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Abstract

The Shapley value assigns, to each game that is adequately represented by its characteristic function, an outcome for each player. An elaboration on the Shapley value that assigns, to characteristic function games, a "partition function" outcome is broadly established and accepted, but elaborations to encompass games with externalities (represented by partition functions) are not. Here, I show that simultaneous consideration of the two elaborations ("generalization" and "extension") obtains a unique Shapley-type value for games in partition function form. The key requirement is that the "Extended, Generalized Shapley Value" (EGSV) should be "recursive": the EGSV of any game should be the EGSV of itself. This requirement forces us to ignore all but the payoffs to bilateral partitions. The EGSV can be conceptualized as the ex ante value of a process of successive bilateral amalgamations. Previous Shapley value extensions, if generalized, are not recursive; indeed, they iterate to the EGSV.

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1. Introduction

Though the potential usefulness of an *extension* of the Shapley value - to encompass games in partition function form - has often been noted, of various extensions that have been proposed none has yet become widely accepted. On the other hand, a *generalization* of the Shapley value - that encompasses the possibility of a prior "coalition structure" - *is* widely accepted. In this paper I argue that if we consider the two problems of extension and generalization at the same time, then the widely accepted solution to the problem of generalization forces a unique solution to the problem of extension. Indeed, it forces us to omit from consideration the additional information about the underlying event that the partition function provides. This finding could be read as an impossibility result: in a certain sense we cannot have an extended and generalized value that makes use of all the information in the partition function. An alternative reading might be that certain information *should* (where the Shapley value is seen as a normative, or rightful, outcome) or *would* (where the Shapley value is seen as a positive, or expected, outcome) be extraneous to the determination of a rightful or expected division of a cooperative surplus.

In cooperative game theory the conventional game primitive is a *characteristic function* which, subject to the "transferable utility" assumption, assigns real numbers -"payoffs" - to coalitions. But a drawback of the characteristic function form is that it cannot differentiate between various situations in which the payoffs that a coalition can obtain depend on the external coalitional arrangement of players. Externalities to coalescence are an important feature of many situations that are of present interest to economists, including many situations to which the cooperative game theory approach otherwise appears to recommend itself: for example, an important feature of environmental treaties is their consequence to non-signatories, and an important feature of mergers in oligopolistic markets is their effect on other remaining firms. The partition function form is one way of preserving information about these externalities. A *partition function* (subject to the "transferable utility" assumption) assigns real numbers - payoffs - to *embedded coalitions*: pairs, comprising a coalition and a partition to which the coalition belongs.

Since the present paper focusses on games in partition function form, these shall generally be referred to as "games". Games in which the payoffs assigned to an embedded coalition depend only on its first element (the coalition) shall be regarded as special cases: *games that can be adequately represented by a characteristic function.*

An important solution concept in the study of cooperative games is that of a *value*: a function that associates utility outcomes with games belonging to some class, where these utility outcomes can be interpreted as the ex ante *expected* (positive), or alternatively as the *rightful* (normative), utilities associated with playing a game. The best-known and most widely used value - the Shapley value - assigns, to each game that can be adequately represented by a characteristic function, a utility outcome for each player. The Shapley value was originally axiomatically grounded, and has since proven to be usefully tractable and robust; the original axioms are in a sense corroborated by the fact that the same value re-emerges from a number of apparently unrelated approaches. The present paper is a contribution to a recently active literature that has been concerned with identifying, by the axiomatic method, an extended Shapley value that accommodates the wider class of games in partition function form while preserving the properties of tractability and robustness associated with the Shapley value itself and of course reducing to the Shapley value for those partition function games that are adequately represented by a characteristic function. Other contributions to this literature include Myerson [8], Bolger [1], Potter [12], Pham Do and Norde [11], Maskin [7], Macho-Stadler, Pérez-Castrillo and Wettstein [6] and de Clippel and Serrano [3]. It should be noted that other solution concepts in cooperative game theory identify outcomes that are reasonable in a different sense, that might be roughly described as ex post plausibility: i.e. outcomes that are not "blocked" or that constitute equilibria of specific bargaining processes. This approach has also been applied to games in partition function form: important contributions include Ray and Vohra [13] and Bloch [2].

Extension to the wider class of games in partition function form is just one kind of elaboration on the Shapley value; a novelty in the approach taken by this paper lies in simultaneous consideration of a second kind of elaboration. The well-established *Owen value*, or *coalition structure value* [5, 10] assigns, to each game that is adequately represented by a characteristic function, a utility outcome for each pair comprising a player and a partition of all the players, and these utility outcomes are interpreted as the expected or rightful utilities associated with playing a game where some *prior* coalition structure exists. A less refined version of the Owen value that assigns, to each game that is adequately represented by its characteristic function, a utility outcome for each embedded coalition (by summing over the Owen values of the players belonging to the embedded coalition) has sometimes been referred to (see for example [4]) as the *generalized Shapley value*. Extending the *generalized* Shapley value to games in partition function form gives us an elegant function that transforms one partition function (that which associates *payoffs* with embedded coalitions: the game) into another (which associates *outcomes* with embedded coalitions: the "extended, generalized value" of the game). We suppose, in effect, that the underlying event is to be preceded by "play" : an unspecified process (which we can view as an expected, or alternatively as a rightful process) of bargaining, arbitration, or allocation. The game itself describes the underlying event, and its extended, generalized value describes (in toto) the combination of the underlying event and the precedent process; both descriptions take the same, partition function, form.

Though there is a reasonable consensus in recent literature that an extended Shapley value should satisfy the original Shapley axioms and also that it should be weakly monotonic (players' outcomes should be non-decreasing in payoffs to embedded coalitions to which they belong), even once these requirements are met the class of candidate values remains quite large. Extending the generalized value gives us a way of narrowing the class to one. It turns out that the axioms that generate the Owen value already also characterize a generalization of any prospective extended value. Furthermore, since the generalized value itself comprises a partition function, there is a very natural requirement - which I shall call recursion - that we would want to impose on it: if one partition function is the extended, generalized value of some other, then it should also be the extended, generalized value of itself. An "extended, generalized value" that is not recursive cannot be regarded as a "solution." The main theorem in this paper isolates a single candidate for the extended Shapley value by requiring that its generalization has the recursion property.

By way of an example consider Game 1 in Figure 1: it is a unanimity game involving three players (a, b and c), which has been perturbed by reducing b's payoff to minus one in the eventuality that no coalitions form. A prospective extension of the Shapley value to games in partition function form must assign, to this game, outcomes for a, b and c subject to the proviso that there are no "prior" coalitions; in essence, it must determine how b's outcome should be affected by the perturbation of the symmetric game. The shaded cells in the "outcome" column of Game 1 are the outcomes that are assigned by the extended value originally proposed by Myerson [8], which is one out of many candidate values that satisfy the original Shapley axioms. If the prospective value is also generalized, then it also assigns outcomes for the remaining embedded coalitions, where these outcomes are interpreted to be the (expected or rightful) utilities assigned to members of an a priori coalition structure. In Figure 1, I follow the well established approach of Owen [10] and Hart and Kurz [5] that treats embedded coalitions approximately as if they were individual, indivisible players: this means that any a priori structure of two coalitions splits the remaining surplus in half. But there are two problems with the resulting outcomes. The first problem is that b's outcome - where there are no prior coalitions - is higher than that of a or c.

Coalition	Partition	Game 1 Payoff	Game 1 Outcome*	Game 2 Payoff	Game 2 Outcome*
$\{a\}$	$\{\{a\},\{b\},\{c\}\}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{12}$
	$\{\{a\},\{b,c\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{b\}$	$\{\{a\},\{b\},\{c\}\}$	-1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
	$\{\{a,c\},\{b\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{c\}$	$\{\{a\},\{b\},\{c\}\}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{12}$
	$\{\{a,b\},\{c\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{a,b\}$	$\{\{a,b\},\{c\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{a,c\}$	$\{\{a,c\},\{b\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{b,c\}$	$\{\{a\},\{b,c\}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{a,b,c\}$	$\{\{a,b,c\}\}$	1	1	1	1

(*generalization of Myerson [8])

Figure 1. Two example games in partition function form, with outcomes generated by the generalization of a prospective extension of the Shapley value.

It is very difficult to see why b should do better in this game than in the symmetric, unanimity game: my contention here (in common with authors of other recent papers listed above) is that, if anything, it should do worse. The second problem emerges when we look at Game 2, the payoffs of which are the outcomes from Game 1. The outcomes that are assigned to the Game 2 (by again using Myerson values) deviate from those assigned to Game 1. My contention in this paper is that a property of any solution concept should be that - once a solution is found - solving the solution should not change it.

The Weak Monotonicity axiom (which is common to many recent papers) eliminates candidate values that exhibit the “first problem” above, and the Recursion axiom (which is original to this paper) eliminates those that exhibit the “second problem”.

The first theorem in the paper derives the established approach to generalization - which will be termed *The Rule of Generalization* - directly from two axioms (*Cohesion* and *Generalized Null Player*) that recollect the axioms of Owen and Hart and Kurz. The second (and main) theorem in the paper characterizes an extended and generalized Shapley value using the conventional *Efficiency*, *Symmetry*, *Null-Player*, and *Linearity* axioms together with *Weak Monotonicity*, *The Rule of Generalization* and *Recursion*. This value assigns symmetrical outcomes to a, b, and c in the game depicted in Figure 1, and more generally it omits from consideration all but the payoffs to the coarsest partition containing any coalition.

The paper also includes further results, which corroborate the *Extended, Generalized Shapley Value* that has been singled out by the axioms. These results can be viewed as tentative indications that the Extended, Generalized Shapley Value shares some of the properties of tractability and robustness that make the Shapley value itself so useful.

The Shapley value of a game is often conceptualized as the expected outcome to each player if the players were to arrive at a meeting point in a random order and to each receive the marginal payoff that their addition brings to the coalition of those players who arrived ahead. The Extended Shapley Value proposed here can be conceptualized in the same way, provided we suppose that this marginal payoff is calculated by assuming there are only two coalitions: one of the players who have arrived, and another of those who have not. Since this proviso seems somewhat ad hoc, I propose an alternative conceptualization. Suppose the "underlying event" is preceded by a number of time periods and that in each time period two existing coalitions are chosen at random to coalesce, with the "gain from coalescence" (which in fact might be negative) split equally between the two. The third theorem in this paper establishes that as the number of time periods becomes large, the partition function that encompasses both the underlying event and the precedent time periods tends to the Extended, Generalized Shapley

Value that was singled out by the axioms.

Finally, it seems natural to consider, in the case of previously proposed extensions to the Shapley Value, how the type of process introduced above in Figure 1 ends. If we again suppose that the "underlying event" is preceded by a number of time periods, but this time that each time period effects a transformation on the partition function that corresponds with the value proposed by Myerson [8] in conjunction with The Rule of Generalization then as the number of time periods is increased, the outcome to b changes in accordance with a sequence that begins $\left\{\frac{2}{3}, \frac{1}{6}, \frac{5}{12}, ...\right\}$. If we use the value proposed by Pham Do and Norde [11] then we generate a sequence beginning $\left\{0, \frac{1}{6}, \frac{1}{4}, ...\right\}$, and if we use the value proposed by Potter [12] or by Macho-Stadler, Pérez-Castrillo and Wettstein [6] we generate then we generate a sequence beginning $\left\{\frac{1}{6}, \frac{7}{24}, \frac{31}{96}, ...\right\}$. All of these sequences converge, and they share a common limit: the outcome of $\frac{1}{3}$ that is assigned to b by the Extended, Generalized Shapley Value, that has been proposed here. The fourth theorem in this paper generalizes this finding to all games and all candidate extended Shapley values that fulfil the four conventional Shapley axioms and a less intuitive but no less compelling alternative to the Weak Monotonicity condition.

The organization of the paper is as follows. In the next section, I formalize the key concepts in the paper. The first and second of the theorems described above are presented and interpreted in section 3. In section 4, I establish the alternative conceptualization of the Extended, Generalized Shapley Value (Theorem 3) described above. In section 5, I set out the extensions of the Shapley value that have been proposed previously in the literature. I show (Theorem 4) that wherever any one of the previously proposed extended values can be represented by a closed form equation then, when generalized, besides not proving recursive, it iterates to the Extended, Generalized Shapley Value. Proofs of the theorems, together with examples that establish independence of the axioms, are provided in the appendix.

2. Preliminaries

2.1. Games

Let N denote a finite set of players and V the set of all mappings $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. We refer to $v \in V$ as a transferable utility (TU) *game in characteristic function form*, on N. We define the set $\mathit{\Pi}$ to be the set of all partitions of N. It should be noted that $\{N\}$ and $\{\{i\}_{i \in N}\}$ are both elements of $\mathit{\Pi}$; the first is the *grand coalition*, and the second is the *finest partition* and shall henceforth be written $[N]$. We define M, sometimes referred to as the *set of embedded coalitions*, to be the set $\{(I, \pi)\colon \pi \in \mathit{\Pi}, I \in \pi\}$; and W to be the set of all mappings $w\colon M \to \mathbb{R}$ with $I = \emptyset \to w(I, \pi) = 0$. We refer to $w \in W$ as a transferable utility (TU) *game in partition function form*, on N. In this paper the term "game," unless otherwise qualified, shall mean a TU game in partition function form.

2.2. Types of Game

It will be convenient to refer later to two subsets of W.

Shubik [17] coined the term "c-games" to describe games that are adequately represented by their characteristic functions, and we shall use the term here in a closely-related way. Let $W^c \subseteq W$ comprise all $w \in W$ such that for some $v \in V$, w is defined by $w(I, \pi) = v(I)$. A *c-game* on N is an element of W^c. We refer to $v \in V$ such that $w \in W^c$ is defined by $w(I, \pi) = v(I)$ as the *correspondent element in V to w*.

All games on N can be constructed by a linear combination of games which shall be referred to here as "β-games." Let $W^\beta \subseteq W$ comprise all $w \in W$ such that for some $(J, \pi\prime) \in M$, w is defined by $w(I, \pi) = 1$ where $(I, \pi) = (J, \pi\prime)$, 0 otherwise. A *$\beta$-game* on N is an element of W^β. We write $w^\beta_{(J,\pi\prime)}$ to denote $w \in W^\beta$ defined by $w(I, \pi) = 1$ where $(I, \pi) = (J, \pi\prime)$, 0 otherwise.

2.3. Value Concepts

We shall depart somewhat from convention in the way that we denote a "value", though not in the meaning that we ascribe to the term. χ shall always denote a mapping from W to W that is referred to as an *extended, generalized value.* If w is an element of W then $\chi(w)$ is also an element of W, so (where (I, π) is an embedded coalition) $w(I, \pi)$ and $\chi(w)(I, \pi)$ are two real numbers of which $w(I, \pi)$ is interpreted as the utility *payoff* prescribed to coalition I given partition π in the game w, and $\chi(w)(I, \pi)$ is interpreted as the (expected or rightful) utility *outcome* associated with coalition I whenever π is the coalition structure *prior to playing* the game w. Formally, a mapping is a set of ordered pairs, so we can write $\chi \equiv \{(w, \chi(w)) : w \in W\}$. This notation enables us to think of *generalized values* (such as that of Gul [4] - which is a less refined version of Owen [10] and Hart and Kurz [5]), *extended values* (such as those of Myerson [8], Bolger [1], Potter [12], Pham Do and Norde [11], Maskin [7], Macho-Stadler, Pérez-Castrillo and Wettstein [6] and de Clippel and Serrano [3]) and *standard values* (such as the original Shapley value itself [15]) as more restricted sets of ordered pairs. A generalized value becomes $\{(w, \chi(w)) : w \in W^c\}$, or $\chi|_{W^c}$. Given $w \in W$, $L \subseteq M$, let $\chi(w)|_L$ denote the restriction of $\chi(w)$ to $L \subseteq M$: so $\chi(w)|_L$ is a mapping from L to $\mathbb{R}$ defined by $\forall \mu \in L$, $\chi(w)|_L(\mu) = \chi(w)(\mu)$. Note that $\{(\{i\}, [N])\colon i \in N\} \subseteq M$ is the set of embedded coalitions comprising singletons and the finest partition. An extended value then becomes $\{(w, \chi(w)|_{\{(\{i\},[N])\colon i \in N\}}) : w \in W\}$, and a standard value becomes $\{(w, \chi(w)|_{\{(\{i\},[N])\colon i \in N\}}) : w \in W^c\}$.

2.4. Carriers and Null-Players

The complementary notions of carrier sets and null-players are well established for games in characteristic function form. Here, we extend these notions to games in partition function form. We shall say that $K \subseteq N$ is a *carrier of* $w \in W$ if and only if $\forall (I, \pi), (J, \pi\prime) \in M, I \cap K = J \cap K, \{(L \cap K)_{L \in \pi}\} = \{(L \cap K)_{L \in \pi\prime}\} \rightarrow w(I, \pi) = w(J, \pi\prime)$. We shall say that $i \in N$ is a *null-player* in $w \in W$ if and only if $N \backslash \{i\}$ is a carrier of w. These definitions are consistent with the definitions used by Bolger [1] and Macho-Stadler, Pérez-Castrillo and Wettstein [6]; it should be noted that alternative definitions have been proposed by Myerson [8], Potter [12], and Pham Do and Norde [11] and that these alternatives class additional sets as "carriers," or additional players as "null" and thereby strengthen axioms that affect carriers or null-players - I discuss this point further in section 5 below.

3. The Extended, Generalized Shapley Value

3.1. Axiomatic Derivation of an Extended, Generalized Shapley Value

Our first four axioms are familiar variants of the original Shapley [15] axioms, except that *Linearity* will be used here instead of "additivity" (the conventional "additivity" axiom entails 4(i) alone); but they are extended to bear upon games in partition function form. All axioms in the paper are assumed to prevail for all $w, w\prime \in W$, and all $j \in N$.

Axiom 1 (Efficiency) $\sum_{i \in N} \chi(w) (\{i\}, [N])$

Let P denote the set of permutations of N, and given $\rho \in P$, $J \subseteq N$, $\pi \in \Pi$, $w \in W$; we define ρJ to be the image under ρ of J; we define $\rho\pi \in \Pi$ to be the set $\{(\rho I)_{I \in \pi}\}$; and we define $\rho w \in W$ by $\rho w(\rho J, \rho\pi) = w(J, \pi)$.

Axiom 2 (Symmetry) $\forall \rho \in P,\ \chi(\rho w) (\{\rho(j)\}, [N]) = \chi(w)(\{j\}, [N])$

Axiom 3 (Null-Player) *If j is a null-player in w, then* $\chi(w)(\{j\}, [N]) = 0$

Axiom 4 (Linearity) *(i)* $\chi(w + w\prime)(\{j\}, [N]) = (\chi(w) + \chi(w\prime))(\{j\}, [N])$

(ii) $\forall \gamma \in \mathbb{R},\ \chi(\gamma w)(\{j\}, [N]) = \gamma\chi(w)(\{j\}, [N])$

Similar extensions of the original Shapley axioms are common to all previous "extensions" of the Shapley value, except that of Maskin [7] who argues that *Efficiency*, for example, should not be assumed when games have externalities. I retain the conventional approach, on the grounds that the Shapley Value is used primarily as a normative solution concept and that there seems no reason to suppose that the normative arguments for Axioms 1-4(i) hold only for c-games. The implicit supposition in the Efficiency axiom that $w(N,\{N\}) = \max_{\pi\in\Pi}\sum_{I\in\pi}\chi(w)(I,\pi)$ is often posited as arising from a superadditivity assumption that was present in von Neumann and Morgenstern [9] and that has commonly been retained since. Though superadditivity seems reasonable for c-games, it is less so for games in which coalescence can generate positive externalities: for example, it is well known (see [14]) that where three or more firms are engaged in Cournot competition then coalescence (or merger) between any two firms can be to their disadvantage in so far as the increased total producer surplus is less than the positive externality that is enjoyed by firms that are not parties to the merger. But the assumption that $w(N,\{N\}) = \max_{\pi\in\Pi}\sum_{I\in\pi}\chi(w)(I,\pi)$ is in itself much weaker than superadditivity, and is no less reasonable with regard to games with externalities - such as the Cournot game - than with regard to c-games, since where the grand coalition arises from any prior partition of the players there are no remaining parties to whom externalities can be spilled. At the point where the grand coalition is formed, externalities to coalescence are necessarily internalized. It should therefore be emphasized that where Maskin [7] drops the Efficiency axiom it is not because he believes the grand coalition is less likely to be efficient in the presence of externalities to coalescence, but it is because he believes the grand coalition is less likely to form. Elsewhere (for example, [6]), Efficiency is sometimes imposed as part of the definition of a “solution” or of a “value”.

While the second part of Axiom 4 is redundant in the characterization of the (standard) Shapley value, Macho-Stadler, Pérez-Castrillo and Wettstein [6] show that it is not redundant in the characterization of an extended value. However, it is hard to imagine - bearing in mind that Axiom 4(i) alone implies $\forall\gamma\in\mathbb{Q}$, $\chi(\gamma w)(\{j\},[N]) = \gamma\chi(w)(\{j\},[N])$ - any intuitive justification for Axiom 4(i) ("additivity") that does not also encompass *Linearity* and therefore also Axiom 4(ii).

We now define the Shapley Value.

Definition 1 (Shapley Value) $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i\in N\}}) : w \in W^c\}$ *is the Shapley Value if and only if* $\forall i \in N, \forall w \in W^c$,

$$\chi(w)(\{i\},[N]) = \sum_{S\subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}(v(S) - v(S\backslash\{i\}))$$

where v *is the correspondent element in* V *to* w*.*

We know (see [15]) that if χ satisfies *Efficiency*, *Symmetry*, *Null-Player,* and *Linearity*, then $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i\in N\}}) : w \in W^c\}$ is the Shapley Value. We also know (see [6]) that the same four axioms do not suffice to uniquely determine $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i\in N\}}) : w \in W\}$.

The next axiom entails that if some game would be a "null-game" except that there is one coalition which in the context of one particular partition obtains a positive payoff (= 1), then the outcome to members of that coalition should not be less than zero. If *Linearity* holds then Axiom 5 is equivalent to the "coalitional monotonicity*"* axiom described by Young [19] whereby "an increase in the value of a particular coalition implies, ceteris paribus, no decrease in the allocation to any member of that coalition." (p. 68). This, as Young (citing [16]) observes, is a necessary property for any value that is to be used in problems such as cost assignment in a firm without creating perverse incentives.

Axiom 5 (Weak Monotonicity) $\forall (I,\pi) \in M, i \in I \rightarrow \chi(w^{\beta}_{(I,\pi)})(\{i\},[N]) \geqslant 0$

Several of the more recently proposed extensions of the Shapley value cite Myerson's [8] contravention of Weak Monotonicity as a motivation for readdressing the extension of the Shapley value to games in partition function form.

It should be emphasized that Axioms 1-5 bear only on the outcomes assigned to individuals in the finest partition (i.e. $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i\in N\}}) : w \in W\}$). The remaining axioms bear more generally on χ.

The following axiom prescribes a generalized value from any standard value. *The Rule of Generalization* entails, for any partition that is coarser than $[N]$, treating one member of each coalition as its "representative," treating all other players as null, and assigning to each coalition the outcome that is assigned to its representative in the game that is thereby created on representatives and Null-Players.

Given $\pi \in \Pi$, let Λ_π denote the set of injective mappings from π to N such that $\forall S \in \pi$, $\lambda \in \Lambda_\pi \to \lambda(S) \in S$. Given $\lambda \in \Lambda_\pi$, we define $\lambda\pi$ to be the image under λ of π, and we define $\lambda w \in W$ by $\lambda w(J, \pi\prime) \equiv w(\bigcup_{i\in(J\cap\lambda\pi)}\lambda^{-1}(i), \{(\bigcup_{i\in(S\cap\lambda\pi)}\lambda^{-1}(i))_{S\in\pi\prime}\})$.

Axiom 6 (The Rule of Generalization) $\forall(I,\pi) \in M, \forall\lambda \in \Lambda_\pi, \chi(w)(I,\pi) = \chi(\lambda w)(\lambda(I),[N])$

Owen's "value of a game with a priori unions" [10] - also known as the Coalition Structure Value [5] - entails *The Rule of Generalization* and goes further, in that it apportions the outcome assigned to an embedded coalition between the members of that coalition. Owen's value has been so widely accepted that a "Generalized Shapley Value" formula (consistent with Definition 2 below) is sometimes adopted without further justification (see especially [4]). Our first theorem (below) establishes that *The Rule of Generalization* is on its own weaker than the conjunction of two axioms (closely related to the axioms used by Owen [10] and by Hart and Kurz [5]) that have their own intuitive appeal.

The first of these axioms is *Cohesion*; it entails that, once some partition of players has formed, *dissolute partitions* - coalition structures that entail one or more existing coalition breaking up - become irrelevant. Put more precisely: in the following axiom we suppose that the outcome to an embedded coalition depends only on the payoffs to embedded coalitions that do not entail dissolute partitions. Given $\pi \in \Pi$, we define the set of embedded coalitions entailing non-dissolute partitions from π, $M_\pi = \{(J, \pi\prime) \in M \colon \forall I \in \pi, \forall I\prime \in \pi\prime, I\prime \cap I \neq \emptyset \rightarrow I \subseteq I\prime\}$.

Axiom 7 (Cohesion) $\forall (I, \pi) \in M, (\forall \mu \in M_\pi, w(\mu) = w\prime(\mu)) \rightarrow \chi(w)(I, \pi) = \chi(w\prime)(I, \pi)$

If we are to ascribe any meaning to a priori coalitions at all, then we must assume that they cohere to some extent: in other words, we cannot suppose that coalitions dissolve at the outset of bargaining, allocation, or arbitration and simply set $\forall w \in W, \forall (I, \pi) \in M, \chi(w)(I, \pi) = \sum_{i \in I} \chi(w)(\{i\}, [N])$. Given this constraint, then the *Cohesion* axiom represents the natural (opposite) position to take: we suppose that coalitions will stay together and therefore in assigning outcomes between coalitions that are presently formed we do not consider payoffs to embedded coalitions that entail such coalitions breaking up. The Cohesion axiom is used by Owen [10] (it is his Axiom A3) and reappears as the slightly weaker "Inessential Game" axiom in Hart and Kurz [5]; the Inessential Game axiom imposes $\forall (I, \pi) \in M, (\forall \mu \in M_\pi, w(\mu) = 0) \rightarrow \chi(w)(I, \pi) = 0$. In the context of other axioms the Inessential Game axiom quickly entails the present Cohesion axiom, the transparency of which is preferred here.

The next axiom entails the Null-Player axiom, plus a requirement that null-players are strategically irrelevant: an a priori coalition neither gains nor loses nor affects other coalitions by merging with a null-player.

Axiom 8 (Generalized Null-Player) *If j is a null-player in w, then j is a null-player in $\chi(w)$*

The original *Null-Player* axiom has an important function: it enables us to concentrate our analysis on any carrier set instead of having to worry about which players matter, or having to worry about a whole universe of players. If this advantage is to be carried over to generalized values, then we require the *Generalized Null-Player* axiom to hold. The Generalized Null-Player axiom is also incorporated into the axiom systems of Owen [10] and Hart and Kurz [5] - where it is incorporated, with Efficiency, into the "Carrier" axiom.

We can now state our first theorem and its corollary.

Theorem 1 *χ satisfies both Cohesion and Generalized Null-Player if and only if χ satisfies both Null-Player and The Rule of Generalization.*

Theorem 1 is proved in the appendix.

Definition 2 (Generalized Shapley Value) *$\chi|_{W^c}$ is the Generalized Shapley Value if and only if $\forall(I,\pi) \in M, \forall w \in W^c$,*

$$\chi(w)(I,\pi) = \sum_{T \subseteq \pi} \frac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!} \left(v\left(\bigcup_{A \in T} A \right) - v\left(\bigcup_{A \in (T \setminus \{I\})} A \right) \right)$$

where v is the correspondent element in V to w.

Corollary 1 *If χ satisfies Efficiency, Symmetry, Linearity, Cohesion, and Generalized Null-Player, then $\chi|_{W^c}$ is the Generalized Shapley Value.*

The final axiom in our system entails that χ does indeed "solve" (we could say "resolve") the underlying game. If one game is the "solution" of another, then "solving" the solution does not change it. We call this property *Recursion.*

Axiom 9 (Recursion) $\chi(\chi(w)) = \chi(w)$

The *Recursion* axiom is the only axiom being proposed here that is not widely used elsewhere, but it is perhaps the easiest of all to justify. The generalized value of a game is, itself, a game in partition function form. It is, as was noted in the introduction, a transformation of the original game that encompasses both the underlying event that the original game describes and the process of (expected or rightful) bargaining, arbitration, or allocation that precedes it.

Our axioms are supposed to describe characteristics of a (positively or normatively) reasonable process, and the transformed game is supposed to represent a "solution" (we could say a "resolution") of the underlying game. If a generalized value is not recursive then we are making the very difficult claim that one "round," or "iteration," of a bargaining, arbitration, or allocation process is reasonable (either positively or normatively), while two rounds of the same process are not. A process does not "solve" (or "resolve") the underlying event unless reapplications of the same process are nugatory; so an "extended, generalized value" is not really a "solution" concept at all unless it satisfies *Recursion.*

We can now state our second (and main) theorem and its corollary.

Definition 3 (Extended, Generalized Shapley Value) *χ is the Extended, Generalized Shapley Value if and only if $\forall (I,\pi) \in M, \forall w \in W$,*

$$\chi(w)(I,\pi) = \sum_{T \subseteq \pi} \frac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!} \left(v\left(\bigcup_{A \in T} A \right) - v\left(\bigcup_{A \in (T \backslash \{I\})} A \right) \right)$$

where $v \in V$ is defined by $v(S) = w(S, \{(N \backslash S), S\})$.

Theorem 2 *χ satisfies Efficiency, Symmetry, Linearity, Weak Monotonicity, Cohesion, Generalized Null-Player, and Recursion (or, equivalently, Efficiency, Symmetry, Null-Player, Linearity, Weak Monotonicity, The Rule of Generalization, and Recursion) if and only if χ is the Extended, Generalized Shapley Value.*

Theorem 2 is proved in the appendix.

Definition 4 (Extended Shapley Value) *$\{(w, \chi(w)|_{\{(\{i\},[N]) :\, i \in N\}}) : w \in W\}$ is the Extended Shapley Value if and only if $\forall i \in N, \forall w \in W$,*

$$\chi(w)(\{i\},[N]) = \sum_{S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} (v(S) - v(S \backslash \{i\}))$$

where $v \in V$ is defined by $v(S) = w(S, \{(N \backslash S), S\})$.

Corollary 2 *If* χ *satisfies Efficiency, Symmetry, Null-Player, Linearity, Weak Monotonicity, The Rule of Generalization, and Recursion, then* $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i\in N\}}) : w \in W\}$ *is the Extended Shapley Value.*

3.2. Interpretation of Theorem 2

The Extended, Generalized Shapley Value of any $w \in W$ equates to the Generalized Shapley Value of a c-game, $w^* \in W^c$ where w^* is obtained by assigning, for all $(I, \pi) \in M$, $w^*(I, \pi) = w(I, \{(N\backslash I), I\})$. It therefore excludes from consideration the payoffs to embedded coalitions that entail partitions of more than two coalitions. This exclusion is forced on us by axioms that do not seem, in any obvious way, to prefigure it.

It should be emphasized that it is not my intention in this paper to suggest that the detail that is excluded from consideration is in any general sense redundant: this detail is very likely to be relevant to more descriptive solution concepts, or to solution concepts that encompass different sets of normative principles. Instead, I am making the milder claim that the payoffs to embedded coalitions that entail partitions of more than two coalitions should or would be ignored in a process of bargaining, allocation, or arbitration that encompasses the sort of normative principles reflected specifically in the Shapley value solution.

4. A Procedural Account of the Extended, Generalized Shapley Value

Shapley [15] provided the following *procedural* account of his value: it is the expected value outcome to each player, if the players are to arrive at a meeting point in a random order and to each receive the marginal payoff that their addition brings to the coalition of those players who arrived ahead. Though the procedure is purely mechanistic (in contrast to modern non-cooperative bargaining models such as that of Gul [4]) it nevertheless provides a useful way of *conceptualizing* the value. It is helpful to be able to consider the reasonableness of axioms as properties of (expected or rightful) bargaining, arbitration or allocation alongside at least one procedure that we know fulfils the axioms.

It is straightforward to *generalize* Shapley's procedure to situations with a priori coalitions - the generalization just entails constraining the order of arrival so that members of any pre-existing coalition arrive consecutively (see [5]) - but a difficulty arises in *extending* the procedure to the wider class of games in partition function form. In order to calculate the "marginal payoff" that a player brings to some coalition, some assumption has to be made about the coalitional configuration of players outside. We obtain the Extended Shapley Value (above) by assuming that the outside players are all coalesced together, so the "marginal payoff that the addition of i brings to C" is $v(C \cup \{i\}) - v(C)$ where $v \in V$ is defined by $v(S) = w(S, \{(N \backslash S), S\})$. Since this assumption is ad hoc, and also since it directly prefigures the distinctive property of the Extended Shapley Value, it is somewhat unsatisfactory.

An alternative procedure - a simplified, mechanistic version of the bargaining procedure proposed by Gul - avoids the need for any ad hoc assumption: we can view the Shapley value, its generalization *and* its extension, as the expected value outcome to each player (or coalition) if the players (or coalitions) are to engage in a sufficiently long process of successive bilateral amalgamations. Suppose that $w_0 \in W$ describes an "underlying event," and then suppose that this event will be preceded by a time period ("one round of bargaining") in which two existing coalitions will be chosen at random to coalesce with the "gain from coalescence" (which in fact might be negative) split equally between the two. We use $w_1 \in W$ to describe (in toto) the combination of w_0 and the round of bargaining: i.e. each $w_1(I, \pi)$ is an expected value outcome to coalition I, given an existing coalition structure π, if there is going to be one round of bargaining followed by a payoff according to the final coalition structure and w_0. More generally we use $w_t \in W$ to describe the combination of w_{t-1} and one round of bargaining, so, $\forall (I, \pi) \in M$,

$$w_t(I,\pi) \equiv \frac{1}{|\pi|\left(|\pi|-1\right)} \left(\sum_{J,K\in\pi\backslash\{I\}} w_{t-1}\left(I,\left(\pi\backslash\{J,K\}\right)\cup\{J\cup K\}\right) \right.$$
$$\left. + \, 2\sum_{J\in\pi\backslash\{I\}} \left(w_{t-1}(I,\pi) + \tfrac{\left(w_{t-1}(I\cup J,(\pi\backslash\{I,J\})\cup\{I\cup J\}) - w_{t-1}(J,\pi) - w_{t-1}(I,\pi)\right)}{2} \right) \right).$$

w_t then describes (in toto) the combination of w_0 and t rounds of bargaining. It transpires that as t becomes large, w_t tends to the Extended, Generalized Shapley Value.

Theorem 3 $\forall w_0 \in W, \lim_{t\longrightarrow\infty}\; w_t$ *is the Extended, Generalized Shapley Value of* w_0.

Theorem 3 is proved in the appendix.

The alternative procedure (Theorem 3) demonstrates that the elimination from consideration of payoffs to embedded coalitions entailing partitions that contain more than two elements, besides arising from *axioms* that in no way seem to prefigure it, arises also from reasonable *procedures* that do not seem to prefigure it either. It can be tentatively viewed as evidence that the Extended, Generalized Shapley Value shares some of the robustness of the original Shapley Value. Another corroboration of the Extended, Generalized Shapley Value emerges at the end of the next section.

5. Alternative Extensions of the Shapley Value

5.1. Previous extensions of the Shapley value

Thrall and Lucas [18] originally proposed the partition function representation; [13] and [6,7] are important recent papers that make a case for its usefulness. In addition to the last two of these, [1,3,8,11,12] propose extensions of the Shapley Value to games in partition function form. In this section I consider the previously proposed extensions.

Maskin's [7] approach to "solving" a game in partition function form stands somewhat apart from both this and other previous papers. Maskin argues that it is inappropriate to impose

Efficiency on the solution of games in which coalescence has positive externalities - so he departs even from Axioms 1-4(i) above. His solution is in effect obtained as an expected value outcome of a procedure that adds an extra dose of reality to the original procedural conceptualization of the Shapley Value described in the first paragraph of section 4 above. On arriving at a meeting point a player can choose either to accept the highest bid from coalitions already formed at the point, or to remain independent and to bid against the existing coalitions in order to attract supervenient players as they arrive. Because existing coalitions bid strategically and simultaneously, there are some underlying games with non-unique outcomes. In this sense, and also since it is neither efficient nor additive, Maskin's solution - while it coincides with the Shapley Value for c-games - cannot be accorded the *normative* interpretation that is more usually accorded to the Shapley Value itself.

Bolger [1] and Macho-Stadler, Pérez-Castrillo and Wettstein [6] take more conventional approaches: they each add, to the standard Shapley axioms, axioms that impose equivalent outcomes in situations that look (from particular standpoints) as if they present individuals with equivalent bargaining power.

Given $\pi \in \mathit{\Pi}$, let $\pi(j)$ denote the coalition in π to which $j \in N$ belongs and let $\pi_j^I \in \mathit{\Pi}$ denote the partition formed from π by moving j from $\pi(j)$ to I. Bolger demonstrates that if, in addition to requiring that χ satisfies Axioms 1-4, we also require, given $w, w\prime \in W$ and $j \in N$ that $\chi(w)(\{j\}, [N]) = \chi(w\prime)(\{j\}, [N])$ whenever $\forall \pi \in \mathit{\Pi}$, $\sum_{I \in (\pi \backslash \{\pi(j)\})}(w(\pi(j), \pi) - w(\pi(j) \backslash \{j\}, \pi_j^I)) = \sum_{I \in (\pi \backslash \{\pi(j)\})}(w\prime(\pi(j), \pi) - w\prime(\pi(j) \backslash \{j\}, \pi_j^I))$ (which is to say that for every partition, the sum of the consequences for the coalition to which i belongs of i leaving to join another coalition is the same in w and $w\prime$), then we obtain an extended value. Bolger does not make the case for his additional requirement for circumstances other than "monotonic, simple" games, though his result only arises once the requirement prevails upon a broader class of games. His additional requirement entails, for example, (where $N = \{a, b, c\}$) $\chi(w^{\beta}_{(\{a,b\},\{\{a,b\},\{c\}\})})(\{a\}, [N]) = \chi(-2w^{\beta}_{(\{b\},\{\{a,c\},\{b\}\})})(\{a\}, [N])$, which - though it would be a plausible property to uncover - is a significant imposition. Bolger's value then has the inconvenient property that when a null-player is added to the set of players, the outcomes to existing players change: so in any given situation the outcomes depend on which null-players *are* players, and which are not.

Macho-Stadler, Pérez-Castrillo and Wettstein demonstrate that if, in addition to requiring that χ satisfies Axioms 1-4, we require for all $(I, \pi) \in M$ and for all $i, j \in N \backslash I$, $\chi(w^{\beta}_{(I,\pi)})(\{i\}, [N]) = \chi(w^{\beta}_{(I,\pi)})(\{j\}, [N])$, and if we also require for all $(I, \pi), (I, \pi\prime) \in M$ that $\{i, j\} \in \pi, \{\{i\}, \{j\}\} \in \pi\prime$, and $(\pi \backslash \{\{i\}, \{j\}\}) = (\pi\prime \backslash \{\{i, j\}\})$ imply $\chi(w^{\beta}_{(I,\pi)})(\{i\}, [N]) = \chi(w^{\beta}_{(I,\pi\prime)})(\{i\}, [N])$ and $\chi(w^{\beta}_{(I,\pi)})(\{j\}, [N]) = \chi(w^{\beta}_{(I,\pi\prime)})(\{j\}, [N])$, we then obtain an extended value. The first requirement turns Symmetry into "Strong Symmetry" and narrows the class of prospective values to those - such as the Extended Shapley Value proposed here - that equate the value of any $w \in W$ to the Shapley value of a c-game, $w^* \in W^c$, obtained by assigning each coalition a weighted average (with the weights themselves constrained by Symmetry and Null-Player) of its payoffs in w across partitions of the other players. The second requirement - "Similar Influence" - is not fulfilled by the Extended Shapley Value proposed here.

In the presence of the *Efficiency* axiom, the *Null-Player* axiom is a necessary (though, as Bolger demonstrates, not sufficient) criterion to absolve the game theorist of the problem of deciding who counts as a "player." Furthermore, it seems intuitively defensible that a player whose position in the coalition structure is irrelevant to payoffs in the underlying event should be assigned a value of zero. But several extensions of the Shapley Value are obtained by classing more sets as "carriers," or more players as "null," than either necessity or intuition warrant.

Myerson [8] demonstrated that if we treat $S \subseteq N$ as a carrier of $w \in W$ whenever $\forall (J, \pi) \in$

$M, w(J, \pi) = w(J \cap S, \{(L \cap S)_{L \in \pi}\} \cup \{(L \cap (N \backslash S))_{L \in \pi}\})$, and then if we impose Axioms 2 and 4, and use (as Shapley did) a "carrier efficiency" axiom in place of Axioms 1 and 3, we then obtain an extended value. Consider a pure externality game, w^*, on $N = \{a, b, c\}$ where in which a payoff of 1 accrues to a if b and c coalesce: i.e. $w^* = w^{\beta}_{(\{a\},\{\{a\},\{b,c\}\})} + w^{\beta}_{(\{a,b,c\},\{\{a,b,c\}\})}$. By the definition given in section 2 above, the only carrier of w^* is $\{a, b, c\}$, but Myerson requires us to treat $\{a\}$ (though, it might be noted, not $\{a, b\}$) as a carrier of $w^*$and therefore to assign $\chi(w^*)(\{a\}, \{\{a\}, \{b\}, \{c\}\}) = 1$. This is a strong imposition as it is not obvious that a is in as strong a position in w^*as in, for example, a "unanimity game" whereby a payoff of 1 accrues to those embedded coalitions that contain a, and of 0 to those that do not. Myerson's extended value has some counter-intuitive properties: for example, it is not weakly monotonic.

In a similar vein, Potter [12] and Pham Do and Norde [11] obtain extended values by classing additional players as "null." If $N = \{a, b, c\}$ then for Pham Do and Norde all three players are null in $w^{\beta}_{(\{a\},\{\{a\},\{b,c\}\})}$, while for Potter all three players are null in $(w^{\beta}_{(\{a\},\{\{a\},\{b\},\{c\}\})} - w^{\beta}_{(\{a\},\{\{a\},\{b,c\}\})})$. These impositions strongly prefigure the extended values that Potter and Pham Do and Norde respectively obtain.

Pham Do and Norde's value counterpoints the Extended Shapley Value by equating (for any $w \in W$) to the Shapley value of a c-game ($w^* \in W^c$) that is obtained by assigning each coalition its payoff in w where the external players are arranged as singletons. The same value emerges in de Clippel and Serrano [3]. De Clippel and Serrano develop Young's ([19]) *marginality* axiomatization of the Shapley value so that it can be applied to games in partition function form: the Linearity and Null-Player axioms are dropped and replaced by progressively stronger versions of "Marginality".

5.2. Convergence to the Extended, Generalized Shapley Value

It is easy to check that if we derive χ by using *any* of the extended values in section 5.1 in conjunction with *The Rule of Generalization* then we contravene *Recursion*: i.e. for some $w \in W$, $\chi(w) \neq \chi(\chi(w))$. It is therefore natural to ask the following: if χ fulfills the original Shapley axioms and the Rule of Generalization, but if χ does *not* necessarily fulfill either Weak Monotonicity or Recursion, then, defining χ_t by $\chi_1 = \chi$, and $\chi_t(w) = \chi(\chi_{t-1}(w))$, what happens to the sequence $\{\chi_t\}_{t=0}^{\infty}$? It turns out that, subject to a magnitude constraint on $\chi(w^{\beta}_{(I,\pi)})(I,\pi)$, the sequence *converges*, and moreover, *in its limit is Weakly Monotonic* with respect to the underlying game; it also preserves the original axioms, so $\lim_{t\longrightarrow\infty} \chi_t$ is the Extended, Generalized Shapley Value. The following theorem formalizes this.

Theorem 4 *Define χ_t by $\chi_1 = \chi$, and $\chi_t(w) = \chi(\chi_{t-1}(w))$ for any positive integer t. If χ satisfies Efficiency, Symmetry, Linearity, Cohesion, and Generalized Null-Player (or, equivalently, Efficiency, Symmetry, Null-Player, Linearity, and The Rule of Generalization) and if for all $(I,\pi) \in M$, $|\pi| > 2$ implies $-\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}$ then $\{\chi_t\}_{t=0}^{\infty}$ converges and $\lim_{t\longrightarrow\infty} \chi_t$ is the Extended, Generalized Shapley Value.*

Theorem 4 is proved in the appendix.

Theorem 4 provides an additional corroboration of the Extended, Generalized Shapley Value. Candidate extensions of the Shapley Value have tended to fulfill the magnitude constraint, in Theorem 4, on $\chi(w^{\beta}_{(I,\pi)})(I,\pi)$. For example, the extended values proposed by [3, 6, 8, 11, 12] can all be represented by straightforward formulae, and it is easy to check that if we derive χ by using any of these extended values in conjunction with *The Rule of Generalization*, then the constraint holds. So, by Theorem 4, if one round or *iteration* of a bargaining, arbitration or allocation process implements χ defined by the generalization of any of these candidate extensions of the Shapley Value then further iterations of the same process implement an extended, generalized value that converges eventually to the Extended, Generalized Shapley Value.

A. Appendix

A.1. Proof of Theorem 1

It is obvious that *Generalized Null-Player* implies *Null-Player*, and that *The Rule of Generalization* implies *Cohesion*. I show first that *Cohesion* and *Generalized Null-Player* together imply *The Rule of Generalization*. Consider any $(I, \pi) \in M$, $w \in W$, and $\lambda \in \Lambda_\pi$. Note that, $\forall \mu \in M_\pi$, $w(\mu) = \lambda w(\mu)$; so by *Cohesion*, $\chi(w)(I, \pi) = \chi(\lambda w)(I, \pi)$. Also note that $\lambda\pi$ is a carrier in λw; and that by *Generalized Null-Player* any carrier in λw must also be a carrier in $\chi(\lambda w)$. $I \cap \lambda\pi = \lambda(I)$, and $\{(L \cap \lambda\pi)_{L\in\pi}\} = \{(L \cap \lambda\pi)_{L\in[N]}\}$, so $\chi(\lambda w)(I, \pi) = \chi(\lambda w)(\lambda(I), [N])$.

I show second that *Null-Player* and *The Rule of Generalization* together imply *Generalized Null-Player*. Consider any $w \in W$, such that $i \in N$ is a null-player in $w \in W$ or, equivalently, $N\backslash\{i\}$ is a carrier of w. Consider any two embedded coalitions, $(I, \pi), (J, \pi\prime) \in M$, such that $I \cap (N\backslash\{i\}) = J \cap (N\backslash\{i\})$ and $\{(L \cap (N\backslash\{i\}))_{L\in\pi}\} = \{(L \cap (N\backslash\{i\}))_{L\in\pi\prime}\}$. Note that if $I \neq \emptyset$ and $J \neq \emptyset$, then there exists $\lambda \in \Lambda_\pi, \lambda\prime \in \Lambda_{\pi\prime}$ such that $\lambda I = \lambda\prime J$ and $\lambda w = \lambda\prime w$. In this case, by *The Rule of Generalization*, $\chi(w)(I, \pi) = \chi(\lambda w)(\lambda(I), [N]) = \chi(\lambda\prime w)(\lambda\prime(J), [N]) = \chi(w)(J, \pi\prime)$. If, on the other hand, either $I = \emptyset$ or $J = \emptyset$, then for any $\lambda \in \Lambda_\pi, \lambda\prime \in \Lambda_{\pi\prime}$ by *The Rule of Generalization*, and *Null-Player* $\chi(w)(I, \pi) = \chi(\lambda w)(\lambda(I), [N]) = 0 = \chi(\lambda\prime w)(\lambda\prime(J), [N]) = \chi(w)(J, \pi\prime)$. So $N\backslash\{i\}$ is a carrier of $\chi(w)$, or, equivalently, i is a null-player in $\chi(w)$. ■

Throughout the remainder of this appendix, *EGSV* shall denote the Extended, Generalized Shapley Value.

A.2. Proof of Theorem 2

It is easy to see that the *EGSV* satisfies *Efficiency*, *Symmetry*, *Null-Player*, *Linearity*, *The Rule of Generalization*, and *Weak Monotonicity* (or equivalently, by Theorem 1, *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, *Generalized Null-Player*, and *Weak Monotonicity*). We prove here that it also satisfies *Recursion*.

Let $w\left(A,\cdot\cdot\right)$ denote $w(A,\{A,N\backslash A\})$. For any $w \in W$, and for any $(I,\pi) \in M$,

$$\begin{aligned} EGSV(w)(I,\pi) &= \sum_{T\subseteq\pi} \tfrac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!}\left[w\left(\bigcup_{A\in T} A,\cdot\cdot\right) - w\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)\right] \\ &= \sum_{T\subseteq\pi} \tfrac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!}\left[\tfrac{w\left(\bigcup_{A\in T} A,\cdot\cdot\right)-w\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)}{2}\right. \\ &\qquad \left. + \tfrac{w\left(N\backslash\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)-w\left(N\backslash\bigcup_{A\in T} A,\cdot\cdot\right)}{2}\right] \end{aligned}$$

$$\begin{aligned} EGSV(EGSV(w))(I,\pi) &= \sum_{T\subseteq\pi} \tfrac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!}\left[EGSV(w)\left(\bigcup_{A\in T} A,\cdot\cdot\right)\right. \\ &\qquad \left. - EGSV(w)\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)\right] \end{aligned}$$

Also, $\forall T \subseteq \pi$,

$$\begin{aligned} &EGSV(w)\left(\bigcup_{A\in T} A,\{.,.\}\right) - EGSV(w)\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right) \\ &\quad = \left(w\left(\bigcup_{A\in T} A,\cdot\cdot\right) + \tfrac{w(N,\{N\})-w\left(\bigcup_{A\in T} A,\cdot\cdot\right)-w\left(N\backslash\bigcup_{A\in T} A,\cdot\cdot\right)}{2}\right) \\ &\quad - \left(w(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot) + \tfrac{w(N,\{N\})-w\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)-w\left(N\backslash\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)}{2}\right) \\ &\quad = \tfrac{w\left(\bigcup_{A\in T} A,\cdot\cdot\right)-w\left(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)}{2} + \tfrac{w\left(N\backslash\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)-w\left(N\backslash\bigcup_{A\in T} A,\cdot\cdot\right)}{2} \end{aligned}$$

So,

$$\begin{aligned} EGSV(EGSV(w))(I,\pi) &= \sum_{T\subseteq\pi} \tfrac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!}\left[\tfrac{w\left(\bigcup_{A\in T} A,\cdot\cdot\right)-w(\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot)}{2}\right. \\ &\qquad \left. + \tfrac{w\left(N\backslash\bigcup_{A\in T\backslash\{I\}} A,\cdot\cdot\right)-w\left(N\backslash\bigcup_{A\in T} A,\cdot\cdot\right)}{2}\right] \\ &= EGSV(w)(I,\pi) \end{aligned}$$

We next prove that only one function satisfies *Efficiency*, *Symmetry*, *Null-Player*, *Linearity*, *The Rule of Generalization*, *Weak Monotonicity*, and *Recursion* (or equivalently, by Theorem 1, *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, *Generalized Null-Player*, *Weak Monotonicity*, and *Recursion*).

Proposition 1 *If χ satisfies Efficiency and The Rule of Generalization, then $\forall w \in W, \forall \pi \in \mathit{\Pi}, \sum_{J\in\pi} \chi(w)(J,\pi) = w(N,\{N\})$.*

Proof. By *The Rule of Generalization*, $\forall w \in W, \forall \pi \in \Pi, \forall \lambda \in \Lambda_\pi, \sum_{J\in\pi} \chi(w)(J,\pi) = \sum_{J\in\pi} \chi(\lambda w)(\lambda(J),[N])$. By *Efficiency*, $\forall w \in W, \forall \pi \in \Pi, \forall \lambda \in \Lambda_\pi, \sum_{J\in\pi} \chi(\lambda w)(\lambda(J),[N]) = w(N,\{N\})$. ■

Proposition 2 *If χ satisfies Linearity and The Rule of Generalization, then* $\forall w \in W, \forall (I,\pi) \in M, \chi(w)(I,\pi) = \sum_{\mu\in M} w(\mu)\chi(w^\beta_\mu)(I,\pi)$.

Proof. From *Linearity* and *The Rule of Generalization* we obtain $\forall w, w\prime \in W, \forall \mu \in M$, $\forall \gamma \in \mathbb{R}$, $\chi(w + w\prime)(\mu) = (\chi(w) + \chi(w\prime))(\mu)$ and $\chi(\gamma w)(\mu) = \gamma\chi(w)(\mu)$. This proposition is then trivial. ■

Proposition 3 *If χ satisfies Cohesion and Generalized Null-Player, then* $\forall (I,\pi) \in M, \forall (J,\pi\prime) \in M\backslash M_\pi, \chi(w^\beta_{(J,\pi\prime)})(I,\pi) = 0$.

Proof. Note that by *Cohesion* $(J,\pi\prime) \notin M_\pi \rightarrow \ \chi(w^\beta_{(J,\pi\prime)})(I,\pi) = \chi(w^0)(I,\pi)$, where $w^0 \in W$ is a game in which all players are null; and *Generalized Null-Player* requires $\chi(w^0)(I,\pi) = 0$. ■

Proposition 4 *If χ satisfies Efficiency, Symmetry, Null-Player, Linearity, The Rule of Generalization, Weak Monotonicity, and Recursion, then* $\forall \pi \in \Pi, \forall I, J \in \pi, |\pi| > 2 \rightarrow \ \chi(w^\beta_{(I,\pi)})(I,\pi) = \chi(w^\beta_{(I,\pi)})(J,\pi) = 0$.

Proof. Suppose that χ satisfies *Efficiency*, *Symmetry*, *Null-Player*, *Linearity*, *The Rule of Generalization*, *Weak Monotonicity*, and *Recursion*. By Proposition 2 and *Recursion*:

$$\forall (\mu), (\mu\prime) \in M, \chi(w^\beta_{\mu\prime})(\mu) = \sum_{\mu\prime\prime\in M} \chi(w^\beta_{\mu\prime})(\mu\prime\prime)\chi(w^\beta_{\mu\prime\prime})(\mu). \tag{1}$$

Consider any $\pi \in \Pi$ such that $|\pi| > 2$, and any $I \in \pi$. Using Proposition 3 and equation (1):

$$\begin{aligned}\chi(w^\beta_{(I,\pi)})(I,\pi) &= \sum_{J\in\pi} \chi(w^\beta_{(I,\pi)})(J,\pi)\chi(w^\beta_{(J,\pi)})(I,\pi) \\ &= \Big(\chi(w^\beta_{(I,\pi)})(I,\pi)\Big)^2 + \sum_{J\in\pi\backslash\{I\}} \chi(w^\beta_{(I,\pi)})(J,\pi)\chi(w^\beta_{(J,\pi)})(I,\pi). \end{aligned} \tag{2}$$

By Proposition 1, *Symmetry* and *The Rule of Generalization*,

$$\forall J \in \pi\backslash\{I\}, \chi(w^{\beta}_{(I,\pi)})(J,\pi) = \chi(w^{\beta}_{(J,\pi)})(I,\pi) = -\frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{(|\pi|-1)}. \tag{3}$$

Using (3), we can rewrite the right hand side of (2) to give

$$\chi(w^{\beta}_{(I,\pi)})(I,\pi) = \left(\chi(w^{\beta}_{(I,\pi)})(I,\pi)\right)^2 + (|\pi|-1)\left(-\frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{(|\pi|-1)}\right)^2. \tag{4}$$

Solving (4) gives

$$\chi(w^{\beta}_{(I,\pi)})(I,\pi) = \text{either } 0 \text{ or } \frac{(|\pi|-1)}{|\pi|}. \tag{5}$$

We can proceed to rule out $\chi(w^{\beta}_{(I,\pi)})(I,\pi) = \frac{(|\pi|-1)}{|\pi|}$. Recall (from section 3.1) the previous definitions of Λ_π and (for $\lambda \in \Lambda_\pi$) of $\lambda\pi$ and λw. Now consider any $\lambda \in \Lambda_\pi$, $i = \lambda(I)$, $j \in (\lambda\pi\backslash\{i\})$. By *The Rule of Generalization,*

$$\chi(w^{\beta}_{(I,\pi)})(I,\pi) = \chi(\lambda(w^{\beta}_{(I,\pi)}))(\{i\},[N]).$$

Also, where $\left\{(J,\pi\prime) \in M : i \in J, \left\{(L \cap \lambda\pi)_{L\in\pi\prime}\right\} = \{\{k\}_{k\in\lambda\pi}\}\right\}$ is the set of embedded coalitions containing i and such that no players in $\lambda\pi$ are coalesced together,

$$\lambda(w^{\beta}_{(I,\pi)}) = \sum_{(J,\pi\prime)\in M: i\in J, \left\{(L\cap\lambda\pi)_{L\in\pi\prime}\right\}=\{\{k\}_{k\in\lambda\pi}\}} w^{\beta}_{(J,\pi\prime)}.$$

By *Linearity*, then

$$\chi(w^{\beta}_{(I,\pi)})(I,\pi) = \sum_{(J,\pi\prime)\in M: i\in J, \left\{(L\cap\lambda\pi)_{L\in\pi\prime}\right\}=\{\{k\}_{k\in\lambda\pi}\}} \chi(w^{\beta}_{(J,\pi\prime)})(\{i\},[N]). \tag{6}$$

Note that $\left(\sum_{(J,\pi\prime)\in M: i\in J} w^{\beta}_{(J,\pi\prime)}\right)$ denotes an inessential game, in which coalitions containing i receive a payoff of 1 and other coalitions receive zero. All players except i are null in this game so, by the *Null-Player* and *Efficiency* axioms $\chi\left(\sum_{(J,\pi\prime)\in M: i\in J} w^{\beta}_{(J,\pi\prime)}\right)(\{i\},[N]) = 1$. By *Linearity*, then

$$\sum_{(J,\pi\prime)\in M: i\in J} \chi(w^{\beta}_{(J,\pi\prime)})(\{i\},[N]) = 1. \tag{7}$$

$\left(\sum_{(J,\pi\prime)\in M:\{i,j\}\subseteq J} w^{\beta}_{(J,\pi\prime)}\right)$ denotes a game in which coalitions containing players i *and* j receive 1 and in which coalitions missing i or j receive zero. This game is symmetric with respect to i and j, and all other players are null. So by *Null-Player*, *Symmetry* and *Efficiency*, $\chi\left(\sum_{(J,\pi\prime)\in M:\{i,j\}\subseteq J} w^{\beta}_{(J,\pi\prime)}\right)(\{i\},[N]) = \frac{1}{2}$. By *Linearity*, then

$$\sum_{(J,\pi\prime)\in M:\{i,j\}\subseteq J} \chi(w^{\beta}_{(J,\pi\prime)})(\{i\},[N]) = \frac{1}{2}. \tag{8}$$

$\left\{(J,\pi\prime)\in M : i\in J, \left\{(L\cap\lambda\pi)_{L\in\pi\prime}\right\} = \left\{\{k\}_{k\in\lambda\pi}\right\}\right\}$ and $\{(J,\pi\prime)\in M : \{i,j\}\subseteq J\}$ are disjoint subsets of $\{(J,\pi\prime)\in M : i\in J\}$ so, using (6), (7), (8), and *Weak Monotonicity*

$$\begin{aligned}\chi(w^{\beta}_{(I,\pi)})(I,\pi) &\leqslant 1-\frac{1}{2}\\ &< \frac{(|\pi|-1)}{|\pi|}.\end{aligned}$$

So, by (5) and (3), $\forall J\in\pi, \chi(w^{\beta}_{(I,\pi)})(I,\pi) = \chi(w^{\beta}_{(I,\pi)})(J,\pi) = 0$. ■

Proposition 5 *If χ satisfies Efficiency, Symmetry, Null-Player, Linearity, The Rule of Generalization, Weak Monotonicity, and Recursion, then* $\forall(I,\pi),(J,\pi\prime)\in M, |\pi|>2 \rightarrow \chi(w^{\beta}_{(I,\pi)})(J,\pi\prime) = 0$.

Proof. Consider any $(I,\pi)\in M$ such that $|\pi|>2$. Suppose the following:

(i) $\forall(J,\pi\prime)\in M, |\pi\prime|<n \rightarrow \chi(w^{\beta}_{(I,\pi)})(J,\pi\prime) = 0$

(ii) $\forall(J,\pi\prime),(K,\pi\prime\prime)\in M, (\pi\prime\prime=\pi\prime, |\pi\prime|>n-1) \rightarrow \chi(w^{\beta}_{(J,\pi\prime)})(K,\pi\prime\prime) = 0.$

Now consider any $(K,\pi\prime\prime)\in M$ such that $|\pi\prime\prime| = n$. Recalling (1),

$$\begin{aligned}\chi(w^{\beta}_{(I,\pi)})(K,\pi\prime\prime) &= \sum_{(J,\pi\prime)\in M} \chi(w^{\beta}_{(I,\pi)})(J,\pi\prime)\chi(w^{\beta}_{(J,\pi\prime)})(K,\pi\prime\prime)\\ &= \sum_{\substack{(J,\pi\prime)\in M:(\pi\prime<n)\\ (J,\pi\prime)\in M:(\pi\prime\prime=\pi\prime,|\pi\prime|>n-1)\\ (J,\pi\prime)\in M:(\pi\prime\notin M_{\pi\prime\prime})}} \chi(w^{\beta}_{(I,\pi)})(J,\pi\prime)\chi(w^{\beta}_{(J,\pi\prime)})(K,\pi\prime\prime)\\ &= 0.\end{aligned}$$

So (i) and (ii) together imply $\forall (J, \pi\prime) \in M, |\pi\prime| < n+1 \rightarrow \chi(w^{\beta}_{(I,\pi)})(J, \pi\prime) = 0$. By Proposition 3, (i) holds for $n = 3$, and, by Proposition 4, (ii) holds for $n > 2$, so (i) holds where n is any positive integer. ■

By Proposition 5, if $\chi(w)$ satisfies *Efficiency*, *Symmetry*, *Null-Player*, *Linearity*, *The Rule of Generalization*, *Weak Monotonicity*, and *Recursion*, then for any $w \in W$, $\chi(w) = \chi(w\prime)$ where $w\prime \in W^c$ is defined by $w\prime(I, \pi) = w(I, \{I, N \backslash I\}))$. Corollary 1 established that *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player* obtain $\chi|_{W^c}$, so by Proposition 5, *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, *Generalized Null-Player*, *Weak Monotonicity*, and *Recursion* obtain χ. ■

A.3. Independence of axioms in Theorem 2

Theorem 2 entails seven axioms: *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, *Generalized Null-Player*, *Weak Monotonicity*, and *Recursion*. Here, I establish that these axioms are independent by providing examples of extended generalized values that satisfy each combination of six from the seven axioms while contravening the seventh.

Let $N = \{1, 2, 3\}$.

Omitting commas - i.e. so that "$(\{12\}\{\{12\}\{3\}\})$" denotes $(\{1, 2\}, \{\{1\}, \{2\}, \{3\}\})$ - we can write:

$$\begin{pmatrix} \chi(w)(\{123\}\{\{123\}\}) \\ \chi(w)(\{12\}\{\{12\}\{3\}\}) \\ \chi(w)(\{3\}\{\{12\}\{3\}\}) \\ \chi(w)(\{13\}\{\{13\}\{2\}\}) \\ \chi(w)(\{2\}\{\{13\}\{2\}\}) \\ \chi(w)(\{23\}\{\{1\}\{23\}\}) \\ \chi(w)(\{1\}\{\{1\}\{23\}\}) \\ \chi(w)(\{1\}\{\{1\}\{2\}\{3\}\}) \\ \chi(w)(\{2\}\{\{1\}\{2\}\{3\}\}) \\ \chi(w)(\{3\}\{\{1\}\{2\}\{3\}\}) \end{pmatrix} = A \begin{pmatrix} w(\{123\}\{\{123\}\}) \\ w(\{12\}\{\{12\}\{3\}\}) \\ w(\{3\}\{\{12\}\{3\}\}) \\ w(\{13\}\{\{13\}\{2\}\}) \\ w(\{2\}\{\{13\}\{2\}\}) \\ w(\{23\}\{\{1\}\{23\}\}) \\ w(\{1\}\{\{1\}\{23\}\}) \\ w(\{1\}\{\{1\}\{2\}\{3\}\}) \\ w(\{2\}\{\{1\}\{2\}\{3\}\}) \\ w(\{3\}\{\{1\}\{2\}\{3\}\}) \end{pmatrix}$$

where

$$A = \begin{pmatrix} \chi\left(w^{\beta}_{(\{123\}\{\{123\}\})}\right)(\{123\}\{\{123\}\}) & \cdots & \chi\left(w^{\beta}_{(\{123\}\{\{123\}\})}\right)(\{3\}\{\{1\}\{2\}\{3\}\}) \\ \chi\left(w^{\beta}_{(\{12\}\{\{12\}\{3\}\})}\right)(\{123\}\{\{123\}\}) & \cdots & \chi\left(w^{\beta}_{(\{12\}\{\{12\}\{3\}\})}\right)(\{3\}\{\{1\}\{2\}\{3\}\}) \\ \chi\left(w^{\beta}_{(\{3\}\{\{12\}\{3\}\})}\right)(\{123\}\{\{123\}\}) & \cdots & \chi\left(w^{\beta}_{(\{3\}\{\{12\}\{3\}\})}\right)(\{3\}\{\{1\}\{2\}\{3\}\}) \\ \vdots & \ddots & \vdots \\ \chi\left(w^{\beta}_{(\{3\}\{\{1\}\{2\}\{3\}\})}\right)(\{123\}\{\{123\}\}) & \cdots & \chi\left(w^{\beta}_{(\{3\}\{\{1\}\{2\}\{3\}\})}\right)(\{3\}\{\{1\}\{2\}\{3\}\}) \end{pmatrix}.$$

By Theorem 2 χ satisfies Efficiency, Symmetry, Linearity, Weak Monotonicity, Cohesion, Generalized Null-Player, and Recursion if and only if $A = A_1$ where

$$A_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{3} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & -\frac{1}{6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

A_1 being given by the Extended, Generalized Shapley Value.

However, χ satisfies the same seven axioms *except Recursion* if, for example, $A = A_2$ where

$$A_2 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & -\frac{1}{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{12} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \end{pmatrix},$$

A_2 being given by any of the extensions of the Shapley value proposed by $[1, 6, 12]$ (these extensions coincide for 3-player games) - I might alternatively have used the extension proposed by $[3, 11]$ - in conjunction with the Rule of Generalization.

χ satisfies the original seven axioms *except Generalized Null-Player* (while still satisfying Null-Player) if, for example, $A = A_3$ where

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \frac{1}{3} \\ 0 & 1 & 0 & \cdots & -\frac{1}{3} \\ 0 & 0 & 1 & \cdots & \frac{1}{3} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & -\frac{1}{6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

A_3 entailing that $\{(w, \chi(w)|_{\{(\{i\},[N]) :\, i \in N\}}) : w \in W\}$ is the Extended Shapley Value, but that $\pi \neq [N] \to \chi(w)(I, \pi) = w(I, \pi)$.

It is worth noting that with 4 or more players $\{(w, \chi(w)|_{\{(\{i\},[N]) :\, i \in N\}}) : w \in W\}$ needn't be the Extended Shapley Value: by setting $\pi \neq [N] \to \chi(w)(I, \pi) = w(I, \pi)$, and $\pi = [N] \to \chi(w^{\beta}_{(i,\pi)})(I, \pi\prime) = 0$ it is relatively easy to construct extended, generalized values that satisfy Efficiency, Linearity, Symmetry, Null-Player, Weak Monotonicity, Cohesion and Recursion without $\{(w, \chi(w)|_{\{(\{i\},[N]) :\, i \in N\}}) : w \in W\}$ being the Extended Shapley Value.

χ satisfies the original seven axioms *except Cohesion* if, for example, $A = A_4$ where

$$A_4 = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \cdots & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \cdots & -\frac{1}{3} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{6} & -\frac{1}{6} & \cdots & -\frac{1}{6} \\ 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & \frac{1}{3} \end{pmatrix},$$

A_4 entailing that $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i \in N\}}) : w \in W\}$ is the extension of the Shapley value proposed by [3, 11] - I might alternatively have used any of the extensions proposed by [1, 6, 12] - and that $\chi(w)$ is inessential.

χ satisfies the original seven axioms *except Weak Monotonicity* if $A = A_5$ where

$$A_5 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & -\frac{1}{3} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \\ 0 & 0 & 0 & \cdots & \frac{1}{6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{3} \\ 0 & 0 & 0 & \cdots & \frac{2}{3} \end{pmatrix}.$$

χ satisfies the original seven axioms *except Linearity* if it is obtained using the Rule of Generalization in conjunction with the following extended value: $\{(w, \chi(w)|_{\{(\{i\},[N]):\, i \in N\}}) : w \in W\}$ is obtained by assigning, for each $w \in W$, an outcome of zero to any null-players, and dividing the the payoff to the grand coalition equally between the remaining players. To see this, note that if χ is obtained in this way then it will self-evidently satisfy Null-Player and the Rule of Generalization, and that by Theorem 1 it will therefore satisfy Generalized Null-Player and Cohesion; it is easy to see that it will also satisfy Efficiency and Symmetry. To see that χ will satisfy Weak Monotonicity note that Weak Monotonicity only entails non-negative outcomes to certain singleton players in a limited class of games, and that if χ is obtained as specified here then it will entail non-negative outcomes to all singleton players in the games belonging to this class. To see that χ will satisfy Recursion, note that $\forall\pi \in \Pi, \forall\lambda \in \Lambda_\pi$, in λw and in $\lambda\chi(w)$ the set of null-players, and the payoff to the grand coalition, remains the same.

Now let $N = \{1, 2\}$. χ satisfies the original seven axioms *except Symmetry* if

$$\begin{pmatrix} \chi(w)(\{12\}\{\{12\}\}) \\ \chi(w)(\{1\}\{\{1\}\{2\}\{3\}\}) \\ \chi(w)(\{2\}\{\{1\}\{2\}\{3\}\}) \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} w(\{12\}\{\{12\}\}) \\ w(\{1\}\{\{1\}\{2\}\}) \\ w(\{2\}\{\{1\}\{2\}\}) \end{pmatrix}$$

and, finally, χ satisfies the original seven axioms *except Efficiency* if $\forall w \in W$, $\chi(w) = w$.

A.4. Proof of Theorem 3

To economize on space, I shall write "AB" instead of "$A \cup B$," "$\pi \sim AB$" instead of "$(\pi\backslash\{A, B\}) \cup \{A \cup B\}$," and $w_0(A, \cdot\cdot)$ instead of $w_0(A, \{A, N\backslash A\})$. Note the following:

$\forall(I, \pi) \in M, \forall w_0 \in W$,

(i) $EGSV(w_0)(I,\pi) \equiv \sum\limits_{T\subseteq\pi\backslash\{I\}} \dfrac{|T|!(|\pi| - |T| - 1)!}{|\pi|!}[w_0(\bigcup_{A\in(T\cup\{I\})} A, \cdot\cdot) - w_0(\bigcup_{A\in T} A, \cdot\cdot)]$

(ii) $\forall t \geqslant 1, |\pi| \leq 2 \rightarrow w_t(I,\pi) = w_1(I,\pi) = EGSV(w_0)(I,\pi)$.

Given any $w_0 \in W$, and supposing $\forall(I,\pi) \in M, |\pi| \leq n-1 \rightarrow \{w_t(I,\pi)\}_{t=0}^{\infty}$ converges to $EGSV(w_0)(I,\pi)$, we seek to prove that $\forall(I,\pi) \in M, |\pi| \leq n \rightarrow \{w_t(I,\pi)\}_{t=0}^{\infty}$ converges to $EGSV(w_0)(I,\pi)$.

Given any $(I,\pi) \in M$, such that $|\pi| \leq n$, rearranging the definition of w_t (see section 4) gives,

$\forall (I, \pi) \in M,$

$$w_t(I,\pi) \equiv \frac{1}{|\pi|\left(|\pi|-1\right)} \left(\sum_{J,K \in \pi \backslash \{I\}} w_{t-1}(I, \pi \sim JK) \right.$$
$$\left. + \sum_{J \in \pi \backslash \{I\}} \left(w_{t-1}(IJ, \pi \sim IJ) - w_{t-1}(J,\pi) + w_{t-1}(I,\pi) \right) \right)$$
$$= \frac{1}{|\pi|\left(|\pi|-1\right)} \left(\sum_{J,K \in \pi \backslash \{I\}} w_{t-1}(I, \pi \sim JK) + \sum_{J \in \pi \backslash \{I\}} w_{t-1}(IJ, \pi \sim IJ) \right.$$
$$\left. - \left(\sum_{J \in \pi} w_{t-1}(J,\pi) \right) + |\pi| . w_{t-1}(I,\pi) \right).$$

By the induction hypothesis,

$\forall J, K \in \pi \backslash \{I\}, \{w_{t-1}(I, \pi \sim JK)\}_{t=0}^{\infty}$ converges to $EGSV(w_0)(I, \pi \sim JK)$

and

$\forall J \in \pi \backslash \{I\}, \{w_{t-1}(IJ, \pi \sim IJ)\}_{t=0}^{\infty}$ converges to $EGSV(w_0)(IJ, \pi \sim IJ))$.

So

$$\left\{ \sum_{J,K \in \pi \backslash \{I\}} w_{t-1}(I, \pi \sim JK) + \sum_{J \in \pi \backslash \{I\}} w_{t-1}(IJ, \pi \sim IJ) \right\}_{t=0}^{\infty} \text{ converges to}$$
$$\left(\sum_{J,K \in \pi \backslash \{I\}} EGSV(w_0)(I, \pi \sim JK) + \sum_{J \in \pi \backslash \{I\}} EGSV(w_0)(IJ, \pi \sim IJ) \right).$$

Using

$$EGSV(w_0)(I, \pi \sim JK)$$
$$= \sum_{T \subseteq \pi \backslash \{I,J,K\}} \frac{|T|!(|\pi|-|T|-3)!}{(|\pi|-1)!} \left[(|\pi|-|T|-2) \left(w_0 \left(\bigcup_{A \in (T \cup \{I\})} A, \cdot\cdot \right) - w_0 \left(\bigcup_{A \in T} A, \cdot\cdot \right) \right) \right.$$
$$\left. + (|T|+1) \left(w_0 \left(\bigcup_{A \in (T \cup \{I,J,K\})} A, \cdot\cdot \right) - w_0 \left(\bigcup_{A \in (T \cup \{J,K\})} A, \cdot\cdot \right) \right) \right]$$

and

$$EGSV(w_0)(IJ, \pi \sim IJ)$$

$$= \sum_{T \subseteq \pi \setminus \{I,J\}} \frac{|T|!(|\pi| - |T| - 2)!}{(|\pi| - 1)!} \left(w_0 \left(\bigcup_{A \in (T \cup \{I,J\})} A, \cdot\cdot \right) - w_0 \left(\bigcup_{A \in T} A, \cdot\cdot \right) \right)$$

a little work gives

$$\sum_{J,K \in \pi \setminus \{I\}} EGSV(w_0)(I, \pi \sim JK) =$$

$$\frac{|T|!(|\pi| - |T| - 1)!}{(|\pi| - 1)!} \left[\sum_{\substack{T \subseteq \pi \setminus \{I\} \\ : |T| \leqslant |\pi| - 3}} (|\pi| - |T| - 2) \left(w_0 \left(\bigcup_{A \in (T \cup \{I\})} A, \cdot\cdot \right) - w_0 \left(\bigcup_{A \in T} A, \cdot\cdot \right) \right) \right.$$

$$\left. + \sum_{\substack{T \subseteq \pi \setminus \{I\} \\ : |T| \geqslant 2}} (|T| - 1) \left(w_0 \left(\bigcup_{A \in (T \cup \{I\})} A, \cdot\cdot \right) - w_0 \left(\bigcup_{A \in T} A, \cdot\cdot \right) \right) \right]$$

and

$$\sum_{J \in \pi \setminus \{I\}} EGSV(w_0)(IJ, \pi \sim IJ) =$$

$$\frac{|T|!(|\pi| - |T| - 1)!}{(|\pi| - 1)!} \left[\sum_{\substack{T \subseteq \pi \setminus \{I\} \\ : |T| \geqslant 1}} w_0 \left(\bigcup_{A \in (T \cup \{I\})} A, \cdot\cdot \right) - \sum_{\substack{T \subseteq \pi \setminus \{I\} \\ : |T| \leqslant |\pi| - 2}} w_0 \left(\bigcup_{A \in T} A, \cdot\cdot \right) \right].$$

Further work gives

$$\sum_{J,K \in \pi \setminus \{I\}} EGSV(w_0)(I, \pi \sim JK) + \sum_{J \in \pi \setminus \{I\}} EGSV(w_0)(IJ, \pi \sim IJ) =$$

$$|\pi| \, (|\pi| - 2) EGSV(w_0)(I, \pi) + w_0(N, \cdot\cdot).$$

It is easy to see that

$$t \geqslant |\pi| \to w_0(N, \cdot\cdot) - \sum_{J \in \pi} w_{t-1}(J, \pi) = 0$$

and that

$$\left\{ \frac{(|\pi| - 2)}{(|\pi| - 1)} EGSV(w_0)(I, \pi) + \frac{1}{(|\pi| - 1)} w_{t-1}(I, \pi) \right\}_{t=0}^{\infty} \text{ converges to } EGSV(w_0)(I, \pi).$$

So $\{w_t(I, \pi)\}_{t=0}^{\infty}$ converges to $EGSV(w_0)(I, \pi)$. ■

A.5. Proof of Theorem 4

Proposition 6 *If χ satisfies Efficiency, Symmetry, Linearity, Cohesion, and Generalized Null-Player, then* $\forall (I,\pi) \in M, |\pi| \leqslant 2 \rightarrow \forall w \in W, \{\chi_t(w)(I,\pi)\}_{t=0}^{\infty}$ *converges.*

Proof. Note that, in fact, given any $(I,\pi) \in M$ such that $|\pi| \leqslant 2$, if χ satisfies *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player*, then $t \geqslant 1 \rightarrow \chi_t(w)(I,\pi) = EGSV(w)(I,\pi)$. ■

Proposition 7 *If χ satisfies Efficiency, Symmetry, Linearity, Cohesion, and Generalized Null-Player, and* $\forall (I,\pi) \in M, |\pi| > 2 \rightarrow -\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}$, *then* $\forall w \in W, \{\chi_t(w)\}_{t=0}^{\infty}$ *converges pointwise.*

Proof. If χ satisfies *Linearity*, *Cohesion*, and *Generalized Null-Player*, then, using Proposition 2 from the proof of Theorem 2,

$\forall w \in W, \forall (I,\pi) \in M, \forall t \geqslant 1,$

$$\begin{aligned}\chi_t(w)(I,\pi) &= \sum_{\mu \in M} \chi_{t-1}(w)(\mu)\chi(w^{\beta}_{\mu})(I,\pi) \\ &= \sum_{\substack{(J,\pi\prime) \in M_\pi \\ :\pi\prime \neq \pi}} \chi_{t-1}(w)(J,\pi\prime)\chi(w^{\beta}_{(J,\pi\prime)})(I,\pi) \\ &\quad + \sum_{J \in \pi} \chi_{t-1}(w)(J,\pi)\chi(w^{\beta}_{(J,\pi)})(I,\pi). \end{aligned} \tag{9}$$

If χ satisfies *Cohesion* and *Generalized Null-Player* then χ satisfies *The Rule of Generalization.* By this, *Efficiency*, and *Symmetry*,

$\forall w \in W, \forall (I,\pi) \in M, \forall J \in \pi \backslash \{I\},$

$$\begin{aligned}\chi(w^{\beta}_{(J,\pi)})(I,\pi) &= \chi(w^{\beta}_{(I,\pi)})(J,\pi) \\ &= \frac{-\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{|\pi|-1}. \end{aligned} \tag{10}$$

Also, if χ satisfies *The Rule of Generalization* and *Efficiency*, then

$\forall w \in W, \forall (I,\pi) \in M, \forall t \geqslant 1,$

$$\sum_{J \in \pi} \chi_t(w)(J,\pi) = w(N,\{N\}). \tag{11}$$

Substituting (11) and (10) in (9) gives

$\forall w \in W, \forall (I,\pi) \in M, \forall t \geqslant 1,$

$$\begin{aligned} \chi_t(w)(I,\pi) = & \sum_{\substack{(J,\pi\prime)\in M_\pi \\ :\pi\prime\neq\pi}} \chi_{t-1}(w)(J,\pi\prime)\chi(w^{\beta}_{(J,\pi\prime)})(I,\pi) \\ & - \frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{|\pi|-1} w(N,\{N\}) \\ & + \left(\chi(w^{\beta}_{(I,\pi)})(I,\pi) + \frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{|\pi|-1}\right)\chi_{t-1}(w)(I,\pi). \end{aligned} \tag{12}$$

By Proposition 6, $|\pi| \leqslant 2 \rightarrow \{\chi_t(w)(I,\pi\}_{t=0}^{\infty}$converges. It is clear from (12) that, provided $-1 < \left(\chi(w^{\beta}_{(I,\pi)})(I,\pi) + \frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{|\pi|-1}\right) < 1,$

$$(\forall (I,\pi) \in M, |\pi| < n \rightarrow \{\chi_t(w)(I,\pi)\}_{t=0}^{\infty} \text{ converges})$$

$$\rightarrow (\forall (I,\pi) \in M, |\pi| < n+1 \rightarrow \{\chi_t(w)(I,\pi)\}_{t=0}^{\infty} \text{ converges}).$$

Note that

$$-\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|} \leftrightarrow -1 < \left(\chi(w^{\beta}_{(I,\pi)})(I,\pi) + \frac{\chi(w^{\beta}_{(I,\pi)})(I,\pi)}{|\pi|-1}\right) < 1.$$

■

Proposition 8 *If χ satisfies Efficiency, Symmetry, Linearity, Cohesion, and Generalized Null-Player, then* $\forall (I,\pi) \in M, |\pi| > 2, -\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|} \rightarrow \left\{\chi_t(w^{\beta}_{(I,\pi)})(I,\pi)\right\}_{t=0}^{\infty}$ *converges to zero.*

Proof.

$\forall (I,\pi) \in M$,

$$\begin{aligned}
\chi_t(w^\beta_{(I,\pi)})(I,\pi) &= \chi_1(w^\beta_{(I,\pi)})(I,\pi)\chi_{t-1}(w^\beta_{(I,\pi)})(I,\pi) \\
&\quad + (|\pi|-1)\frac{-\chi_1(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1}\frac{-\chi_{t-1}(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1} \\
&= \left(\chi_1(w^\beta_{(I,\pi)})(I,\pi) + \frac{\chi_1(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1}\right)\chi_{t-1}(w^\beta_{(I,\pi)})(I,\pi)
\end{aligned}$$

So it is easy to see that

$$\begin{aligned}
&-1 < \left(\chi_1(w^\beta_{(I,\pi)})(I,\pi) + \frac{\chi_1(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1}\right) < 1 \\
&\quad \to \left\{\chi_t(w^\beta_{(I,\pi)})(I,\pi)\right\}_{t=0}^{\infty} \text{ converges to zero.}
\end{aligned}$$

Finally,

$$\begin{aligned}
&-1 < \left(\chi_1(w^\beta_{(I,\pi)})(I,\pi) + \frac{\chi_1(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1}\right) < 1 \\
&\quad \leftrightarrow -\frac{|\pi|-1}{|\pi|} < \chi(w^\beta_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}.
\end{aligned}$$

■

If χ satisfies *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player*, and for all $(I,\pi) \in M, \pi > 2$ implies $-\frac{|\pi|-1}{|\pi|} < \chi(w^\beta_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}$, then by Proposition 7, $\lim_{t\longrightarrow\infty}\chi_t$ exists, and by construction $\lim_{t\longrightarrow\infty}\chi_t$ satisfies *Recursion.* Notice that in the proof of Theorem 2, *Weak Monotonicity* was only required in order to impose (by Proposition 4) the condition $\forall\pi \in \Pi, \forall I,J \in \pi, |\pi| > 2 \to\ \chi(w^\beta_{(I,\pi)})(I,\pi) = \chi(w^\beta_{(I,\pi)})(J,\pi) = 0$. We know that, by *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player* $\pi \in \Pi, I \in \pi, J \in \pi, I \neq J \to \chi(w^\beta_{(I,\pi)})(J,\pi) = \frac{-\chi(w^\beta_{(I,\pi)})(I,\pi)}{|\pi|-1}$, and therefore that, by Proposition 8, if χ satisfies

Efficiency, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player*, then

$$\forall \pi \in \Pi, \forall I, J \in \pi, |\pi| > 2, -\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}$$
$$\rightarrow \lim_{t \longrightarrow \infty} \chi_t(w^{\beta}_{(I,\pi)})(I,\pi) = \lim_{t \longrightarrow \infty} \chi_t(w^{\beta}_{(I,\pi)})(J,\pi) = 0$$

So if χ satisfies *Efficiency*, *Symmetry*, *Linearity*, *Cohesion*, and *Generalized Null-Player*, and for all $(I,\pi) \in M, |\pi| > 2$ implies $-\frac{|\pi|-1}{|\pi|} < \chi(w^{\beta}_{(I,\pi)})(I,\pi) < \frac{|\pi|-1}{|\pi|}$, then $\lim_{t \longrightarrow \infty} \chi_t$ is the Extended, Generalized Shapley Value. ■

REFERENCES

1. E.M. Bolger, A set of axioms for a value for partition function games, Int. J. Game Theory 18 (1989) 37-44.
2. F. Bloch, Sequential formation of coalitions in games with externalities and fixed payoff division, Games Econ. Behav. 14 (1996) 19-123.
3. G. de Clippel, R. Serrano, Marginal contributions and externalities in the value, Econometrica, forthcoming.
4. F. Gul, Bargaining foundations of Shapley value, Econometrica 57 (1989) 81-95.
5. S. Hart, M. Kurz, Endogenous formation of coalitions, Econometrica 51 (1983) 1047-64.
6. I. Macho-Stadler, D. Pérez-Castrillo, D. Wettstein, Sharing the surplus: an extension of the Shapley value for environments with externalities, J. Ec. Theory 135 (2007) 339-356.
7. E. Maskin, Bargaining, coalitions, and externalities, Presidential address of the Econometric Society, 2003.
8. R.B. Myerson, Values for games in partition function form, Int. J. Game Theory 6 (1977) 23-31.

9. J. Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1944.

10. G. Owen, Values of games with a priori unions, in: R. Henn, O. Moschlin (Eds.), Essays in Mathematical Economics and Game Theory, Springer Verlag, New York, 1977, pp. 76-88.

11. K.H. Pham Do, H. Norde, The Shapley value for partition function form games, Int. Game Theory Rev. 9 (2007) 353-360.

12. A.J. Potter, A Value for Partition Function Form Games, Working paper: Dept. of Mathematics, Hardin-simmons University, Abilene, Texas, 2000.

13. D. Ray, R. Vohra, A theory of endogenous coalition structures, Games Econ. Behav. 26 (1999) 286-336.

14. S.W. Salant, S. Switzer, R.J. Reynolds, Losses from horizontal merger: the effects of an exogenous change in the industry structure on Cournot-Nash equilibrium, Q. J. Econ. 98 (1983) 185-200.

15. L.S. Shapley, A value for n-person games, in: H.W. Kuhn, A.W. Tucker (Eds.), Contributions to the Theory of Games, II, Princeton University Press, Princeton, 1953, pp. 307-317.

16. M. Shubik, Incentives, decentralized control, the assignment of joint costs, and internal pricing, Manage. Sci. 8 (1962) 325-343.

17. M. Shubik, Game Theory in the Social Sciences: Concepts and Solutions, MIT Press, Cambridge, Mass., 1982.

18. R.M. Thrall, W.F. Lucas, n-person games in partition function form, Naval Logistics Quarterly 10 (1963) 281-293.

19. H.P. Young, Monotonic solutions of cooperative games, Int. J. Game Theory 14 (1985) 65-72.