When risk weights increase the risk: some concerns for capital regulation

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Abstract

In this chapter I argue that as a response to the introduction of capital requirements in the form of risk weights investors might potentially choose riskier portfolios than before the regulation – this is, presumably, not what the regulation intends to achieve. That is, while regulation most likely diverts investors from their optimum decision it does not guarantee that the new optimum has a lower risk. The effect of the regulation depends on several things, most importantly the correlation between individual investments, investor preferences and the relative size of risk weights.

I. Introduction

In this paper I examine the question whether introducing capital requirement by the application of risk weights it is true that the new optimal portfolio or investment mix of regulated entities is necessarily of lower risk than before the regulation. I show that it is not the case under all circumstances and that the only affect of the regulation that can be taken as granted is that it imposes a new, worse optimum on investors (of course, whenever it is binding). Moreover, I argue that this problem may arise even with risk weights that are derived directly from a loss distribution (rather than being more exogenously determined). Thus, regulation through risk weights is not "absolute" in the sense that its effects and effectiveness have to be monitored in a changing environment.

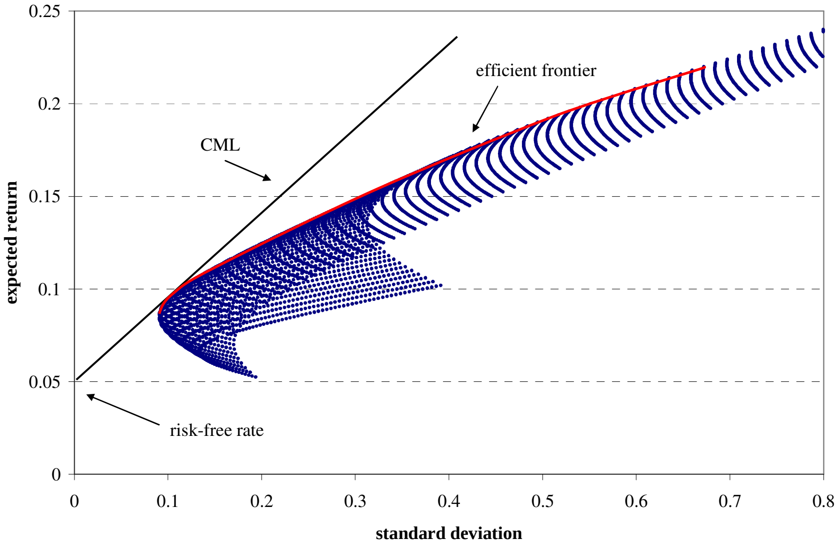
In the first part of the paper I am mainly talking about risk weights in general; in the second part I also try to be more practical: I will interpret the results with an eye on the Basel II regulation.

There is already a large body of critical literature on Basel II (see, for example, Danielsson et al. (2001)). One can say that the rules are too complex, the model behind the advanced risk measurement approach is too simplistic, the Value-at-Risk approach embedded in the way risk weights are set is inappropriate, the regulation is pro-cyclical, and so on.
Here I won't join the critics in any of these above issues but try to give a fresh perspective of the Basel II risk weighting scheme. A fresh and, in a sense, deeper insight than some of the existing ones: I address the fundamental principle of the regulation, the idea, that simply by imposing risk weights on individual investment assets the system as a whole is necessarily stabilised. The question here to start with is this: 'what can we expect from risk weights?'. While there is a lot of talk about regulation and Basel I/II this simple, fundamental question seems to be forgotten. There are related questions like, for example, 'how do risk weights change institutions' portfolio selection?', or 'how should risk weights be set to achieve the regulatory purpose?'.

The paper is structured as follows. In the second and third parts I show how a simple framework can be applied to the question. It comes from modern portfolio theory and simply consists of a utility maximising investor in the risk-return space. In this part of the paper – since the applied modelling framework is so specific and is not applied too commonly to such problems – I will intentionally omit any reference to any existing regulation. By this I try to avoid the situation that the reader raises scepticism at every sentence of this first part. On the other hand, I will also argue that this framework does not restrict the validity of the findings in more general setups and, eventually, that my findings can be valid for real-life financial institutions, as well. Thus, in the fourth part of the paper I discuss some implications for actual financial regulation. Section five concludes.

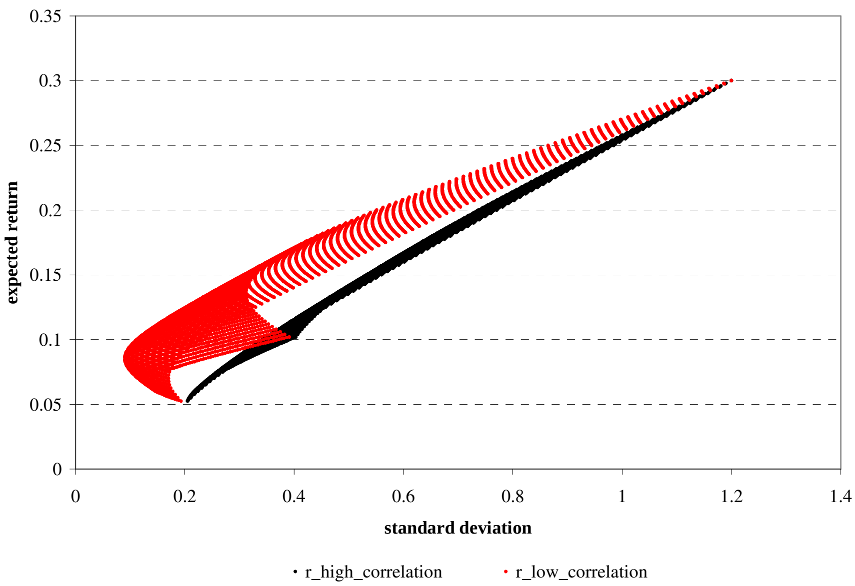
II. Portfolio theory

Our starting point is the mean-variance framework of Markowitz. Here, given the individual investment options along with their expected return and risk (standard deviation) it is possible to draw the efficient frontier in the mean-standard deviation space representing for each level of return the attainable lowest volatility over all portfolios (combinations of the individual investment options). Then, a straight line can be drawn starting from an appropriate point on the y-axis (the risk-free rate) that is tangential to the efficient frontier (and extends beyond this tangential point). Under certain restrictions, each point on this line represents a portfolio that is superior to even portfolios on the efficient frontier in the sense that it yields higher return for a given level of volatility (except for the tangential point where there's equality in returns). This line is called the CML (capital market line). Each point on this line represents an investment in the combination of the risk-free asset and the optimal mix of the risky assets (which is the tangent itself). For portfolios on the CML that are in the section to the left of the tangent exactly 100 percent of the capital is invested and no more (i.e., there is no leverage); to invest in portfolios on the right side of the tangent it is necessary that the investor borrows additional funds (so that it invests over 100 percent where the excess is financed by borrowing). The following figure shows these basics (here and throughout the paper I have three individual assets):



II.1 Correlations: the shape of the efficient frontier

Correlations have a pronounced effect on the set of investment opportunities: the more correlated the assets the more difficult it is to diversify; with lower correlations, on the other hand, the same level of risk is rewarded by a higher return. A high- and a low-correlation scenario are shown in the figure below:

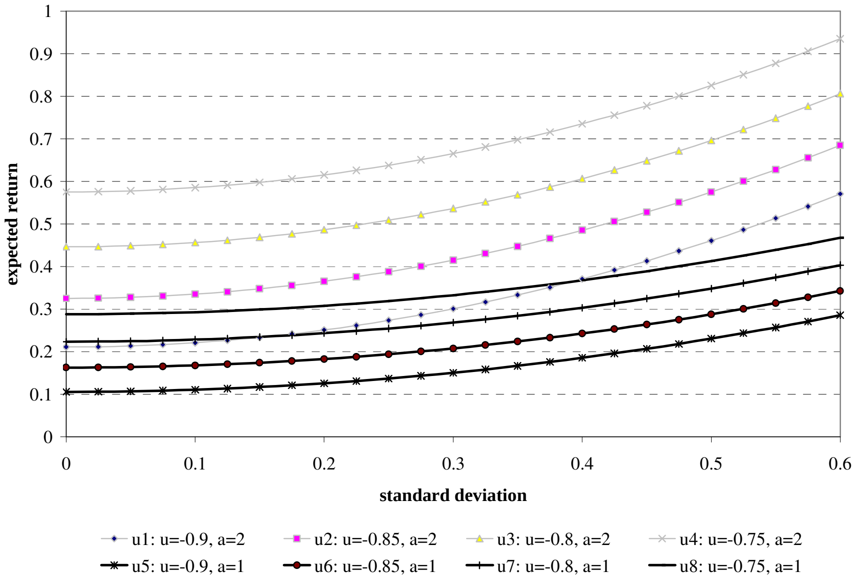


II.2 Investor preferences

We know that the optimal investment must lie on the CML (since it "dominates" all the other portfolios). Investors' ultimate choice is determined by their preference in the risk-return space. A more risk averse investor can be expected to chose a portfolio on the CML closer to the risk-free asset, while a less risk averse one will chose a portfolio closer to the optimal risky asset-mix, or even beyond that point.
A common and easy way of modelling investor's preferences is the exponential utility. I will use it, too; the reason for it is that it is easy to work with and, besides, I don't think that other functional forms would change the result meaningfully.
The functional form to be used here is:

$$E\left(U(x)\right) = -\exp\left(-\mu + a\sigma^2\right), \tag{1}$$

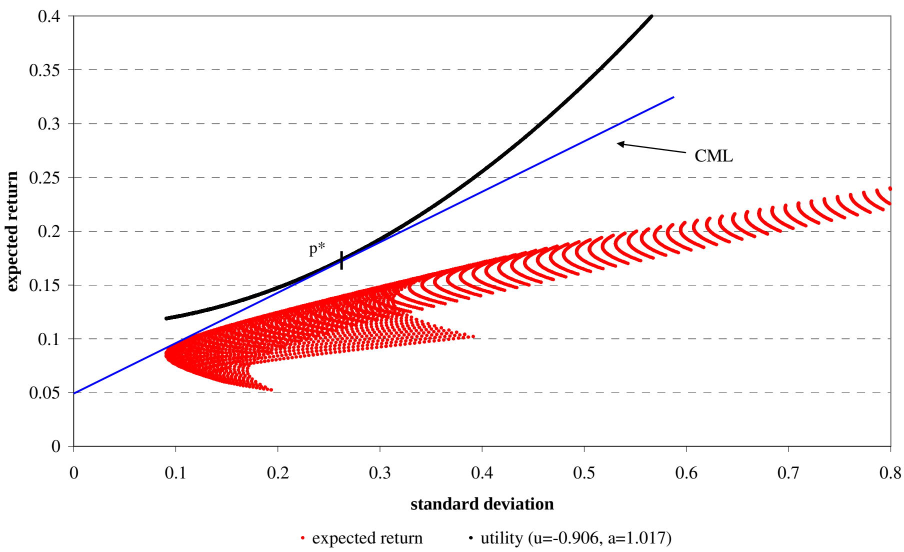
where μ is the expected return on the portfolio and σ is the volatility of the portfolio return. Parameter a represents risk aversion; the higher it is, the more risk averse the investor: for a given level of volatility it requires higher (expected) return to stay at the same level of utility. The following figure shows utility functions with different risk aversions (a) and utility values (u):



The first four curves (u1–u4) represent an investor with higher risk aversion than the second four curves (hence these curves are in general above the second five curves). For each level of risk aversion we can see four different, increasing utility levels: with the same aversion parameter and the same level of risk if the investor receives a higher return he feels better. With their lower level of risk aversion investors with utilities u5–u8 can at a given risk level reach the same level of satisfaction with lower return than the ones with higher risk aversion.

II.3. Choosing the optimal portfolio

Now we can apply the preference curve over the set of portfolios to find the investment that brings about the highest utility to the investor. This optimal point is shown in the figure below:



In this case the investor chooses portfolio p*. As we see this point is beyond the tangent – this means that the investor has to borrow money (in fact, even more than its own share) to realize that investment. By this investment the investor can expect a much higher return than without borrowing and can achieve maximal (unconstrained) utility. Of course, with additional constraints on the optimisation things will change, as we shall see in the next section.

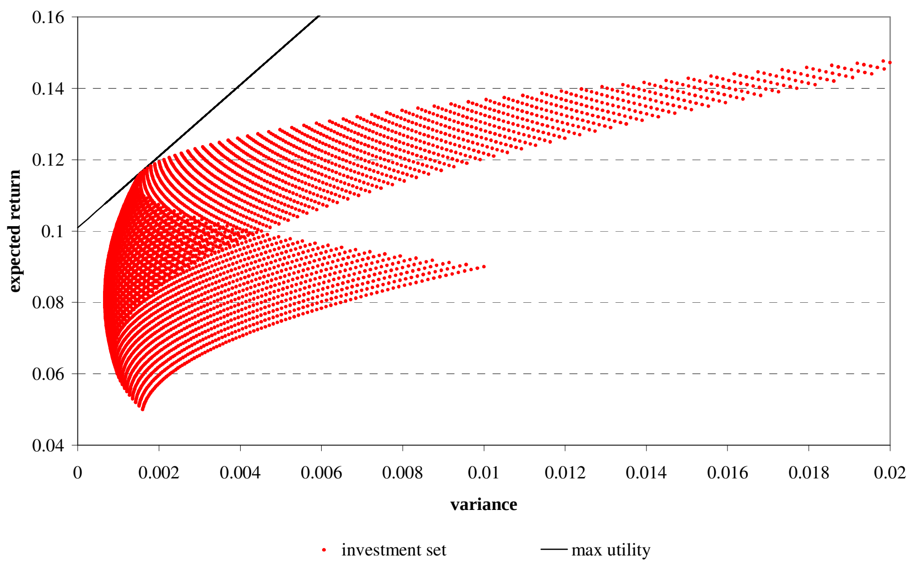
III. Optimal investments without borrowing and other restrictions

In Markowitz's model the efficient frontier is made up of those portfolios that have the highest return for a given level of risk (standard deviation). Furthermore, in that model it is possible to define the capital market line (CML) which is the straight line starting from the risk free rate and being tangential to the efficient frontier.

There are two reasons, however, why this approach would not be feasible here. First, the CML beyond the tangential point represents investments over 100% of the capital. In our problem it makes no sense to allow this extension since, by definition, we cannot model more than 100% of assets – we assume that an institution wants to optimally invest its total assets and not its total equity with additional borrowing.

The other reason is purely technical: finding first the efficient set, then the CML and, finally, the optimal point on the CML using the utility function would lead to algebraic expressions that would be highly complex and this would make the interpretation of the results hugely (and, in my view, unnecessarily) difficult.

For these reasons I will modify the above representation of the problem by adding the risk free asset to the individual asset pool (thereby making it possible to solve the optimisation in one single step) and assuming that for all feasible portfolios the asset weights are positive (i.e. there is no borrowing). The following figure shows a typical situation under these circumstances with the set of feasible portfolios and a utility function:



It also has to be clarified that in graphical presentations from now on I will have the variance of the portfolio on the horizontal axis, instead of the often used standard deviation (that was used even here in previous sections). This has an effect on the shape of the set of investment options and also the shape of the utility function which is a straight line in our case and a strictly convex function when standard deviation is used.

As I will detail in the Appendix this approach leads to the following formalisation of the problem of finding the optimal portfolio:

$$\begin{aligned} & kxQx^T - cx^T \quad \to \min \\ & s.t. \quad Ax \le b \end{aligned}, \qquad (2)$$

where k is the investor's risk-aversion parameter (a greater k means greater risk-aversion), x contains the portfolio weights, Q and c are the covariance matrix and the expected return vector of the individual assets, respectively. Vector b represents the constraints and the elements of matrix A show how each individual asset contributes to each constraint.
In our case, without risk weights, A is simply a vector of ones:

$$A = [1 \quad 1 \quad 1],$$

and b = [1].

Formally, the requirement that all the asset weights be positive does not appear here because that would make the analysis again analytically intractable. Instead, I will concentrate on solutions (optimums) in which this constraint is satisfied by itself.

III.1 Introducing risk weights

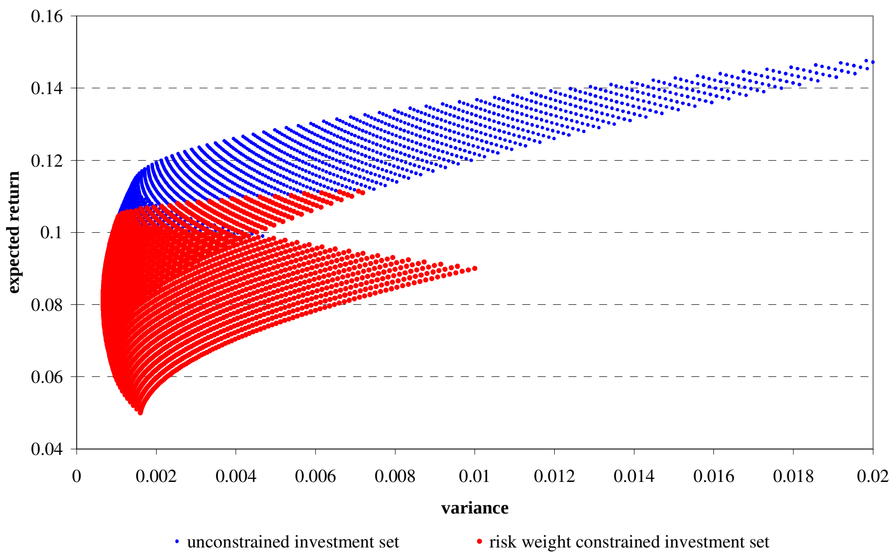
Once we have formulated the problem introducing risk weights is straightforward: the only thing to be done is to modify the appropriate vectors and matrices in (2) by the inclusion of the new constraint: the weighted sum of the risk weights can't be larger than the available capital. This means that now A and b look as follows:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 \\ T \end{bmatrix},$$

there t_i is the risk weight for asset i and T is the available capital.

Perhaps, however, it is best to look at the effect of risk weights graphically at first. In the following figure it can be seen that risk weights directly change the set of feasible portfolios by cutting off a part of the original set:



In the above figure the blue area denotes the feasible set without risk weights and the red one the feasible set with risk weights (the total blue area includes the red area, they are not complementers). As we can see the effect of risk weights is simply that they make a part of the unconstrained feasible set unfeasible.

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As we shall see the effect of the risk weights on the optimisation depends on the following factors:

- the absolute and relative size of risk weights
- the investor's preferences
- asset returns and variances/covariances
- the available capital

All of these factors (but preferences) appear in some form in the above figure since they are the determinants of the portion of the originally feasible portfolios that has to be cut off as a result of introducing risk weights. For example, as the available capital decreases, the red area representing the new feasible set shrinks; increasing the risk weights has similar effect.

In the Appendix I derive some formulas to be able to check these effects analytically. There are two reasons, however, why I won't use too much algebra to present the ideas. First, – to my knowledge – the optimisation has an exact analytical solution only if equality-constraints are used. The requirement that the asset weights must all be non-negative implies an inequality constraint given which the problem can only be solved numerically. Second, even the analytical solution with merely equality constraints can be rather complex and I would like to place the emphasis on the ideas rather than on mathematics. Thus, my purpose here is to demonstrate scenarios in which "anomalies" can arise rather than carrying out a "full scale" analysis of the issue; in simpler cases I will refer to the formulas, though, and I will also use them to double check some of the intuitive insights.

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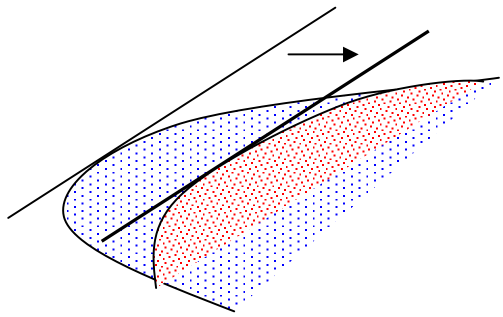
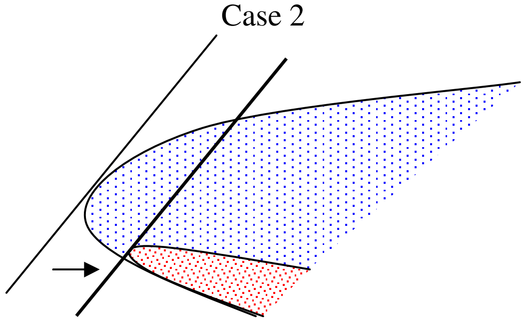
As I show in this section there can be three basic types of situations where introducing risk weights can lead to an optimum with increased variance. These are, schematically, as follows:

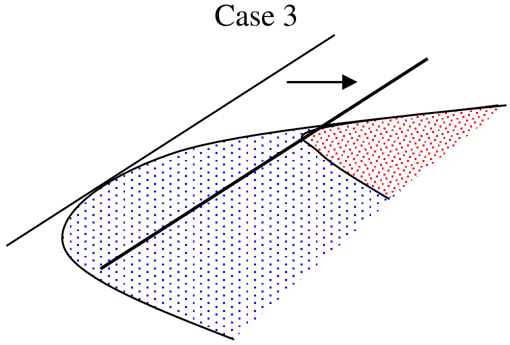
Case 1



Case 2



Case 3



In all of the above figures the blue and the red (more densely dotted) area designate the unconstrained and the constrained investment set, respectively. The thin straight line

shows the maximal utility in the unconstrained scenario and the thicker line that in the risk weight constrained scenario.
For further analysis it is useful to separate two broad set of scenarios: in the first we do have a riskless asset (with zero risk weight) and in the second we don't (and all the risk weights are strictly larger than zero).

III.1.1 Optimisation including zero-risk asset

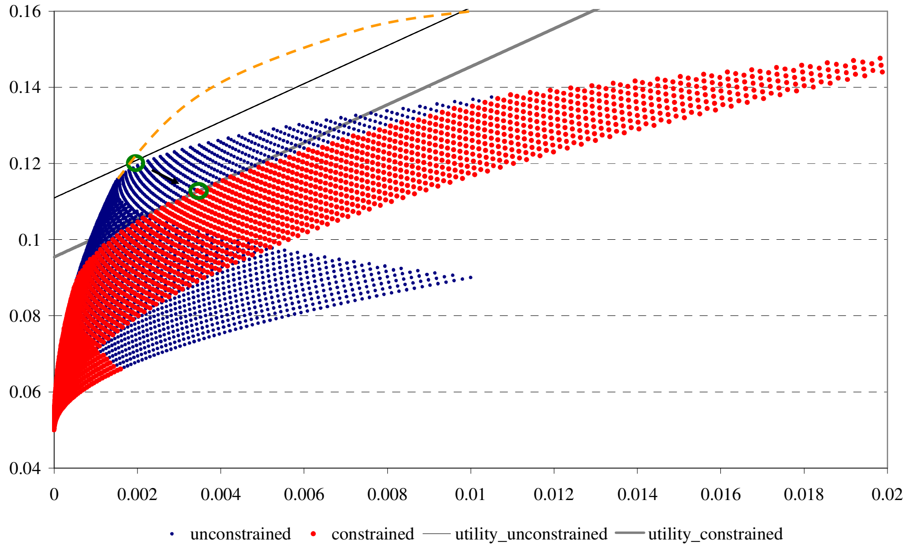
A riskless asset does not only change the results because its variance is zero, but also because its covariances with the other assets are also zero. This latter fact simplifies the analytical solution so much that in this subsection I will use some of my formulas derived in the Appendix.

The question whether introducing risk weights can lead to an optimal portfolio with a higher variance can be answered by formally examining to what direction a change in the available capital changes the variance. The rationale behind this idea is as follows. We can calculate the portfolio risk weight (which is the same as the required capital) even without formally introducing asset risk weights in the optimisation problem (i.e. the risk weight that *would be* assigned to the unconstrained optimum had risk weights been introduced). If the derivative of the variance of the (unconstrained) optimal portfolio with respect to the available capital is negative then as we start to decrease the available capital from the level implied by this unconstrained optimum after introducing asset risk weights we will see the variance of the optimal portfolio increase. Formally, it can be shown that:

$$\frac{\partial s^{2*}}{\partial b} = b^T \left(AQ^{-1}A^T\right)^{-1} < 0 \Leftrightarrow \frac{a_{21}(q_{33}q_{22} - q_{23}^2) + a_{22}q_{11}(q_{33} - q_{32}) - a_{23}q_{11}(q_{23} - q_{22})}{q_{33}q_{22} - q_{23}^2 + q_{11}q_{33} + q_{11}q_{22} - 2q_{11}q_{32}} > b_2 , \quad (3)$$

where a_{2i} denotes the risk weight of the ith asset (the second row of matrix A), q_{ij} contains the ijth element of the covariance matrix of the asset returns, Q, and b_2 is the available capital.
One can observe that the expression to the left of the second inequality sign is the weighted average of the risk weights (a_{2i}, where, in fact, a_{21} is zero), where the weights are different expressions including elements of the covariance matrix. The first element in the numerator is zero, since the risk weight of the riskless asset is zero. The other two elements contain the variance of the riskless asset – which, for technical reasons we can't set exactly to zero so we use a very small positive number – so these will be very close to zero. This implies that the quotient is generally very, very low – so low that if we decrease the available capital (b_2) below that level it will enable hardly any risk-taking. This result will be independent of the other parameters of the problem (e.g. the correlation between the risky assets). That means that if we introduce risk weights in this case the volatility of the optimal portfolio will decrease – most probably even if there are "problems" or inconsistencies with the risk weight (see below).

Notice, however, that by making use of the analytical results here, implicitly, we didn't apply the restriction that each asset-weight must be non-negative (we only have analytical results without such a constraint). This restriction should indeed be applied (it naturally follows from the way we approached the problem) and if we do so it has a very remarkable effect on the feasible investment set. To analyse this effect I won't use any algebra but will demonstrate the ideas graphically.
The following figure shows a typical situation in which risk weights restrict the feasible investment set in such a way that the constrained optimum has a higher variance than the unconstrained one:



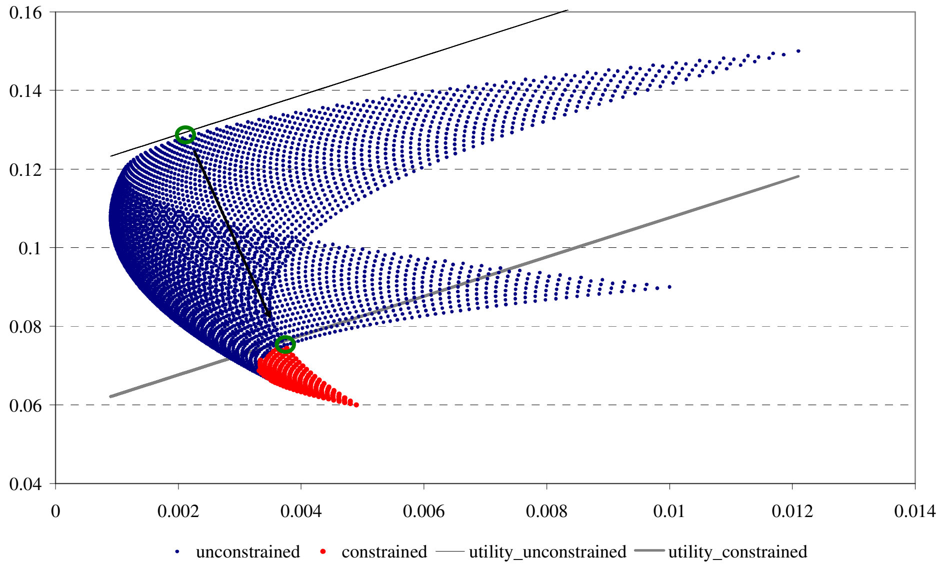
In the above figure green circles mark the optimums in the constrained and unconstrained cases. We can see that as risk weights are introduced the new optimum has a higher variance and lower expected return (and, of course, utility). The orange curve is a rough indication of the border of the feasible set when we allow the asset weights to go negative to a certain extent.
Under what circumstances (parameters) can such a situation arise? First, and most importantly, the risk weights have to be set in an inconsistent manner, i.e. the risk weight, for at least two assets, should be inversely related to the risk of the assets (a riskier asset receives lower risk weight than a less risky one). At the same time, an inconsistency is not sufficient to lead to such an adverse effect: this discrepancy has to exceed a 'certain level'. In the example above the risk weight of the second asset was much higher than that of the third asset while it was less risky.
Second, the correlation between the risky assets should be low – the more it is true the higher the increase in the variance can be expected when applying risk weights. A lower (negative) correlation enables lower variance; and the higher the proportion of these risky assets with low correlation in the portfolio the higher the decrease in the variance – thus, in the figure above the area in red will be pulled towards the vertical axis to a lesser extent than the blue area (that contains more of the second and the third assets) above it.
Third, investors can't be 'too' risk-averse: in this case the utility curve would touch the unconstrained feasible set at a point where the border of the unconstrained set is steeper

than that of the constrained set which would mean that the variance in the constrained optimum is lower than in the unconstrained one.

III.1.2 Optimisation without zero-risk asset

In our formal derivation of the solution to the portfolio selection problem we couldn't impose the condition that all the asset weights must be non-negative (i.e. there is no borrowing); as a consequence, we have many parameter settings where the optimal solution violates this requirement. More exactly, it is notable that a low level of return on the riskless asset and a moderate level of risk aversion at the most will probably lead to situations where there is a high negative weight on the riskless asset meaning it is in the best interest of the investor to borrow substantially at that low rate. Now, if we don't allow for negative weights the optimal weight of the riskless asset in the above situations can be expected to be zero – this allows the investor to invest in the risky asset mix to the maximum extent. If risk weights are introduced, there will still be optimal solutions with a negative weight for the risk-free asset – with the same reasoning we can leave the non-negativity-constrained optimal weight of this asset at zero. Thus, the investor still invests only in the risky asset-mix, and the introduction of or a change in the risk weights will only result in a realignment of the weights within this mix. This reasoning underpins the idea that we examine the portfolio selection problem without a riskless asset.

When we lock out the existence of a risk-free asset the above results change somewhat. Above I identified three situations in which the introduction of risk weights leads to an optimum with increased variance. We saw an example for Case 1 above in a situation where the zero-risk asset is a possible investment choice. Of course, we can have the same situation when we do not allow investment in the risk-free asset.
Case 2 and Case 3 are specific to situations where there is no risk-free asset. To get into the situation represented by Case 2 it is not necessary that the asset risk weights be adversely related to risk; it is enough that the available capital is very low: in this case only a small part at the 'bottom' of the unconstrained feasible set remains feasible after the introduction of risk weights:



As we can see from the above figure in this case even if investors' risk aversion varies over a wide range the adverse effect of risk weights can be observed. The problem is related to low correlations again: given these (and without risk weights) investors can find a very good trade-off between risk and return containing high proportions of risky assets. When these assets receive high risk weights (relative to available capital) investors will be forced by the capital constraint to invest increasingly into low-risk assets – the risks of which are higher though than that of an appropriately chosen portfolio of more risky assets without risk weights. In other words, investors cannot benefit from the risk moderating effect of low correlations.

In Case 3 we have a more serious breach of the natural requirement that more risky assets should receive a higher risk weight than in Case 1: now the system of risk weights is turned "upside-down", with riskier assets receiving smaller risk weights.

IV. Some implications

In this section I will touch on some important practical questions that can be related to the above analysis. I discuss and argue in favour of the model's applicability to credit portfolios; I interpret the findings in light of the Basel II regulation; and I discuss some policy implications.

IV.1 The application of the model to credit risk related portfolios

One might argue that virtually none of the assumptions of the mean-variance model applies to banks' credit portfolios. We can be pretty sure that none of the assets' payoff in which banks invest has a standard normal payoff; none of the banks have a simple exponential utility; banks won't use standard deviation as a risk measure, for example.
Still, such the assets have *some kind* of payoffs and the bank *somehow* ranks these assets based on their return and risk profile; and banks probably have *some kind* of a utility

function. Thus, even if preferences, when depicted in the risk-return space, are not as smooth as in the model doesn't necessarily invalidate our findings – rather, additional aspects in the portfolio selection together with a regulation that is based on risk might create further room for anomalies.

Our finding that risk weights can alter portfolio choice adversely could be translated using more general terms. We could use 'risk' instead of 'variance', 'reward' instead of 'expected return', we could omit the reference to 'utility' and simply refer to the 'choice' of a bank. But let's move to an even more general level instead and summarize our conclusion as follows. *When the introduction of risk weights makes a bank's so far optimal choice unfeasible, under certain conditions, the best feasible choice will have a higher risk.* The above analysis shows that one important (although insufficient) condition is the existence of diversification opportunities which banks might be prevented to benefit from after the introduction of risk weights. Another such condition is the relatively low level of capital in which case banks can't diversify into higher risk assets. This is not to say that each bank always wants to diversify; but the banks that do want to diversify might have such problems.

Another argument against the application of the model to banks is that banks' investment horizon (at least, in credit risk) is much longer than what the model was originally created for. A bank can't change its (credit) risk profile from one day to another thus can't carry out such a fine-tuning of its optimal portfolio as the investor does in the model. Banks have a more straightforward option: they would rather raise their capital level. This might be true, but what if banks find it difficult to do so?[1]

IV.2 Regulation

The current Basel II regime is a successor to Basel I which was the first comprehensive, international initiative for harmonised capital requirement rules (see BCBS (1988)). It applied five categories of fixed risk weights on credit risk related assets – this relative lack of risk sensitivity was one major criticism of the regime; for example most corporate assets would have come under 100 percent risk weight irrespective of the credit quality of the corporate. This simplistic approach led to another problem of the regulation, regulatory arbitrage: institutions had the incentive to circumvent the rules by, for example, securitising low-risk loans (which were 'over-penalised' by receiving the same risk weight as higher-risk loans).[2] This situation, in fact, can be reflected in the above model simply in the form of inappropriately calibrated risk weights (and we saw that such situations can lead to an 'adverse' portfolio selection).

Basel II brought about large changes compared to Basel I. From our point of view one of the most important of these is the introduction of a completely new approach: the "internal ratings-based", or "IRB" method for credit risk and a similar, internal model approach for market risk in which banks can make use of their internally estimated risk

[1] On the other hand, if one argues that risk weights are not capable of modifying the portfolio selection of banks than one should question the effectiveness of such efforts in Basel II (e.g. preferential risk weights for SME credit exposures in the form of lower correlations).

[2] It is interesting that while regulation (and especially Basel I) hardly gives any benefit to the yield of the assets, this, in turn, is a crucial component in securitisations!

parameters in calculating risk weights and capital requirement. Perhaps even more importantly, there is a big shift in the approach itself, in the meaning of risk weights: while in Basel I these were set 'exogenously' with the aim of risk differentiation only between the broad risk categories (and without any reference to an actual estimated risk measure of the assets), in the internal model based methods the risk weights equal an (estimated) high percentile of the loss distribution, that is, a percentage loss that can be exceeded with only a low probability. This difference is important from our point of view: the whole logic of the problem changes now since the 'correctness' of the risk weights can't be questioned any more; the risk weight – at least at the individual asset level and assuming that the model behind the approach is more or less accurate – becomes in line with the risk.

Generally speaking, the correlation argument applies here, too, *since we still have individual asset-level risk weights while the investment strategy is implemented on a portfolio basis*.[3]

One could argue that the risk measurement in the IRB approach is not necessarily accurate thus leading to situations where risk weights are not in line with the risk of assets thus having adverse effects on portfolio selection. However, if banks use the same models to calculate risk weights and to decide on their investment strategy (i.e. they accept that risk weights are representative of the true risks) there is no such problem.

The other major approach for credit and market risk – besides the internal model based approach – in Basel II still contains simpler risk weighting schemes. These are now linked to ratings of exposures by rating agencies. Since ratings are updated over time (most importantly, they are updated more often than the frequency by which banks can substantially adjust their balance sheet) we can't expect long-standing discrepancy between risk weights and ratings.

Finally, for most non-rated assets still a single 100% (of 8%) risk weight is applied – which, as we saw earlier, might change portfolio selection in an undesired way.

IV.3 Policy issues

Throughout the paper the major concern was whether and how the introduction of risk weights can actually lead to riskier (optimal) portfolios, that is, how regulation changes investors' choice. But once we have some ideas as to the effect of (introducing/changing) the risk weights on the optimal portfolio choice we can start to deal with policy questions such as, for example, how investment in certain sectors could be encouraged. This is not to say that capital regulation *should* directly be used to change investors' preferences but we know that it actually does.

One example of such intention can be found in the CRD itself whereby small enterprises receive more favourable risk weights compared to larger companies with comparable risk parameters.

[3] On the one hand the IRB approach contains a kind of diversification: it assumes that the asset pool is perfectly fine-grained. On the other, to assets in this diversified pool it assigns risk weights based on a specific correlation assumption (one that is derived from the default probability of the asset).

The analysis above is not tailored to answer such questions since it has more to do with the risk of the optimal portfolio globally than with the composition of the optimal portfolio. However, the model can be used to track the changes in the composition of the optimal portfolio resulting from the application of risk weights. As an example, it may sound strange at first hearing that increasing the risk weight of an asset can lead to an increase in the weight of the same asset in the optimal portfolio. But this is the case when the risk weight of the lowest risk asset increases and the capital constraint is binding, since higher risk assets (with higher risk weights) have to be substituted with this asset with the lowest risk weight. This is quite obvious a case; however, depending on the actual parameters, trying to influence portfolio selection by risk weights may not always be successful in more general settings either.

V. Conclusion

In this chapter we examined some potential effects of capital requirements in the form of risk weights on portfolio selection. The analysis was carried out in the mean-variance framework of modern portfolio theory. Given the specific nature of this model we had to argue that we can use it to our more general problem. Indeed, the model is just a helpful tool for demonstrating the ideas; we can word our conclusions without reference to the model's key notions.

Our most important finding is that when risk weights are introduced investors might – under certain conditions – choose a new optimum that is riskier than the one before the introduction of risk weights. The primary reason for it is that risk weights – that are applied over the individual asset level – are not sensitive to the (risk-reducing) effect of low correlations at the portfolio level.

Reconciling the findings that were based on a simplified model with actual financial market regulation is far beyond the cope of this paper. Thus, throughout a large part of the chapter I tried to avoid direct and strong language regarding actual financial markets or regulations; for example, I didn't examine the reasonability of assuming substantial negative correlations between asset classes – I simply pointed out that if this is the case, regulation might have undesired effects.[4]

Unintended outcomes can be the results of violations of otherwise correct rules-of-thumb, for example, identical risk weights for a broad range of assets; and can be reinforced by features of the model behind the regulation that are not reflective of the underlying reality (e.g. correlations). In any case, the results lead to an important conclusion regarding regulation: it is not only what regulation wants to achieve, but also what it actually achieves taking into account the possibility of errors. A good regulation not only achieves its purpose effectively and efficiently, but also minimizes the potential for damages stemming from any problems arising after its implementation.

[4] One might note that during stressful periods correlations have a tendency to increase substantially. However, Value at Risk – the current 'best practice' of market risk and credit risk measurement, forming also the basis of a large part of regulation – assumes 'normal' market conditions (see, for example Jorion (2000), preface, page xxii).

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APPENDIX – a formal analysis of the change in the optimal investment

Our intention here is to demonstrate formally how the optimization leads to a different solution as risk weights are introduced. The investor wants to maximise its utility, that is, wants to solve the following quadratic programming problem:

$$\begin{aligned} & kxQx^T - cx^T \quad \to \min \\ & s.t. \quad Ax \leq b \end{aligned} , \tag{A.1}$$

where k is the investor's risk-aversion parameter (a greater k means greater risk-aversion), x contains the portfolio weights, Q and c are the covariance matrix and the expected return vector of the individual assets, respectively. Vector b represents the constraints and the elements of matrix A show how each individual asset contributes to each constraint. In our case there are 3 assets and 1 constraint when there are no risk weights (the portfolio weights sum to unity) and 1 additional constraint when there are risk weights (the portfolio risk weight should be less then or equal to a pre-set level).

It has to be noted that I found three common formalisation of the portfolio selection problem in the literature one of which is what I presented above. It has the advantage that it is directly aimed at finding the optimum, whereas the other two can be only used to find the efficient frontier (either by finding the portfolio with minimum variance for a given level of expected return or the portfolio with the maximum expected return for a given level of variance).

Now, it can be shown that with equality constraints (A.1) has the following solution:

$$\begin{aligned} & \lambda = -k\left(AQ^{-1}A^T\right)^{-1}\left(AQ^{-1}c/k + b\right) \\ & x = \frac{1}{k}\left(-Q^{-1}A^T\lambda + Q^{-1}c\right) \end{aligned} \tag{A.2}$$

Using (A.2) it can further be shown that the variance of the optimum can be expressed as:

$$s^{2*} = x^T Qx = \frac{1}{k^2}\left[-c^T Q^{-1}A^T\left(AQ^{-1}A^T\right)^{-1}AQ^{-1}c + c^T Q^{-1}c\right] + b^T\left(AQ^{-1}A^T\right)^{-1}b \tag{A.3}$$

From (A.3) it follows that the effect of the change in the risk weight constraint on the variance of the new optimum is determined by the following expression:

$$\frac{\partial s^{2*}}{\partial b} = 2b^T\left(AQ^{-1}A^T\right)^{-1} ,$$

or, more precisely, its second element (it's a 2x1 vector). Calculating it is straightforward though requires some complicated algebra. First, the inverse of the covariance matrix is (with three investments assets):

$$Q^{-1} = \frac{1}{\det_Q}\begin{bmatrix} q_{33}q_{22}-2q_{23} & -(q_{33}q_{12}-q_{32}q_{13}) & q_{23}q_{12}-q_{22}q_{13} \\ -(q_{33}q_{12}-q_{32}q_{13}) & q_{33}q_{11}-2q_{13} & -(q_{23}q_{11}-q_{21}q_{13}) \\ q_{23}q_{12}-q_{22}q_{13} & -(q_{23}q_{11}-q_{21}q_{13}) & q_{22}q_{11}-2q_{12} \end{bmatrix},$$

where

$$\det_Q = q_{11}(q_{33}q_{22}-2q_{23}) - q_{21}(q_{33}q_{12}-q_{13}q_{23}) + q_{31}(q_{23}q_{12}-q_{13}q_{22}).$$

Introducing the notation q_{ij}^{-} for the ij-th element of Q^{-1} and a_{ij} for the ij-th element of matrix A and making use of the fact that the first row of A consists of ones we can write $AQ^{-1}A^T$ as:

$$\Omega = AQ^{-1}A^T = \frac{1}{\det_Q}\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}, \text{ where}$$

$$\omega_{11} = (q_{11}^{-} - q_{21}^{-} + q_{31}^{-}) + (-q_{12}^{-} + q_{22}^{-} - q_{32}^{-}) + (q_{13}^{-} - q_{23}^{-} + q_{33}^{-})$$
$$\omega_{12} = \omega_{21} = (q_{11}^{-} - q_{21}^{-} + q_{31}^{-})a_{21} + (-q_{12}^{-} + q_{22}^{-} - q_{32}^{-})a_{22} + (q_{13}^{-} - q_{23}^{-} + q_{33}^{-})a_{23}$$
$$\omega_{22} = (a_{21}q_{11}^{-} - a_{22}q_{21}^{-} + a_{23}q_{31}^{-})a_{21} + (-a_{21}q_{12}^{-} + a_{22}q_{22}^{-} - a_{23}q_{32}^{-})a_{22} + (a_{21}q_{13}^{-} - a_{22}q_{23}^{-} + a_{23}q_{33}^{-})a_{23}$$

Finally, inverting Ω we obtain:

$$\Omega^{-1} = \left(AQ^{-1}A^T\right)^{-1} = \frac{1}{\det_Q \det_\Omega}\begin{bmatrix} \omega_{22} & -\omega_{12} \\ -\omega_{21} & \omega_{11} \end{bmatrix}, \text{ where}$$
$$\det_\Omega = \omega_{11}\omega_{22} - 2\omega_{12}$$

Now, the change in the variance of the optimal portfolio following a small change in the capital constraint is:

$$\frac{\partial s^{2*}}{\partial b_2} = \frac{2}{\det_Q \det_\Omega}\left(-b_1\omega_{12} + b_2\omega_{11}\right)$$

I wouldn't go into the element-by-element interpretation of this expression; however, we are now in a position to point out the factors on which the effect of changing the risk weights depends and the direction of the relationship. Recall that we are interested in such situations where the above derivative is negative, i.e. increasing the available capital (a lower b_2) leads to an increase in standard deviation. Thus, we are interested in the sign of the above derivative, first of all.
The three groups of factors are: the available capital, the individual asset risk weights and the covariance matrix. The effect of the available capital is straightforward: roughly

speaking, as it decreases the derivative also decreases thus getting closer to an 'adverse' situation.

Since b_1 is fixed at 1 (the portfolio weights sum to unity) and ω_{11} does not depend on the risk weights the only item of interest (apart from the determinants) is ω_{12}. It can be shown that the derivative is negative if and only if the following relationship is satisfied:

$$\frac{\partial s^{2*}}{\partial b} = b^T \left(AQ^{-1}A^T\right)^{-1} < 0 \Leftrightarrow \frac{(q^{-}_{11} - q^{-}_{21} + q^{-}_{31})a_{21} + (-q^{-}_{12} + q^{-}_{22} - q^{-}_{32})a_{22} + (q^{-}_{13} - q^{-}_{23} + q^{-}_{33})a_{23}}{(q^{-}_{11} - q^{-}_{21} + q^{-}_{31}) + (-q^{-}_{12} + q^{-}_{22} - q^{-}_{32}) + (q^{-}_{13} - q^{-}_{23} + q^{-}_{33})} > b_2$$

This is a more complicated expression than what I use in the main text for the case where the first asset is risk-free (thus having no correlation with the other assets).