Reappraising the classics - the case for a dynamic reformulation of the labour theory of value

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REAPPRAISING THE CLASSICS–THE CASE FOR A DYNAMIC REFORMULATION OF THE LABOUR THEORY OF VALUE

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Abstract

This article argues that simultaneous equation systems, widely regarded as a standard formalisation of labour value theory, import equilibrium assumptions which rule out a realistic or consistent theory of price formation. An alternative, dynamic formalisation exists yielding time-varying or dynamic labour values free of such assumptions.

We show that simultaneous equation systems, formally equivalent to neoclassical general equilibrium systems, cannot represent technical change or economic growth and apply only to hypothetical static economies in which neither the scale of output nor the technology changes. The resulting static values are a special, limiting case of dynamic values, which converge to them only tendentially and in the absence of technical change. Under conditions of technical change, dynamic values — and the prices and profit rates derived from them — differ systematically from those derived from simultaneous equation systems, and therefore provide a different foundation for economics.

We show that the behaviour of dynamic values corresponds more closely to observed reality than either neoclassical equilibrium prices or static labour values by showing how, in a dynamic framework, the rate of profit can, and in general does, fall despite productivity-enhancing technical change.

These results provide a rigorous foundation for the study of capital movements and technical change which is superior to conventional neoclassical price theory, calling for a radical reappraisal of the debate on value in this century.

REAPPRAISING THE CLASSICS–THE CASE FOR A DYNAMIC REFORMULATION OF THE LABOUR THEORY OF VALUE

Introduction

Of all basic concepts of mainstream economics, the most questionable and questioned is equilibrium. The self-evident facts that neither quantities produced nor consumed stabilise, and that goods do not trade at equilibrium prices, have provoked countless offshoots of neoclassical economics including Harrod-Domar growth theory, post-Keynesianism, business cycle theory and many variants of expectation theory.

Yet equilibrium is arguably the least dispensable concept of neoclassical economics. If it were irrefutably and finally proven that there is neither a real nor a logical link between observed economic magnitudes and those predicted by general equilibrium, little would remain of the rigorous foundations so painstakingly assembled by Walras, Arrow and Debreu.

It is hence a paradox of twentieth century economic thought that the labour theory of value, the most critical alternative to the neoclassical system, has – we shall show – embedded the equilibrium approach into its own foundations. In effect it has been grafted onto a neoclassical root. We hold that the hybrid is barren and a superior alternative exists. If value theory is rigorously reformulated without equilibrium, its most heavily-criticised inconsistencies vanish. We conclude that these result not from the theory but the stem onto which it has been transplanted.

During the last decade, a growing minority of writers has turned from this mainstream synthesis to study the dynamic behaviour of labour values and prices. (See for example Ernst[1982], Carchedi [1984], Andrews [1984], Naples [1985], Kliman and McGlone[1988], Kliman[1988], Freeman [1990, 1991], and Walker's [1988] survey). Inspired by Langston's[1984] pioneering work, Mandel & Freeman [1984] brought together a number of seminal works in which equilibrium assumptions were dissected and rejected.[1] The recent debate between Kliman [1993] and Naples [1993] is testimony to the vitality of this alternative tradition.

We are convinced that this new emerging framework is one of the most critical developments in economics of recent decades. It has the potential to re-establish the theoretical validity of labour value theory and, moreover, to settle a series of outstanding debates in economics as a whole to which neoclassical economics has failed as yet to give a decisive reply. For this very reason we believe the debate needs to be approached with exceptional rigour but also with exceptional openness. We have to admit the serious new problems to be faced but recognise the great advances which the new approach yields. It is therefore not constructive, in our view, to take up entrenched positions at this point in the debate on issues which properly deserve much more study.

For this reason, although Kliman, Naples and Giussani have already opened up the topic of price formation in a dynamic framework, here we confine ourselves largely to the value realm. Elsewhere [Freeman 1991, 1993] we discuss a fully general framework for prices of production in a labour value system, demonstrating that Marx's two equalities can both be recovered in a fully general context. In a postscript to the present article we indicate how the Kliman/Giussani approach may be generalised to the case of unequal profit rates and to continuous time. Here we have three aims: to lay a thoroughly rigorous foundation for the work; to show how this leads to a treatment of continuous time, and show how dramatic a change follows from a dynamic approach.

Simultaneous equation systems: the twentieth century synthesis

Economics being an exact science, its theorists since Aristotle have tried to express their conclusions in a precise mathematical manner — to formalise them. The modern formalisation of labour value theory, however, is not the work of its authors but of twentieth-century writers redressing their alleged inconsistencies, in particular Marx's presentation of the quantitative relation between values and prices of production.

The most widely accepted derivation of prices from values – mooted by Dmitriev[1905], developed by Tugan-Baronovsky[1905], and expounded by von Bortkiewicz[1907] – is a 'correction' of Marx in which any input to production is purchased at the same price at which it is sold as an output. Input prices cannot therefore equal values; but can be calculated from the very fact that they must equal output prices, given some additional – allegedly realistic – assumptions such as an equal profit rate.

Originally formalising one aspect of classical theory, the resulting equation systems have become a standard representation of the theory as a whole. Von Bortkiewicz's approach was developed successively by Winternitz[1951], May[1951] and Seton[1957], extending his three-sector system to the multi-sector case. It became accepted that values themselves could be directly determined from the physical quantities of inputs and outputs using the same technique.[2] Promoted by many Marxists such as Sweezy[1948], Dobb[1972] and Meek[1973], simultaneous equation systems became accepted as *the* correct mathematical expression of the labour theory of value.

In parallel, non-Marxist formalisations of general equilibrium theory emerged from which marginal concepts, often considered a neoclassical hallmark, were absent. Von Neumann[1937] determined a unique set of prices and output magnitudes that maximises the growth rate and minimises the interest rate. Linear production models derived from Wassily Leontieff's[1953] input-output analysis, though not originally proposed as a model of price formation, were used to predict those price levels which can reproduce the economy under an equal profit rate (Pasinetti[1977]). Under the influence of Piero Sraffa[1960] this methodology was promoted by the surplus approach or neoricardian school as a comprehensive approach to price formation. (Steedman[1977]).

The formal identity between linear production models and simultaneous equation labour value systems was soon recognised and linear production models gradually became the bedrock of an approach common to both marxist and non-marxist schools (see Cameron [1952], Georgescu-Roegen [1950], Morishima and Seton [1961], Bronfenbrenner [1968], Pasinetti [1977]).

We contend that the outcome of a century's theoretical debates has therefore been a synthesis between neoclassical general equilibrium theory and labour value theory. Within this synthesis, there are wide-ranging and even vitriolic disputes about interpretation; but equally wide agreement on the techniques, and above all on the simultaneous equation approach.

This has two consequences. First, if the apparent contradictions of labour value theory are in fact imported from the simultaneous equation approach, then it should be possible to reconstruct the theory without contradiction. And if the simultaneous equation approach is inherently contradictory then *all* systems of this type are invalid – including neoricardian and neoclassical general equilibrium systems. This we now set out to show.

An underrated alternative: iterative solutions to the transformation problem

The fundamental flaw in the approach is the passage of time. It is true, and recognised by Marx, that inputs are not purchased at their values. But neither in Marx nor the real world do outputs at the end of a period of production command the same prices, or possess the same values, as inputs did at the beginning. Outputs emerge after inputs are consumed, and no being known to science eats what it has not yet produced.[3] Today's inputs are purchased, not at the prices of tomorrow but of yesterday. Value theory has been cast into the timeless fantasy world of the Midgard serpent, which spent eternity consuming its tail.

Work cast in a labour value framework but rejecting equilibrium assumptions remained largely at the level of criticism and explored isolated aspects of an alternative, rather than a systematic new approach. Nevertheless the germ of a complete alternative existed in the shape of 'iterative' solutions to the transformation problem first proposed, to our knowledge, by Okishio [1972] and Shaikh [1973].

A neglected work by Morishima and Catephores [1979] explores this in some detail. The authors explore a system in which both values prices at any given time were derived from values and prices in the *previous* period. They say

> *'The simultaneous equation approach ... is based on von Bortkiewicz's criticism of Marx's transformation formula, to the effect that it only transformed outputs from value to price, retaining inputs intact. This allegation, however, is completely wrong. It is indeed true that Marx was aware that both inputs and outputs had to be transformed from those in terms of values into the ones in terms of prices. But he did not transform them simultaneously; instead, he used an alternative approach transforming inputs and outputs in a successive way, according to an iteration formula.'*
> *[Morishima and Catephores 1979, p160]*

The authors do not seem to have explored this avenue further and the approach was never accepted into the mainstream of labour value theory. It has hardly figured in the subsequent debate with the neoricardians.

In particular, the authors deny their solution a real historical character, reducing it to a convenient fiction as do most references to Shaikh's solution. This negates the most important conclusion from an iterative approach – that iteratively-calculated prices and values are a *better* approximation to the real world than their hypothetical equilibrium counterparts. The vital conceptual step is to understand that the succession of iterated prices and values are not successive approximations to 'correct' equilibrium prices and values: correctly calculated, they *are* correct prices and values.

A simple illustration of iteratively calculated values

An iterative calculation brings the logical contradictions of the simultaneous equation system to the fore. We illustrate this with a simple example, which we hope is accessible to the non-mathematician. This deals only with values but the principles will later be extended to prices.

Suppose two producers, **P1** and **P2,** make two commodities, **C1** and **C2**. Consumption and production between two times which we shall call *t* and *t*+1 is given in the table below, in which **L** represents labour time.

		C1	C2	L		C1	C2
P1	*used*	10	5	4	*to produce*	18	
P2	*used*	8	15	2	*to produce*		50

Table 1 Quantities consumed and produced

The simultaneous equation approach proceeds as follows: input values must be equal to output values. Use the symbols v_1 and v_2 to represent unit values of **C1** and **C2**. Then the following two equations must be satisfied:

1) $$10v_1+5v_2+4=18v_1$$

2) $$8v_1+15v_2+2=50v_2$$

These equations have only one solution, giving the only unit values compatible with the assumption of equilibrium.

3) $$v_1 = \tfrac{5}{8} \quad v_2 = \tfrac{1}{5}$$

These values comply with the basic tenet of labour value theory: the value of any output is equal to the value transferred by non-labour inputs, plus the labour time spent on it. **P1,** for example, transfers 6¼+1 hours of dead labour from its non-labour inputs, and adds 4 hours of live labour, embodying a total of 6¼+1+4=11¼ hours of abstract labour in the 18 units of output. Each unit of output of **C1** therefore contains 11¼÷18 $=\tfrac{5}{8}$ hours of abstract labour.

Conveniently, this is the same amount of value as each unit of input of **C1**. This convenient fiction is to say the least implausible. Technical change or even fluctuations in production conditions (eg bad harvests, changes in working practices) ensure that the inputs consumed in producing a given output vary over time. This output's unit value must fluctuate accordingly.[4]

The limits of comparative statics

The only equilibrium-based way of dealing with time-varying technology is comparative statics. In each period, a new simultaneous equation system is derived in which economic magnitudes are derived from the coefficients of that period, independent of all previous periods. Economic movement is reduced to a sequence of instantaneous equilibria.

This approach is fundamentally flawed. It recognises that values and prices change from one time period to the next, but demands that they adjust *instantaneously* to the new conditions of production, which is logically impossible and never happens.

To illustrate this suppose that in a previous cycle of production different conditions of production resulted in unit values of, say

$$v_1=1, v_2=2.$$

At time *t* technical conditions change to those given by table 1. The simultaneous equation solution asserts that values instantaneously change to those given by equation (3), namely

$$v_1=\tfrac{5}{8}, \ v_2=\tfrac{1}{5}$$

The logical problem involved now stands out. The simultaneous equation approach dictates that inputs at time *t*-1 must be the same as outputs at time *t*, the end of the production period which began at time *t*-1. But the outputs of this cycle are consumed as inputs in the next cycle. The values of inputs for the time period which begins at time *t* must therefore be $v_1=1, v_2=2$. But the simultaneous equation approach insists that these inputs have the same value as at time *t*+1, that is $v_1=\tfrac{5}{8}, v_2=\tfrac{1}{5}$. The static derivation is contradictory: commodities cannot have the same unit value at the beginning and at the end of production unless the economy is in equilibrium.

Superiority of the iterative approach

The iterative solution, even though originally proposed as an explanation of equilibrium prices instead of an alternative to them, avoids this contradiction. It proceeds as follows: during the time period [*t*-1, *t*] producer **P1** transfers 1×10 + 2×5 labour hours from its non-labour inputs to its output and adds 4 hours of live abstract labour, embodying a total of 24 hours in the 18 units of output of **C1.** The new unit value of **C1** at the end of the period is thus $^{24}/_{18}=^4/_3$. Similarly the unit value of **C2** must become $^4/_5$.

This procedure lets us calculate unit values at any time, given only a set of initial values at some arbitrary starting point, and the quantities of inputs consumed during each successive cycle of production. Note that these values are independent of distribution; it makes no difference whether the surplus **C2** is consumed as luxury goods by capitalists or as wage goods by labourers.

Such a solution is extremely common in the physical sciences and typically arises when a problem is investigated in such a way as to produce a difference or differential equation. The calculation we have just given can in fact be expressed as a first-order linear difference equation.

Simple and valid though the calculation may be, however, it changes much. First, we can no longer speak of v_1 and v_2 without saying when they are measured. They are time-varying magnitudes $v_1^{(t)}$ and $v_2^{(t)}$. Secondly, it reverses the traditional critique of Marx. The value of inputs at time t need not equal the value of outputs at time $t+1$ — the outputs of the next period of production. They must, however, equal the value of outputs at time t as they emerge from the previous period of production. This takes us outside an equilibrium framework, and simultaneous equations can no longer help.

Iteration, convergence and equilibrium

Shaikh saw the iterative solution as vindicating Marx's historical approach, showing that the value of a commodity is made up of successive 'layers' of embodied labour. Sraffa also refers to layers of dated labour. But the intermediate magnitudes have generally been treated as steps in the calculation rather than real historical prices and values. Iterative calculations remained little more than a mathematical curiosity for some years. In particular neither Sraffa nor his followers seem to recognise the vital fact that if a price is built up from dated layers of labour and profit, no justification remains for assuming that profits remain constant during this process. The programme of devising a measure of value founded in 'pure' technology is irrelevant if prices, values and profits can all vary while technology (and the real wage) remains fixed.

Perhaps the main reason for this blind spot is an important property of this calculation known as the convergence property. This is best illustrated with a table of unit values from successive iterations, starting from $v_1=1$, $v_2=2$ as before.

t	v_1	v_2
0	1.0000	2.0000
1	1.3333	0.8000
5	0.8059	0.2776
10	0.6502	0.2108
20	0.6255	0.2002
30	0.6250	0.2000

Table 2: convergence of dynamic values

Provided technology does not change, successive unit values approach closer and closer to the hypothetical equilibrium. This is a well-known result (see Gantmacher, Seneta) which holds for any technical coefficients representing an economy that satisfies the so-called Hawkins-Simons conditions – more or less any economy which produces a net physical surplus.[5]

Technical change and the rate of convergence

The convergence property makes it easier to dismiss the iterative solution as a mere technique for calculating equilibrium values. It is widely held that, in the medium or long run, prices – and, if they exist, values – converge to their equilibrium magnitudes, so that temporary fluctuations do not matter. According to this view there is a kind of 'sea level' around which real prices fluctuate like waves on the ocean.

But if technology changes continuously, there is no guarantee of convergence. Moreover since technical change is itself a function of capital movements and hence price levels, the coefficients of our difference equation are variable functions of price, and the entire system is complex, potentially non-linear, and offers no guarantee of a comparative static approximation.

The question 'how fast does technology change?' is thus decisive. If prices converge quickly, a hypothetical equilibrium might reasonably be treated as a first approximation to real market prices. But if they converge slowly compared with the rate at which technology changes, then prices will *never approximate to* a hypothetical equilibrium. In this case 'sea-level' prices simply do not exist. Economic movement is better compared with the flow of a river, which has no definite level at all; in order to know the height of a river, one must also know whereabouts on the river you are.[6].

'Joint technology' and the process of technical change

The question 'how *fast* does technology change?' raises a prior problem: *how* does technology change? Linear production models such as von Neumann's, Sraffa's and Okishio's[1972] compare the results of two global alternatives. If there are two ways to make a given commodity, then either one or the other is adopted — but never both. In von Neumann's system, the coefficients of inefficient processes are assumed to be zero. In Sraffa's system

'switching' is an instantaneous transition from one set of technical coefficients to another. Okishio compares profit rates before and after a technical change is made — not while it is going on.

This is a necessary consequence of simultaneous equation price determinations, which depend on the well-known condition that the number of equations must equal the number of variables. Since each variable corresponds to a commodity and each equation to a process of production, this dictates that there must be as many producers as commodities. As Savran has pointed out, this gives rise to a peculiarly cramped treatment of 'joint production' in which a capital can make two products, but a product cannot be made by two capitals, save for the exceptional case where it is jointly produced by at least one of them. 'Joint technology' is ruled out.

This is profoundly unrealistic. In real life no new technology is adopted instantaneously. Its spread is governed by the speed at which capital is invested in it. Few innovations are installed before they are paid for and no invention, however compelling, is installed before it is produced. Every commodity is therefore produced using a spectrum of technologies, the complete replacement of an old technology taking decades. Work by the IASSA group of researchers (Nakicenovic[1988]) has shown that key technologies are replaced over the same time span as a 'long wave' of capitalist production – hardly a short-run fluctuation.

This does not at all mean that technical change is a slow or long-run process. It simply shows that it is an incessant one. New production methods are introduced neither instantaneously, nor so slowly that the rate of change is insignificant. Even as yesterday's technology bows out and today's takes the stage, the scene shifts for the entry of tomorrow's. There is, as Marx puts it, an never-ending revolution in the methods of production — to which prices *never* finally adjust.

Surplus Profit — the Foundation of Economic Dynamics

Any theory of price formation must therefore explain how a single, uniform price is arrived at when costs and therefore profits are not uniform. Ricardo never resolved this problem, to the end maintaining that a single rate of profit must rule — that 'capital cannot have more than one price'. General Equilibrium theory rules it out, since in equilibrium there is no reason for any producer to adhere to a less productive technique than the best available, except for some special arbitrary condition such as a permanent monopoly.

Marx regarded the existence of multiple profit and cost structures as self-evident and addressed the resulting problems of price formation in *Theories of Surplus Value,* in the section on the disintegration of the Ricardian school, in chapter 10 of *Capital* Volume III, and in the unpublished 'sixth chapter' of *Capital* Volume I, now generally known as *Results of the Immediate Process of Production.* (Marx [1977b])

It has been left to a small minority of writers such as Mandel, Carchedi and Savran to reiterate this approach to price formation, which distinguishes between the 'market' and 'individual' value of a commodity. The first is the average value of the total amount of a given commodity in circulation; the latter is the value added by a single producer. The difference is a technical rent or surplus appropriated by efficient producers, generally referred to as a surplus or super-profit.

The 'market price of production' is a more complex construct but it remains the case that individual producers, with different costs, must sell their outputs at the same ruling market price. It follows that value is transferred within each branch of the economy from backward to efficient producers and the latter realise more profit than the former. The restless drive to invest in new technology derives from this surplus profit[7]. Thus there cannot ever be a single uniform rate of profit.[8] As soon as a new technology begins to dominate one branch of production, capital decamps for the latest advance in another.

The central concept in Marx's price dynamics is that investment is driven by this quest for superprofit. Since investment necessarily takes place over time, this means that price formation and accumulation are indissolubly linked in an intimate ebb and flow from which derive the drive to innovate, the tendency of the profit rate to fall, the business cycle, and the coexistence of extreme poverty and gross wealth which we today call underdevelopment. In an equilibrium framework the very idea of superprofit is either nonsensical or must be explained by recourse to arbitrary features such as monopoly. It is hardly surprising, therefore, that no equilibrium theory has given an even passable integrated account of these phenomena.

The formation of market values through abstraction — an example

We use the term 'abstraction' to refer to the formation of market values by averaging in circulation. Suppose, for example, inputs and outputs in a given time period, for two different circuits of capital, are

		Corn	Bread	L		Corn	Bread
P1	*used*	2 tons	0	1	*to produce*	0	1 tons
P2	*used*	2 tons	0	3	*to produce*	0	2 tons

Table 3 Quantities consumed and produced by two techniques

Suppose that initially $v_{corn} = 2$ and $v_{labour} = 1$. **P1**'s value contributions give

4) $$2v_{corn} + 1v_{labour} = 5 = 1v_{bread}$$

P2's value contributions give

5) $$2v_{corn} + 3v_{labour} = 7 = 2v_{bread}$$

So is the unit value of bread 5 or 3½? If we cannot distinguish one producer's output from another's, we cannot treat them as separate goods; we must study the total result of the two processes. Together these create 3 tons of bread whose total value is 12; so the new unit value of bread is 4 per ton.[9] The new unit value is thus the *average* value of the total social product.

Stocks of capital, fixed capital and abstraction

Marx argues that the market value of any commodity is the sum of the values added by all producers of that commodity, divided by the total amount of that commodity in circulation. What happens, however, to the pre-existing stocks from *previous* cycles of production? We argue that they enter the formation of market values. Suppose our bakers, in the example above, find that society already has 10 tons of bread valued at 24, left over from yesterday. Assuming society is willing to eat yesterday's bread, there will now be 12 tons of bread with a total value of 36; the new unit value of bread is therefore 3.

The difficulty with any other proposal is that we cannot draw an arbitrary line in time. It would be odd to deny that bread produced at 2am will meet up with bread produced at 3am in determining new values. So why not bread produced yesterday? Only if it has gone stale or rotten – that is, if use value is destroyed – will it cease to affect value formation.[10]

In short, consistent value accounting must bring together Marx's treatment of market values and his treatment of the *turnover of capital*. Marx did not perform this integration but it is perfectly logical and natural to do so, and treat all usable stocks as part of the pool of social values which enter the formation of market values and market prices of production. Indeed it is illogical and inconsistent not to. Moreover, with this extension Marx's value accounting framework falls into place and becomes internally consistent.

Stocks and flows: the necessary distinction between invested and turned over capital

Marx dealt with fixed capital through the concept of the turnover time of capital, which was central to the accounting practice of the time. If, for example, the period of production is considered to be a year and the capitalist lays out wages weekly, then wage-capital turns over fifty-two times during a period of production. If the production process requires the factory to lay up stocks of raw materials three months in advance, then this turns over four times in the same period. The capital tied up in stocks is correspondingly greater than the capital tied up in wages.

A more tractable though formally equivalent approach is to distinguish between advanced, or invested capital and capital turned over per unit time. In short, we recognise the existence of stocks of capital, which are constantly being used up and replenished by the processes of circulation.

We first consider, to illustrate this point, the same example as in our first section. Let us study, however, a period of production in which, say, one-half of the invested capital is used up. Outputs in this time period are exactly half what they were, namely 18 units of **C1** and 25 units of **C2**. We assume that the turnover time of capital is the same for all commodities and for variable capital (labour power). We can draw up a table of the capital invested, or stocks of productive capital, in each circuit (table 4).

	C1	C2	L
P1 *invested*	10	5	4
P2 *invested*	8	15	2

Table 4 Quantities invested

We can draw up a second table of quantities consumed and produced in a given time period, which is exactly half what it was before:

		C1	C2	L		C1	C2
P1	*used*	5	2.5	2	*to produce*	9	
P2	*used*	4	7.5	1	*to produce*		25

Table 5 Quantities consumed and produced

At the end of such a cycle productive stocks are reduced by the amount turned over. But now new stocks have been produced amounting to 9 units of **C1** and 25 units of **C2**. The total quantity of **C1** is now 10+8-5-4+9=18.

The value calculation is now slightly more complex. Assuming, as before, initial values of $v_1^{(0)}=1, v_2^{(0)}=2$, the value transferred or added to the output of **C1** is 5×1+2.5×2+2=12. The value 'conserved' as unused productive stocks is 5+4=9. The total value of the 18 units of output therefore comes to 12+9=21, and the new unit value $v_1^{(1)}$ is $^{21}/_{18}=^{7}/_{6}$. Similarly in the same period 25 units of **C2** are produced and 10 are conserved as unused productive stocks, making 35 in all; their total value is 4×1+7.5×2+1+20=40. The new unit value[11] $v_2^{(1)}$ is therefore $^{40}/_{35}=^{8}/_{7}$.

Note that since at this point we consider only simple reproduction we can assume that the entire surplus of **C2** is consumed either by capitalists or workers. The distribution of this surplus makes no difference to the value calculation, which is as it should be.
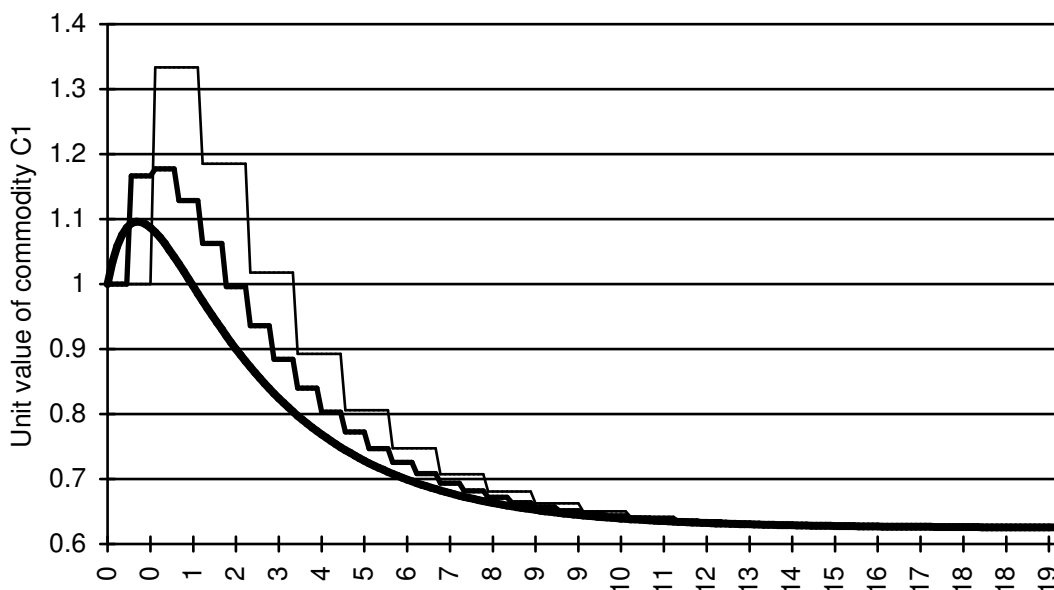
Fixed capital and circulation

The only question mark over this procedure is the role of fixed capital itself, which, it could be argued, has left the sphere of circulation and is irrevocably bound up with production. This criticism has been levelled at Sraffa's and von Neumann's treatment of fixed capital as a joint product.

It is true that capital once purchased is not in general re-exchanged. Nevertheless, a machine does not cease to be a commodity simply because it is not up for sale. Nor does it lose its price, which appears in company balance sheets as long as it exists physically, as assets, as part of the worth of the company. If the company is broken up and sold off, productive capital is liberated and re-enters circulation. Even productive capital, therefore, always exists potentially as circulating capital. In a slump, the falling average social value of productive capital imposes itself forcibly.

However a problem which the Sraffian treatment does not address is that different generations of fixed capital are produced using different technology. On the one hand this finally and irrevocably rules out the idea that all capital in existence is the product of a single, unified technology. In fact all capital in existence is the product of infinitely many technologies, each from different time periods. This in turn, however, raises another issue: it would appear on the face of it that our measure of value depends on the unit of time used for the period of production.

Fig 1: Successive approximations to time-varying values

If, for example, we consider a period of production lasting one month, we will produce a different value measure from the same calculation with a period of one year. A 'cycle of reproduction' is an arbitrary period of time, an

accounting fiction. In the general case where all inputs have different turnover times there is no 'natural' period of production during which all inputs are consumed.
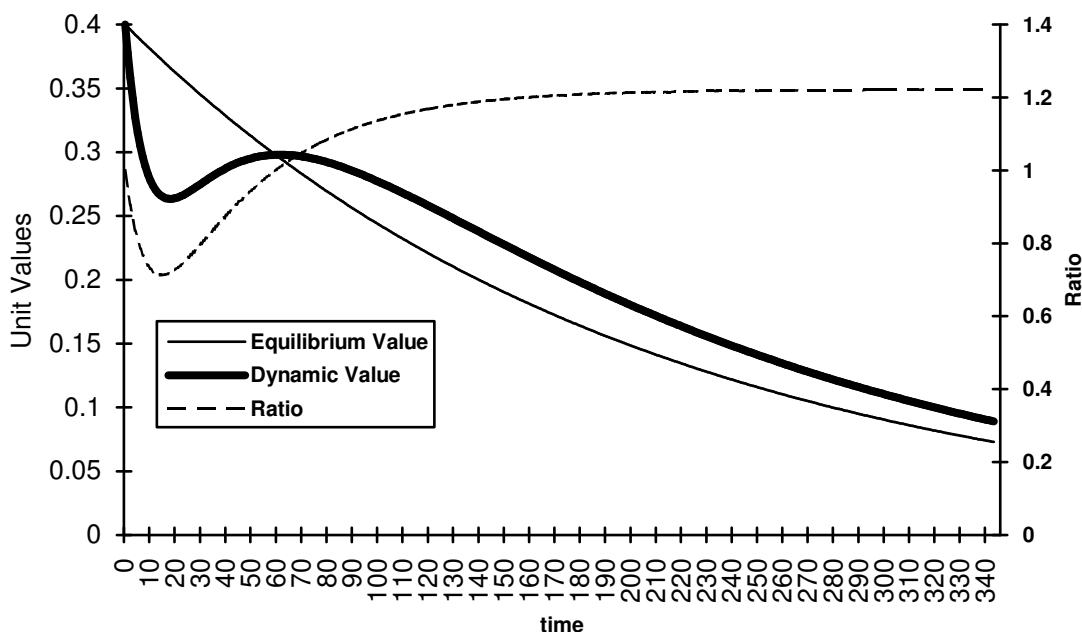
This leads us to propose a consistency condition which, it can be proved, the value calculation adheres to. As the period of production is reduced, the successive time-paths of calculated values should converge uniformly[12] to a path which, in the limit (for an infinitesimally small period of production), can be said to represent the actual movement of value, to which all the preceding calculations are an approximation.

Figure 1 shows the path followed by the unit value of **C1** as the period of production is successively reduced from 100% to 50% and then 10% of the turnover time of capital. to a smooth time-path which, we argue, represents the real movement of value.

Expanded reproduction: technical change, depreciation and surplus value

We are now in a position to assess the effects of technical change on the time sequence of labour values. Figure 2 shows what happens when, in our simple example above, we introduce expanded reproduction involving secular, labour-saving technical change. We introduce a small surplus of commodity **C1** and assume that the surplus of both **C1** and **C2** is invested at a steady rate of 1% per time period. However this investment is conducted in such as way that while the proportions of capital inputs and outputs remains constant, the proportion of labour employed grows less fast — in this example, at 0.5% per time period.

Dynamic and Equilibrium values

The result is a slow secular fall in unit values, as we would expect. However, it can be seen from figure 2 that the dynamic, that is, correctly-calculated value *ceases to converge on the equilibrium value*. As can be seen, in this case the ratio between the two converges on a steady 1.21. *The hypothetical equilibrium has simply ceased to provide a correct account of the real movement of values.*

It would now seem perfectly reasonable to enquire what the implications of this difference are for the rate of profit and for surplus value. But how should these two quantities be calculated in a dynamic framework?

What is profit?

In a static framework, profit is the difference between revenue and cost. But when prices vary, the value of stock also changes. If my company buys a computer for £3,000 in 1990, and in 1993 an equivalent costs £600, then its profits are reduced by £2,400, the difference between the historic and current price of new equipment. This is not to be confused with wear and tear, which is equal to the difference between the new and the second-hand price.

All companies correct their profit figures for the depreciation (or appreciation) of stock. However it is common to write down the value of fixed assets in some more or less arbitrary conventional fashion because their true value is notoriously difficult to estimate. Nevertheless this true value asserts itself if the company is liquidated and has to

realise its assets by sale, or catches fire and has to replace them. In a dynamic framework, therefore, we must account for changes in the value of all stock including fixed capital. Profit, therefore, is equal to the *difference in the worth of a capital before and after production*.

Consistency requirements, profit rate and surplus value

This definition is consistent in several important respects with our dynamic value calculation. First of all it is independent of the time interval. The profit generated in January, February and March clearly add up to the profit of the first quarter, that is, the difference between the worth of the company on January 1st and its worth on April 1st.

	C1	C2	L
P1 *invested*	10	5	4
P2 *invested*	8	15	2

Use-values invested in production (from Table 4)

But it is consistent in a second sense which illustrates the internal coherence of labour value calculations: the *change in value of the total social capital of any commodity over time is exactly equal to the total live labour embodied in it, less the unproductive consumption (by workers and capitalists) of that same commodity*. We illustrate this using the same example as above. How much profit is made during one half of a turnover period? Assume, as before, initial values (1, 2). Now, for the first time, we have to take into account the level of the wage. At the start of production the entrepreneur owns capital as specified in table 4.

The value of labour power

What is this company worth? This depends on the value of the commodity labour power, that is, the value of the labourer's consumption. This is not a technical condition of production, as Sraffian models tend to assume, but a socially-determined quantity. There is no *a priori* reason for any particular assumption such as a constant rate of exploitation or a constant real wage. Elsewhere we demonstrate the validity of the results in this paper under completely general circumstances.[13] In this example, where we want to illustrate the calculation, we shall assume a constant real wage equal to ¼ units of **C1** per hour.

What is the value of this real wage? It makes no difference when the wage is paid. What is important is that goods consumed in this period are produced in previous time periods. Consumed wage-goods should thus be accounted for in the values of time *t*, regardless of when wages are paid — just as the values of inputs to production are measured at time *t* regardless of when the bills are paid. The initial value of **C1** is 2; the real wage is ¼ and so the value of the wage, per hour of labour power, is 2×¼=½. The initial worth (value) of the company can now be calculated from table **4** and the initial values $v_1=1, v_2=2$

$$\textbf{P1 is worth } 10 \times 1 + 5 \times 2 + 4 \times \tfrac{1}{2} = 22$$

$$\textbf{P2 is worth } 8 \times 1 + 15 \times 2 + 2 \times \tfrac{1}{2} = 39$$

Total social capital is the sum of these, namely 61. Note that since the wage-goods (total value 3, total quantity 1½) which are about to be consumed must by now be in existence, having been preserved from the last period of production, the total quantity of commodity **C2** in existence is 5+15+1½=21½, and its total value is 43. The total quantity of **C1** in existence is 10+8 and its total value is 18×1=18.

When half the capital has turned over, the amount of each commodity in existence is the sum of unconsumed stock and newly-produced outputs. As regards **C1**, from table 5 it can be seen that 9 out of 18 units have been consumed, and 9 fresh units produced, so that the total remains at 18. As regards **C2** we must consider both the inputs to production and wage goods. In production 10 out of 20 units have been consumed and 25 produced, making 35 in all. As regards wage goods, $\tfrac{3}{4}$ units have been consumed by the workers leaving $\tfrac{3}{4}$ still in the possession of the capitalists.

What is the value of the 18 units of **C1** and the $35\tfrac{3}{4}$ units of **C2** now in the possession of the capitalists? The internal consistency of abstraction over time now becomes clear. We could use the calculation in the previous section, to derive new unit values of **C1** and **C2**. In fact $v_1^{(1)}=\tfrac{7}{6}$ as before, but $v_2^{(1)}$ is slightly different because now stocks of wage-goods enter the abstraction. The total value of **C2** is 41½ and so the new unit value is $41\tfrac{1}{2} \div 35\tfrac{3}{4} = \tfrac{166}{143}$. We would need this to calculate the individual profits of producers **P1** and **P2**. But as regards total social capital we don't need to recalculate the total value of **C1** as being $18 \times \tfrac{7}{6} = 21$, the total value of **C2** as being $35\tfrac{3}{4} \times \tfrac{166}{143} = 41\tfrac{1}{2}$ and the total combined value as 21+41½=62½: the very manner in which unit values were calculated guarantees that the total value of each commodity is equal to the sum of conserved, transferred and new labour contained in it, *less the value consumed by the labourers*. Therefore the total worth of the capitalists *must* be equal to this sum of conserved, transferred and new live labour, less the value consumed by the labourers. In short, this total worth must have grown by the difference between the value added by the labourers and the value consumed

by them – surplus value, that is, 3–1½ = 1½. This can be confirmed by subtracting the initial company worth, 61, from its new worth, 62½.

Value as an invariant quantity

This vital conclusion has two central consequences. First, it provides a mathematical justification for treating value as a 'substance' like water or energy, which is *conserved in reproduction*, that is, an *invariant quantity of economics*. It can be increased by labour, decreased by loss or use, and redistributed by circulation but no other economic process can alter it.

Second, although this is beyond the scope of this article, it explains very simply the much disputed differences between total price, total value, total surplus and total profit. It can be shown that Marx's two equalities both hold under very general conditions within a dynamic framework. (See Giussani [1992], Kliman and McGlone [1988], Freeman [1992]).
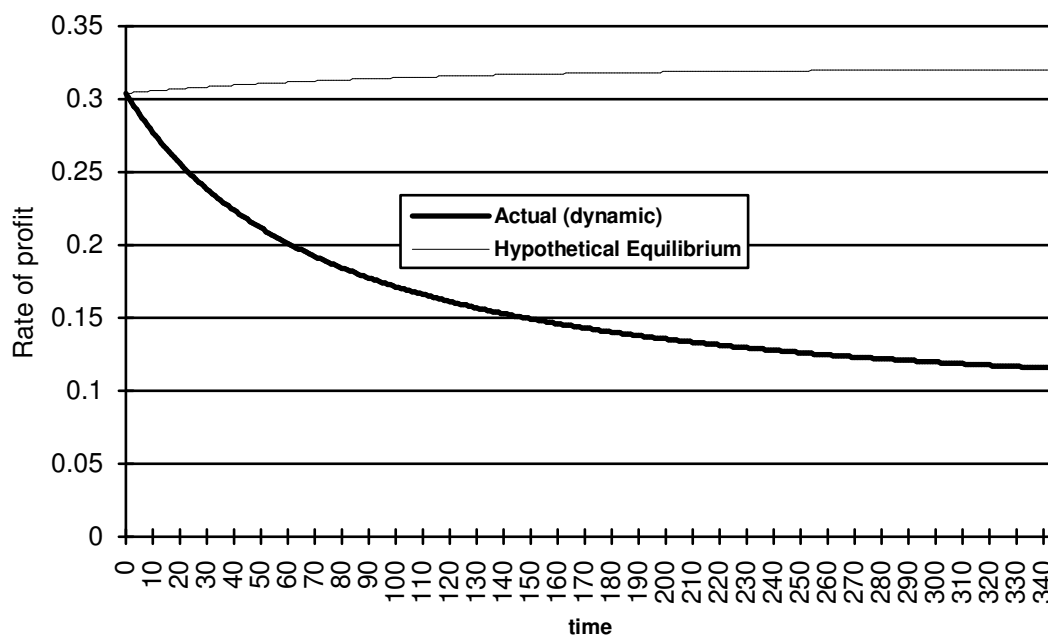
The falling rate of profit revisited

We can illustrate the power of this approach by revisiting one of the classic issues of value theory. Was Marx justified in assuming that the rate of profit would fall due to technical change which improves productivity? As yet we cannot calculate the price rate of profit because we have not yet shown how values are transformed into prices. We can, however, respond to one of the most important arguments levelled against Marx's view. This is the argument that improvements in productivity can lower the value of fixed capital. If so, the denominator in the expression for profit rate falls, the organic composition likewise falls and the profit rate rises.

Since dynamic values diverge quite strongly from the hypothetical equilibrium price, it is reasonable to ask whether the dynamically-calculated profit also diverges from the hypothetical equilibrium rate. The result of this calculation is displayed in figure 3. In this case we set the initial value to be equal to the hypothetical equilibrium so that there can be no question that the choice of initial value is responsible for the difference. We assume a real wage of 0.25 units of **C2**, and a steady growth of all inputs and outputs except labour, equal to 20% for each complete turnover of capital.

As can be seen there is a startling difference. The hypothetical equilibrium profit rate rises because inputs are continuously getting cheaper. *But the dynamically-calculated profit rate falls continuously*. There are three reasons: firstly the dynamically-calculated value of wage-goods is persistently higher as shown in Figure 1. Secondly the value of **C1** also falls more slowly than the predicted equilibrium. And thirdly the depreciation of fixed assets due to the steady fall in their value affects *current* profits, reducing the numerator as well as the denominator of the rate of profit, and this continues as long as the innovation continues.

It might seem that the total movement of these three factors is extremely complex. In fact it is extremely simple. The numerator in the rate of profit expression is always equal to the rate of surplus value; and the denominator is always equal to the initial value of stocks, plus the *total live labour which has been expended in society, less the consumption of the workers and less the consumption of the capitalists*. In short it is, exactly as described by Marx, the accumulated dead labour of society.[14]

Figure 3: Dynamic and Equilibrium Profit Rates

This calculation offers a far more realistic account of the actual movement of profits both during a business cycle and secularly over longer periods of time. As suggested by Marx, according to this calculation the effect of innovation is to raise the organic composition of capital by the accumulation of dead labour. This suggests that during periods of intensive accumulation the rate of profit will indeed tend to fall. The profit rate can only be restored to its equilibrium value if the investment and innovation *stops* so that capital can depreciate downwards towards its equilibrium value and profits can begin to move back up to their equilibrium rate. The effect of this is shown in figure 4, where we assume that new investment ceases halfway through the cycle, the surplus being consumed by the capitalists from then on.
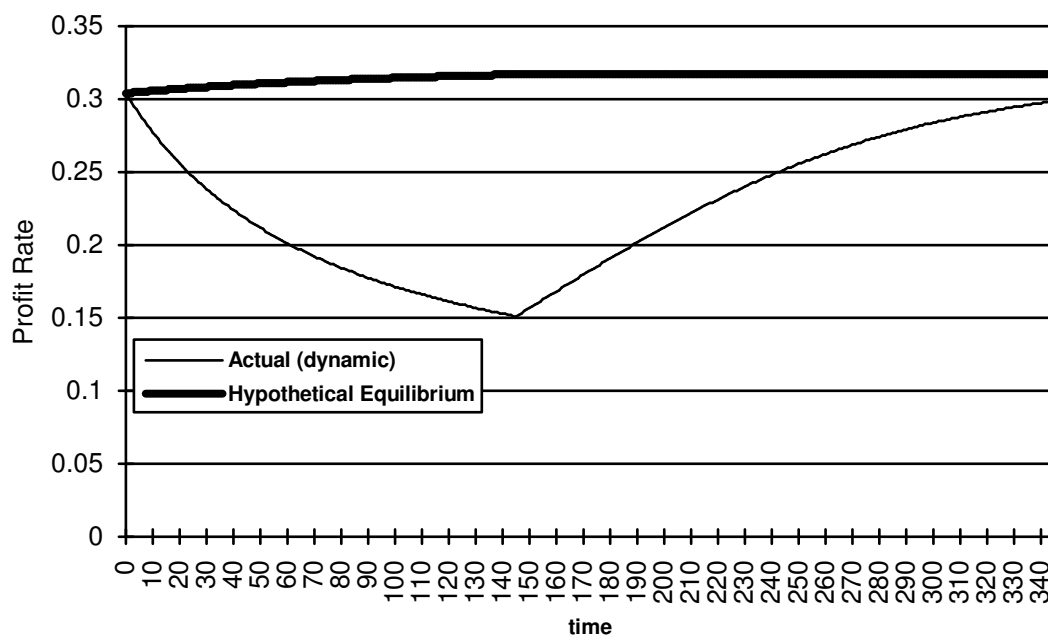
The mechanism of the recovery is this; new means of production are constantly being produced using a more productive average technology. The individual value of this new output is lower than the average of existing, invested, old capital and therefore steadily devalues this old capital through the averaging process. However, as long as investment continues, this is offset by the deployment of *more* fixed capital. It is only when the accumulation of new fixed capital is temporarily suspended or reduced, and indeed when old capital is actually written down to its new value through liquidations and bankruptcy, that profits can recover.

But this is exactly what happens in a slump, which is brought on by the declining rate of profit and whose function is the accomplish the devaluation of fixed capital to levels close to the socially necessary labour time implied by the new, more advanced, average technology.

To the extent that output itself declines, the recovery will of course be slower. Only if non-productive consumption can absorb the extra output will the slump have its full corrective effect. This offers a realistic and reasonable explanation both for the mechanism of the slump and the recovery itself, and for the role which Keynesian manipulation of aggregate demand can play in the recovery. It also demonstrates that such manipulation cannot overcome the underlying cause of the slump, which is the process of investment itself.

Figure 4: Dynamic and Equilibrium profit rates

Innovation stops halfway through

Value as an invariant quantity: the basis for a reappraisal of price theory

This article is limited to showing that dynamically-calculated values provide a superior foundation for economic theory both to static labour values and to neoclassical general equilibrium prices. Nevertheless it is worth establishing why a dynamic concept of value and above all of abstract labour remains central to a dynamic theory of price.

The classical vision consists in explaining and representing the movement of prices as the distribution and redistribution of an economic substance common to all commodities – abstract labour. This is not just a theological issue. All schools of economics possess, each in its own language, a concept of value independent of price. Without it they could not entertain or such ideas as price level, GDP deflator, or any distinction between real and nominal price. The fact that neoclassical growth theory appears to possess a circular concept of real value is at the heart of the famous capital controversy (see Harcourt [1972]).

If a group of commodities is considered to possess a quantity of value which is invariant with respect to changes in the price *level*, it is legitimate to ask how this fact is reflected when there are changes in *relative* price, that is, in the rate at which commodities exchange for each other.

It is almost trivial, but often overlooked, that if prices are measured relative to one particular commodity – a money commodity – then the total money price of this aggregate of commodities will not vary provided that the value contained in each individual commodity is considered to be redistributed through exchange. For example, suppose that there are in society 100 tons of corn valued at 200 units and 50 units of gold valued at 300 units, giving 500 units of value in all. Now suppose that in exchange 100 units are transferred from the corn to the gold due to an exchange favouring the gold producers. The 50 units of gold are now worth 400, so that value contained in one unit of gold is now 8 instead of 6. The value contained, however, in one unit of corn, is now 1. Suppose now that the value contained in a commodity regulates its exchange with every other commodity. One unit of gold will exchange for 8 units of corn. Now the total price of the commodities is given by the following calculation:

Gold	50 gold units	Each gold unit is worth 8 value units.	400 value units
Corn	100 units; each exchanges for 1/8 gold units giving 100/8 gold units in all	Each gold unit is worth 8 value units	$100/8 \times 8 = 100$ value units
Total	$50 + 100/8 = 62^1/_2$ gold units	Each gold unit is worth 8 value units	500 value units, as before

This is what gives rise to Marx's insistent assertion, often with italics for emphasis, that the total value of commodities *and money* in circulation is equal to its total price. The assertion must be true for any commodity measure of value, provided the money commodity measures value relative to the value contained in it, not relative to the absolute quantity of money.

Therefore on the one hand it demonstrates that private exchange can mediate a redistribution of value in society as a whole. It is reasonable and necessary to treat the market as transferring value from commodity to commodity and this cannot be done without understanding the role which money plays in this.[15] But the purpose of explaining the transformation of values into prices is to explain the transfer of value; to explain that the exchange of commodities, an essentially private act, is at one and the same time a redistribution of value, an essentially public phenomenon. Therefore, on the other hand, it is misleading to insist, as von Bortkiewicz tried to do, on the equality of total values and total prices expressed as a quantity of gold or indeed of a quantity of any physical commodity. The issue is the value which the gold represents, not the gold itself.

Marx's proposed transformation of values into prices begins from the social redistribution of value in exchange. This transformation is therefore valid whether or not profits are equalised and *must in fact remain valid for any set of relative prices and any set of profit rates.*

If at each point in time a definite quantity of value, albeit redistributed through the price mechanism, enters the process of production; and if through production a definite quantity of new value (socially necessary abstract labour time) is added, then the total quantity of value reaching the market in the *next* period is equal to the sum of these two quantities. The price mechanism may then redistribute the resulting value still further, resulting in a new set of prices *which, just like the values of the next period, is different from the prices of the previous period.*

The only truly general approach to price is therefore to consider an *arbitrary* redistribution of value and pose the question: if total value remains invariant under a general redistribution of value, are there any other quantities which also remain invariant? Marx's transformation establishes that there is such a quantity, namely profit. In a static framework this is an unworkable proposition, since a change in relative prices necessarily implies a change in the value of the wage and therefore the value of those commodities appropriated as profit. In a dynamic framework this contradiction vanishes. A change in relative prices in this period cannot possibly modify the value consumed in a previous period. In [Freeman 1993] we establish that, provided fixed capital is properly accounted for, including value transfers between different periods of production, the equality of total price and total profit is valid for any arbitrary redistribution of values.

This is the relevance of Marx's well-known assertion that supply and demand cannot explain the *average* rate of profit. They can explain deviations from it, but there is nothing, and can be nothing, in any theory of distribution based on supply and demand which can explain why the average should be 10%, 20% or 2000%.

We have already seen that, if a dynamic framework is adopted for the calculation of values, then value does indeed act as a conserved substance. By maintaining this framework for the analysis of price, we believe the classical vision can prove more accurate and more relevant in its portrayal of real economic phenomena than all that has followed.

MATHEMATICAL APPENDIX

This section presents some mathematical results based on a linear production model. This is a mathematical and not an economic model. It does not specify the properties of a value system but shows how they work and proves they are not internally contradictory.

Notation

We begin with a notation common for linear production systems.

$\mathbf{A}^i_j \geq 0$, the technical production matrix, represents the quantity of commodity i consumed by capital j, in unit time

$\mathbf{X}^i_j > 0$, the output matrix, represents the quantity of commodity i produced by capital j, in unit time.

$L_j > 0$, the labour use vector, represents the quantity of labour used by capital j in unit time

N stands for total labour inputs per unit time or $\Sigma^j L$

$\mathbf{K}^i_j \geq 0$, the fixed capital matrix, represents the quantity of capital which must be deployed in order to produce at output levels given by the matrix $\mathbf{X}$.

p_j, the price vector, represents the unit price of commodity j.

v_j, the value vector, represents the unit value of commodity j.

w^i, the wage vector, represents the quantity of commodity i which the wage purchases in unit time

The concept of an economic trajectory

We study the succession of states of the economy between two points in time. This is characterised fully by the set of all magnitudes at every point in time between the start and finish of this interval. As is normal in the physical sciences, we consider that we have a full and unambiguous description of the economy if we can express p, v, and if necessary $\mathbf{K, A, X}$ and L uniquely as functions of time. Such a set of functions, which may be given by a difference or differential equation, will be termed an *economic trajectory* over the interval under study.

To indicate the time when something is measured we shall use a superscript in brackets thus

$p^{(t)}$ represents unit prices at time t.

$\mathbf{Q}^{(t)}$ represents the output of goods between t and $t+1$

Where this is unambiguous, the time superscript will be omitted.

We consider only *continuous* trajectories, that is, where there are no sudden jumps in p, v or the technical coefficients.

Simplifying assumptions and further notation

For ease of exposition in this article we introduce a number of simplifications. Freeman [1992] contains a general linear model with less simplifying assumptions. Here we assume:

- ❑   No alternative technologies: $\mathbf{A, K}$, and $\mathbf{X}$ are square matrices
- ❑   No joint production: $\mathbf{X}$ is diagonal
- ❑   No consumer durables: all consumption of wage goods, and of luxury goods by capitalists, takes place in the period following that in which they are produced.
- ❑   No heterogeneous labour; labour coefficients represent hours of labour of average intensity and skill

Initially we assume $\mathbf{A, K}$ and $\mathbf{X}$ fixed; subsequently this assumption is dropped.

An iterative formula ignoring fixed capital

The values produced during period [t, $t+1$] equal the sum of values transferred from consumed inputs and added by direct labour:

$$v^{(t+1)} \mathbf{X} = v^{(t)}\mathbf{A} + L \qquad\qquad (6)$$

i.e. $\qquad\qquad\qquad\qquad\qquad v^{(t+1)} = v^{(t)}\mathbf{a} + 1$

Where $\mathbf{a} = \mathbf{AX}^{-1}$ is the normal technical coefficient matrix. This is a difference equation with the solution

$$v^{(t)} = v^{(0)}\mathbf{a}^t + l(I + \mathbf{a} + \mathbf{a}^2 + ... + \mathbf{a}^{t-1})$$

This is valid whatever $\mathbf{a}$ is. It is positive provided provided the initial value $v^{(0)}$ is strictly positive and $\mathbf{a}$ is non-negative, which is true by convention since a negative input is in fact an output. If the Leontieff inverse $(I - \mathbf{a})^{-1}$ exists[16] it can be written more simply as

$$= v^{(0)}\mathbf{a}^t + l(I - \mathbf{a}^t)(I - \mathbf{a})^{-1}$$

It can be shown (Freeman[1992]) that if there is a static equilibrium (simultaneous equation) solution then the dynamic solution converges to it[17], namely

$$v = l(I-\mathbf{a})^{-1}$$

However this solution is far more general than the convergent case. It is valid and positive whether or not there is a net surplus, including when **a** varies over time, as generally happens. In fact:

1. For all economic trajectories, dynamic values exist wherever the economy is defined;

2. Whenever the equilibrium value exists, dynamic values will converge on it, no matter what its initial starting point;

3. When the fixed-point value does not exist dynamically-determined values exist but do not converge. The economic interpretation of these circumstances is that either there is negative net output of some commodity or zero net output of a non-basic commodity.

4. Given positive initial values, values can never become negative, whether or not there is a physical surplus of anything.

The rate of surplus value

According to Marx, the surplus value generated in unit time is the total time worked less the value consumed by workers. We do not propose to modify this definition here but it is instructive to derive it in a slightly different manner, essential when dealing with fixed capital.

At the beginning of a period of production, the aggregate value in the economy is the value of all the inputs to production, plus the value of consumer goods generated in the last period.

This is equal to $v^{(t)}\mathbf{X}$, the value output of the last production period. We can treat this as if distributed at the start of period [t, $t+1$] between workers and capitalists. One portion, $v^{(t)}wL$, is consumed as the wage, advanced at the start of the current period. A second portion, consumed by the capitalists, is the profit generated in the previous period. The remainder, $v^{(t)}\mathbf{A}$, serves as inputs to production.[18] The capital advanced, precisely speaking, is therefore

$$v^{(t)}\mathbf{A} + v^{(t)}wL$$

or $\qquad\qquad\qquad C + V \qquad\qquad\qquad$ in Marx's terminology

A portion of profit is also accumulated but this case is not considered here.

After production the capitalists own a new quantity of value $v^{(t+1)}\mathbf{X}$ which differs not only from $v^{(t)}\mathbf{A}$ but also from $v^{(t)}\mathbf{X}$, because of changes in both v and **X**. The value they gain is therefore the difference between the value had when they started and when they finished, namely

$$v^{(t+1)}\mathbf{X} - v^{(t)}\mathbf{A} - v^{(t)}wL \qquad\qquad (7)$$

or $\qquad\qquad\qquad C' - C - V \qquad\qquad\qquad$ in Marx's terminology

This is Marx's S, the mass of surplus value generated in the period of production under study. This expression can be simplified using equation (6). Substituting in (7) gives

$$S = v^{(t)}\mathbf{A} + L - v^{(t)}\mathbf{A} - v^{(t)}wL$$

which reduces to $\qquad\qquad S = L - v^{(t)}wL$

or $\qquad\qquad\qquad L - V \qquad\qquad\qquad$ in Marx's terminology

the conventional expression for the rate of surplus value. This shows that the dynamic reformulation yields the conventional, expected results.

We now show that this generalises to fixed capital. This, along with the convergence results, justifies the claim that static systems are a special limiting case of the more general dynamic systems.

Fixed Capital

Fixed capital is an essential component of price and value analysis and cannot be dispensed with. Results which have not been extended to include it cannot be considered general. There are decisive reasons for this aside from the fact that in real life, it exists.

Though most treatments of value theory do not question the length of the period of production, this is not justified. There is no 'natural' period; it is an arbitrary, accounting convention to adopt a period lasting a day, week, month, or year. Moreover to study the continuous case we must reduce the production period to an infinitesimal one and *all* inputs to production then persist as fixed capital to one degree or another, so that the issue cannot be avoided.

A continuous formulation dispenses with the arbitrary simplification that periods of production and distribution alternate. In real life production, consumption and distribution take place contemporaneously and capital exists in parallel in all three forms of work-in-progress, output awaiting sale and purchased inputs entering production or private consumption.

However the distinction between fixed and circulating capital based on a period of production of one year is common accounting practice and is of fundamental use in the study of the business cycle. We shall therefore maintain this convention and introduce a new term, *conserved* capital, to refer to stocks of anything not fully used up in the period of production under study.

First consider *conserved circulating capital*. How should the value equation be modified if there is an unused portion $(\mathbf{K}_i - \mathbf{A}_i)$ of the inputs of i to all production processes that consume i? At the end of a period of production the total social stock of i is to be found in two places:

1   Producer i will own $\mathbf{X}^i_j$ of this commodity which has just been produced
2   Every producer who consumes i will own $(\mathbf{K}^i - \mathbf{A}^i)$ as work-in-progress.

What is the connection between the value, at time $t+1$, of these two components? The simplest, most rational, and consistent solution is that the unit value of every aliquot part of commodity i is the same, that is, $v^{(t+1)}$.

Digression: more notation

$(\mathbf{K}^i - \mathbf{A}^i)$ represents the amount of i conserved in each separate circuit. We are interested, however, in the amount of i conserved in society as a whole. This is the *row sum* of $(\mathbf{K}^i - \mathbf{A}^i)$. It is very convenient to form a diagonal matrix by putting each row sum of $(\mathbf{K}^i - \mathbf{A}^i)$ in the corresponding diagonal element, that is, by adding across rows and diagonalising the result. We use the notation

$$\mathbf{<K>}$$

to represent this operation, in this case as applied to the matrix $\mathbf{K}$.

The modified value difference equation

At time $t+1$, the total stock of each commodity in society is thus $\mathbf{<K–A+X>}$, the sum of conserved and produced commodities. The total value of each commodity in society is therefore given by

$$v^{(t+1)}\mathbf{<K - A + X>}$$

But this is the sum of values from three sources: value conserved from the previous period, value transferred from the inputs to production, and value added by live labour. This comes to

$$v^{(t)}\mathbf{<K - A >} + v^{(t)}\mathbf{A} + L$$

These two amounts must be equal, giving the more general difference equation

$$v^{(t+1)}\mathbf{<K - A + X>} = v^{(t)}\mathbf{<K - A>} + v^{(t)}\mathbf{A} + L \tag{8}$$

Note that when there is no fixed capital, that is when all stocks are used in the current period, then $\mathbf{K} = \mathbf{A}$ and this difference equation reduces to

$$v^{(t+1)}\mathbf{<X>} = v^{(t)}\mathbf{A} + L$$

which is the same as we derived earlier since $\mathbf{X}$ is already diagonal so $\mathbf{<X> = X}$

Passage to the continuous case

Consider an interval $[t, t+\Delta t]$ in which v increases by Δv. $\mathbf{A}$, $\mathbf{X}$ and L are now treated as *rates* of consumption and production, rather than absolute amounts. Therefore in time Δt, $\mathbf{A}\Delta t$ is used up, $\mathbf{X}\Delta t$ is produced, and $L\Delta t$ live abstract labour is added to the product. Equating values at the beginning and end of production gives:

$$(v+\Delta v)\mathbf{<K - A}\Delta t + \mathbf{X}\Delta t\mathbf{>} = v\mathbf{<K - A}\Delta t\mathbf{>} + (v\mathbf{A} + L)\Delta t$$

which can be simplified to

$$v \mathbf{<X- A>}\Delta t + \Delta v\mathbf{<K>} + o(2) = v\mathbf{<A>}\Delta t + (v\mathbf{A} + L)\Delta t$$

where $o(2)$ represents terms of order 2. Omitting these yields,

$$v\mathbf{X}\Delta t + \Delta v\mathbf{<K>} = (v\mathbf{A} + L)\Delta t$$
$$\Delta v\mathbf{<K>} = (v(\mathbf{A} - \mathbf{X}) + L)\Delta t$$
$$\Delta v/\Delta t\mathbf{<K>} = v(\mathbf{A} - \mathbf{X}) + L$$

and hence in the limit $\qquad v'\mathbf{<K>} + v\mathbf{X} = v\mathbf{A} + L.$

This introduces an entirely new term which does not appear in static calculations, namely $v'\mathbf{<K>}$, which we term the *stock revaluation factor*. This expresses the depreciation or appreciation of conserved capital as a result of changes in values. The left hand side of this equation therefore represents the *rate at which new value enters the economy*, equivalent to Marx's C' in the reproduction schemas. Since $v\mathbf{A}$ is Marx's C, we therefore have the classic identity

$$C' = C + L$$

Surplus value

The *mass of surplus value generated* in period [t, $t+\Delta t$] is the difference between the worth, in value terms, of the company at time t and time $t+1$. The worth at time t is $v^{(t)}(\mathbf{K} + w\mathbf{L})$. After production at time $t+1$ the capitalists own use-values amounting to $\mathbf{K} + (\mathbf{X} - \mathbf{A})\Delta t$. Their value is therefore

$$v^{(t+1)}(\mathbf{K} + (\mathbf{X} - \mathbf{A})\Delta t) = (v + \Delta v)(\mathbf{K} + (\mathbf{X} - \mathbf{A})\Delta t)$$
$$= v\mathbf{K} + v(\mathbf{X} - \mathbf{A})\Delta t + \Delta v.\mathbf{K} + o(2)$$

The change in value of stocks, ignoring terms of order 2, is thus

$$v(\mathbf{X} - \mathbf{A} - w\mathbf{L})\Delta t + \Delta v.\mathbf{K}$$

The *rate of surplus value generation per unit time,* given by dividing by Δt is thus, in the limit

$$S = v(\mathbf{X} - \mathbf{A} - w\mathbf{L}) + v'.\mathbf{K}$$

This likewise involves a stock revaluation term absent from static calculations, namely $v'.\mathbf{K}$. This, however, is not in all cases equal to $v'.\langle\mathbf{K}\rangle$ since $\mathbf{K}$, which expresses the distribution of each commodity among different producers, is not equal to $\langle\mathbf{K}\rangle$ which expresses the total of each commodity in society regardless of who owns it. In a full treatment of fixed capital we therefore find that value may be redistributed between capitalists as a result of stock value changes, even before we consider the effect of price variations. However in aggregate the sum of $v'\mathbf{K}$ is equal to the sum of $v'.\langle\mathbf{K}\rangle$ and so

$$\Sigma S = \Sigma v(\mathbf{X} - \mathbf{A} - w\mathbf{L}) + \Sigma v'.\mathbf{K}$$
$$= \Sigma v(\mathbf{X} - \mathbf{A} - w\mathbf{L}) = \Sigma v'\langle\mathbf{K}\rangle$$

Now the differential equation for prices already furnishes an expression for $v'\langle\mathbf{K}\rangle$, namely

$$v'\langle\mathbf{K}\rangle = v(\mathbf{A} - \mathbf{X}) + \mathbf{L}$$

Summing this gives

$$\Sigma v'\langle\mathbf{K}\rangle = \Sigma v(\mathbf{A} - \mathbf{X}) + N$$

giving

$$\Sigma S = N - vwN$$

that is, in aggregate

$$S = L - V$$

where now S, L and V refer to social aggregates rather than sectoral magnitudes. Total surplus value created per unit time is therefore total direct labour expended, less the total wage. Thus in the continuous case, the basic Marxian formulae remain valid in aggregate.

The value profit rate and the effect of accumulation

Our calculation also bears on the rate of profit through its effect on the mass of invested capital, that is, the denominator in the profit rate expression. What happens if we drop the simplification that $\mathbf{K}$ is fixed, that is, if we study accumulation? The value of invested capital is $\Sigma v\langle\mathbf{K}\rangle$. Differentiating gives the rate at which this total mass of accumulated value grows:

$$\frac{d}{dt}\Sigma v\langle\mathbf{K}\rangle = \Sigma v'\langle\mathbf{K}\rangle + v\langle\mathbf{K}'\rangle$$
$$= \Sigma\,[v(\mathbf{A} - \mathbf{X}) + \mathbf{L} + v\langle\mathbf{K}'\rangle] \tag{9}$$

Now $\mathbf{K}'$ is the rate at which use-values accumulate. These come from the surplus $\mathbf{X}$–$\mathbf{A}$, along with wages and capitalist consumption. It is instructive to consider *maximum expanded reproduction* in which the whole of $\mathbf{X}$–$\mathbf{A}$ is either consumed as the wage or invested. In this case

$$\mathbf{K}' = \mathbf{X} - \mathbf{A} - w\mathbf{L}$$

and equation (9) for the value rate of accumulation becomes

$$\Sigma[v(\mathbf{A} - \mathbf{X}) + \mathbf{L} + v(\mathbf{X} - \mathbf{A} - w\mathbf{L})]$$
$$= N(1 - vw)$$

That is, invested stock grows at *exactly* the rate of surplus value, regardless of changes to physical productivity. This, finally, leads to a mathematically exact formula for the tendency of the rate of profit to fall. If we use S to stand for aggregate surplus value and F for $\Sigma v\mathbf{K}$, then we have just established that F' = S. Now r, the value rate of profit, is given by S/F. Differentiating gives

$$r' = \frac{FS' - SF'}{F^2} = \frac{S}{F}\left(\frac{S'}{S} - \frac{F'}{F}\right) = r\left(\frac{d}{dt}(\log S) - r\right)$$

The profit rate will therefore fall as long as S'/S fails to rise at a rate greater than the profit rate; but since S can in no circumstances exceed N, r must eventually fall.

These relations are exact. They involve no heterogeneous quantities, no leets, putty or clay. The quantities are not estimated or assumed but can be measured with a stopwatch, a weighing machine, a tape measure and patience, using no approximations other than the usual limitations of measurement. They have no arbitrary constants requiring complex econometric investigation, no accelerators, multipliers or fundamental psychological laws.

They show that we can indeed treat value as substance as the classical vision suggests. This leads naturally to a treatment of accumulation and technical change as movements of this substance from one part of the economy to another, and invites a business cycle model based on this movement.

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[1]Many writers of the Post-Keynesian school, although not working in a labour value framework, have been building dynamic formalisations of price theory which effectively renounce a general equilibrium framework. See the comprehensive survey by Lee [1993]. Though (in our view) this replaces equilibrium theory with an eclectic and nonrigorous foundation, it must be said that Marxists, most of whom are still working in an equilibrium framework, are in a poor position to complain.

[2]'Hence from knowledge of the physical conditions of production and the real wage, one can determine values, the value of labour power and surplus value' Steedman[1977], p40

[3]Indeed, within an economy without time even circulation is impossible. Since the owner of a capital has to dispose of the goods which compose it on the market, acquire a new set and then manufacture yet a new set, a time dimension is essential to demarcate the succession of different forms which this capital takes.

[4]'...all equalisations are *accidental*, and although the proportionate use of capitals in the various spheres is equalised by a continuous process, nevertheless the continuity of this process itself equally presupposes the constant disproportion, which it has continuously, often violently, to even out.' (Marx, 1968a, p368)

[5]Strictly speaking, the iteration will converge if the coefficient matrix is positive and irreducible. In economic terms, it means the net production of every commodity must be non-negative, and every group of non-basic commodities must produce a strictly positive net surplus. See the mathematical appendix.

[6]'The economists say that the *average price* of commodities is equal to the cost of production; that this is a *law*. The anarchical movement in which rise is compensated by fall and fall by rise, is regarded by them as chance. With just as much right one could regard the fluctuations as the law and the determination by the cost of production as chance.' (Marx [1955] p87)

[7]"Whether or not [average profit is obtained] and whether it is higher or lower than the profit corresponding *to the market price* – that is, corresponding to the direct result of the [production] process – determines the reproduction process, or rather the scale of reproduction; it determines whether more or less of the capital existing in this or that sphere of production is withdrawn or invested; it also determines the ratio in which newly accumulated capital flows into these particular spheres, and finally, to what extent these particular spheres act as buyers in the money market." – Marx [1968c], p513 (emphasis and addenda in the original)

[8]One of the several elisions offered by Marx's erudite followers is replacing the term 'general rate of profit' which Marx uses throughout Volume III with the equilibrium concept of a *uniform* rate of profit (see Albarracín[1984]), which Marx neither proposed nor subscribed to.

[9]Marx (1968b; 1977b)

[10]If large items of fixed capital are embedded in production with no secondary market this could modify the analysis. The old machine is the a different commodity from the new. But in this case we have to be consistent. The equivalent replacements have to be regarded as a new stock of machines with their own, independent valuation.

[11]In this section, to simplify the explanation, we leave wage-goods destined for consumption by workers out of the calculation. Total stocks of **C2** are in fact greater than 35 because extra amounts of **C2** are still on hand awaiting purchase by labourers. This is corrected in the next section.

[12]For a formal definition of uniform convergence see, for example, Philips [1960], p74

[13] Freeman, A [1993.

[14]'But the capital which is not used up (machinery, etc) retains its value (for the fact that it is not used up means precisely that its value has not been used up); it retains the same value after the conclusion of the production process as it had before this process started.' Marx[1968c] p372

[15]and indeed to exchangeable objects that do not possess intrinsic value such as paper money – Marx's famous 'objects with a price and no value'. All that must be defended is that the process of exchange cannot create new social value.

[16]That is, if 1 is not an eigenvalue of $\mathbf{a}$.

[17]The conditions for convergence are that the maximal eigenvalue of the matrix $\mathbf{a}$ be less than 1, which leads the term $\mathbf{a}^t$ to become vanishingly small as t becomes large. This is guaranteed if $\mathbf{a}$ is non-negative, viable (no negative net production), if there is a physical surplus of at least one commodity, and if $\mathbf{a}$ is irreducible. Economically this last condition means that there is a positive net output of every non-basic commodity.

[18]It has been suggested that the results of this approach may depend on an arbitrary assumption, that wages are paid at the end of the period instead of the beginning. But the assumption is not arbitrary, and the results do not depend on it. Values as such are independent of the wage and unaffected by when it is paid. The issue is therefore the magnitude of surplus value. Whether the employers pay the wage now or tomorrow it is unreasonable to assume that consumer goods do not exist at the start of the current cycle, or the labourers will die and fail to produce the output. The wage is therefore valued in terms of $v^{(t)}$. The substantive issue is not this magnitude but when it is paid. If paid at the end of the period, then when the employers begin production they are worth $v^{(t)}\mathbf{A}$ and when they finish, having paid the wage, they are worth $v^{(t+1)}\mathbf{Q} – v^{(t)}wL$ The surplus, being the difference between the two, is just the same. Even if the wage is paid at values $v^{(t+1)}$ it makes no qualititive difference and the small difference vanishes in the passage to the continuous case. The only substantive difference is that the capital advanced is smaller, being C instead of C+V, which will raise the rate of profit slightly but cannot offset its tendency to fall.