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Abstract

A model is presented to characterise the (optimal) demand for cash balances in deregulated markets. After the model of James Tobin, 1958, net balances are determined in order to maximise the expected return of a certain portfolio combining risk and capital. Unlike the model of Tobin, however, the price of the underlying exposures are established in actuarial terms. Within this setting, the monetary equilibrium determines the rate at which a unit of capital is exchange by a unit of exposure to risk, or equivalently, it determines the market price of risk. In a Gaussian setting, such a price is expressed as a mean-to-volatility ratio and can then be regarded as an alternative measure to the Sharpe ratio. The effects of credit and monetary flows on money and security markets can be precisely described on these grounds. An alternative framework for the analysis of monetary policy is thus provided.

Key words: Liquidity-preference; Money demand; Monetary equilibrium; Market price of risk; Sharpe ratio.

JEL-Classification: E41, E44, E52, G11.

1 Introduction

The primary role of money is to allow the exchange of goods and services in the economy. The *transactions motive* for holding money is usually justified on these grounds, which claims that the demand for money is in proportion to the volume of transactions, which in turn is considered as proportional to the level of income. Individuals that hold portfolios containing assets and liabilities with different maturities are obliged to maintain some stock of cash in order to fulfil their outstanding balances. They are accordingly said to demand money for *precautionary* motives. Finally, the presence of *unknown* capital *profits and losses* ($P\&L$) in the balance sheets of the pursuers of investment projects producing *random* outcomes causes them to additionally demand cash balances for *speculative* motives.

A a matter of fact, individuals that expect to obtain capital *profits* prefer to buy securities instead of keeping cash provisions, for in this way they assure to themselves a *sure* gain. Plenty of cheap credit is likely to be found in markets where such a mood predominates. By contrast, credit is likely to become scarce and expensive in markets where most of the public believe their assets will produce capital *losses* in the near future — people prefer to reduce the exposition to risk in their portfolios and to raise their stocks of reserves in this case, as a means of protection against unexpected shortfalls and bankruptcy.

Within this context, the well-known Keynes's *liquidity preference* proposition is enunciated, according to which the demand for cash balances is positively affected by the level of income and negatively affected by the return offered by a certain class of *money substitutes* (see Keynes, 1937a and 1937b, and also Howells and Bain, 2005).

Two important models at the core of economic theory are connected to the liquidity-preference proposition.

In the first place, the *Capital Asset Pricing Model (CAPM)*, originally and independently developed by William Sharpe (1964, 1966) and John Lintner (1965), establishes the *price* at which some *risky* asset must be exchanged under conditions of *equilibrium*. The derivation of the model depends on the assumption that the expected return and the volatility of every *efficient* portfolio combining risk and cash must be related to each other according to a linear schedule. The collection of such portfolios is known as the *capital market line*. The *optimal* combination of risk and cash, which is determined at the tangency point of intersection between the capital market line and the curve representing the preferences of the decision-maker, ultimately determines the *preference for liquidity*.[1]

A more explicit role is played by the liquidity-preference function in macroeconomic analysis. Recall that the *monetary equilibrium* of the economy is determined in such a way that the total demand for cash holdings is equal to the total *stock of money* supplied by the central bank. Within this context, the liquidity preference function, which explicitly measures the proportion of nominal income that is spent on cash holdings, corresponds to a *property* of the economy that determines the extent to which monetary interventions affect economic and financial conditions — as described by the level of prices P, the real output y and the interest rate r.

An *alternative* theoretical setting will be proposed in this paper for the characterisation of the preference for liquidity of the economy. The main departures from the *classical* setting is that in the alternative model national income is regarded as a random variable and people are supposed to face restrictions when looking for funding in financial markets.

The alternative model is based on the approach of James Tobin, 1958. The money demand is accordingly corresponded to the maximisation of the expected value of some portfolio that contains cash holdings and a mutual fund delivering random payments. The main bibliographical references supporting the model are, on the one hand, Tobin, 1958, and Sharpe, 1964, for the characterisation of liquidity-preference and the *CAPM*,

[1] The liquidity-preference function is first derived in this way by Tobin, 1958. See specially Sharpe, 1964, and also *Section* 4 in this paper.

and Keynes, 1937a, and Friedman, 1970, for the description of the monetary equilibrium and the monetary mechanism (also Tobin, 1947, can be considered in this respect).

2 Liquidity-Preference in the Monetary Equilibrium

The liquidity-preference proposition is commonly represented by the following functional expression (see e.g. *Equation* (6) in Friedman, 1970):

$$L(r) \;=\; Y \cdot \lambda(r) \;=\; P\, y \cdot \lambda(r) \quad \textit{with} \quad \frac{d\lambda(r)}{dr} < 0 \tag{1}$$

where $L(r)$ represents the aggregate cash balance demanded by the economy and the *liquidity-preference* function $\lambda(r)$ expresses the ratio between demanded cash balances and nominal income. The inverse ratio $v(r) = 1/\lambda(r)$ is known as the *velocity of money.* The level of prices P establishes the connection between *real* and *nominal* incomes, respectively denoted as y and Y, with $Y = P \cdot y$. Recall that *nominal* magnitudes represent flows expressed in monetary units, while *real* quantities are expressed in terms of the goods and services that money can purchase (see e.g. Romer, 1996, and Blanchard, 2005).

The aggregate *money supply,* on the other hand, refers to the total amount or *stock* of money held by the public in the economy. It is traditionally related to a class of *narrow* money denoted as $M1$, which mostly contains currency held by non-banking institutions and householders. Other monetary aggregates have been proposed as well, such as $M2$, which includes small-denomination time deposits and retail mutual funds, and $M3$, which adds mutual funds, repurchase agreements and large-denomination time deposits (see Edwards and Sinzdak, 1997, and also Howells and Bain, 2005).

Letting M denote the total stock of money supplied by the monetary authority, we obtain from *Equation* 1 that at equilibrium the following equation must necessarily hold:

$$M \;=\; Y \cdot \lambda(r) \;=\; P\, y \cdot \lambda(r) \quad \textit{with} \quad \frac{d\lambda(r)}{dr} < 0 \tag{2}$$

Within this context, any change in the nominal quantity of money M induces a variation in any of the variables determining the money demand, P, y or r, in order to reestablish the monetary equilibrium. Since the level of real income y is expressed in terms of goods and services, it is normally assumed to depend on economic fundamentals and hence, it is normally regarded as a *stable* variable in the short-run. Short-term fluctuations are then expected to mostly affect the level of prices and interest rates.

On these grounds, if the level of prices and real output were pegged to some determined paths of variation (respectively corresponded to some determined rates of *inflation* and *growth*), the monetary authority would be able to provide, in principle, the amount of money that is *consistent* (in the sense that *Equation* 2 is satisfied) with some *target* level of the interest rate. The efficacy of this mechanism depends, however, on how much of the response of the economy is performed through adjustments in the level of prices P, and how much is performed by modifying the demand for balances.

Indeed, assuming that the demand for money is *perfectly elastic*, i.e. assuming that $|\lambda(r)| \to \infty$, implies that the amount of money can vary while both the levels of nominal income and interest rates remain unchanged. Under such circumstances, expansions and contractions of the money supply must be respectively followed by increments and reductions of the same magnitude in the stock of cash, in such a way that the monetary mechanism proves to be *useless* for dealing with short-run fluctuations. The preference for liquidity is said to be *absolute* in this situation.[2]

By contrast, if liquidity-preference is *non-absolute*, i.e. if $|\lambda(r)| < \infty$, every change in the money stock affects (at least partially) the level of nominal income — in such a way that every monetary expansion and every monetary contraction respectively stimulates and contracts the level of nominal output in the short-run.[3]

3 Why should We Stop Relying on Linear Specifications of the Money Demand?

The main difficulty faced by monetary authorities when applying the monetary mechanism in practice is the lack of a well established functional expression characterising the preference for liquidity of the economy.

In this respect, a large majority of scholars and central bankers assume the demand for cash balances varies *constantly* with respect to the interest rate. Accordingly, *log-log* and *semi-log* functions are normally used in empirical investigations of the money demand — such that $\lambda(r) = A \cdot r^{-\eta}$ and $\lambda(r) = B \cdot e^{-\epsilon \cdot r}$ respectively, where A and B are constants.[4]

Although these specifications lead to a satisfactory description of the money demand for the most of the recorded paths of monetary aggregates and interest rates, there are times when their predictions have failed to anticipate the actual liquidity needs of the economy. Multiple revisions of the model have intended to explain these results, but there is still no agreement on the subject, and there is still no alternative theoretical setting that can simultaneously incorporate all the scenarios observed during the last

[2]Keynes (1936, 1937a) and his disciples claim that *firmly* convinced investors will necessarily absorb any increment or reduction of the stock of money without changing their perceptions about the level of interest rates. Thus if individuals share expectations about the level of the interest rate, variations in the amount of money must be totally transmitted to the demand for balances, or in other words, the aggregate money demand must be *perfectly elastic* (see also Tobin, 1947).

[3]Any monetary expansion then leads to a new equilibrium involving higher prices for the same quantity, the higher this response the more inelastic the money demand. In short-terms, production is encouraged until prices are reestablished to their original levels. In the long-run, new producers enter the market and existing plants are expanded. Throughout the process, it may take time for output to adjust, but no time for prices to do so. See Friedman, 1968, 1970.

[4]Such functional expressions can be justified on the grounds of a model of general equilibrium, where people allocate their funds to cash holdings and consumption. The money demand is derived in this framework by maximising the utility of a representative agent. See Lucas, 2000, and Holmstrom and Tirole, 2000.

forty years.[5]

In the theoretical setting that will be soon presented, it is assumed that the level of income Y is a *random* variable and hence, that individuals do not know with certainty the level that this variable will take in the future. However, they can observe the series of percentage income returns and estimate its parameters with respect to some class of *probability distributions*.

It is possible to prove, within this framework, that an *optimal liquidity principle* exists, which explicitly depends on the *riskiness* of national income (see *Sections* 6 and 8). This implies, in the first place, that the stock of money determined by the central bank is not corresponded to a unique level of the interest rate (as stated in *Section* 9), and in the second place, that the liquidity-preference is not necessarily a *linear* function of the interest rate (as shown in *Section* 10). The consequences of these results to macroeconomic analysis are presented in *Section* 12.

First in *Section* 4, the model of liquidity-preference of James Tobin will be presented, which is taken as a reference (and a comparison basis) for the construction of the optimal liquidity principle in *Section* 6. The basic idea is that the preference for liquidity is determined by an optimal combination of a certain *risky* fund and some *non-risky* security.

4 Preference for Liquidity as Behaviour towards Risk and the Capital Market Line

The theory of liquidity-preference, as stated by Tobin, 1958, is exclusively concerned with the problem of building efficient portfolios combining two different kind of financial products: some *risky* aggregate exposure (delivering some *random* payment at the maturity date) and a certain *non-risky* security (which provides some cash flow that is known with *certainty* at any moment before the instrument expires).

Non-risky securities are related to a class of *monetary assets*, with no risk of default, which offer some *fixed* return delivered at the maturity date of the instrument. Cash holdings and non-risky bonds belong to this class. The class of risky assets, on the other hand, contains individual securities as well as *diversified* portfolios and *mutual funds*. Every portfolio is supposed to be *efficient*, in the sense that it maximises the *utility* attained by its holder — in other words, every portfolio is built after the utility maximisation approach of Harry Markowitz, 1952. In this setting, portfolio decisions are taken *before* the level of cash reserves is decided.

Before formally establishing a problem leading to the *optimal* stock of cash, both the notion of *risk* and the *preferences* of individuals are characterised in mathematical terms.

[5]The classic reference on this issue is Goldfeld, 1976, who examines the failures of the model of money demand occurred during the 1970s. Duca, 2000, and also Teles and Zhou, 2005, investigate whether a stable specification can be obtained if alternative monetary aggregates are considered. Choi and Oh, 2003, on the other hand, propose a model of utility maximisation that incorporates output uncertainty. Calza and Sousa, 2003, examine the effects of additional variables, such as the degree of aggregation of national income.

Thus, in the first place, *risks* are uniquely corresponded to *probability distributions.* More specifically, individuals are supposed to assess the *riskiness* of investments based on the empirical frequencies of the price movements of the alternative securities. The series of price returns are additionally supposed to follow *Gaussian* probability distributions.

Therefore, every risk is completely characterised by a unique pair of *expected return* and *volatility*, in such a way that if the market participants share their expectations about the future performance of securities, every portfolio is represented by a single pair of expected return and volatility in the market. Under such conditions, it is possible to prove that the mean return and the volatility of every *hedged* or *insured* portfolio (as we will refer in the following to every portfolio that combines some risky asset with a certain cash balance) must be related to each other according to a *linear* schedule.

Indeed, consider some portfolio that allocates a proportion λ of wealth to some non-risky bond offering the return r_0 per unit of investment, and the rest to a certain mutual fund providing the aggregate percentage profit and loss X. Then the *capital return* of the *hedged* or *insured* portfolio at maturity, per unit of wealth, is determined by the random variable $Z = (1 - \lambda) \cdot X + \lambda \cdot r_0$, whereas its *expected return* is given by:

$$\mu_Z = E[Z] = (1 - \lambda) \cdot \mu_X + \lambda \cdot r_0 \tag{3}$$

Besides, the volatility of the insured portfolio Z can be expressed in terms of the volatility σ_X of the risky fund:

$$\sigma_Z = \sqrt{E\left[(Z - \mu_Z)^2\right]} = (1 - \lambda) \cdot \sigma_X$$

Solving for λ in both equations, a linear relationship is established between the expected return and the volatility of every insured portfolio:

$$\mu_Z \;=\; r_0 \;+\; \frac{\mu_X - r_0}{\sigma_X} \cdot \sigma_Z \tag{4}$$

This relationship determines the set of *efficient* portfolios, in the sense that for any combination contained in the set of investment opportunities and outside the line, it is always possible to build a new fund providing the same expected return and a lower risk, or the same risk but a higher return. Accordingly, only increasing the borne risk is possible to raise the expected return of the portfolio.[6]

The locus of portfolios satisfying *Equation* 4 in the plane (μ_Z, σ_Z) is known as the *capital market line* (Sharpe, 1964). The slope of the curve is regarded as the *market price of risk*, for it determines the rate at which a unit of expected return is exchanged by a unit of risk in the market:

$$S_X \;=\; \frac{\mu_X - r_0}{\sigma_X} \tag{5}$$

[6]Tobin considers a class of *pure* cash instruments, with $r_0 = 0$. See *Equation* (3.4) in the paper of Tobin, 1958.

The coefficient S_X is also known as the *Sharpe ratio*. Since the expected return and the volatility of the price returns of securities are *observable* variables — which can be estimated based on historical figures — the Sharpe ratio can be empirically determined (Sharpe, 1966).

Regarding the *preferences* of decision-makers, in the model of Tobin they are represented by *utility* functions depending on the return of the portfolio Z, which satisfy the axioms of Von Neumann and Morgenstern (1944).[7] Therefore, given any level of *expected* utility:

$$E[U(Z)] = \int U(z)\ dF_Z(z) \tag{6}$$

an *indifference* curve is determined in the plane (μ_Z, σ_Z), containing all the portfolios that provide the expected utility $E[U(Z)]$, characterised in such a way that for a certain function φ:

$$\varphi(\sigma_Z, \mu_Z) = E[U(Z)]$$

As long as *risk-lover* decision-makers are always willing to accept a lower expected return if there is some chance of obtaining additional profits, their indifference curves must show *negative* slopes. *Averse-to-risk* decision-makers, on the other hand, do not accept to increase their exposure to risk unless they are compensated by a greater expected return and consequently, their indifference curves have positive slopes. Besides, as long as *more* is regarded as *better*, the indifference curves located to the upper left corner of the plane are related to higher utilities. An implicit relationship is ultimately determined, which expresses the expected return of every hedged portfolio in terms of its volatility, i.e. $\mu_Z = \mu(\sigma_Z)$.

Within this theoretical setting, every *rational* decision-maker must choose, among those portfolios contained in the market capital line (*Equation* 4), the combination of risk and cash that maximises her or his expected utility. Such combination is determined at the point of tangency between the line of efficient portfolios (which represents the frontier of the set of investment opportunities) and the *indifference curve* representing the individual's preferences (Sharpe, 1964).

In other words, the optimal portfolio Z is determined at the point where the slope of the tangent to the indifference curve is equal to the slope of the capital market line:

$$\frac{d\mu(\sigma_Z)}{d\sigma_Z} = \frac{\mu_X \ - \ r_0}{\sigma_X}$$

[7] This means, in particular, that the preferences of *averse-to-risk* individuals are represented by *concave* utility functions (which satisfy $d^2U(z)/dz^2 < 0$, $\forall z$), whereas the preferences of *risk-lovers* are represented by *convex* utility functions (which satisfy $d^2U(z)/dz^2 > 0$, $\forall z$). In this context, averse-to-risk individuals must receive a *greater* expected return than risk-lovers in compensation for every additional unit of risk. Moreover, imposing that *more* is always regarded as *better*, implies that the marginal utility must by positive over the whole range (such that $dU(z)/dz > 0$, $\forall z$), both for averse-to-risk and risk-lover individuals.

In this way, an expression for the liquidity preference schedule in terms of the risk-free interest rate, $\lambda = \lambda(r_0)$, can be obtained. However, a tangency point of intersection between the capital market line and some indifference curve will only occur if the later has a *positive* slope, i.e. if $d\mu(\sigma_Z)/d\sigma_Z > 0$. As already stated, this is only true in the case of *averse-to-risk* individuals, for *risk-lovers* are precisely characterised by indifference curves with *negative* slopes.[8] On these grounds, Tobin (1958) regards liquidity-preference as *behaviour towards risk.*

5 Major Limitations of the Utility Maximisation Approach

As shown in the previous section, the utility maximisation approach provides a well established theoretical setting to derive the demand for cash holdings as a function of the interest rate.[9] A model for the pricing of financial securities under conditions of equilibrium is built on this basis, which is known as the *Capital Asset Pricing Model* in the literature (abbreviated as *CAPM*, see Sharpe, 1964 and 1966, and also Lintner, 1965).

The utility maximisation approach requires, in the first place, that decision-makers show *aversion-to-risk*, and secondly, that the transactions of cash and securities are carried out under conditions of *perfect competition.*

Markets are said to run under conditions of *perfect competition* if the following conditions are satisfied (see e.g. Sharpe, 1964). **(PM1)** The series of capital returns of the security prices follow Gaussian probability distributions. Hence only two measures completely describe risks: the *expected return* and the *volatility* of the series of capital returns, respectively corresponded to the *mean return* and the *standard deviation* of the underlying series of capital profit and losses. **(PM2)** Lending and borrowing are allowed at any moment for a common risk-free interest rate, at least up to some desired extent. **(PM3)** At any point of time, investors share expectations concerning the future performance of securities and thus portfolios.

Some consequences of the utility maximisation approach, however, are not fully convincing economically speaking. For example, *drastic* state transitions are sometimes observed in capital markets, manifested as *drastic* variations in the level of the interest rate and the risk-parameters μ and σ, which ultimately induce adjustments in the market prices of securities, as deduced from *Equation* 5. These transitions can be only the consequence of sudden changes in the expectations of individuals, for these (apart from historical information) are the *only* determinants of their estimations of the risk-parameters. But *drastic* expectations and price adjustments are difficult to explain on the grounds of financial and economic theory.[10]

[8] Tobin actually distinguishes between two kind of averse-to-risk individuals: those characterised by *convex* indifference curves, which satisfy $d^2\mu(\sigma_Z)/d\sigma_Z^2 > 0$, and *plungers*, who are characterised by *concave* indifference curves, which satisfy $d^2\mu(\sigma_Z)/d\sigma_Z^2 < 0$. The former include both cash and risk in their hedged portfolios, while the later maintain all their wealth in cash. See *Figures* 3.1, 3.2 and 3.3 in the paper of Tobin, 1958.

[9] Some recent contributions where this approach is adopted are those of Holmstrom and Tirole (2000), Lucas (2000) and Choi and Oh (2003).

[10] It is difficult to accept, in particular, that such adjustments reflect the behaviour of *efficient* markets,

On these grounds, many researchers have questioned the hypothesis (PM1) of perfect competition. They have pointed out instead that drastic state transitions do not necessarily require of drastic adjustments in the underlying risk-parameters if the series of capital returns are statistically modelled by means of *heavy tailed* probability distributions.

As a matter of fact, *heavy tailed* distributions assign greater probability to *big* price movements than the *Gaussian* — whilst the *Gaussian* assigns greater probability to *small* price movements. Hence, *big* price movements do not necessarily correspond to *structural* adjustments when risks are modelled by heavy tailed probability distributions.[11] Unfortunately, such models have not been satisfactorily integrated with economic theory and accordingly, the paradigm of *perfect* markets has predominated.

Finally, notice that the hypothesis (PM2) and (PM3) of perfect competition are crucial to guaranteeing the existence of the market equilibrium in the *CAPM*.

Indeed, recall that every *hedged* portfolio Z is related to a single proportion of cash λ. Hence assuming that every portfolio in the capital market line can be attained by performing the appropriate transactions of securities and cash balances, necessarily implies that such transactions can be performed *at any moment* and *without restrictions* in capital markets. Conversely, if *hedging* were only possible *up to some extent*, *some* portfolios satisfying *Equation* 4 might require of lending or borrowing operations involving amounts that are *not* available in the market.

The hypothesis (PM3), on the other hand, implies that individuals agree on the estimations of the risk-parameters μ, σ and hence, on the market prices of securities. Only if this hypothesis is satisfied the transactions of assets can be performed at a *unique* price in the market, in such a way that the market is found *at equilibrium* (Sharpe, 1964).

The alternative model of equilibrium that will be presented in the following sections is built on a framework of *imperfect* competition — where hypotheses (PM1), (PM2) and (PM3) are *not* satisfied. Consequently, the set of risks will be corresponded to a general class of probability distributions. It will be additionally assumed that the transactions in the markets of assets and cash holdings are only possible if the quantities involved do not surpass certain limits and that individuals do not necessarily agree on their expectations about the future performance of securities.

Within this framework, the influence of liquidity restrictions in the funding strategies followed by decision-makers can be described by considering that the only substitute to borrowing in capital markets, apart from cash holdings, is deposit *insurance*. This approach is suggested by Robert Merton (1974, 1977) for the pricing of liability guarantees. Merton, however, assumes that individuals can trade securities and cash balances at their will — in other words, he assumes that investors can hedge *continuously* — concluding that the price of every guarantee must be equal to the price of a put option on the value

as *every* market is assumed to be according to the *efficient markets hypothesis* (in the terms that is formulated by Fama, 1970 and 1998).

[11] Mandelbrot (1963) is among those that first attempted to introduce *heavy tailed* distributions for the statistical characterisation of the movements of stock prices. Merton (1976) and Cox and Ross (1976) reformulate the Black and Scholes' option pricing model, in order to consider stochastic processes with jumps.

of the underlying claim.

The Merton's model of deposit insurance can be naturally extended if the price of the guarantee is related to the *actuarial* price of the underlying *residual* exposure — equal to the excess of loss over the level of reserves.[12] As demonstrated later in *Section* 6, then an *optimal* surplus exists, which ensures that the value of the *hedged* portfolio is maximised. The optimal *liquidity principle* thus obtained can be naturally aggregated to account for the preference for liquidity of markets, economic sectors and the economy as a whole, as shown in *Section* 8. On these grounds, the monetary equilibrium of the economy can be characterised, see *Section* 9.

As shown in *Sections* 10 and 11, an alternative to the *CAPM's* characterisation of market equilibrium is obtained applying this model to the particular case of *Gaussian* risks and *homogeneous* expectations. The model also leads to an extended approach to the monetary equilibrium (see *Section* 12). Market *vulnerabilities*, manifested as *peaks* in the sensibility of the liquidity-preference function with respect to the interest rate, are then consistent with some market scenarios, and can then be regarded as *natural* transitions in markets where individuals face borrowing restrictions.

6 The Optimal Liquidity Principle

Let the parameter θ denote the state of information of some firm or individual investor that holds a mutual fund whose percentage return is represented by the *random* variable $X = \Delta Y/Y$, where Y denotes the level of *income* of the fund. Because of the *precautionary* motive, a guarantee L is maintained until maturity in order to avoid bankruptcy, whose magnitude, on account of the *transactions* motive, is expressed as a proportion λ of the level of income, i.e. $L = Y \cdot \lambda$. In the following, this surplus will be treated as an additional liability that induces the cost $r_0 \cdot L$.

The total payment per unit of investment delivered by the *hedged* or *insured* portfolio (which combines the risky fund X and the guarantee L) at maturity is then equal to the claim $Z = X - \lambda - r_0 \cdot \lambda$. Hence the *expected* return $Y \cdot \mu_{\theta,Z}$ of the *insured* portfolio is given by:

$$Y \cdot \mu_{\theta,Z} = Y \cdot E_\theta[Z] = (Y \cdot \mu_{\theta,X} - L) - r_0 \cdot L = Y \cdot [\ (\mu_{\theta,X} - \lambda) - r_0 \cdot \lambda\ ]$$

Financing decisions are thereby affected by the *percentage* return on income:

$$\mu_{\theta,Z} = E_\theta[Z] = (\mu_{\theta,X} - \lambda) - r_0 \cdot \lambda \tag{7}$$

[12] It can be actually demonstrated that the expected value of the excess of loss satisfy a set of basic mathematical properties and hence, that it can be regarded as a *fair* insurance price, see Goovaerts et al., 1984. In Dhaene et al., 2003, and Goovaerts et al., 2005, this principle is used as a tool for allocating capital inside financial institutions. In fact, a model of economic capital can be formally established on this basis, which I present in details in Mierzejewski, 2006a, 2006b, 2008a and 2008b.

Comparing *Equations* 3 and 7 we notice that the rules determining the *insured* portfolio in the derivation of the *CAPM* in *Section* 4 are different from the prescription considered in the alternative model.

Indeed, as established in *Equation* 3, in the former setting the proportion λ simultaneously determines the amount of funds allocated to risk and cash holdings. As long as *any* combination of assets and cash balances can be attained in the market, *any* proportion λ is corresponded to some portfolio that can be built by performing the appropriate transactions.

In order to incorporate the possibility that some combinations cannot be attained due to liquidity restrictions, in *Equation* 7 the exposition to risk is *fixed*, although individuals can modify their cash holdings by borrowing or lending at the interest rate r_0. Assuming this setting makes sense if the portfolio X is regarded as a *non-standarised* fund that cannot be continuously transacted in the market. The holders of such portfolios are obliged to perform a complete reallocation of resources if they want to change their exposition to risk — in other words, they are obliged to implement again the Markowitz's procedure to find a *new* optimal portfolio. Such adjustments are seen as *structural* changes by creditors, which might lead to increments or reductions in the market price of the fund. These price returns, in turn, might eventually lead to changes in the premiums (over the risk-free interest rate) the holders of the fund have to pay to borrow in the markets of cash balances.[13]

Maximising the expected return of the hedged portfolio as defined in *Equation* 7 actually leads to the *trivial* solution $\lambda = 0$, because in this case demanding cash holdings only produces an additional loss. To obtain this result, the hypothesis is implicit that individuals are indifferent between holding *positive* or *negative* balances.

However, if individuals face liquidity restrictions, transacting positive and negative balances might induce to some *net* profit or loss.

In fact, the total returns obtained in each case can be explicitly measured in terms of the expected values of the claims $(X-\lambda)_+ = max(0, X-\lambda)$ and $(X+\lambda)_- = -min(0, X+\lambda)$, which respectively represent the *surplus* and the *excess of loss* with respect to the cash stock. Then the expected return (per unit of income) of the insured portfolio should be written as:

$$\mu_{\theta,Z} = E_\theta\left[(X-\lambda)_+\right] - E_\theta\left[(X+\lambda)_-\right] - r_0 \cdot \lambda = \Delta(\lambda) - r_0 \cdot \lambda \tag{8}$$

The term $\Delta(\lambda) := E_\theta[(X-\lambda)_+] - E_\theta[(X+\lambda)_-]$ represents the *economic margin* obtained because of financial intermediation, while $E_\theta\left[(X+\lambda)_-\right]$ accounts for the *cost of assuming bankruptcy*, a role that can be adopted by the own investor, an insurance company or

[13] According to Billet and Garfinkel, 2004, such premiums depend explicitly on the difference between the costs of *internal* and *external* financing, and thereby reflect the degree of *financial flexibility* of the institution. Thus, institutions with *greater* flexibility have access to cheaper funding sources, have greater market values and carry less cash holdings. Kashyap and Stein, 2000, analyse the effects of monetary policy over financial decisions under such circumstances.

some governmental division.[14]

From the actuarial point of view, the terms $E_\theta[(X-\lambda)_+]$ and $E_\theta[(X+\lambda)_-]$ represent the *fair* or *actuarial* prices of the corresponding claims. This means that these terms represent the prices at which the underlying exposures $(X-\lambda)_+$ and $(X+\lambda)_-$ should be transacted in some *insurance market* free of arbitrage (see Goovaerts et al., 1984, Venter, 1991, and Wang et al., 1997).

Within this context, the expected return $\mu_{\theta,Z}$ represents the *fair* price of the portfolio Z when capital and insurance markets are found *at equilibrium* (see Mierzejewski, 2008b). Hence, as implied by *Equation* 8, the market value of the insured portfolio certainly depends on the proportion of funds λ invested on cash reserves. We can thereby postulate that *rational* decision-makers choose the proportion λ in order to maximise the expected return $\mu_{\theta,Z}$ — for in this way they maximise the market valorisation of their portfolios — but the question then arises of under which conditions the existence of such an *optimal* proportion can be assured.

In order to give an answer to this question, it is necessary, in the first place, to provide an explicit expression for the *distorted* expectation operator $E_\theta[\cdot]$. For this purpose, let us consider the *proportional hazards* transformation,[15] introduced by Wang (1995) as an insurance principle:

$$E_\theta[X] = \int x \; dF_{\theta,X}(x) = \int T_{\theta,X}(x) \; dx \quad with \; T_{\theta,X}(x) := T_X(x)^{\frac{1}{\theta}} \; \forall x \tag{9}$$

The *distorted cumulative* and *distorted tail* probability distribution functions appear in *Equation* 9, respectively defined as $F_{\theta,X}(x) = P_\theta\{X \le x\}$ and $T_{\theta,X}(x) = P_\theta\{X > x\}$, with $F_{\theta,X}(x) = 1 - T_{\theta,X}(x)$, $\forall x$. Whenever $\theta > 1$ the expected value of risk is *overestimated*, and *underestimated* when $\theta < 1$, in this way respectively accounting for the behaviour of *risk-averse* and *risk-lover* investors.

Applying *Lagrange optimisation*, leads the *optimal* proportion λ^*, which maximises the criterion of *Equation* 8, to be characterised by the *first-order condition*, determined at the point where the derivative of $\mu_{\theta,Z}$ with respect to λ is equal to zero:[16]

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[14] Froot et al., 1993, propose a similar model to characterise the *optimal* demand for capital, which is also based on the expected values of the positive part of the surpluses of the underlying portfolio. Unlike the model presented in this paper, Froot et al. propose to *add* some random perturbation to the income of the portfolio — and do not *multiply*, as suggested in this paper. Besides, they simultaneously maximise over the level of capital and the level of investment. See also Froot and Stein, 1998, and Froot, 2007.

[15] So called since it is obtained by imposing a safety margin to the *hazard rate* $h_X(x) := dln \; T_X(x)/dx$ in a multiplicative fashion: $h_{\theta,X}(x) = (1/\theta) \cdot h_X(x)$, with $\theta > 0$. Other distortions can be used instead. In the general case, a *distortion function* is defined over the unit interval, and an axiomatic description is provided for the distorted price (see Wang et al., 1997 and Wang & Young, 1998). *Averse-to-risk* and *risk-lover* investors are then respectively characterised by *concave* and *convex* transformations. All the analysis that follows is maintained in the same terms under this general setting (see also Mierzejewski, 2006b).

[16] Applying the Leibnitz's rule:

$$\frac{d}{dy}\int_{u(y)}^{v(y)} H(y,x)dx = \int_{u(y)}^{v(y)} \frac{\partial H(y,x)}{\partial y}dx + H\left(y, v(y)\right) \cdot \frac{dv(y)}{dy} - H\left(y, u(y)\right) \cdot \frac{du(y)}{dy}$$

$$\frac{dE_\theta\left[(X-\lambda^*)_+\right]}{d\lambda} - \frac{dE_\theta\left[(X+\lambda^*)_-\right]}{d\lambda} - r_0 \;=\; -T_{\theta,X}(\lambda^*) + F_{\theta,X}(-\lambda^*) - r_0 \;=\; 0$$

Since $F_{\theta,X}(-\lambda) = P_\theta\{X \le -\lambda\} = P_\theta\{-X > \lambda\} = T_{\theta,-X}(\lambda)$, $\forall\lambda$, the following equivalent characterisation is obtained:

$$T_{\theta,-X}(\lambda^*) - T_{\theta,X}(\lambda^*) = r_0 \tag{10}$$

The *rational* liquidity demand is thus determined in such a way that the marginal gain minus the marginal loss on capital (i.e. the instantaneous benefit of liquidity) equals the marginal return of the sure investment. Within this context, the optimal proportion of cash is corresponded to an optimal exchange of a sure return and a flow of probability, and it is the mass accumulated in the tails of the distribution what matters. No explicit relationship is obtained for the cash demand, but some numerical procedure could be implemented to find the solution.

The existence of some optimal proportion λ^* can be mathematically assured as long as, for any proportion level below the optimal, i.e. for any $\lambda < \lambda^*$, the expected income per unit of investment, equal to the term $\Delta(\lambda) - r_0 \cdot \lambda$, is an *increasing* and *concave* function on the liquidity preference coefficient λ. This requirement actually corresponds to the *second-order condition* of Lagrange optimisation (see Froot et al., 1993).

In other words, an optimal proportion of cash exists as long as the following inequalities are simultaneously satisfied:

$$\begin{aligned}
\frac{d\Delta(\lambda)}{d\lambda} - r_0 > 0 \quad &\Leftrightarrow \quad T_{\theta,-X}(\lambda) - T_{\theta,X}(\lambda) > r_0 \quad \forall\, \lambda < \lambda^* \\
\frac{d^2\Delta(\lambda)}{d\lambda^2} < 0 \quad &\Leftrightarrow \quad \frac{d\,T_{\theta,-X}(\lambda)}{d\lambda} - \frac{d\,T_{\theta,X}(\lambda)}{d\lambda} < 0 \quad \forall\, \lambda < \lambda^*
\end{aligned}$$

The first inequality implies that, for any given liquidity preference ratio λ lower than the optimal level λ^*, the marginal loss due to financial intermediation is greater than the total cost of the guaranty and accordingly, that there are incentives to maintain some cash surplus. The second condition ensures *concavity.* In fact, recalling that $T_{\theta,X} = 1 - F_{\theta,X}$, this condition can be written in terms of the *density* probability distribution $f_{\theta,X}(x) := dF_{\theta,X}(x)/dx = P_\theta\{X = x\}$:

$$P_\theta\{X = \lambda\} \;<\; P_\theta\{X = -\lambda\} \quad \forall\, \lambda < \lambda^*$$

The *second-order condition* thereby implies that an optimal liquidity ratio λ^* exists as long as the probability of obtaining a certain capital gain is always lower than the probability of obtaining a capital loss of the same magnitude.

the relationship is obtained by noticing that, from *Equation* 9, the following expressions are respectively obtained for the *expected surplus* and the *expected excess of loss*: $E_\theta\left[(X-\lambda)_+\right] = \int_\lambda^\infty (x-\lambda)\; dF_{\theta,X}(x)$ and $E_\theta\left[(X+\lambda)_-\right] = -\int_{-\infty}^{-\lambda} (x+\lambda)\; dF_{\theta,X}(x)$.

7 The Optimal Liquidity Principle as the Optimal Insurance Retention

One additional condition has to be satisfied, however, for the optimal cash balance to be determined by *Equation* 10. Indeed, recall that for the market price of the hedged portfolio to be characterised by the expected return $\mu_{\theta,Z}$ defined in *Equation* 8, individuals must be able to sell their *surpluses* at the price $E_\theta[(X-\lambda)_+]$, in such a way that the benefit they have to resign (in average) for holding the proportion of capital λ is equal to:

$$\begin{aligned} r_{\theta,X}(\lambda) &= \tfrac{E_\theta[X_+]-E_\theta[(X-\lambda)_+]}{\lambda} \\ \Leftrightarrow \; E_\theta\left[(X-\lambda)_+\right] &= E_\theta[X_+] - r_{\theta,X}(\lambda)\cdot\lambda \end{aligned} \tag{11}$$

Combining *Equations* 8 and 11:

$$\mu_{\theta,Z} = E_\theta\left[X_+\right] - E_\theta\left[(X+\lambda)_-\right] - (r_0 + r_{\theta,X}(\lambda))\cdot\lambda \tag{12}$$

In this context, the return $r_{\theta,X}(\lambda)$ can be interpreted as an extra *premium* paid for keeping the balance $L = Y\cdot\lambda$ as a cash stock, instead of investing it in the mutual fund X.

Equivalently, we can say that the *total* cost of capital for the holders of the hedged portfolio is equal to:

$$r(\lambda) = r_0 + r_{\theta,X}(\lambda) \tag{13}$$

Since the risk-free interest rate r_0 does not depend on the cash proportion λ, deriving *Equation* 11 with respect to λ and rearranging terms, we obtain that the marginal change of the cost of capital with respect to the proportion of cash can be explicitly calculated:

$$\frac{dr(\lambda)}{d\lambda} = -\frac{1}{\lambda}\cdot\left(r_{\theta,X}(\lambda) + \frac{dE_\theta\left[(X-\lambda)_+\right]}{d\lambda}\right) \tag{14}$$

Under such circumstances, maximising the expressions of *Equations* 8 and 12 lead to the same optimal cash balance. Therefore, only if the cost of capital is determined according to *Equations* 13 and 14, the optimal cash balance is characterised in order to satisfy *Equation* 10.

Notice that *Equations* 13 and 14 determine the cost of capital *as perceived* by the *holders* of the hedged portfolio. But as we have assumed that individuals borrow the cash balance L in some open market of capital, the cost of capital should rather reflect the perceptions of *lenders*.

As a matter of fact, debt can be implemented by issuing a bond promising to pay a certain interest rate r at maturity. As long as the market regards this deposit as *riskier* than the risk-free security, the issuers of the bond have to offer some return higher than the risk-free interest rate in order to make it attractive to investors. Hence the condition $r > r_0$ must be satisfied. On the other hand, the bond issuers are not willing to pay a premium greater than $r_{\theta,X}(\lambda)$, for then the alternative of providing these founds themselves (whose cost is measured by the premium $r_{\theta,X}(\lambda)$) would be cheaper. Hence, also the condition $r \leq r_0 + r_{\theta,X}(\lambda)$ must hold.

Provided that the previous conditions are satisfied, the cost of capital r must be determined by the *credit quality* of borrowers. Consequently, it can be only affected by events that change the perception of investors about the willingness and capability to pay of the bond issuers. It can then be assumed as *constant* in practice, as long as the issuers of bonds do not drastically change their capital structures — i.e. as long as the proportion of reserves λ is not drastically modified.

Replacing the return r in the place of $(r_0 + r_{\theta,X}(\lambda))$ in *Equation* 12, the following expression is obtained for the expected percentage income:

$$\mu_{\theta,Z} = E_\theta\left[X_+\right] - E_\theta\left[(X+\lambda)_-\right] - r \cdot \lambda \tag{15}$$

Applying Lagrange optimisation, we obtain that individuals attract funds until the marginal return on risk equals the total cost of holding capital:

$$-\frac{d\, E_\theta\left[(X+\lambda^*)_-\right]}{d\lambda} - r \;=\; T_{\theta,-X}(\lambda^*) - r \;=\; \left[\, T_{-X}(\lambda^*)\,\right]^{\frac{1}{\theta}} - r \;=\; 0$$

Equivalently, it can be said that investors stop demanding money at the level at which the marginal expected gain in solvency equals its opportunity cost. The optimal *liquidity principle* is thereby given by:

$$\lambda_{\theta,X}(r) \;=\; T^{-1}_{\theta,-X}(r) \;=\; T^{-1}_{-X}\left(r^\theta\right) \tag{16}$$

From this expression, the optimal demand for cash balances always follows a non-increasing and (as long as the underlying probability distribution is continuous) continuous path, whatever the kind of risks and distortions, because the tail probability function, and hence its inverse, are always *non-increasing* functions of their arguments. The minimum and maximum levels of surplus are respectively demanded when $r \geq 1$ and $r \leq 0$. Besides, *averse-to-risk* and *risk-lover* individuals (respectively characterised by $\theta > 1$ and $\theta < 1$) systematically demand *higher* and *lower* amounts of cash holdings — for they respectively *under-* and *over-estimates* the cost of capital.[17]

[17] Consequently, one of the main advantages of the actuarial-based liquidity principle defined in *Equation* 16 is its *functionality*. Such result crucially depends on the choice of the *distorted probability* insurance principle of *Equation* 9. Indeed, if the *expected utility* operator (defined in *Equation* 6) were used instead to evaluate the hedged portfolio's return of *Equation* 15, then the first-order condition would lead to:

$$\int_{-\infty}^{-\lambda} u'(x+\lambda)\, dF_X(x) \;-\; u(0)\cdot f_X(-\lambda) \;-\; r \;=\; 0$$

where u' denotes the first derivative of the utility function. Then no explicit expression would be obtained

8 The Aggregate Liquidity Principle

The *aggregate* liquidity-preference of some industry or economic sector will be now characterised, where each firm can borrow at a single interest rate r. As stated in the previous section, such rate depends on the *credit quality* of borrowers and is supposed to remain unchanged as long as firms do not drastically alter their capital structures. In other words, firms are supposed to remain in the same credit class (i.e. the return at which firms can borrow in the market is supposed to remain the same) as long as the levels of income and reserves in their portfolios are kept more or less invariant.

Let us additionally assume that firms hold securities and combinations of securities (or are involved in venture projects) producing capital returns represented by the random variables $X_1, \ldots, X_n$. The levels of income and the liquidity preference functions corresponding to each of the funds will be respectively denoted as $Y_1, \ldots, Y_n$ and $\lambda_1(r), \ldots, \lambda_n(r)$. The total surplus accumulated in the industry must then be equal to:

$$Y \cdot \lambda(r) \;=\; \sum_{i=1}^{n} Y_i \cdot \lambda_i(r) \quad \textit{with} \quad Y = \sum_{i=1}^{n} Y_i$$

where Y and $\lambda(r)$ respectively denote the level of income and the preference for liquidity accumulated in the industry. Dividing by Y we obtain that:

$$\lambda(r) \;=\; \sum_{i=1}^{n} \omega_i \cdot \lambda_i(r) \quad \textit{with} \quad \omega_i = \frac{Y_i}{Y} \;\; \forall i \;\; \textit{and} \;\; \sum_{i=1}^{n} \omega_i = 1 \tag{17}$$

Accordingly, at any level of the interest rate, the liquidity-preference of the industry is equal to the sum of the liquidity-preferences of the different firms weighted by their relative magnitudes in terms of the levels of income.

Notice that the level of aggregation plays no role in *Equation* 17. Indeed, the random variables $X_1, \cdots, X_n$ could be assumed to represent the capital $P\&L$ of the totality of firms belonging to the class, as well as the aggregates of some predetermined groups or *clusters*. Then the liquidity-preferences of the different economic sectors could be summed up in order to obtain the preference for liquidity of the economy as a whole. Alternatively, the function $\lambda(r)$ could be represented in terms of the incomes and the cash balances demanded by each of the individuals participating in it, from householders and small companies, to big holdings and rich private investors. Although the functional specification and the evolution of $\lambda(r)$ are certainly expected to depend on the level of aggregation, there is no *formal* difference in applying any of the alternative representations. They are all *equivalent* characterisations of the same property of the economy.

Actually, from the mathematical point of view, *Equation* 17 can be treated as an *invariance* condition leading to a certain set of functional specifications. Imposing that the different liquidity-preference functions $\lambda_1(r), \cdots, \lambda_n(r)$ follow the same functional

for the optimal liquidity principle — except for some restricted class of utility functions.

expression and that this expression is always preserved at different levels of aggregation (whatever the number of components or the relative magnitudes of incomes), necessarily leads to accept only a limited set of functions.

Let me illustrate the meaning of this claim by examining the case when individuals choose their balances according to the liquidity principle of *Equation* 16 and share expectations about the probability distributions describing risks — i.e. they agree on the informational type θ. Then the aggregate surplus must be equal to the sum of the distorted quantiles of the individual exposures:

$$\lambda(r) \;=\; \sum_{i=1}^{n} \omega_i \cdot T^{-1}_{\theta,-X_i}(r) \quad with \quad \omega_i = \frac{Y_i}{Y} \;\; , \;\; \sum_{i=1}^{n} \omega_i = 1$$

Now define:

$$\lambda_i = \omega_i \cdot T^{-1}_{\theta,-X_i}(r)$$

$$\Leftrightarrow \;\; r = T_{\theta,-X_i}\left(\tfrac{\lambda_i}{\omega_i}\right) = P_\theta\left\{-X_i > \tfrac{\lambda_i}{\omega_i}\right\} = P_\theta\left\{-\omega_i \cdot X_i > \lambda_i\right\}$$

$$\Longrightarrow \qquad \lambda_i = T^{-1}_{\theta,-\omega_i \cdot X_i}(r)$$

Hence the contributions of firms and individuals to the aggregate liquidity-preference can be equivalently expressed as the optimal principles corresponded to the weighted capital returns $\omega_1 \cdot X_1, \cdots, \omega_n \cdot X_n$:

$$\lambda(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta,-\omega_i \cdot X_i}(r)$$

Therefore, for the aggregate liquidity-preference to be expressed as the quantile of the aggregate capital $P\&L$, we must necessarily impose the sum of the quantiles of the underlying risks to be equal to the quantile of the aggregate exposure.

In fact, as demonstrated by Dhaene et al. (2002), the property of the sum of the quantiles mathematically characterises the *comonotonic dependence structure*. A random vector $(X_1^c, \ldots, X_n^c)$ is said to be *comonotonic* if a random variable ζ exists, as well as a set of *non-decreasing* functions $h_1, \ldots, h_n$, such that the realisation of any joint event is entirely determined by ζ, i.e.:

$$(X_1^c, \ldots, X_n^c) \;=\; (h_1(\zeta), \ldots, h_n(\zeta))$$

Hence the realisation of any joint event is uniquely related to some event contingent on the single exposure ζ. Besides, since the functions $h_1, \ldots, h_n$ are all *non-decreasing*, all the components of the random vector $(X_1^c, \ldots, X_n^c)$ *move in the same direction.* On these grounds, it is said that *comonotonicity* characterises an *extreme* case of dependence, when no benefit can be obtained from diversification.

Let $(X_1^c, \ldots, X_n^c)$ denote the random vector described by the same marginal probability distributions as $(\omega_1 \cdot X_1, \ldots, \omega_n \cdot X_n)$ and let $X^c = X_1^c + \ldots + X_n^c = \omega_1 \cdot X_1 + \ldots + \omega_n \cdot X_n$ denote the *comonotonic aggregate* (or *comonotonic sum*) of the individual capital returns. Then the quantile $T^{-1}_{\theta,-X^c}$ of the comonotonic sum is equal to the sum of the quantiles of the weighted exposures $(\omega_1 \cdot X_1, \ldots, \omega_n \cdot X_n)$, in such a way that the preference for liquidity of the economy can be written as:

$$\lambda(r) \;=\; T^{-1}_{\theta,-X^c}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta,-\omega_i \cdot X_i}(r) \quad \textit{with} \;\; X^c = \sum_{i=1}^{n} \omega_i \cdot X_i \tag{18}$$

The *comonotonic aggregate* X^c thereby characterises liquidity-preference in economies where individuals rely on the optimal liquidity principle of *Equation* 16.

When differing expectations are allowed in the economy, the aggregate money demand is given by:

$$\lambda(r) \;=\; T^{-1}_{\theta_1,\ldots,\theta_n,-X^c}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta_i,-\omega_i \cdot X_i}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{-\omega_i \cdot X_i}\left(r^{\theta_i}\right) \tag{19}$$

where $\theta_1, \ldots, \theta_n$ denote the different informational types and $T_{\theta_1,\ldots,\,\theta_n,-X^c} = \left(\sum_{i=1}^{n} T^{-1}_{\theta_i,-\omega_i \cdot X_i}\right)^{-1}$ denotes the distribution function of the comonotonic sum when the marginal distributions are given by $T_{\theta_1,-\omega_1 \cdot X_1}, \ldots, T_{\theta_n,-\omega_n \cdot X_n}$.

Comparing *Equations* 18 and 19, we observe that there is no formal difference between assuming equal and different expectations: in both cases, the aggregate liquidity-preference is determined by the quantile function of the probability distribution of the sum of the underlying exposures. Moreover, as long as the proportions $\omega_1, \ldots, \omega_n$ and the riskiness of the capital returns $X_1, \ldots, X_n$ remain constant, the *instability* of both functional expressions depends alone on the *instability* of the expectations firmly maintained by individuals, and not on whether individuals agree or not on these expectations. Hence the difference between the *homogeneous* and the *non-homogeneous expectations* settings is not relevant in explaining the *instability* of the money demand of the economy.[18]

Endowed with an expression for the aggregate liquidity-preference of the economy, we can now proceed to characterise the *monetary equilibrium* when individuals determine their cash holdings according to the optimal liquidity principle defined in *Equation* 16.

[18]This conclusion contradicts the Keynes's argument, that the money demand of the economy must be *absolute* (and so, that monetary policy is useless) in the case of *homogeneous* expectations (see Keynes, 1937a and 1937b). As explained later in *Section* 12, the preference for liquidity can indeed be *absolute* under certain circumstances, but as a consequence of the *riskiness* of national income.

9 The Monetary Equilibrium with the Optimal Liquidity Principle

Replacing *Equations* 16 and 18 into *Equation* 2, we obtain that in the case of *homogeneous* expectations the monetary equilibrium is determined by the following equation:

$$M \; = \; Y \cdot \lambda(r) \; = \; Y \cdot T^{-1}_{\theta,-X^c}(r) \; = \; Y \cdot T^{-1}_{-X^c}\left(r^{\theta}\right) \tag{20}$$

where M denote the total stock of money in the economy. Hence both the riskiness of national income (determined by the random variable X^c) and the market expectations (characterised by the informational type θ) explicitly affect the monetary equilibrium.

This means, in particular, that the monetary policy choosen by the central bank (characterised by the money supply M) is not corresponded to a *unique* level of the interest rate, as obtained from *Equation* 2. In fact, given any money stock M, multiple interest rates can satisfy *Equation* 20, depending on the probability distribution describing the riskiness of national income and the informational type θ corresponded to the market expectations.

The influence of expectations over the monetary equilibrium can be actually more precisely described.

Indeed, notice, on the one hand, that since the cost of capital is *under-estimated* in *averse-to-risk* economies (characterised by $\theta > 1$), the interest rate attained at equilibrium in this case is always *greater* than the levels attained in *risk neutral* and *risk-lover* economies (respectively characterised by $\theta = 1$ and $\theta < 1$) for the same money supply M and the same aggregate exposure X^c.

On the other hand, the level of interest rates attained at equilibrium in *risk-lover* economies is always *lower* than the levels attained in *risk neutral* and *averse-to-risk* economies, because risk-lover individuals systematically *over-estimate* the cost of capital. As a consequence, in economies where both the riskiness of the percentage return of national output and the monetary policy implemented by the central bank remain constant, changes in expectations must be necessarily followed by adjustments in the rate of interest.

When individuals maintain different expectations about risks, the equilibrium interest rate depends on the particular combination of the informational parameters $\theta_1, \ldots, \theta_n$.

The other determinant of the monetary equilibrium in *Equation* 20 is the *riskiness* of national income. In the particular case when some analytical expression is available for the probability function describing the random variable X^c, such dependence can be investigated in terms of the underlying risk-parameters.

A careful examination of the model under the different families of probability distributions found in the statistical literature is out of the scope of this paper. Instead, the case of the Gaussian distribution will be analysed in the following sections. In this way, the classic theoretical framework supporting the *CAMP* and the classical analysis of the monetary equilibrium will be naturally *extended.*

Indeed, as established in *Section* 11, an *extended* version of the *capital market line* is obtained when the Gaussian quantile function is replaced in *Equation* 20. Later in *Section* 12, an *extended* theoretical framework for the conduction of monetary policy will be presented, based on the fact that the slope of the money demand (or the *semi-elasticity* of the preference for liquidity) explicitly depends on the mean return and the volatility of the aggregate exposure X^c when the Gaussian liquidity principle is introduced.

10 The Gaussian Liquidity Principle

In the particular case when the aggregate percentage return X is represented by a *Gaussian* probability distribution with *mean return* μ and *volatility* σ, the optimal liquidity principle takes the form (see *Equation* 16):

$$\lambda_{\mu,\sigma}(r) \;=\; \sigma \cdot \Phi^{-1}\left(1 - r^{\theta}\right) - \mu \tag{21}$$

where Φ denotes the cumulative probability distribution of a *standard* Gaussian random variable, whose mean and volatility are respectively equal to zero and one (see e.g. De Finetti, 1975, and also Dhaene et al., 2002):

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \exp\left(\frac{-y^2}{2}\right) dy \quad \forall x$$

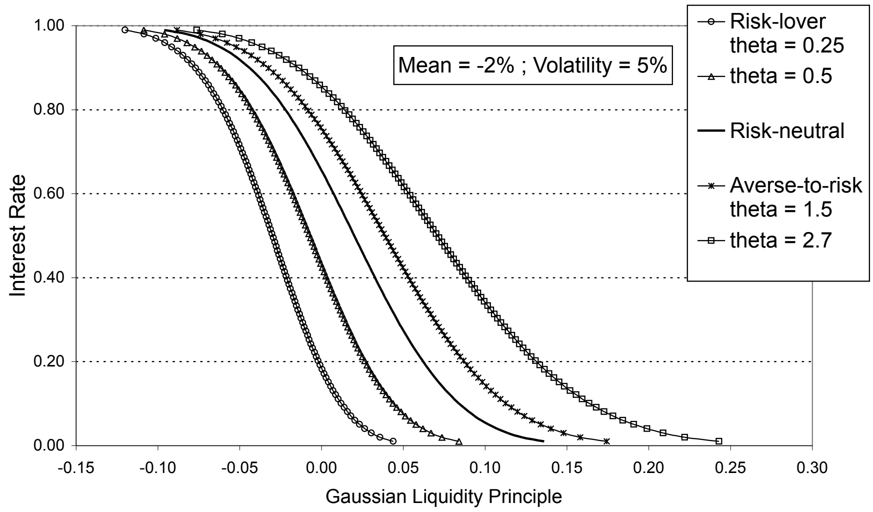
As depicted in *Figures* 1 and 2, the Gaussian liquidity principle always follows a decreasing and continuous path, independently of the levels of the risk parameters μ and σ and the informational type θ.

This implies that the derived demand for cash holdings $L(r) = Y \cdot \lambda_{\mu,\sigma}(r)$ always follows a decreasing and continuous path — for every fixed level of income Y — and consequently, that the derived money demand $L(r)$ is *well defined.*

As depicted in *Figure* 1, the dependence of the Gaussian liquidity principle on the informational type θ follows indeed the patterns described in *Section* 6 for general probability distributions. Accordingly, given fixed levels of expected return and volatility, and at any level of the interest rate, *averse-to-risk* individuals (characterised by $\theta > 1$) always demand higher cash balances than *neutral* or *risk-lover* individuals. By contrast, *risk-lovers* individuals (characterised by $\theta < 1$) always prefer to maintain lower surpluses than *neutral* and *averse-to-risk* individuals. Besides, ceteris paribus, the size of the cash stock always increases with the informational type θ.

Regarding the dependence of the Gaussian liquidity principle on income uncertainty, notice in the first place, that given any fixed level of volatility, raising the mean return always implies that the demand curve is *moved to the left* (see the upper graph of *Figure* 2). More specifically, when $\mu < 0$ and when $\mu > 0$ the cash requirement $\lambda_{\mu,\sigma}(r)$ per unit of income respectively *increases* and *decreases* with the *magnitude* of the mean return.

Figure 1: The Gaussian Liquidity Principle with Different Distortions



Consequently, the amount of reserves always rises with the magnitude of realised expected *losses* (when $\mu < 0$). By contrast, when positive returns are obtained (and $\mu > 0$), at least part of the losses are cancelled by realised profits, in such a way that the *higher* the magnitude of the expected capital gain, the *lower* the required cash balance and vice versa.

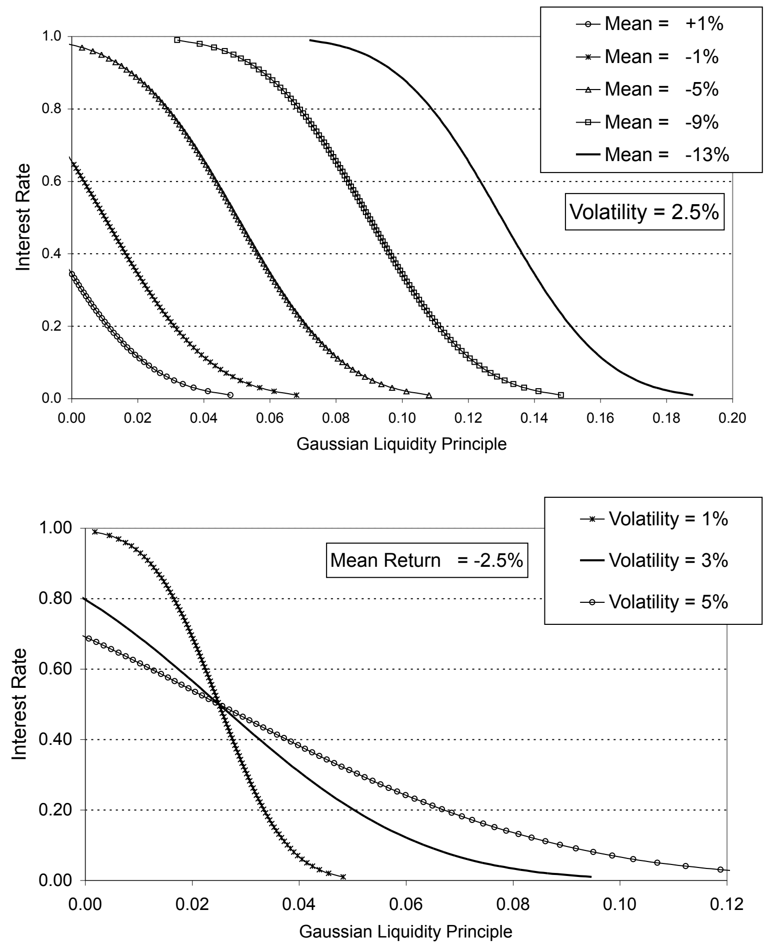
Secondly, as depicted in the lower graph of *Figure* 2, for every fixed level of expected return, the slope of the Gaussian liquidity principle always increases with volatility, which means that the *higher* the variability of the underlying series of percentage returns, the *more* sensible are individuals to the interest rate and vice versa. This result makes economic sense, as long as the parameter σ measures the variability (and hence the *riskiness*) of income. Moreover, as a consequence of the symmetry of the Gaussian distribution, all the demand curves intersect at the point $r = 0.5$ in the lower graph of *Figure* 2. At this level, there is an equal chance of obtaining a capital gain or a capital loss, no matter the level of volatility, and hence the same balance is demanded, equal to the expected value of the fund.

In fact, the *sensibility* of the Gaussian liquidity principle with respect to the interest rate can be explicitly measured by the *semi-elasticity* of the Gaussian liquidity-preference function with respect to the interest rate, equal to the percentage variation in the proportion of reserves with respect to the interest rate. From *Equation* 21, the following expression is obtained for this coefficient:

$$\eta\left(r, \frac{\mu}{\sigma}\right) = \frac{1}{\lambda_{\mu,\sigma}(r)} \cdot \frac{d\lambda_{\mu,\sigma}(r)}{dr} = \frac{-\sqrt{2\pi} \cdot \theta \ r^{\theta-1}}{\Phi^{-1}(1 - r^{\theta}) - \frac{\mu}{\sigma}} \cdot \exp\left(\frac{\left[\Phi^{-1}\left(1 - r^{\theta}\right)\right]^2}{2}\right) \tag{22}$$

The sign of the variation thereby depends on the interest rate and the risk parameters μ

Figure 2: The Gaussian Liquidity Principle with Different Combinations of Mean Returns and Volatilities



and σ, in such a way that:

$$\eta\left(r, \tfrac{\mu}{\sigma}\right) < 0 \quad \Leftrightarrow \quad \Phi^{-1}(1 - r^{\theta}) - \tfrac{\mu}{\sigma} > 0 \quad \Leftrightarrow \quad \lambda_{\mu,\sigma}(r) > 0$$

$$\eta\left(r, \tfrac{\mu}{\sigma}\right) > 0 \quad \Leftrightarrow \quad \Phi^{-1}(1 - r^{\theta}) - \tfrac{\mu}{\sigma} < 0 \quad \Leftrightarrow \quad \lambda_{\mu,\sigma}(r) < 0$$

Therefore, a curve with negative slope (as every demand curve should look like, according to classic economic analysis) is always obtained, no matter the risk and informational parameters (as depicted indeed in *Figures* 1 and 2).

Regarding the magnitude of the semi-elasticity, it can be easily verified from *Equation* 22 that:

$$
\begin{aligned}
|\Phi^{-1}(1-r^\theta) - \tfrac{\mu}{\sigma}| \uparrow +\infty \;\; \textit{and} \;\; |\Phi^{-1}(1-r^\theta)| < +\infty \implies \left|\eta\left(r, \tfrac{\mu}{\sigma}\right)\right| \downarrow 0 \\
|\Phi^{-1}(1-r^\theta) - \tfrac{\mu}{\sigma}| \downarrow 0 \;\; \textit{or} \;\; |\Phi^{-1}(1-r^\theta)| \uparrow +\infty \implies \left|\eta\left(r, \tfrac{\mu}{\sigma}\right)\right| \uparrow +\infty
\end{aligned}
\tag{23}
$$

Hence the semi-elasticity of the Gaussian liquidity principle is actually *undefined* when $\Phi^{-1}(1-r^\theta) = \mu/\sigma$, since it converges to magnitudes with opposite signs depending on whether the term $\Phi^{-1}(1-r^\theta) - \mu/\sigma$ approaches to zero *from the right* or *from the left*:

$$
\begin{aligned}
\eta\left(r, \tfrac{\mu}{\sigma}\right) \;\downarrow\; -\infty \quad \textit{when} \quad \Phi^{-1}(1-r^\theta) - \tfrac{\mu}{\sigma} \;\downarrow\; 0 \\
\eta\left(r, \tfrac{\mu}{\sigma}\right) \;\uparrow\; +\infty \quad \textit{when} \quad \Phi^{-1}(1-r^\theta) - \tfrac{\mu}{\sigma} \;\uparrow\; 0
\end{aligned}
$$

On these grounds, we can say that the condition $\Phi^{-1}(1-r^\theta) = \mu/\sigma$ determines a *critical* point, since at this point the sign of the liquidity principle is *undefined* and its magnitude converges to infinite. Liquidity-preference becomes *absolute* under such circumstances.

Let us finally verify whether the sum of income returns preserves the Gaussian liquidity principle. Indeed, consider a series of Gaussian exposures $X_1, \ldots, X_n$ with means $\mu_1, \ldots, \mu_n$ and volatilities $\sigma_1, \ldots, \sigma_n$. Let the individual and aggregate income levels be respectively denoted by $Y_1, \ldots, Y_n$ and Y, with $Y = Y_1 + \cdots + Y_n$.

Replacing the liquidity principles $\lambda_1(r), \ldots, \lambda_n(r)$ according to *Equation* 21, we obtain that the optimal individual cash balances are given by:

$$
L_i(r) \;=\; Y_i \cdot \lambda_i(r) \;=\; Y_i \cdot \Big[\; \sigma_i \;\Phi\left(1 - r^\theta\right) - \mu_i \;\Big] \quad \forall i = 1, \ldots, n
$$

and summing up the individual cash contributions, the following expression is obtained for the aggregate cash balance:

$$
L(r) \;=\; \sum_{i=1}^{n} L_i(r) \;=\; Y \cdot \sum_{i=1}^{n} \omega_i \cdot [\; \sigma_i \;\Phi\left(1 - r^\theta\right) - \mu_i \;] \quad \textit{with} \;\; \omega_i = \frac{Y_i}{Y} \;\forall i
$$

Hence the *aggregate* Gaussian liquidity principle is equal to the optimal liquidity principle related to a Gaussian exposure whose mean and volatility are respectively given by the *weighted average* means and volatilities:

$$
\mu \;=\; \sum_{i=1}^{n} \omega_i \cdot \mu_i \quad \textit{and} \quad \sigma \;=\; \sum_{i=1}^{n} \omega_i \cdot \sigma_i \tag{24}
$$

Dhaene et al. (2002) actually demonstrate that the *comonotonic* sum of Gaussian random variables is also a Gaussian random variable, whose mean and volatility are defined as in *Equation* 24. Then the aggregation of the Gaussian liquidity principle according to *Equations* 21 and 24 is consistent with the aggregation condition established in *Equation* 17.

11 The Capital Market Line Extended

Let us consider some industry whose aggregate capital return is represented by a Gaussian random variable X with mean return μ and volatility σ. As stated in *Sections* 8 and 10, the *optimal* cash balance demanded at the aggregate level must then be given by:

$$L_{\mu,\sigma}(r) = Y \cdot \lambda_{\mu,\sigma}(r)$$

where Y, r and $\lambda_{\mu,\sigma}$ respectively denote the level of income produced by the industry, the cost of capital and the aggregate Gaussian liquidity principle (as determined by *Equation* 21).

Let us additionally assume that when covering their short-term imbalances firms and investors rely on some *secondary* market of capital, and let M denote the total cash balance supplied by lenders.[19] Since the aggregate balance demanded by the industry must be necessarily equal to the total money stock M, the level of the interest rate must adjust at equilibrium in order to satisfy the condition $M = L_{\mu,\sigma}(r)$, i.e.:

$$M = Y \cdot \lambda_{\mu,\sigma}(r) \quad \Leftrightarrow \quad m := \frac{M}{Y} = \sigma \cdot \Phi^{-1}\left(1 - r^{\theta}\right) - \mu$$

where $m = M/Y$ denotes the *relative money supply* or the *relative stock* of cash in the industry. Rearranging terms leads to the equation:

$$\mu \;=\; -m + \Phi^{-1}\left(1 - r^{\theta}\right) \cdot \sigma \tag{25}$$

Therefore, the equilibrium in the market of cash balances implies that the mean return and the volatility of the underlying fund must be related to each other according to a *linear* schedule.

Recall that the balance $L_{\mu,\sigma}(r)$ has been defined as the sum of the surpluses preferred by individuals who seek to maximise the expected return of their *insured* portfolios (as determined by *Equation* 15). In other words, the demand function $L_{\mu,\sigma}(r)$ corresponds to the sum of the stocks of reserves maintained by individuals that build *efficient* portfolios — i.e. that choose *efficient* combinations of risk and cash holdings.

On these grounds, *Equation* 25 can be regarded as an alternative relationship to the *capital market line* (abbreviated *CML*) presented in *Equation* 4. It will then be known as the *capital market line extended* (abbreviated *CML-extended*) in the following. Some important discrepancies regarding the interpretation of the variables and parameters that appear in *Equations* 4 and 25 should be pointed out, however.

In fact, notice in the first place that while in *Equation* 4 the risk-parameters (μ_Z, σ_Z) of the *insured* portfolio are related to each other, the risk-parameters (μ, σ) of the *underlying*

[19] Later in *Section* 12 the monetary equilibrium will be investigated, where Y represents the national output, μ and σ respectively represents the mean return and the volatility of the series of capital returns of Y, and M represents the stock of money determined by the central bank.

fund, which in *Equation* 4 are denoted as (μ_X, σ_X), are related to each other in *Equation* 25. This fact reflects a fundamental disagreement between both theoretical frameworks. Indeed, while the rate of interest and the risk-parameters of the underlying series of capital returns are regarded as *exogenous* variables in the *CML*, which are supposed to remain *unchanged* at least in short-terms, the same variables are *endogenously* determined in the *CML-extended*, in order to equalise the incoming and outgoing cash flows. In other words, the *CML* and the *CML-extended* are respectively corresponded to *static* and *dynamic* approaches to the market equilibrium.

Secondly, recall that in the *CML* the cost of capital is corresponded to the return r_0 offered by a class of risk-free securities, which is supposed to remain unaltered as long as individuals are *price takers* and their preferences — as well as the market conditions in general — are more or less *stable*. In the *CML-extended*, on the other hand, the cost of capital is equal to the risk-free interest rate plus some liquidity *premium*, established by creditors to compensate for the additional risk of *default* (see *Equation* 13).

Having explained the meaning of the variables involved in the *CML* and the *CML-extended*, let us now analyse how the line is determined in both settings. We must then analyse the *slope* of the line and its *intercept* with the mean return's axis.

We already know that the *intercept* of the *CML* is equal to the risk-free interest rate r_0. This is consistent with the fact that in the *CAMP* individuals can always lend their balances to some free-of-default counterpart to obtain the return r_0, independently of the size of the loan.

The *intercept* of the *CML-extended* with the mean return axis, on the other hand, is equal to the additive inverse of the relative stock m. This result is connected to the fact that in the extended model the level of the interest rate may actually be affected when the amount of demanded balances surpasses certain levels. In this context, the relative stock of money (equal to the level of reserves per unit of income) plays the role of a *compensation* for the expected capital loss of the nominal income. In fact, in the particular case when $\sigma = 0$, we obtain that:

$$\mu + m = 0$$

This equation explicitly establishes a *balance* of expected return and cash reserves. It reflects the fact that when $\mu < 0$ the relative stock m can be used to pay back at least some of the realised losses. When $\mu > 0$, by contrast, a pressure exists to sell every outstanding cash balance.

Regarding the *slope* of the line of efficient portfolios, recall that it determines the rate at which a unit of volatility is exchanged by a unit of expected return in the market, or equivalently, it determines the *market price of risk*. Besides, the slope of the *CML* is equal to the *Sharpe ratio* (see *Equations* 4 and 5).

The slope of the *CML-extended*, on the other hand, is equal to the term $R = \Phi^{-1}(1 - r^{\theta})$, which means that the market price of risk and the equilibrium interest rate are simultaneously determined in the extended model:

$$R = \Phi^{-1}\left(1 - r^{\theta}\right) \quad \Leftrightarrow \quad r^{\theta} = 1 - \Phi(R) \tag{26}$$

On these grounds, every *risk* can be interpreted as a *zero coupon* or *discount* bond, which promises a *discounted* payment at some maturity date. In this context, the level of income Y represents the *face* or *nominal* value of the investment, while the interest rate r represents the *internal return* earned by the holder of the instrument. As with zero coupon bonds, the *market price* of risk and its *internal return* are *inversely* related to each other (see Hull, 2000).

Additionally, from *Equation* 25 we obtain that *at equilibrium* the market price of risk R must be simultaneously determined in terms of the risk-parameters of the underlying portfolio and the relative stock m:

$$R = \frac{\mu + m}{\sigma} = \frac{\mu + M/Y}{\sigma} \tag{27}$$

Comparing *Equation* 27 to *Equation* 5 we see that the coefficient R can be indeed regarded as an *extended* measure of risk to the *Sharpe ratio.* However, while the Sharpe ratio is expressed as a reward over the level of the risk-free interest rate r_0, the discount factor R is expressed as a reward over the relative stock $m = M/Y$.

These differences are consistent with the different roles that cash holdings play in both models. Indeed, while in the *CAPM* individuals can always attract deposits if they offer the interest rate r_0 (no matter the size of the deposits), in the extended model the relative stock represents a guarantee maintained in order to compensate for the average capital return μ.

From *Equations* 26 and 27 we conclude that the market price of risk R is actually determined by the equilibrium of two different markets.

Thus, on the one hand, as established in *Equation* 26, the market price of risk is related to the *return* (r) at which short-term loans are offered to the firms in the class and the expectations (θ) of lenders about the credit quality of borrowers. On the other hand, as established in *Equation* 27, the market price of risk determines a *reward* (m) over the level of expected return per unit of volatility. In other words, the market price of risk (and hence the market interest rate) determines the exchange rate of capital for risk that implies the markets of cash balances and securities to be *at equilibrium.*

Accordingly, if the market is found in a certain state of equilibrium, changing the relative stock m necessarily implies that all or some of the variables θ, r, μ, and σ must vary until a new equilibrium is attained. While changes in m and r respectively correspond to *quantity* and *price* adjustments, changes in θ, μ and σ should be more properly interpreted as *structural* adjustments, for they respectively involve changes in the expectations of individuals and in the *riskiness* of the underlying portfolio X.

In conclusion, unlike the *CML* and the *CAPM*, the *CML-extended* (as defined by *Equations* 25, 26 and 27) provides a theoretical framework that is intimately connected to the monetary equilibrium.

In fact, since the total stock of money demanded by the economy is obtained in the extended model by summing up the aggregate balances demanded by the different industries and economic sectors, *Equation* 25 can be used as well to characterise the monetary equilibrium of the economy.

For this purpose, the involved variables should be defined accordingly: thus, the money stock M should be corresponded to some monetary aggregate controlled by the central bank; the level of income Y should be related to the output obtained at the national level, and finally, the risk-parameters μ, σ should be related to the series of capital returns of the national income. Such a model will be next analysed in *Section* 12.

12 Extended Macroeconomic Analysis

Consider some economy that produces the income percentage return $X = \Delta Y/Y$, where Y denotes the level of national income. Let us additionally assume that the return X is distributed as a Gaussian random variable with mean μ and volatility σ. Since in this case the function $\lambda_{\mu,\sigma}(r)$ (defined in *Equation* 21) determines the *optimal* aggregate cash holding $L_{\mu,\sigma}(r) = Y \cdot \lambda_{\mu,\sigma}(r)$ demanded at the aggregate level,[20] the level of the interest rate r and the risk parameters μ, σ must be related to each other at equilibrium, in such a way that the optimal aggregate cash balance is equal to the total stock of money M supplied by the monetary authority:

$$M \ = \ Y \cdot \lambda_{\mu,\sigma}(r) \ = \ P \ y \cdot \Big[\ \sigma \ \Phi^{-1} \Big(1 - r^{\theta} \Big) - \mu \ \Big] \tag{28}$$

where, as in *Equation* 2, the variables P and y respectively denote the level of prices and the level of *real* income. Accordingly, variations in the amount of money M must be followed by changes in any of the variables P, y, r, μ and σ in order to reestablish the monetary equilibrium.

Hence the main difference between the *classic* and the *alternative* theoretical settings describing the monetary equilibrium (respectively characterised by *Equations* 2 and 28) is that national income is regarded as a *random* variable in the alternative setting. Then the risk-parameters μ and σ (which describe the riskiness of the series of capital returns of national income) explicitly affect the preference for liquidity of the economy and are thereby determinants of the monetary equilibrium.

On these grounds, the alternative model of equilibrium can be regarded as an *extended* model.

Let us now investigate how the monetary equilibrium is established in the short-run in the *extended* model. More precisely, we would like to know how the level of the interest rate r adjusts in the short-run in response to variations in the money stock M, assuming that the risk-parameters μ, σ remain unchanged. Applying differences to *Equation* 28 we actually obtain that:

[20] This must be the balance demanded by the economy if it *efficiently* allocates resources, for only in this way the expected output of the economy, as defined in *Equation* 15, is maximised.

$$\frac{\Delta M}{M} = \pi + \xi + \frac{\Delta\lambda_{\mu,\sigma}(r)}{\lambda_{\mu,\sigma}(r)} \quad with \quad \pi := \frac{\Delta P}{P} \quad and \quad \xi := \frac{\Delta y}{y}$$

where π denotes the *rate of inflation*, equal to the percentage variation in the level of prices, and ξ denotes the *growth rate* of the economy, equal to the percentage variation in the level of real output. The equation above can be equivalently expressed in terms of the *semi-elasticity* $\eta(r, \mu/\sigma)$ of the Gaussian liquidity principle with respect to the interest rate (see *Equations* 21 and 22):

$$\frac{\Delta M}{M} - \xi = \pi + \eta\left(r, \frac{\mu}{\sigma}\right) \cdot \Delta r \qquad with \quad \eta\left(r, \frac{\mu}{\sigma}\right) = \frac{1}{\lambda_{\mu,\sigma}(r)} \cdot \frac{d\lambda_{\mu,\sigma}(r)}{dr} \tag{29}$$

Hence the monetary policy chosen by the central bank can be related to some monetary trend that assures a certain path ξ of economic growth (consistent with the rate of growth of productivity in the economy) together with some predetermined (and preferably *low*) level of inflation π (see Friedman, 1968 and 1970, Romer, 1996, Edwards and Sinzdak, 1997, Blanchard, 2005, and also, Howells and Bain, 2005).

Within this context, the levels of inflation and interest rates are respectively corresponded to the *instrument* and the *target* of monetary policy.

Accordingly, when inflation is *above* its target level, the central bank must react by reducing the amount of money M. As long as $\eta(r, \mu/\sigma) < 0$, such policy has the effect of raising the level of interest rates and *cooling* the economy, which are conditions that ultimately reduce inflation. Conversely, when inflation is *below* its target, the central bank must take actions conducting to lowering interest rates, i.e. it must increase the amount of money M. This usually has the effect of accelerating the economy and raising inflation.

During the process, individuals are informed about what the central bank considers the target inflation rate. In this way, the efficiency of the mechanism is increased — eventually leading to increased economic stability.[21]

However (as already stated in *Section* 2), the *efficacy* of the mechanism depends on the *magnitude* of the semi-elasticity $\eta(r, \mu/\sigma)$.

Indeed, notice from *Equation* 29 that the portion of the variation of the money supply that is explained by inflation decreases with the magnitude of $\eta(r, \mu/\sigma)$. In other words, given some fixed rate of economic growth ξ, the *lower* the term $|\eta(r, \mu/\sigma)|$, the *more* monetary interventions are transmitted to inflation — and hence the *more* effective is monetary policy. In the limit when $|\eta(r, \mu/\sigma)| \to 0$ the whole effect is transmitted to the level of prices:

[21] It should be emphasised that *inflation targeting* policies are based on the assumption that inflation is a good estimator of the growth of money supply. Unfortunately, this is not always the case. The most serious exception occurs when price increments are produced by external factors, such as oil and commodity prices in general. Under such conditions, strictly adjusting interest rates may restrict economic growth when it is not necessary to do so.

$$\frac{\Delta M}{M} - \xi \;=\; \pi \qquad \textit{with} \quad \eta\left(r, \frac{\mu}{\sigma}\right) = 0$$

Monetary policy performs *at its best* under such circumstances.

By contrast, the *greater* the term $|\eta(r, \mu/\sigma)|$ in *Equation* 29, the *more* the variations in the money stock are explained by means of changes in the liquidity preference of individuals and hence, the *less* effective is monetary policy to induce the desired inflation rate. In the limit when $|\eta(r, \mu/\sigma)| \to \infty$, variations in the amount of money have no effect on interest rates and hence, monetary policy is *useless* under such circumstances — recall that liquidity preference is *absolute* in this case.

The magnitude of the semi-elasticity can be precisely determined in the case of the Gaussian liquidity principle.

In fact, as stated in *Section* 10, when the series of income returns follows a Gaussian probability distribution, *low* and *high* semi-elasticities are corresponded to specific states of the market characterised by the level of the interest rate and the risk-parameters μ, σ.

Thus, on the one hand, as established in *Equation* 23, if $|\Phi^{-1}(1 - r^\theta)| < +\infty$, i.e. if $r^\theta > 0$ and $r^\theta < 1$, then:

$$\left|\Phi^{-1}\left(1 - r^\theta\right) - \frac{\mu}{\sigma}\right| \to +\infty \quad \Longrightarrow \quad \left|\eta\left(r, \frac{\mu}{\sigma}\right)\right| \to 0$$

Therefore, as long as $0 < r^\theta < 1$, the magnitude of the semi-elasticity is diminished both when the magnitude of the income's expected return is increased (no matter the sign of the expected return) and when the volatility of income is reduced. Accordingly, the monetary mechanism is *more effective* in economies that produce *higher* expected returns (both when *positive* and *negatives* returns are obtained) and show *lower* variability.

On the other hand, from *Equation* 22 we obtain that the magnitude of the semi-elasticity converges to infinite when the level of the *corrected* interest rate converges to zero or one:

$$r^\theta \downarrow 0 \;\; \textit{or} \;\; r^\theta \uparrow 1 \quad \Longrightarrow \quad \left|\Phi^{-1}\left(1 - r^\theta\right)\right| \to +\infty \quad \Longrightarrow \quad \left|\eta\left(r, \frac{\mu}{\sigma}\right)\right| \to +\infty$$

The same result is obtained when $|\Phi^{-1}(1 - r^\theta) - (\mu/\sigma)| \to 0$, i.e.:

$$\Phi^{-1}(1 - r^\theta) = \frac{\mu}{\sigma} \quad \Longrightarrow \quad r^\theta = 1 - \Phi\left(\frac{\mu}{\sigma}\right) \quad \Longrightarrow \quad \left|\eta\left(r, \frac{\mu}{\sigma}\right)\right| = \infty$$

Then the magnitude of the semi-elasticity is equal to infinite when the *corrected* interest rate attains any of the values $r^\theta = 0$, $r^\theta = 1$ or $r^\theta = 1 - \Phi(\mu/\sigma)$. Consequently, in any of these states the preference for liquidity of the economy is *absolute* and hence, monetary policy is useless for dealing with price and output fluctuations. On these grounds, these interest rates values are corresponded to *critical* states of the economy.

Other complications may arise when implementing the monetary mechanism due to the dependence of the cost of capital on the market expectations and the riskiness of national output.

Indeed, recall that the market interest rate r must lie in the interval determined by the risk-free interest rate r_0 and the liquidity premium $r_{\theta,X}$ (where the liquidity premium depends on the benefit lost from maintaining cash holdings instead of investing on risk, see *Equations* 11 and 13 and the related discussion in *Section* 6), in such a way that:

$$r_0 \;\leq\; r \;\leq\; r_0 + r_{\theta,X}$$

In this context, the returns r and $r_0+r_{\theta,X}$ denote the cost of capital as perceived by lenders and borrowers respectively. Accordingly, individuals prefer to maintain cash holdings and do not rely on capital markets to fit their balances when $r > r_0 + r_{\theta,X}$, because in this case the cost imposed by lenders is *too expensive* for them.

As a consequence, if the premium $r_{\theta,X}$ is diminished (i.e. if the income surplus over the level of reserves is reduced, see *Equation* 11) until the borrowers' perceptions of the cost of capital is under the lenders' estimations of it (i.e. until $r_0 + r_{\theta,X} < r$), people will be induced to modify their funding strategies, moving from *external* to *internal* financing — i.e. moving from *debt* to *capital*. By contrast, if the premium $r_{\theta,X}$ is increased, then the profit that is obtained from relying on capital markets instead of keeping cash holdings (equal to $r_0 + r_{\theta,X} - r$) will be augmented, and hence the incentives to replace capital by debt will be incremented.

In other words, in the extended model the monetary equilibrium can be affected by changes in the expectations of individuals and in the riskiness of national output, which in the case of Gaussian risks are reflected in the risk-parameters μ, σ. Such adjustments are manifested as fluctuations in the amount of funds *demanded* at the aggregate level.[22]

Finally, recall that in the extended model the riskiness of national income is expressed in terms of the riskiness of the outputs produced by the portfolios held by individuals at different aggregation levels (as stated in *Equations* 17, 18 and 19). Consequently, the variability of income at the economic level might be induced by a single industry or economic sector — in such a way that, in particular, the volatility and the mean return of national income might be determined by a single industry or economic sector. Hence, the possibility of *contagion* naturally arises in the model.[23]

[22]Recall that in the model creditors are regarded as *price takers*, who set the price of loans based on the *credit class* the borrower belongs to according to the market, see *Equation* 15 and the related discussion.

[23]Some recent studies emphasise the role of *aggregation* in explaining macroeconomic and financial *stability*. Thus, for example, Calza and Sousa (2003) postulate that considering aggregation effects it is possible to explain why the money demand has been *more stable* in the euro are than in other large economies. The fact that Germany has a large weight in the $M3$ aggregate for the euro area and that the money demand has been historically stable in that country contributes to support such hypothesis. In other words, the stability of the German economy is supposed to be shared by the rest of the economies in the block, as a positive externality.

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