{i+2}} \leq \frac{w(ap^*_{i+1}) - w(ap^*_{i+2})}{ap^*_{i+1} - ap^*_{i+2}}$$

Rearranging, we obtain $w(ap^*_{i+1}) - w(ap^*_{i+2}) \geq a\left(w(p^*_{i+1}) - w(p^*_{i+2})\right)$.

(II) $w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a) \leq a\left(w(p^*_i) - w(p^*_{i+1})\right)$
Note that $ap^*_i + 1 - a > p^*_i$, since $a, p^*_i \in (0,1)$ implies that $1 - a > p^*_i(1-a)$. Similarly, $ap^*_{i+1} + 1 - a > p^*_{i+1}$, and we know that $p^*_i > p^*_{i+1}$. Using lemma 2t, it follows that:

$$\frac{w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a)}{(ap^*_i + 1 - a) - \left(ap^*_{i+1} + 1 - a\right)} \leq \frac{w(p^*_i) - w(p^*_{i+1})}{p^*_i - p^*_{i+1}}$$

Rearranging, we obtain $w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a) \leq a\left(w(p^*_i) - w(p^*_{i+1})\right)$

Combining (I) and (II) (noting that both sides of (II) are greater than zero), it follows that

$$\frac{w(ap^*_{i+1}) - w(ap^*_{i+2})}{w(ap^*_i + 1 - a) - w(ap^*_{i+1} + 1 - a)} \geq \frac{w(p^*_{i+1}) - w(p^*_{i+2})}{w(p^*_i) - w(p^*_{i+1})} \tag{11}$$

which is a contradiction of (10).

(b) f, f' have exactly 2 elements:
Let $f = (z_1, 1-p; z_2, p)$, $f' = (z'_1, 1-p; z'_2, p) \in \mathfrak{L}_o$ such that $f \sim_N f'$, and $(z'_1, z'_2) \subset (z_1, z_2)$. Suppose there exists some $a \in (0,1)$ and some $z \in (z_1, z_2)$ such that $af' + (1-a)\delta_z \succ_N$

$af + (1-a)\delta_z$. Using the derivation of theorem 3, it follows that

$$\frac{w(ap)}{1-w(ap+1-a)} < \frac{w(p)}{1-w(p)} \tag{12}$$

We now show:

(I) $w(ap) \geq aw(p)$
$a \in (0,1)$ and so $p > ap > 0$. It follows from the well-known result (9) used in proving lemma 2t that:

$$\frac{w(p)-w(0)}{p} \leq \frac{w(ap)-w(0)}{ap-0}$$

Using $w(0)=0$ and rearranging, we obtain $w(ap) \geq aw(p)$

(II) $1 - w(ap+1-a) \leq a\,(1-w(p))$
Note that $1 > ap+1-a > p$, since it is immediate from $a, p \in (0,1)$ that $a > ap$ and $1-a > p(1-a)$.

Using (9) again,

$$\frac{w(1)-w(ap+1-a)}{1-(ap+1-a)} \leq \frac{w(1)-w(p)}{1-p}$$

Using $w(1)=1$ and rearranging, we obtain that $1-w(ap+1-a) \leq a\,(1-w(p))$.

Combining (I) and (II), we obtain

$$\frac{w(ap)}{1-w(ap+1-a)} \geq \frac{w(p)}{1-w(p)} \tag{13}$$

which contradicts (12).

Proof of (B) Suppose that for any $f = (z_1, p_1; ...z_i; p_i; z_{i+1}, p_{i+1}; ...; z_n, p_n)$, $f' = (z_1, p_1; ...z'_i; p_i; z'_{i+1}, p_{i+1}; ...; z_n, p_n) \in \mathfrak{L}_{\mathbf{o}}$ such that $f \sim_N f'$, and $(z'_i, z'_{i+1}) \subset (z_i, z_{i+1})$, and for all $a \in (0,1)$ and $z \in (z_i, z_{i+1})$,

$$af + (1-a)\delta_z \succeq_N af' + (1-a)\delta_z$$

We proceed as follows: (a) we first show that there is no interval $[\underline{p}, \overline{p}] \subseteq [0,1]$ on which w is strictly convex; (b) we then show that there is no interval $[\underline{p}, \overline{p}] \subseteq [0,1]$ such that for all $p \in [\underline{p}, \overline{p}]$, $w(p)$ is 'under the diagonal', i.e. $\frac{w(\overline{p})-w(p)}{\overline{p}-p} > \frac{w(\overline{p})-w(\underline{p})}{\overline{p}-\underline{p}} > \frac{w(p)-w(\underline{p})}{p-\underline{p}}$ (note that with stronger smoothness assumptions this would be sufficient for concavity); (c) we use results (a) and (b) to prove that w must be concave. We first note that it follows from the claim and from the derivation of theorem 3 that:

$$\frac{w(ap_1)-w(ap_2)}{w(ap_0+1-a)-w(ap_1+1-a)} \geq \frac{w(p_1)-w(p_2)}{w(p_0)-w(p_1)} \tag{14}$$

for all $0 \leq p_2 < p_1 < p_0 \leq 1$ and $a \in (0,1)$.

(a) We proceed by contradiction: suppose there does exist an interval $[\underline{p}, \overline{p}] \subseteq [0,1]$ on which w is strictly convex. Let $\underline{p} < p_2 < p_1 < p_0 < \overline{p}$, and let $\{\frac{\underline{p}}{p_2}, \frac{1-\overline{p}}{1-p_0}\} < a < 1$. It follows that $\underline{p} < ap_2 < ap_1 < ap_1 + 1 - a < ap_0 + 1 - a)\overline{p}$. Using lemma 2t, it follows that:

$$\frac{w(p_1) - w(p_2)}{p_1 - p_2} > \frac{w(ap_1) - w(ap_2)}{ap_1 - ap_2} \tag{15}$$

$$\frac{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)}{(ap_0 + 1 - a) - (ap_1 + 1 - a)} > \frac{w(p_0) - w(p_1)}{p_0 - p_1} \tag{16}$$

Rearranging and combining (15) and (16), it follows that

$$\frac{w(ap_1) - w(ap_2)}{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)} < \frac{w(p_1) - w(p_2)}{w(p_0) - w(p_1)}$$

which contradicts (14).

(b) We proceed again by contradiction: suppose that there does exist an interval $[\underline{p}, \overline{p}] \subseteq [0,1]$ such that $\frac{w(\overline{p})-w(p)}{\overline{p}-p} > \frac{w(\overline{p})-w(\underline{p})}{\overline{p}-\underline{p}} > \frac{w(p)-w(\underline{p})}{p-\underline{p}}$ for all $p \in [\underline{p}, \overline{p}]$.
Let $a = 1 - (\overline{p} - \underline{p}) + \epsilon$, for an arbitrarily small ϵ. Let $\tilde{p} = \underline{p}/a$. Using result (a), $[\tilde{p}, \tilde{p} + \delta]$ cannot be strictly convex, for any $\delta \in (0, 1 - \tilde{p}]$. We can therefore find $\{p_0, p_1, p_2\} \in [\tilde{p}, \tilde{p} + \delta]$ such that $p_2 < p_1 < p_0$ and

$$\frac{w(p_1) - w(p_2)}{p_1 - p_2} \geq \frac{w(p_0) - w(p_1)}{p_0 - p_1} \tag{17}$$

As δ, ϵ become arbitrarily small (and $a\delta \leq \epsilon$), $ap_2 \to \underline{p}$, $ap_0 + 1 - a \to \overline{p}$ and $\{ap_2, ap_1, ap_1 + 1 - a, ap_0 + 1 - a\} \in [\underline{p}, \overline{p}]$. We therefore have that for small enough δ, ϵ,

$$\frac{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)}{(ap_0 + 1 - a) - (ap_1 + 1 - a)} > \frac{w(\overline{p}) - w(\underline{p})}{\overline{p} - \underline{p}} \tag{18}$$

and

$$\frac{w(\overline{p}) - w(\underline{p})}{\overline{p} - \underline{p}} > \frac{w(ap_1) - w(ap_2)}{a(p_1 - p_2)} \tag{19}$$

Combining (18) and (19):

$$\frac{w(ap_1) - w(ap_2)}{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)} < \frac{p_1 - p_2}{p_0 - p_1} \tag{20}$$

Combining (17) and (20), we obtain:

$$\frac{w(ap_1) - w(ap_2)}{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)} < \frac{w(p_1) - w(p_2)}{w(p_0) - w(p_1)}$$

which contradicts (14).

(c) We now prove that w is concave. Suppose not, i.e. suppose there exist $0 \leq p < q < r < 1$ such that

$$\frac{w(r) - w(q)}{r - q} > \frac{w(q) - w(p)}{q - p} \tag{21}$$

Let $a = 1 - (r - q) + \epsilon$, for an arbitrarily small ϵ. Let $\tilde{p} = q/a$. Using result (a), $[\tilde{p} - \delta, \tilde{p}]$ cannot be strictly convex, for any $\delta \in (0, \tilde{p}]$. We can therefore find $\{p_0, p_1, p_2\} \in [\tilde{p} - \delta, \tilde{p}]$ such that $p_2 < p_1 < p_0$ and

$$\frac{w(p_1) - w(p_2)}{p_1 - p_2} \geq \frac{w(p_0) - w(p_1)}{p_0 - p_1} \tag{22}$$

As δ, ϵ become arbitrarily small (and $a\delta \leq \epsilon$), $ap_1 \to q$, $ap_0 + 1 - a \to r$, $\{ap_2, ap_1\} \in (p, q]$ and $\{ap_1 + 1 - a, ap_0 + 1 - a\} \in [q, r]$.
Using result (b), we have can find some (small enough) δ, ϵ such that

$$\frac{w(ap_1) - w(ap_2)}{a(p_1 - p_2)} \leq \frac{w(q) - w(p)}{q - p} \tag{23}$$

$$\frac{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)}{(ap_0 + 1 - a) - (ap_1 + 1 - a)} \geq \frac{w(r) - w(q)}{r - q} \tag{24}$$

Combining (21, (23) and (24) we have

$$\frac{w(ap_1) - w(ap_2)}{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)} < \frac{p_1 - p_2}{p_0 - p_1} \tag{25}$$

Combining (22) and (25), we have

$$\frac{w(ap_1) - w(ap_2)}{w(ap_0 + 1 - a) - w(ap_1 + 1 - a)} < \frac{w(p_1) - w(p_2)}{w(p_0) - w(p_1)}$$

which contradicts (14), and completes the proof.

Theorem 5. *Proof.* Axioms **A.1-A.4** imply that $\succeq_N$ is a weak order and that Jensen-continuity holds. The proof for the RDU representation then follows from Wakker (1994).
As in theorem 2, for any $f \in \mathfrak{L}_o$, $\delta_H(f) \sim_N f$. Since $w(1) = 1$, it follows that $v(H(f)) = v^{-1}(V_{RDU}(f))$, and hence $H(f) = v^{-1}(V_{RDU}(f))$.

Theorem 6. *Suppose that axioms **A.1** through **A.4** and the RDU axioms hold, and let u and v be the utility functions associated with the resolved and unresolved lotteries, respectively, and w be the decision weight associated with the unresolved lotteries. In addition, suppose that u, v are both differentiable. Then:*
(i) If there exists $p \in (0,1)$ such that $p < w(p)$, then there exists an $f \in \mathfrak{L}_o$ such that $\delta_f \succ f$. Similarly, if there exists $p' \in (0,1)$ such that $p' > w(p')$, then there exists an $f' \in \mathfrak{L}_o$ such that $f' \succ \delta'_f$.
(ii) If $\succeq$ exhibits doubt-aversion, then $p \geq w(p)$ for all $p \in (0,1)$. Moreover, if u exhibits stronger diminishing marginal utility than v (i.e. $u = \lambda \circ v$ for some continuous, weakly concave, and increasing λ on $v([\underline{z}, \bar{z}])$), then $\succeq_N$ violates quasi-concavity. (that is, there exists some $f', f'' \in \mathfrak{L}_o$, and $\alpha \in (0,1)$ such that $f' \succ f''$ and $f'' \succ_N \alpha f' + (1-\alpha) f''$).
Similarly, if $\succeq$ exhibits doubt-proneness, then $p \leq w(p)$ for all $p \in (0,1)$. Moreover, if v exhibits stronger diminishing marginal utility than u , then $\succeq_N$ violates quasi-convexity. (that is, there exists some $f', f'' \in \mathfrak{L}_o$, and $\alpha \in (0,1)$ such that $f' \succ f''$ and $\alpha f' + (1-\alpha) f'' \succ_N f'$).

Proof. (i) Suppose not, i.e. suppose that there exists $p \in (0,1)$ such that $p < w(p)$, and that $f \succeq \delta_f$ for all $f \in \mathfrak{L}_o$. Let $f_\epsilon = (z; 1-p; z+\epsilon, p)$ for some $z \in \mathcal{Z}$, $p \in \mathfrak{L}_o$, $0 < \epsilon < \bar{z} - z$. Since $f \succeq \delta_f$, by continuity (and using the certainty axiom), there exists a $\tilde{z}_\epsilon \in (z, z+\epsilon)$ such that $f \succeq \left[\delta_{\tilde{z}_\epsilon} \sim \delta_{\delta_{\tilde{z}_\epsilon}}\right] \succeq \delta_f$. Hence:

$$(1-p)u(z) + pu(z+\epsilon) \geq u(\tilde{z}_\epsilon)$$

$$w(p)\left(v(z+\epsilon) - v(z)\right) + v(z) \leq v(\tilde{z}_\epsilon)$$

Rearranging:

$$p \geq \frac{u(\tilde{z}_\epsilon) - u(z)}{u(z+\epsilon) - u(z)}$$

$$w(p) \leq \frac{v(\tilde{z}_\epsilon) - v(z)}{v(z+\epsilon) - v(z)}$$

Hence:

$$\frac{u(\tilde{z}_\epsilon) - u(z)}{u(z+\epsilon) - u(z)} - \frac{v(\tilde{z}_\epsilon) - v(z)}{v(z+\epsilon) - v(z)} \leq p - w(p)$$

But as $\epsilon \to 0$, $\frac{u(\tilde{z}_\epsilon)-u(z)}{u(z+\epsilon)-u(z)} \to \frac{u'(z)}{u'(z)}$, and $\frac{v(\tilde{z}_\epsilon)-v(z)}{v(z+\epsilon)-v(z)} \to \frac{v'(z)}{v'(z)}$, by differentiability. Since the left-hand-side goes to $1 - 1 = 0$ in the limit, while the right-hand-side does not change, it must be that $0 \leq p - w(p)$. But this is a contradiction, since $p < w(p)$.
The second part of the result can be proved in a similar manner, for the case $p' > w(p')$.

(ii) The result is only shown for doubt-aversion; a similar reasoning holds for doubt-proneness. By the contrapositive of (i), it is immediate that if $f \succeq \delta_f$ for all $f \in \mathfrak{L}_o$, then $w(p) \leq p$ for all $p \in (0,1)$. Now suppose that $f \succ \delta_f$ for some f, and that u is a (weakly) concave transformation

of v. If w is not concave, then $\succeq_N$ cannot be quasi-concave, by Wakker (1994) theorem 25. Since $w(0) = 0$, $w(1) = 1$, $w(p) \geq p$ for a concave function. We have that $w(p) \leq p$, and so it suffices to show that $w(p) < p$ for some p. Suppose not. That is, $w(p) = p$ for all p. Since u is more concave than v, it must be that $u^{-1}(EU(f)) \leq v^{-1}(EV(f))$(that is, the certainty equivalent of f for the informed lotteries is not bigger than the certainty equivalent of f for the unresolved lotteries, by a well known result). However, since $f \succ \delta_f$, it must also be that $u^{-1}(EU(f)) > v^{-1}(EV(f))$, which is a contradiction.
Note that if $f \sim \delta_f$ for all $f \in \mathfrak{L}_o$, than trivially, u is a linear transformation of v, and $w(p) = p$.

Corollary. *Proof.* If $\succeq_N$ displays mean-preserving risk-aversion, then $w(p)$ is convex, by Chew, Epstein and Safra (1986) or Grant, Kajii and Polak (2000). Since $w(0) = 0$, $w(1) = 1$, it must be that $p \geq w(p)$. Since $\delta_f \succeq f$, it follows from result (ii) that $p \leq w(p)$. Hence $w(p) = p$, implying that $\succeq_N$ satisfies expected utility.
Since $\delta_f \succeq f$ for all $f \in \mathfrak{L}_o$, and both u and v are of EU form, u must be a concave transformation of v. This is well-known, see for instance Kreps-Porteus (1978).

Preservation of self-image. For an agent who is doubt-prone and risk-averse for both resolved and unresolved lotteries, the following holds:

$$C(e) \equiv u \circ v^{-1}(Ev(t)) - \sum_m p(m|e) u \circ v^{-1}(Ev(t|m,e)) \geq 0$$

Proof. Note that $u \circ v^{-1}(\cdot)$ is concave. Hence

$$\begin{aligned}
\sum_m p(m|e) u \circ v^{-1}(Ev(t|m,e)) &\leq u \circ v^{-1}\left(\sum_m p(m|e)(Ev(t|m,e))\right) \\
&\leq u \circ v^{-1}\left(\sum_m p(m|e) \sum_t \frac{p(m|t,e)p(t)}{p(m|e)} v(t)\right) \\
&\leq u \circ v^{-1}\left(\sum_m \sum_t p(m|t,e)p(t)v(t)\right) \\
&\leq u \circ v^{-1}\left(\sum_t \sum_m p(m|t,e)p(t)v(t)\right) \\
&\leq u \circ v^{-1}\left(\sum_t p(t)v(t)\right) = u \circ v^{-1}(Ev(t))
\end{aligned}$$

Doubt-neutrality result. *Proof.* If (i) holds, then it is trivial that (ii) and (iii) hold as well. To show that (ii) $\Rightarrow$ (i):
Suppose not. Then there exists an $f \in \mathfrak{L}_o$ such that either $f \succ \delta_f$ or $\delta_f \succ f$. Suppose

$f \succ \delta_f$. Then by lemma 1, there exists an $H(f) \in \mathcal{Z}$ such that $\delta_f \sim \delta_{H(f)}$. By transitivity, $f \succ \delta_f \Leftrightarrow f \succ \delta_{H(f)}$, and so by (ii), $\delta_f \succ \delta_{\delta_{H(f)}}$. By transitivity again, $\delta_{H(f)} \succ \delta_{\delta_{H(f)}}$, but this violates the certainty axiom **A.1**. Now suppose that $\delta_f \succ f$. Then $\delta_{H(f)} \succ f$, and by (ii), $\delta_{\delta_{H(f)}} \succ \delta_f \Leftrightarrow \delta_{\delta_{H(f)}} \succ \delta_{H(f)}$, which violates **A.1**.

To show that (iii) $\Rightarrow$ (i):
Suppose not. Then there exists an $f \in \mathfrak{L}_o$ such that either $f \succ \delta_f$ or $\delta_f \succ f$. Suppose that $f \succ \delta_f$. Note that by continuity, it is also the case that there exists an $\tilde{H} \in \mathcal{Z}$ such that $f \sim \delta_{\tilde{H}(f)}$. By the certainty axiom **A.1**, $\delta_{\tilde{H}(f)} \sim \delta_{\delta_{\tilde{H}(f)}}$. By transitivity, $\delta_{\delta_{\tilde{H}(f)}} \succ \delta_f$, and by (iii), $\delta_{\tilde{H}(f)} \succ f$. But this is a contradiction. Now suppose that $\delta_f \succ f$. Then $\delta_f \succ \delta_{\delta_{\tilde{H}(f)}} \Leftrightarrow f \succ \delta_{\tilde{H}(f)}$ which is a contradiction.

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