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Suen, Richard M. H.

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Abstract

The constant-relative-risk-aversion (CRRA) utility function is now predominantly used in quantitative macroeconomic studies. This function, however, is not bounded and thus creates problems when applying the standard tools of dynamic programming. This paper devises a method for "bounding" the CRRA utility functions. The proposed method is based on a set of conditions that can establish boundedness among a broad class of utility functions. These results are then used to construct a bounded utility function that is identical to a CRRA utility function except when consumption is very small or very large. It is shown that the constructed utility function also satisfies the Inada condition and is consistent with balanced growth.

Keywords: Utility Function – Elasticity of Marginal Utility – Boundedness
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*Department of Economics, Sproul Hall, University of California, Riverside, CA 92521-0427. Email: *richard.suen@ucr.edu*.

1 Introduction

It is now a standard practice in macroeconomics to postulate that preferences over consumption streams are additively separable over time, and in each time period consumption is evaluated by a constant-relative-risk-aversion (CRRA) utility function.[1] Besides analytical simplicity, this particular functional form has two other appealing features. First, it satisfies the Inada condition which states that marginal utility of consumption approaches infinity when consumption approaches zero. This condition ensures that it is never optimal to have zero consumption at any point of time and hence one can focus on interior solutions. Second, for a large class of economic growth models, a constant intertemporal elasticity of substitution (IES) is sufficient to ensure the existence of balanced growth equilibria. Since this type of equilibria is consistent with the stylized facts of economic growth, this feature is often cited as the main justification for using the CRRA utility function.[2]

One drawback of the CRRA utility function is that it is not bounded. This creates problems in applying the tools of dynamic programming to models using this kind of utility function. The standard theory of dynamic programming establishes that a unique solution to the Bellman equation exists and can be obtained through successive iterations of a contraction mapping. These results, however, are based on the assumption of bounded utility function.[3] Various attempts have been made to extend these results to the unbounded case in deterministic models (Boyd 1990; Alvarez and Stokey 1998; Durán 2000; Le Van and Morhaim 2002; and Rincón-Zapatero and Rodríguez-Palmero 2003). The upshot of these efforts is a set of additional conditions under which the basic results of dynamic programming are valid when the utility function is unbounded. However, it remains unclear whether these results would hold in stochastic models. When the utility function is unbounded, expected utility and the expectation of the value function may be unbounded as well.[4] In order to avoid this problem, theoretical studies on stochastic economies typically adopt a bounded utility function (see, among others, Aiyagari 1993; Huggett 1997; Huggett and Ospina 2001; and Miao 2006). But the CRRA utility function is still predominantly used in quantitative studies.

In order to bridge this gap between theoretical and quantitative analysis. this paper

[1] The standard CRRA utility function is given by $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$. In a dynamic stochastic setting, σ is both the coefficient of relative risk aversion and the inverse of the intertemporal elasticity of substitution. Since $\sigma = -cu''(c)/u'(c)$, it is also the elasticity of marginal utility for consumption. These three terms are used interchangeably in this paper.

[2] To ensure the existence of these paths, it actually suffice to have a constant IES when consumption is large. See Palivos, Wang and Zhang (1997) for a formal proof using the AK growth model. See, also, Steger (2007) for an analytical example.

[3] More precisely, the utility function is required to be bounded over the set of feasible choices. This assumption permeates all the results presented in Stokey, Lucas and Prescott (1989) Section 4.2 and Section 9.2, which are now considered standard by macroeconomists.

[4] See Geweke (2001) for some examples.

proposes a method to “bound” the CRRA utility functions. More specifically, this paper proposes a method to construct a bounded utility function that is identical to a CRRA utility function except when consumption is very small or very large. The current study begins by deriving a set of conditions under which any twice continuously differentiable utility function is bounded, satisfying the Inada condition and consistent with balanced growth. The theoretical analysis is motivated by the following observations. First, a strictly increasing utility function is either unbounded at the origin or unbounded at infinity. Second, the Inada condition is a property of the utility function when consumption is very close to zero. Finally, in order to ensure the existence of balanced growth equilibrium paths, it suffice to have a constant IES *along those paths* where consumption is growing indefinitely. In other words, all these are properties of the utility function when consumption is either very small or very large. It is thus possible to formulate two sets of assumptions, one governing the utility function when consumption is small and the other when consumption is large, so as to obtain all the desired properties. To achieve this, the current study begins with a broad class of utility functions which share the same elasticity of marginal utility. This elasticity can be any bounded, continuous, nonnegative function of consumption which converges to a positive constant when consumption approaches infinity. The last condition ensures that the underlying utility functions are consistent with balanced growth. It is shown that, under some mild additional restrictions, these utility functions are bounded and satisfy the Inada condition. This class of utility functions thus shares the same appealing features as the CRRA utility functions. Since these new utility functions are bounded, they will not cause any problem when applying the standard tools of dynamic programming. This offers a wide range of alternative utility functions that one can use in quantitative analysis.

The theoretical results presented in this paper also provide a set of guidelines for constructing upper bounds and lower bounds for the CRRA utility functions. Consider as an example a CRRA utility function with coefficient of relative risk aversion greater than or equal to one. Suppose now the constant-relative-risk-aversion assumption is relaxed in the vicinity of the origin, say over a certain interval $[0, \bar{c}]$, but remains intact beyond this range. It is shown that such a specification would generate a class of utility functions that are bounded at the origin, satisfy the Inada condition and display CRRA for $c \geq \bar{c}$. With a suitable normalization, a utility function that coincides with the original CRRA utility function for $c \geq \bar{c}$ can be obtained. Most importantly, the value of $\bar{c}$ can be made arbitrarily small so that the new utility function is “almost identical” to the original CRRA utility function.

The rest of this paper is organized as follows. Section 2 presents the theoretical results underlying the proposed method. Section 3 provides a demonstration of this method. This is followed by some concluding remarks in section 4.

2 Theoretical Results

Let $\sigma(c)$ be a real-valued function defined on the positive real line. Let $\mathcal{U}(\sigma)$ be the class of functions that are (i) twice differentiable for all $c > 0$, (ii) strictly increasing and (iii) solves the following second-order differential equation:

$$u''(c) + \frac{\sigma(c)}{c} u'(c) = 0, \qquad \text{for all } c > 0. \tag{1}$$

If $u(c)$ is an element of $\mathcal{U}(\sigma)$, then any linear transformation $v(c) = au(c) + b$, with $a > 0$, is also an element of $\mathcal{U}(\sigma)$. If $u(c)$ is a utility function, then $\sigma(c)$ is the elasticity of marginal utility. In a stochastic environment, $\sigma(c)/c$ is the coefficient of absolute risk aversion and $\sigma(c)$ is the coefficient of relative risk aversion. It is a common practice to characterize utility functions through the elasticity of marginal utility (or through the absolute risk aversion). The most well-known example is the CRRA utility function which can be derived from (1) by imposing the restriction: $\sigma(c)$ is constant for all $c \geq 0$. More generally, the class of HARA utility functions can be obtained from (1) by imposing

$$\frac{\sigma(c)}{c} = \frac{1}{\alpha + \gamma c}, \qquad \text{for } \alpha + \gamma c > 0.$$

A similar approach is adopted in the current analysis. The task at hand is to formulate a set of conditions on $\sigma(c)$ under which any function in $\mathcal{U}(\sigma)$ is bounded, satisfying the Inada condition and consistent with balanced growth. Throughout this section, the function $\sigma(c)$ is assumed to have the following properties:

Assumption A1 $\sigma(c)$ is bounded, continuous and non-negative for all $c \geq 0$.

Assumption A2 $\lim_{c \to \infty} \sigma(c)$ exists and is given by $\widetilde{\sigma} \in (0, \infty)$.

Assumption A1 has a number of implications. First, it ensures that the initial value problem (1), with initial conditions $u(c_0) = u_0$ and $u'(c_0) = z_0 > 0$ for any $c_0 > 0$, has a unique solution on $[c_0, \infty)$.[5] Thus $\mathcal{U}(\sigma)$ is non-empty under Assumption A1. Second, it ensures that any function in $\mathcal{U}(\sigma)$ is twice continuously differentiable and strictly concave. This means under Assumption A1, all the functions in $\mathcal{U}(\sigma)$ satisfy the "usual" properties of a utility function, including continuity, monotonicity, concavity and differentiability. Third, this assumption implies that, in the vicinity of the origin, the marginal utility function $u'(c)$ is bracketed between two strictly decreasing power functions.[6] This result is summarized in

[5] The existence and uniqueness theorem for linear second-order initial value problems can be found in almost any textbook on ordinary differential equations. See, for instance, Boyce and DiPrima (1997) Theorem 3.2.1. Note that only continuity of the function $\sigma(c)/c$ is required in order to apply this theorem.

[6] The bracketing functions are specific to the underlying utility function. In other words, different functions in the same class, for example $u(c)$ and $v(c) = au(c) + b$, would have different bracketing functions. This point is made clear in the proof of Lemma 1.

Lemma 1. All proofs can be found in the Appendix.

Lemma 1 *For any $\varepsilon > 0$, define $\overline{\sigma}_\varepsilon \equiv \sigma(0) + \varepsilon$ and $\underline{\sigma}_\varepsilon \equiv \sigma(0) - \varepsilon$. Under Assumption A1, there exists $x > 0$ such that for all $c \in (0, x]$,*

$$\eta_1(x, \varepsilon) c^{-\underline{\sigma}_\varepsilon} \leq u'(c) \leq \eta_2(x, \varepsilon) c^{-\overline{\sigma}_\varepsilon}, \tag{2}$$

where $\eta_1(x, \varepsilon)$ and $\eta_2(x, \varepsilon)$ are two strictly positive constants depending on x and ε.

If $\sigma(0)$ is strictly positive, then it is possible to set $\underline{\sigma}_\varepsilon > 0$ in Lemma 1. It follows that the lower bound in (2), and hence $u'(c)$, would become infinite when c approaches zero. Thus Assumption A1 and $\sigma(0) > 0$ are sufficient to establish the Inada condition. This result is summarized in Proposition 2.

Proposition 2 *Suppose Assumption A1 is satisfied. If $\sigma(0) > 0$, then every utility function $u(c)$ in $\mathcal{U}(\sigma)$ satisfies the Inada condition, i.e., $\lim_{c \to 0} u'(c) = +\infty$.*

The next proposition states the conditions under which any utility function in $\mathcal{U}(\sigma)$ is bounded or unbounded below.

Proposition 3 *Suppose Assumption A1 is satisfied.*

(i) If there exists $x > 0$ such that $\sigma(c) \geq 1$ for $c \in [0, x]$, then every utility function $u(c)$ in $\mathcal{U}(\sigma)$ is not bounded below, i.e., $u(0) = -\infty$.

(ii) If there exists $x > 0$ such that $\sigma(c) < 1$ for $c \in [0, x]$, then every utility function $u(c)$ in $\mathcal{U}(\sigma)$ is bounded below, i.e., $u(0) > -\infty$.

The first part of this proposition states that a utility function is unbounded below if there exists a neighborhood of the origin in which the elasticity of marginal utility is greater than or equal to one. Any CRRA utility function with $\sigma \geq 1$ clearly satisfies this condition and is thus unbounded below. More generally, if $\sigma(c)$ satisfies Assumption A1 and $\sigma(0) > 1$, then the condition in part (i) is satisfied and consequently the underlying utility functions are unbounded below. This result is summarized in the first part of Corollary 4. The second part of Proposition 3 states that a utility function is bounded below if the elasticity of marginal utility is strictly less than one when c is very close to zero. This includes all the CRRA utility functions with $\sigma < 1$. It also includes any utility function with $\sigma(c)$ that satisfies Assumption A1 and $\sigma(0) \in (0, 1)$. This result is stated in the second part of Corollary 4.

Corollary 4 *Suppose Assumption A1 is satisfied.*

(i) If $\sigma(0) > 1$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is not bounded below.*

(ii) If $\sigma(0) \in (0,1)$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is bounded below.*

Proposition 3 and its corollary together imply the following. Suppose Assumption A1 is satisfied and $\sigma(0) > 0$. Then any utility function in $\mathcal{U}(\sigma)$ is bounded below only if either (i) $\sigma(0) \in (0,1)$, or (ii) $\sigma(0) = 1$ and $\sigma(c) < 1$ within a neighborhood of the origin. In order words, the assumption of bounded utility function would imply certain restrictions on the altitude towards risk and intertemporal substitution when consumption is small.

We now turn to the properties of $u(c)$ when c is large. Proposition 5 states the conditions under which $u(c)$ is bounded or unbounded above. Part (i) of this proposition establishes that a utility function is bounded above if the elasticity of marginal utility is strictly greater than one when c is sufficiently large. This includes any CRRA utility function with $\sigma > 1$. The second part of this proposition states that a utility function is unbounded above if the elasticity is no greater than one when c is large. This includes any CRRA utility function with $\sigma \leq 1$.

Proposition 5 *Suppose Assumption A1 is satisfied.*

(i) If there exists $x \geq 0$, *such that* $\sigma(c) > 1$ *for* $c \geq x$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is bounded above.*

(ii) If there exists $x \geq 0$, *such that* $\sigma(c) \leq 1$ *for* $c \geq x$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is not bounded above.*

Clearly if $\lim_{c\to\infty} \sigma(c)$ exists and is strictly greater than one, then the condition in the first part of Proposition 5 is satisfied. On the other hand, if the limit is no greater than one then the condition in the second part is fulfilled. These results are summarized in Corollary 6. Similar to Propositions 3 and its corollary, these results show that the assumption of bounded utility function would imply certain restrictions on the altitude towards risk and intertemporal substitution when consumption is large.

Corollary 6 *Suppose Assumptions A1 and A2 are satisfied.*

(i) If $\widetilde{\sigma} > 1$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is bounded above.*

(ii) If $\widetilde{\sigma} \in (0,1]$, *then every utility function* $u(c)$ *in* $\mathcal{U}(\sigma)$ *is not bounded above.*

We conclude this section by proposing a class of bounded utility functions that inherit the appealing features of the CRRA utility functions. Let $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a real-valued function that satisfies Assumptions A1 and A2. In addition, suppose the condition $\widetilde{\sigma} > 1 > \sigma(0) > 0$ is satisfied. Then according to the theoretical results presented above, any utility function in $\mathcal{U}(\sigma)$ is bounded, satisfies the Inada condition and displays constant IES when consumption approaches infinity.

3 Bounding the CRRA Utility Functions

Consider the typical CRRA utility function,

$$v(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \text{for all } c \geq 0. \tag{3}$$

This function is not bounded below when $\sigma \geq 1$ and is not bounded above when $\sigma \leq 1$. This section describes a method for constructing a lower bound for the case when $\sigma \geq 1$. Empirical studies on IES typically report an estimate that falls within this range.[7] For this reason, these values of σ are commonly used in calibrated models. The same approach can be used to construct an upper bound for the case when $\sigma \leq 1$.

The goal in here is to construct a utility function u that has the following properties: (i) it is bounded below, (ii) it satisfies the Inada condition and (iii) there exists $\overline{c} > 0$ such that $u(c) = v(c)$ for $c \geq \overline{c}$. In particular, the value of $\overline{c}$ can be made arbitrarily small so that $u(c)$ is "almost identical" to $v(c)$.[8] The third property also ensures that u is consistent with balanced growth. In order to establish the third property, the utility function $u(c)$ must display constant relative risk aversion over the range $[\overline{c}, \infty)$. The next proposition provides the necessary and sufficient condition for this to be true. The proof is standard and is thus omitted.

Proposition 7 *The following statements are equivalent:*

(i) $\sigma(c) = \sigma$ *for all* $c \geq \overline{c} > 0$.

(ii) The utility functions in $\mathcal{U}(\sigma)$ *can be expressed as*

$$u(c) = \eta_1 + \eta_2 \frac{c^{1-\sigma}}{1-\sigma}, \qquad \textit{for } c \geq \overline{c},$$

where η_1 *is an arbitrary constant and* $\eta_2 = u'(\overline{c})\,\overline{c}^{\sigma} > 0$.

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[7] See, for instance, Ogaki and Reinhart (1998) and Vissing-Jorgensen (2002).

[8] It is tempting to call $u(c)$ an approximation of $v(c)$. We choose not to do so because at the origin, $v(0) = -\infty < u(0)$. This means the approximating error at the origin is always infinitely large.

Begin with a real-valued function $\widehat{\sigma}(c)$ that is defined on the positive real line. According to Proposition 2 and Corollary 4, if $\widehat{\sigma}(c)$ satisfies Assumption A1 and $\widehat{\sigma}(0) \in (0,1)$, then any utility function $u(c)$ in $\mathcal{U}(\widehat{\sigma})$ is bounded below and satisfies the Inada condition. If, in addition, $\widehat{\sigma}(c)$ is constant over the range $[\overline{c}, \infty)$ then $u(c)$ is a CRRA utility function over this range. Clearly, this condition implies Assumption A2. Thus, $\mathcal{U}(\widehat{\sigma})$ is a subclass of the utility functions defined at the end of section 2. In general, $\widehat{\sigma}(c)$ can be non-monotonic or non-differentiable over the interval $(0, \overline{c})$. A simple piecewise linear function is adopted in here for illustrative purposes. Select two parameter values $\sigma_0 \in (0,1)$ and $\overline{c} > 0$. The function $\widehat{\sigma}(c)$ is assumed to be given by

$$\widehat{\sigma}(c) = \begin{cases} \sigma_0 + \psi c & \text{for } c \in [0, \overline{c}] \\ \sigma & \text{for } c \geq \overline{c}, \end{cases} \tag{4}$$

where $\sigma \geq 1$ and $\psi \equiv (\sigma - \sigma_0)/\overline{c} > 0$.

As shown in Proposition 7, the utility function $u(c)$ can be expressed as

$$u(c) = \eta_1 + \eta_2 \frac{c^{1-\sigma}}{1-\sigma}, \qquad \text{for } c \geq \overline{c}.$$

Set $\eta_1 = 0$ and $\eta_2 = u'(\overline{c})\,\overline{c}^{\sigma} = 1$ so that $u(c)$ is equivalent to the CRRA utility function in (3) for $c \geq \overline{c}$. For $c \in [0, \overline{c}]$, the utility function is given by

$$u(c) = \zeta - \lambda \int_c^{\overline{c}} \exp(-\psi z)\, z^{-\sigma_0} dz, \tag{5}$$

where $\lambda \equiv \exp(\sigma - \sigma_0)(\overline{c})^{-(\sigma-\sigma_0)} > 0$. The derivations of (5) are shown in the Appendix. Continuity of $u(c)$ at $c = \overline{c}$ then requires $\zeta = \overline{c}^{1-\sigma}/(1-\sigma)$. It is straightforward to check that $u'(c)$ and $u''(c)$ are both continuous at $\overline{c}$. According to Proposition 2, this utility function is bounded below at the origin. In particular, $u(0)$ is given by

$$u(0) = \frac{\overline{c}^{1-\sigma}}{1-\sigma} - \lambda \int_0^{\overline{c}} \exp(-\psi z)\, z^{-\sigma_0} dz,$$

which is finite.

4 Final Remarks

The main contribution of this paper is two-fold. First, this paper proposes a class of utility functions that are bounded, satisfying the Inada condition and consistent with balanced growth. This class of utility functions represents a wide range of alternatives to the CRRA utility functions for quantitative analysis. Second, this paper devises a method for "bounding" the CRRA utility functions. The proposed method involves constructing a bounded utility function that is identical to a given CRRA utility function except when consumption is very small or very large.

The proposed method is particularly useful for constructing a lower bound for CRRA utility functions with coefficient of relative risk aversion strictly greater than one. In this case, a bounded utility function can be obtained by relaxing the CRRA assumption within the vicinity of the origin. In other words, the proposed method would incur a modification in the altitude towards risk and intertemporal substitution when consumption is very close to zero. Since the range of consumption values affected can be made arbitrarily small, such a modification should be innocuous in most applications. In principle, the same method can be used to construct an upper bound for CRRA utility functions with coefficient of relative risk aversion strictly less than one. However, in most applications an upper bound for this type of CRRA utility functions is not necessary. In this case, the standard results of deterministic dynamic programming can be recovered by using the weighted contraction mapping theorem described in Boyd (1990). Becker and Boyd (1997, p.135-138) explain how these results can be used to solve an one-sector neoclassical growth model with CRRA utility function and the coefficient of relative risk aversion is strictly less than one.

Appendix

Preliminaries

The following theorems regarding improper integrals are useful in our proofs. The proofs of these theorems can be found in Widder (1989) Chapter 10.

Theorem 8 *Let f be a real-valued continuous function defined on the interval $(a, b]$. If* $\lim_{c \to a+} (c - a) f(c) = A$, *for some* $A \neq 0$, *then* $\int_a^b f(c)\, dc = +\infty$.

Theorem 9 *Let f be a real-valued continuous function defined on the half-line $[a, \infty)$. If* $\lim_{c \to \infty} cf(c) = A$, *for some* $A \neq 0$, *then* $\int_a^\infty f(c)\, dc = +\infty$.

Theorem 10 *Let f be a real-valued continuous function defined on the interval $(a, b]$. If* $\lim_{c \to a+} c^\alpha f(c)$ *exists for some* $\alpha \in (0, 1)$, *then* $\int_a^b f(c)\, dc$ *is finite.*

Proof of Lemma 1

The following proof is similar to the proof of Lemma 1 in Barelli and Pessôa (2003). For any $\varepsilon > 0$, define $\overline{\sigma}_\varepsilon \equiv \sigma(0) + \varepsilon$ and $\underline{\sigma}_\varepsilon \equiv \sigma(0) - \varepsilon$. Since $\sigma(c)$ is continuous at $c = 0$, there exists $x > 0$ such that

$$\underline{\sigma}_\varepsilon \leq \sigma(c) \leq \overline{\sigma}_\varepsilon, \qquad \text{for all } c \in (0, x],$$

which implies

$$\frac{\underline{\sigma}_\varepsilon}{c} \leq -\frac{1}{u'(c)} \frac{du'(c)}{dc} = \frac{\sigma(c)}{c} \leq \frac{\overline{\sigma}_\varepsilon}{c}, \qquad \text{for all } c \in (0, x].$$

Since $\sigma(c)/c$ is bounded and continuous on $(0, x]$, it is also integrable. Integrating the above inequalities over the interval $[c, x]$ gives

$$\ln\left[\left(\frac{c}{x}\right)^{-\underline{\sigma}_\varepsilon}\right] \leq \ln\left[\frac{u'(c)}{u'(x)}\right] \leq \ln\left[\left(\frac{c}{x}\right)^{-\overline{\sigma}_\varepsilon}\right],$$

which implies

$$u'(x)\left(\frac{c}{x}\right)^{-\underline{\sigma}_\varepsilon} \leq u'(c) \leq u'(x)\left(\frac{c}{x}\right)^{-\overline{\sigma}_\varepsilon}, \qquad \text{for all } c \in (0, x], \tag{6}$$

Note that $\eta_1(x, \varepsilon) \equiv u'(x)\, x^{\underline{\sigma}_\varepsilon}$ and $\eta_2(x, \varepsilon) \equiv u'(x)\, x^{\overline{\sigma}_\varepsilon}$ are both strictly positive and specific to the function $u(c)$. This completes the proof of Lemma 1.

Proof of Proposition 3

Part (i) Suppose there exists $x > 0$ such that $\sigma(c) \geq 1$ for all $c \in [0, x]$. Then following the same steps as in the proof of Lemma 1, one can obtain

$$-\frac{du'(c)}{u'(c)} \geq \frac{dc}{c}, \qquad \text{for all } c \in (0, x].$$

Integrating this over the range $[c, x]$ gives

$$u'(c) \geq u'(x) \left(\frac{c}{x}\right)^{-1}, \qquad \text{for all } c \in (0, x].$$

Since $\lim_{c \to 0} \left[c \cdot u'(x) \left(\frac{c}{x}\right)^{-1}\right] = u'(x)\, x \neq 0$, it follows from Theorem 8 that

$$\lim_{c \to 0} \int_c^x u'(x) \left(\frac{y}{x}\right)^{-1} dy = +\infty.$$

Hence

$$\lim_{c \to 0} \int_c^x u'(y)\, dy = u(x) - \lim_{c \to 0} u(c) = +\infty.$$

Since $u(\cdot)$ is continuous on $[0, \infty)$, $u(x)$ must be of finite value for any $x > 0$. Thus $u(0) = -\infty$. This completes the proof of part (i).

Part (ii) Suppose there exists $x > 0$ such that $\sigma(c) < 1$ for all $c \in [0, x]$. Then there exists $\widehat{\sigma} < 1$ such that $\sigma(c) \leq \widehat{\sigma}$ for all $c \in (0, x]$. Following the same steps as above, we have

$$u'(c) \leq u'(x) \left(\frac{c}{x}\right)^{-\widehat{\sigma}}, \qquad \text{for all } c \in (0, x].$$

It follows that

$$\lim_{c \to 0} \int_c^x u'(y)\, dy \leq \lim_{c \to 0} \int_c^x u'(x) \left(\frac{y}{x}\right)^{-\widehat{\sigma}} dy = \frac{u'(x)\, x}{1 - \widehat{\sigma}} < +\infty.$$

The desired result follows immediately from the following

$$\lim_{c \to 0} \int_c^x u'(y)\, dy = u(x) - \lim_{c \to 0} u(c) < +\infty.$$

This completes the proof of Proposition 3.

Proof of Proposition 5

Part (i) If there exists $x \geq 0$ such that $\sigma(c) > 1$ for all $c \geq x$, then there must exist $\widehat{\sigma} > 1$ such that $\sigma(c) \geq \widehat{\sigma}$ for all $c \geq x$. This implies

$$u'(c) \leq u'(x) \left(\frac{c}{x}\right)^{-\widehat{\sigma}}, \qquad \text{for all } c \geq x. \tag{7}$$

Since

$$\lim_{c \to \infty} \int_x^c u'(x) \left(\frac{y}{x}\right)^{-\widehat{\sigma}} dy = \frac{u'(x)\, x}{\widehat{\sigma} - 1} < +\infty.$$

It follows from (7) that

$$\lim_{c \to \infty} \int_x^c u'(y)\, dy = \lim_{c \to \infty} u(c) - u(x) < +\infty.$$

Hence the desired result.

Part (ii) If there exists $x \geq 0$ such that $\sigma(c) \leq 1$ for all $c \geq x$, then the following inequality holds for all $c \geq x$,

$$u'(c) \geq u'(x) \left(\frac{c}{x}\right)^{-1}.$$

Since $\lim_{c \to \infty} \left[c \cdot u'(x) \left(\frac{c}{x}\right)^{-1}\right] = u'(x)\, x \neq 0$, it follows from Theorem 9 that

$$\lim_{c \to \infty} \int_x^c u'(x) \left(\frac{y}{x}\right)^{-1} dy = +\infty.$$

Hence

$$\lim_{c \to \infty} \int_x^c u'(y)\, dy = \lim_{c \to \infty} u(c) - u(x) = +\infty$$

and the desired result. This completes the proof of Proposition 5.

Derivations of Equation (5)

Consider the following initial value problem,

$$u''(c) + \frac{\sigma(c)}{c} u'(c) = 0, \qquad \text{for } c \geq c_0 > 0,$$

with $u'(c_0) > 0$ given. The choice of c_0 and $u'(c_0)$ are immaterial. Since $\sigma(c)/c$ for $c > 0$ is continuous, a unique solution exists and is given by

$$u'(c) = u'(c_0) \exp\left[-\int_{c_0}^c \frac{\sigma(z)}{z} dz\right], \qquad \text{for } c \geq c_0.$$

For any $c \in [c_0, \overline{c}]$,

$$\begin{aligned} u'(\overline{c}) &= u'(c_0)\exp\left[-\int_{c_0}^{c}\frac{\sigma(z)}{z}dz\right]\exp\left[-\int_{c}^{\overline{c}}\frac{\sigma(z)}{z}dz\right] \\ &= u'(c)\exp\left[-\int_{c}^{\overline{c}}\frac{\sigma(z)}{z}dz\right]. \end{aligned}$$

Hence for any $c \in (0, \overline{c}]$,

$$u'(c) = u'(\overline{c})\exp\left[\int_{c}^{\overline{c}}\frac{\sigma(z)}{z}dz\right].$$

According to (4), we have

$$\int_{c}^{\overline{c}}\frac{\sigma(z)}{z}dz = (\sigma - \sigma_0) - \psi c + \sigma_0(\ln\overline{c} - \ln c),$$

$$\Rightarrow \exp\left[\int_{c}^{\overline{c}}\frac{\sigma(z)}{z}dz\right] = \exp(\sigma - \sigma_0)(\overline{c})^{\sigma_0}\exp(-\psi c)\, c^{-\sigma_0},$$

where $\psi \equiv \frac{\sigma - \sigma_0}{\overline{c}} > 0$. Hence

$$u'(c) = \exp(\sigma - \sigma_0)(\overline{c})^{\sigma_0} u'(\overline{c})\exp(-\psi c)\, c^{-\sigma_0}.$$

Since we adopt the normalization $\eta_2 = u'(\overline{c})\,\overline{c}^{\sigma} = 1$, thus

$$u'(c) = \lambda\exp(-\psi c)\, c^{-\sigma_0},$$

where $\lambda \equiv \exp(\sigma - \sigma_0)(\overline{c})^{-(\sigma - \sigma_0)} > 0$. Integrating this over the range $[c, \overline{c}]$ for any $c > 0$ gives

$$u(c) = \frac{(\overline{c})^{1-\sigma}}{1-\sigma} - \lambda\int_{c}^{\overline{c}}\exp(-\psi z)\, z^{-\sigma_0}dz.$$

Note that $\int_0^{\overline{c}}\exp(-\psi z)\, z^{-\sigma_0}dz$ is another improper integral as $\exp(-\psi z)\, z^{-\sigma_0} \to \infty$ when $z \to 0$. However, $\lim_{z\to 0}\left[z^{\sigma_0}\cdot\exp(-\psi z)\, z^{-\sigma_0}\right] = 0$ and $\sigma_0 \in (0, 1)$. Hence it follows from Theorem 10 that the integral

$$\int_{0}^{\overline{c}}\exp(-\psi z)\, z^{-\sigma_0}dz$$

is finite.

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