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2008

Online at https://mpra.ub.uni-muenchen.de/13674/
MPRA Paper No. 13674, posted 28 Feb 2009 11:57 UTC

A Schumpeterian Growth Model with Heterogenous Firms

Antonio Minniti[*] Carmelo Pierpaolo Parello[†] Paul S. Segerstrom[‡]

Current version: August 7, 2008

Abstract: A common assumption in the Schumpeterian growth literature is that the innovation size is constant and identical across industries. This is in contrast with the empirical evidence which shows that: (i) the innovation size is far from being identical across industries; and (ii) the size distribution of profit returns from innovation is highly skewed toward the low value side, with a long tail on the high value side. In the present paper, we develop a Schumpeterian growth model that is consistent with this evidence. In particular, we assume that when a firm innovates, the size of its quality improvement is the result of a random draw from a Pareto distribution. This enables us to extend the class of quality-ladder growth models to encompass firm heterogeneity. We study the policy implications of this new set-up numerically and find that it is optimal to heavily subsidize R&D for plausible parameter values. Although it is optimal to tax R&D for some parameter values, this case only occurs when the steady-state rate of economic growth is very low.

JEL classification: E10, L16, O31, O38.

Keywords: Schumpeterian Growth, R&D, optimal policy

Acknowledgments: Antonio Minniti acknowledges the financial support of the University of Bologna through the Marco Polo program and of the Belgian research programmes "Poles d'Attraction interuniversitaires" PAI P5/21, and "Action de Recherches Concertée" 03/08-302. Paul S. Segerstrom acknowledges financial support from the Wallander Foundation.

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1 Introduction

Starting from the seminal contribution of Grossman and Helpman (1991), much research has been devoted to exploring the normative implications of R&D-based growth models with quality improvements. This literature has largely investigated the question of whether to tax or subsidize R&D, and has shown that the direction of the optimal R&D policy is sensitive to the nature of the innovation, drastic (with a big skip separating two consecutive quality vintages) or non drastic (with a small skip separating two consecutive quality vintages). In most of these contributions, R&D policy recommendations are drawn and discussed with respect to the size of quality improvement, the latter being usually treated as constant and identical across industries (see, e.g. Grossman and Helpman, 1991; Segerstrom, 1998; Li, 2001 and 2003).[1]

However, assuming a constant innovation size is in contrast with the empirical evidence which shows that the probability distribution of R&D outcomes, as well as the size distribution of profit returns from innovations, are highly skewed toward the low value side, with a very long tail into the high value side. This suggests that R&D projects are not equally profitable, with innovations of lower size having a higher probability to be achieved.

In this paper, we propose an R&D growth model which is consistent with this evidence. In our setting, firms address two different types of uncertainty when engaging in R&D. The first one is the traditional uncertainty related to the outcome of the R&D race. The second one is new in the literature and is related to the size of innovation; more specifically, we model the idea that there is an increasing difficulty to realize innovations of greater size by assuming that the probability distribution of quality improvements (denoted also as quality jumps) is Pareto. Now, since a patent is granted to each innovator and the size of a quality increment determines the profitability of an innovation, our modeling strategy is in accordance with the evidence reported by Scherer (1965) where it is shown that R&D profits measured through a survey of U.S. patents conform quite well to a Pareto-type distribution.

It is important to observe that firms are heterogenous in our model. In fact, once we allow the quality jumps to vary, it is possible to record a drastic innovation subsequent to a non-drastic one and so forth, with the result that the size of each industry leader, as well as its strategic behavior, may differ across industries and over time. We build upon the non-scale growth model of Segerstrom (1998), recently discussed and extended by Li (2003); such a setting, characterized by diminishing technological opportunities and knowledge spillovers within and across industries, is thus generalized to the case of heterogeneous firms. The model has a unique steady-state equilibrium, with firms operating either as a price setting oligopolist or as a monopolist, depending on the type of innovation pursued, drastic or just incremental.

The research objectives of this paper are twofold. First, we are interested in building a tractable analytical framework that introduces firm heterogeneity into the class of quality-ladder growth models. Second, we explore the policy implications of this new set-up and assess the robustness of the normative results of the previous literature on the topic. In this respect, it is worth noticing

[1] Differences in R&D policy recommendations often emerge in this literature. Such discrepancies arise because of interactions between different externalities at work in these models. For instance, while Grossman and Helpman (1991) find that it is optimal to subsidize R&D when the size of quality improvement is very small or vary large and to tax R&D for innovations of intermediate size, Segerstrom (1998), which extends Grossman and Helpman's work to the case of increasing R&D difficulty, shows that for small-sized innovations R&D subsidies are optimal, whereas for sufficiently large-sized innovations R&D taxes are welfare maximizing. More recently, Li (2003), relaxing the assumption of unitary elasticity of substitution between goods and allowing for positive across-industry R&D spillovers, finds that R&D should be subsidized when innovations are sufficiently large, and taxed when innovations are of intermediate size. These results hold when across-industry R&D spillovers are not too large. Instead, if these spillovers are sufficiently big, it is optimal to subsidize R&D regardless of the size of innovation.

that the optimal R&D policy we focus on does not lead to the first-best outcome. The reason is that, differently from the canonical quality-ladder growth model, not all the industry leaders operate under the same competition regime in our setting. In fact, in the presence of heterogeneous firms, the markup of price over marginal cost differs across industries; in particular, goods are sold at higher markups in sectors where leaders operate as price setting monopolists than in ones where they behave like price setting oligopolists. This means that the first-best optimum, which requires identical markups across firms, can not be achieved.

In order to determine the direction and magnitude of the optimal R&D policy, we calibrate the model by using data from the U.S. economy. Our benchmark simulations show that it is optimal to subsidize R&D at a rate of 56 percent, with the share of resources optimally allocated to R&D being roughly twice the laissez-faire share. We check the robustness of this result by considering high and low values for some key parameters. We find that the optimal R&D subsidy rate remains positive and fluctuates between 18 and 56 percent when the steady-state rate of economic growth is kept within the interval [0.5%, 2%]. Since the optimal R&D tax case only occurs when economic growth is unreasonably low, our overall analysis suggests that over-investment in R&D, although possible in principle, is very unlikely to occur in a decentralized economy. In earlier research by Segerstrom (1998) and Li (2003), the possibility of optimal R&D taxes is discussed at length but the importance of this case for plausible parameter values is not assessed since the models are not solved numerically.[2]

The paper develops as follows. In Section 2 we report some empirical evidence on the distribution of innovation size and we discuss some recent theoretical work that treats the results of R&D activity as uncertain. Section 3 sets up the model. Section 4 solves for the balanced-growth path and analyzes the steady-state properties of the model. Section 5 performs the welfare analysis and numerically explores the relevant policy implications. Finally, Section 6 concludes.

2 Uncertainty in the R&D activity

As mentioned before, the main departure of this paper from the existing literature is that the realization of each R&D race is uncertain; in our contribution, such an uncertainty follows a Pareto distribution. The concern that the innovation size is far from being identical in all industries is quite acknowledged in the empirical literature on this topic. As there is no direct measure concerning the value of innovation activity, economists have often made use of some proxies. For instance, the standard patent-design literature uses the notion of patent claims to mean the size of innovation.[3] The claims of a patent determine the "scope" or the "breadth" of an innovation and define, in technical terms, the extent of the protection conferred by a patent. As a result, the number of claims accorded to a patent may be taken as a proxy for the innovation size and, in our view, its pattern over-time and across industries provides evidence on the fact that quality jumps vary substantially both within the same industry line and across sectors.

Figure 1 shows the time evolution of the average number of claims made by technological fields for the U.S. economy. The across industries differences are very large; in 1984, for instance, the

[2] Our finding that it is optimal to heavily subsidize R&D is in line with the results in Jones and Williams (2000). They calibrate a growth model to U.S. data, obtaining under-investment in R&D for a wide range of parameter values. However, Jones and Williams (2000) do not allow for firm heterogeneity; the markup of price over marginal cost is the same for all firms in their model. Also, they study the case where innovations are increases in product variety (instead of product quality).

[3] See among others, Klemperer (1990), O'Donoghue (1998), O'Donoghue et al. (1998) and O'Donoghue and Zweimuller (2004).

industry sectors with the larger patent scope are Drugs & Medical while Computer and Communications records the larger number of claims in 1997.

Another potential measure of the width of the innovation is the number of citations received by a patent. Silverberg and Verspagen (2004), for instance, point out that forward patent citations can be taken as a possible proxy for the innovation size. Hall et al. (2000) find out that citation-weighted patent stocks are highly correlated with firm's market value because of the high valuation placed on firms that hold very highly cited patents.

Figure 2 is taken from Hall et al. (2000) and shows the estimated citation lag distribution per technology field. The vertical axis measures the relative probability of citation, whereas the area under each curve is the estimated overall relative citation intensity for a given field. Citation lag differs across industries emphasizing a high heterogeneity in terms of innovation sizes. On average, the most cited patents are those in Computers and Communications, followed by the patents in Drug and Medical sector.

As regards the probability distribution governing R&D's outcomes, as well as the size distribution of profit returns from innovations, it is widely recognized that it is highly skewed toward the low value side, with a very long tail into the high value side. In a seminal work Scherer (1965) reports that the profits measured through a survey of U.S. patents conforms quite well to the Pareto-type distribution with a slope coefficient of less than 0.5. Successive studies have tested whether the patent values, profitability and citations obey a Pareto distribution. It is interesting to observe that in all these tests the Pareto distribution is taken as a benchmark; this literature either supports the Pareto distribution or concludes that it is hard to distinguish between the Pareto and the lognormal distributions. For instance, Harhoff et al. (1997) examine the distribution of the value of patents in Germany and the U.S.. The Authors find out that for patents worth more than $500,000 or more than 100,000 Deutsche Marks, a Pareto distribution accurately describes patent values, although for the entire range of patent values a lognormal distribution seems to fit better. Bertran (2003) finds that the distribution for ideas is very close to the Pareto distribution by using data on the number of citations to value patents.

In our paper we model the uncertainty concerning R&D outcomes by assuming that quality jumps are drawn independently from a Pareto distribution. Since a patent is granted to each innovator and, as we will see, the size of a quality increment determines the profitability of an innovation, our model is consistent with the evidence that the distribution of patent values is close to the Pareto distribution.

It is interesting to observe that other authors have recently used the Pareto distribution to model the uncertainty regarding the results of R&D projects. Bental and Peled (1996) describe R&D activity as a sequential search involving random draws - also called technology draws - from a population of untried technologies characterized by a Pareto distribution. Kortum (1997) and Eaton and Kortum (1999) also use the Pareto distribution to characterize the R&D novelty distribution. As observed by Kortum (1997), a nice property of the Pareto distribution is that the distribution function for new inventions does not depend on the current state of knowledge.[4] In addition to this, a key-advantage of using a Pareto distribution is its simplicity; among all the probability distributions, it is probably the easiest to work with. This also explains the reason why recently there has been a wide utilization of the Pareto distribution in economic applications.[5]

[4]Later, we will interpret accurately this property in our model.

[5]See for instance Bental and Peled (2002) and Popp (2005). In addition to the literature on R&D and growth, the recent literature of trade largely use the Pareto distribution for modeling uncertainty concerning the future productivity levels of firms after the process of entry into the industry (see for instance Melitz, 2003; Melitz and Ottaviano, 2003).

3 The model

We consider a closed economy with a continuum of industries indexed by $\omega \in [0,1]$. In each industry ω, firms are distinguished by the quality of the products they produce. Better quality products cannot be manufactured until they have been invented, and firms in each industry ω engage in R&D races. As in the standard quality ladders model by Grossman and Helpman (1991), when the top quality in an industry is the product of vintage j, the next winner of an R&D race becomes the sole producer of vintage $j+1$ quality product. In contrast with the standard model though, the quality-jumps realized by the winners of each R&D race are not fixed and are drawn from a probability distribution with support $[1,+\infty)$.

3.1 Consumers

There is a representative household that is modeled as a dynastic family that grows over time at an exogenous rate n. Normalizing the initial number of members of this family to equal one, the economy' population size at time t amounts to $L(t) = e^{nt}$. Each individual provides labor services in exchange for wages and choose from the continuum of products $\omega \in [0,1]$, where each product ω can potentially be supplied in a countably-infinite number of qualities. Quality vintage j of product ω provides quality $q(j,\omega,t)$. By the definition of quality improvement, new generations are better than the old ones; i.e., $q(j,\omega,t) > q(j-1,\omega,t)$. At time $t=0$, the state-of-the-art quality q for each product ω equals one.

The representative household has additively separable intertemporal preferences given by the following lifetime utility function:

$$U = \int_0^\infty e^{-(\rho-n)t} \ln u(t) \ \mathrm{d}t, \tag{1}$$

where $\rho > n$ denotes the subjective discount rate and $u(t)$ is the static utility of each household member. The static utility function $u(t)$ takes the following Dixit-Stiglitz form:

$$u(t) = \left\{ \int_0^1 \left[\Sigma_j q(j,\omega,t)^{\frac{1}{\sigma-1}} \cdot d(j,\omega,t) \right]^{\frac{\sigma-1}{\sigma}} \ \mathrm{d}\omega \right\}^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $d(j,\omega,t)$ is consumption by an individual of a product ω with quality vintage j at time t and $\sigma > 1$ is the elasticity of substitution between products. Note that goods of different vintages are perfect substitutes.[6]

The representative household maximizes lifetime utility (1) subject to the following intertemporal budget constraint:

$$\int_0^\infty e^{ns} c(s) e^{-R(s)} \ \mathrm{d}s = A(0) + \int_0^\infty e^{ns} w(s) e^{-R(s)} \ \mathrm{d}s, \tag{3}$$

where $c(t)$ is the flow of individual consumer spending, $R(t) \equiv \int_0^t r(\tau) \ \mathrm{d}\tau$ is the cumulative interest rate (with $R'(t) = r(t)$), $r(t)$ is the market interest rate, $A(0)$ is the initial value of asset holdings

[6] The specification (2) has been used by Thompson and Waldo (1994), Dinopoulos and Thompson (1998), Thompson (1999), Li (2003) and Segerstrom (2007).

of the representative household and $w(t)$ is the wage rate earned by each household member. Households own firms in equal shares and receive profits as dividends. Individual consumer spending is given by:

$$c(t) = \int_0^1 \left[\Sigma_j p(j,\omega,t)\, d(j,\omega,t)\right] \, \mathrm{d}\omega,$$

where $p(j,\omega,t)$ is the price of product ω with quality vintage j at time t.

The household maximization problem can be broken into three stages: the allocation of expenditure at each instant for each product across available quality levels, the allocation of expenditure at each instant across products, and the allocation of expenditure across time. The first two subproblems are two static maximization problems while the last subproblem is essentially a dynamic or intertemporal maximization problem.

3.1.1 The allocation of expenditure for each product across available quality levels

In the first stage, each household member solves a within-industry static optimization problem which consists in selecting the vintage for each product. Formally, this within-industry static maximization reads:

$$\max_{d(\cdot)} \; \Sigma_j q(j,\omega,t)^{\frac{1}{\sigma-1}} \cdot d(j,\omega,t)$$

subject to:

$$c(\omega,t) = \Sigma_j p(j,\omega,t)\, d(j,\omega,t),$$

where ω and t are fixed and $c(\omega,t)$ is consumer expenditure in industry ω at time t.

It is straightforward to verify that the representative household member is indifferent between quality vintage j and quality vintage $j-1$ if:

$$\frac{p(j,\omega,t)}{p(j-1,\omega,t)} = \lambda(\omega,t)^{\frac{1}{\sigma-1}},$$

where $\lambda(\omega,t) \equiv \frac{q(j,\omega,t)}{q(j-1,\omega,t)}$ is the quality jump separating two consecutive vintages of product ω at time t. When consumers are indifferent between two vintages, we restrict attention to equilibria where consumers only buy the higher quality product, with the result that only the highest quality level available is sold in equilibrium.

3.1.2 The allocation of expenditure across products

In the second stage, each household member decides the allocation of expenditure $c(\omega,t)$ across the unit measure of products. Formally, for fixed time t, this across-industry static optimization problem is:

$$\max_{d(\cdot)} \left[\int_0^1 q(j_\omega,\omega,t)^{\frac{1}{\sigma}} \cdot d(j_\omega,\omega,t)^{\frac{\sigma-1}{\sigma}} \, \mathrm{d}\omega\right]^{\frac{\sigma}{\sigma-1}},$$

subject to:

$$c(t) = \int_0^1 p(j_\omega,\omega,t)\, d(j_\omega,\omega,t) \, \mathrm{d}\omega,$$

where $q(j_\omega, \omega, t)$ is the quality level, j_ω is the vintage, and $d(j_\omega, \omega, t)$ is the individual quantity demanded for the product with the lowest quality-adjusted price in industry ω at time t. Solving this problem yields the individual consumer demand function:

$$d(j_\omega, \omega, t) = \frac{p(j_\omega, \omega, t)^{-\sigma} q(j_\omega, \omega, t) c(t)}{P(t)^{1-\sigma}}. \tag{4}$$

where

$$P(t) \equiv \left[\int_0^1 q\left(j_{\omega'}, \omega', t\right) p\left(j_{\omega'}, \omega', t\right)^{1-\sigma} \, \mathrm{d}\omega' \right]^{1/(1-\sigma)} \tag{5}$$

is the relevant quality-adjusted price index. According to (4), the representative household member demands the amount $d(j_\omega, \omega, t)$ of product ω with a quality level $q(j_\omega, \omega, t)$, and no units of lower quality versions of that product. Other things being equal, there is stronger consumer demand for products of higher quality. Because preferences are homothetic, aggregate demand equals $D(j_\omega, \omega, t) = d(j_\omega, \omega, t) L(t)$ in each industry ω.

3.1.3 The allocation of lifetime wealth across time

In the first stage, the representative household allocates its lifetime expenditure across time. The optimal control problem consists of maximizing discounted utility (1) given (2), (3), (4) and (5). The solution to this optimal control problem yields the well-known Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = r(t) - \rho, \tag{6}$$

which says that individual consumer expenditure c grows over time if and only if the market interest rate r exceeds the subjective discount rate ρ.

3.2 Product Markets

In each industry, firms compete in prices. Labor is the only input used in production and there are constant returns to scale. One unit of labor is required to produce one unit of output, regardless of quality. The labor market is perfectly competitive and the wage is normalized to unity throughout time. Consequently, each firm has a constant marginal cost of production equal to one.

The pricing decision of each state-of-the-art good producer (henceforth, industry leader) will depend on the kind of innovation obtained: drastic (radical) or non-drastic (incremental). When $\lambda(\omega, t) < \left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}$, the innovation are non-drastic and the industry leader uses the limit-price $p = \lambda(\omega, t)^{1/(\sigma-1)}$ to ensure that consumers purchase his good instead of the second-highest quality good. On the other hand, when $\lambda(\omega, t) \geq \left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}$, charging the unconstrained monopoly price $p = \sigma/(\sigma-1)$ is profit-maximizing for the industry leader. As a result, the firm's pricing decision can be summarized as follows:

$$p(j_\omega, \omega, t) = \begin{cases} \lambda(\omega, t)^{1/(\sigma-1)} & \text{if } \lambda(\omega, t) < \left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1} \\ \frac{\sigma}{\sigma-1} & \text{if } \lambda(\omega, t) \geq \left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1} \end{cases} \tag{7}$$

The instantaneous flow of profits earned by each industry leader will depend on the size of the quality increment. Using (4) and (7), the flow of profits when innovation is drastic is:

$$\pi_D\left(j_\omega, \omega, t\right) = \frac{\sigma^{-\sigma}\left(\sigma-1\right)^{\sigma-1} q\left(j_\omega, \omega, t\right) c\left(t\right) L\left(t\right)}{P(t)^{1-\sigma}},$$

whereas the flow of profits when innovation is non-drastic is:

$$\pi_I\left(j_\omega, \omega, t\right) = \frac{\left[\lambda\left(\omega, t\right)^{\frac{1}{\sigma-1}} - 1\right] \lambda\left(\omega, t\right)^{-\frac{\sigma}{\sigma-1}} q\left(j_\omega, \omega, t\right) c\left(t\right) L\left(t\right)}{P(t)^{1-\sigma}}.$$

3.3 Research and development

There is free entry into each R&D race, so firms may target their research effort at any of the continuum of state-of-the-art quality products. Labor is the only input used in R&D and all firms have the same R&D technology. Any firm i that hires $\ell_i\left(\omega, t\right)$ units of R&D labor in industry ω at time t is able to discover the next higher quality product $j_\omega + 1$ with instantaneous probability (or Poisson arrival rate):

$$\iota_i\left(\omega, t\right) = \frac{Q\left(t\right)^{\psi} \ell_i\left(\omega, t\right)}{aX\left(\omega, t\right) q\left(j_\omega, \omega, t\right)}, \tag{8}$$

where $Q\left(t\right) \equiv \int_0^1 q\left(j_\omega, \omega, t\right) \mathrm{d}\omega$ is the average quality across industries at time t, $\psi > 0$ is an across-industry R&D spillover parameter and $a > 0$ is a R&D productivity parameter. $X(\omega, t)$ is a R&D difficulty index which takes on the initial value $X(\omega, 0) = 1$ for all ω and grows over time according to:

$$\frac{\dot{X}\left(\omega, t\right)}{X\left(\omega, t\right)} = \mu \cdot \iota\left(\omega, t\right), \tag{9}$$

where $\mu > 0$ a R&D difficulty growth parameter and $\iota\left(\omega, t\right) = \sum_i \iota_i\left(\omega, t\right)$ is the industry-wide instantaneous probability of R&D success.

This R&D technology was introduced by Li (2003). It captures two reasons why innovating can become more difficult over time and one reason why innovating can become less difficult over time. First, innovating can become more difficult over time because $X(\cdot)$ increases. During each R&D race, researchers start off pursuing the most promising projects and if they fail, they try less promising projects. Equation (9) captures increasing R&D difficulty arising from a series of research failures, as in Segerstrom (1998). Second, innovating can become more difficult over time because $q\left(j_\omega, \omega, t\right)$ increases. Since $q\left(j_\omega, \omega, t\right)$ only increases when innovation occurs, this term highlights another source of increasing R&D difficulty, namely research successes. As products improve in quality and become more complex, the creation of the next vintage quality becomes more difficult. Third, innovating can become less difficult over time because $Q(t)^{\psi}$ increases. This term captures the possibility of positive across-industry knowledge spillovers. As other industries experience R&D successes and $Q(t)$ increases over time, this contributes to increasing the likelihood of research success by individual firms. These positive R&D spillovers have been found to be significant in many empirical studies (see Griliches (1992) and Sveikauskas (2007)).

Once a firm wins a R&D race, it observes its realized quality jump λ and decides whether to charge the unconstrained monopoly price or the limit price. We assume that the size of the quality jump is drawn from a Pareto distribution with a shape parameter $1/\kappa$ and a scale parameter equal to 1. The probability density function for this Pareto distribution is given by:

$$g(\lambda) = \frac{1}{\kappa}\lambda^{-(1+\kappa)/\kappa}, \qquad \lambda \in [1, \infty). \tag{10}$$

The parameter $\kappa \in (0, 1)$ governing the shape of the Pareto distribution can be interpreted as a measure of dispersion or heterogeneity. A lower value of κ corresponds to a thinner upper tail of the distribution of quality jumps. As κ increases, the expected value of λ increases, in fact, it equals $1/(1 - \kappa)$.

As in Grossman and Helpman (1991), Segerstrom (1998) and Li (2003), we solve the model for symmetric equilibrium behavior where the industry-level innovation rate $\iota(\omega, t)$ is the same in all industries ω at each point in time t. This allows us to simplify notation since both ι and X are simply functions of t, that is, $\iota(\omega, t) = \iota(t)$ and $X(\omega, t) = X(t)$ for all ω. Furthermore, to simplify the transitional dynamics analysis, we assume that the initial distribution of λ values is given by $g(\lambda)$ at time $t = 0$. Then as firms innovate and draw new values of λ, the distribution of λ values does not change over time.

3.4 Quality dynamics

Consider now how average quality $Q(t)$ evolves over time. In industry ω, the quality index $q(j_\omega, \omega, t)$ jumps to $q(j_\omega + 1, \omega, t) = \lambda q(j_\omega, \omega, t)$ at the rate $\iota(t)$ when an innovation occurs. Since this process of quality improvement is common to all industries in the economy, the time derivative of $Q(t)$ can be written as:

$$\dot{Q}(t) = \int_0^1 (\lambda - 1) q(j_\omega, \omega, t)\, \iota(t)\ \mathrm{d}\omega.$$

Using the law of large numbers, the previous equation becomes:

$$\dot{Q}(t) = \iota(t) \int_0^1 q(j_\omega, \omega, t) \left[\int_1^\infty \lambda g(\lambda)\ \mathrm{d}\lambda - 1\right]\ \mathrm{d}\omega,$$

which can be rewritten as:

$$\frac{\dot{Q}(t)}{Q(t)} = \frac{\kappa}{1-\kappa}\iota(t). \tag{11}$$

The growth rate of the average quality is an increasing function of the Pareto parameter κ (which measures the dispersion of the innovation size distribution) and the industry-level innovation rate $\iota(t)$.

Given that the expected number of R&D successes before time t is $\Phi(t) \equiv \int_0^t \iota(\tau)\, d\tau$, solving the differential equation (11) with initial condition $Q(0) = 1$ yields $Q(t) = e^{\kappa\Phi(t)/(1-\kappa)}$. Likewise, solving the differential equation (9) with initial condition $X(0) = 1$ yields $X(t) = e^{\mu\Phi(t)}$. It follows that $\frac{Q(t)^\psi}{X(t)} = Q(t)^\phi$ for all t, where $\phi \equiv \psi - \mu(1-\kappa)/\kappa$. The net-spillover parameter ϕ can be either positive or negative depending on whether the positive spillovers associated with ψ dominate the negative spillovers associated with μ, or vice versa. Using this insight, the R&D technology (8) can be rewritten as

$$\iota_i(\omega,t) = \frac{Q(t)^{\phi}\,\ell_i(\omega,t)}{aq(j_\omega,\omega,t)}. \tag{12}$$

We use this simplified R&D technology in the rest of the paper since the parameter values ψ and μ only influence equilibrium outcomes through their combined effect on ϕ. An increase in ψ is equivalent to an increase in ϕ and an increase in μ is equivalent to a decrease in ϕ. The only restriction that we impose on ϕ is that $\phi < 1$, to guarantee that the equilibrium rate of economic growth is finite.

3.5 R&D Optimization

Now, let us consider a firm's choice of industry in which to target its R&D effort. The prize for a research success in an industry is a flow of profits that will last until the next success is achieved in the same industry. At the beginning of each R&D race, a firm faces two different types of uncertainty. The first one is related to the outcome of the race, because the firm may fail to win the R&D race. The second one is related to the size of the quality jump and determines whether the firm will practice limit-pricing or pure monopoly pricing.

Let $v^e(j_\omega+1,\omega,t)$ denote the expected value of the uncertain profit stream for winning a R&D race and discovering the next higher quality product $j_\omega+1$ in industry ω at time t. Let s_R denote the fraction of the firm's R&D cost subsidized by the government. As in Segerstrom (1998) and Li (2003), we assume that the government finances the chosen R&D subsidy s_R by means of lump-sum taxation. By hiring $\ell_i(\omega,t)$ units of R&D labor for a time interval dt, firm i expects to realize $v^e(j_\omega+1,\omega,t)$ with probability $\iota_i(\omega,t)\,dt$. Thus, at each point in time t, firm i will choose its R&D employment ℓ_i in order to solve:

$$\max_{\ell_i}\left\{v^e(j_\omega+1,\omega,t)\frac{Q(t)^{\phi}\,\ell_i(\omega,t)}{aq(j_\omega,\omega,t)} - \ell_i(\omega,t)(1-s_R)\right\}.$$

The first order condition for an interior solution is:

$$v^e(j_\omega+1,\omega,t) = aq(j_\omega,\omega,t)\,Q(t)^{-\phi}(1-s_R). \tag{13}$$

If $v^e(j_\omega+1,\omega,t) < aq(j_\omega,\omega,t)\,Q(t)^{-\phi}(1-s_R)$, then the marginal cost of R&D exceeds the marginal benefit and it is profit-maximizing for firms to devote no labor to R&D. In contrast, if $v^e(j_\omega+1,\omega,t) > aq(j_\omega,\omega,t)\,Q(t)^{-\phi}(1-s_R)$, then the marginal benefit of R&D exceeds the marginal cost and it is profit-maximizing for firms to devote infinite resources to R&D. Only if (13) holds for all ω can a symmetric equilibrium exist where the innovation rate $\iota(t)$ is positive, finite and the same in all industries. Then marginal cost equals marginal benefit in all R&D activities.

3.6 The stock market

There is a stock market that channels consumer savings to R&D projects and helps consumers to diversify the risks of holding stocks issued by firms. The stock market valuation of each innovation is the expected discounted profits that the innovation generates. We now solve for these expected discounted profits.

Regardless of whether an innovation is drastic or non-drastic, over a time interval dt the shareholder receives an expected dividend $\pi^e(j_\omega+1,\omega,t)dt$ and the value of the quality leader appreciates by $\dot{v}^e(j_\omega+1,\omega,t)dt$. Because each quality leader is targeted by other firms that conduct R&D to discover the next higher quality product, the shareholder suffers an expected loss of $v^e(j_\omega+1,\omega,t)$ if further innovation occurs. This event occurs with probability $\iota(t)dt$, whereas no innovation occurs with probability $1-\iota(t)dt$. Efficiency in the stock market requires that the expected rate of return from holding a stock of a quality leader is equal to the riskless rate of return $r(t)$ that can be obtained through complete diversification. Taking limits as dt approaches zero, we get the following no-arbitrage condition for the stock market:

$$\frac{\pi^e(j_\omega+1,\omega,t)}{v^e(j_\omega+1,\omega,t)}+\frac{\dot{v}^e(j_\omega+1,\omega,t)}{v^e(j_\omega+1,\omega,t)}=r\left(t\right)+\iota(t).$$

In equilibrium, the dividend rate plus the expected rate of capital gains equals the riskless interest rate plus a risk premium (since quality leaders risk being driven out of business by further innovation). Using the free entry condition (13) and taking into account that $q\left(j_\omega,\omega,t\right)$ is fixed during a R&D race, $\frac{\dot{v}^e(j_\omega+1,\omega,t)}{v^e(j_\omega+1,\omega,t)}=-\phi\frac{\dot{Q}(t)}{Q(t)}$. Consequently, the expected dividend rate is:

$$\frac{\pi^e(j_\omega+1,\omega,t)}{v^e(j_\omega+1,\omega,t)}=r\left(t\right)+\iota(t)+\phi\frac{\dot{Q}\left(t\right)}{Q\left(t\right)}.$$

Solving for the expected profit flow of the firm that wins the R&D race and produces the top quality $q\left(j_\omega+1,\omega,t\right)$, we get:[7]

$$v^e(j_\omega+1,\omega,t)=\frac{\frac{q(j_\omega,\omega,t)c(t)L(t)}{Q(t)}\frac{\kappa(1+\kappa)\chi}{\sigma+\kappa-1}}{r\left(t\right)+\iota(t)+\phi\dot{Q}\left(t\right)/Q\left(t\right)},\tag{14}$$

where $\chi\equiv\left[1+\frac{\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-(\sigma-1)/\kappa}}{1-\kappa}\right]/\left[1+\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-(\sigma-1)(1+\kappa)/\kappa}\right]$ is completely determined by parameter values. Combining (14) with (13), we get the following research equation:

$$\frac{\frac{\kappa(1+\kappa)\chi}{\sigma+\kappa-1}c\left(t\right)}{r\left(t\right)+\iota\left(t\right)+\phi\dot{Q}\left(t\right)/Q\left(t\right)}=ax\left(t\right)\left(1-s_R\right),\tag{15}$$

where $x\left(t\right)\equiv\frac{Q(t)^{1-\phi}}{L(t)}$ is a new endogenous variable that measures relative R&D difficulty in the economy.

The economic intuition behind (15) is straightforward. The left-hand side is proportional to the benefit (expected discounted profits) of winning each R&D race and the right-hand side is proportional to the cost. The benefit from innovating increases when $c\left(t\right)$ increases (the representative consumer buys more), when $r\left(t\right)$ decreases (future profits are discounted less), when $\iota\left(t\right)$ decreases (the industry leader is less threatened by further innovation), and when $\phi\dot{Q}\left(t\right)/Q\left(t\right)$ decreases (there are higher capital gains from remaining in business). The cost of innovating increases when a increases (R&D workers become less productive at generating innovations), when $x\left(t\right)$ increases

[7] See Appendix A.1 for further details.

(innovating becomes relatively more difficult), and when s_R decreases (the government subsidizes R&D less).

3.7 The labor market

Labor is perfectly mobile across industries and between production and R&D activities. In each industry ω, consumers only buy from the current quality leader. Employment in the manufacturing sector is given by:[8]

$$L_M(t) = \int_0^1 D\left(j_\omega, \omega, t\right) \, \mathrm{d}\omega = \frac{(\sigma-1)(1+\kappa)}{\sigma(\kappa+1)-1} c(t) L(t).$$

Total employment in the R&D sector is given by:

$$L_I(t) = a\iota(t) Q(t)^{-\phi} \int_0^1 q\left(j_\omega, \omega, t\right) \, \mathrm{d}\omega = a\iota(t) Q(t)^{1-\phi}.$$

Consequently, it follows from the full employment of labor condition $L(t) = L_M(t) + L_I(t)$ that:

$$1 = \frac{(\sigma-1)(1+\kappa)}{\sigma(1+\kappa)-1} c(t) + a\iota(t) x(t). \tag{16}$$

The two terms on the right-hand-side of (16) are the shares of labor in production and R&D activities, respectively. The production employment share increases when $c(t)$ increases (the representative consumer buys more). The R&D employment share increases when a increases (more R&D labor is needed to generate any given innovation rate), when $\iota(t)$ increases (firms innovate at a faster rate), and $x(t)$ increases (innovating is relatively more difficult).

This completes the description of the model. In the next section we analyze the dynamic properties of the model.

4 The balanced-growth equilibrium

We are interested in solving the model for a balanced growth (or steady-state) equilibrium where all endogenous variables grow over time at constant (not necessarily the same) rates and the innovation rate ι is the same in all industries.

Differentiating the identity $x(t) \equiv Q(t)^{1-\phi}/L(t)$ with respect to time, and plugging $\iota(t)$ from the full employment of labor condition (16) into (11), it is easy to verify that the time evolution of relative R&D difficulty $x(t)$ is governed by the following differential equation:

$$\frac{\dot{x}(t)}{x(t)} = \frac{B}{a(1-\kappa)x(t)} \left[1 - \frac{(\sigma-1)(1+\kappa)}{\sigma(\kappa+1)-1} c(t)\right] - n, \tag{17}$$

where $B \equiv (1-\phi)\kappa$ and the assumption $\phi < 1$ guarantees that $B > 0$.

[8] See Appendix A.2 for further details.

Next we derive a second differential equation governing the time evolution of consumer expenditure $c(t)$. Using the labor full-employment condition (16) to substitute for the innovation rate $\iota(t)$, and then using (15) and (11) to substitute for the interest rate $r(t)$, the Euler equation (6) becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{\kappa(1+\kappa)\chi c(t)}{(\sigma+\kappa-1)\,ax(t)(1-s_R)} - \frac{1-B}{(1-\kappa)\,ax(t)}\left[1-\frac{(\sigma-1)(1+\kappa)c(t)}{\sigma(\kappa+1)-1}\right]-\rho. \tag{18}$$

Equations (17) and (18) form a system of two non-linear differential equations in the two endogenous variables $x(t)$ and $c(t)$.

Equation (17) implies that $\frac{\dot{x}(t)}{x(t)}$ is constant over time only if both x and c are constant over time. Thus we solve the model for balanced growth equilibria by solving for when both $\frac{\dot{x}(t)}{x(t)}=0$ and $\frac{\dot{c}(t)}{c(t)}=0$. This yields the following pair of equations that are linear in (x,c) space:

$$1=\frac{(\sigma-1)(1+\kappa)}{\sigma(\kappa+1)-1}c+\frac{a(1-\kappa)n}{B}x, \tag{19}$$

$$\frac{\frac{\kappa(1+\kappa)\chi}{\sigma+\kappa-1}c}{\rho+\frac{1-B}{a(1-\kappa)x}\left[1-\frac{(\sigma-1)(1+\kappa)}{\sigma(\kappa+1)-1}c\right]}=ax(1-s_R). \tag{20}$$

It is easily verified that (19) is downward-sloping in (x,c) space, (20) is upward-sloping in (x,c) space, and they have a unique intersection in the strictly positive orthant given by:

$$c^*=\frac{(1-s_R)\left[\rho+\frac{1-B}{B}n\right]\frac{\sigma(\kappa+1)-1}{(\sigma-1)(1+\kappa)}}{(1-s_R)\left[\rho+\frac{1-B}{B}n\right]+\frac{n(1-\kappa)}{B}\frac{\kappa\chi[\sigma(\kappa+1)-1]}{(\sigma+\kappa-1)(\sigma-1)}}, \tag{21}$$

$$x^*=\frac{\frac{\kappa\chi[\sigma(\kappa+1)-1]}{a(\sigma+\kappa-1)(\sigma-1)}}{(1-s_R)\left[\rho+\frac{1-B}{B}n\right]+\frac{n(1-\kappa)}{B}\frac{\kappa\chi[\sigma(\kappa+1)-1]}{(\sigma+\kappa-1)(\sigma-1)}}. \tag{22}$$

Furthermore, this balanced growth equilibrium is saddle-path stable; starting from any initial value of x, if c jumps immediately to the saddle-path value, then over time both x and c converge to the balanced growth values given by (21) and (22).

Next, we solve for the innovation rate ι^* corresponding to this balanced growth equilibrium. From the definition $x(t)\equiv\frac{Q(t)^{1-\phi}}{L(t)}$ and (11), $\frac{\dot{x}(t)}{x(t)}=(1-\phi)\kappa\iota/(1-\kappa)-n$. Solving for when $\frac{\dot{x}(t)}{x(t)}=0$ yields the unique balanced growth innovation rate:

$$\iota^*=\frac{n(1-\kappa)}{(1-\phi)\kappa}. \tag{23}$$

Equation (23) has two implications that are standard and one implication that is new. As in Li (2003), the steady-state innovation rate ι^* is an increasing function of the population growth rate n and an increasing function of the strength of R&D spillovers parameter ϕ. What is new is that the steady-state innovation rate ι^* is also a decreasing function of κ, the Pareto distribution parameter that determines the expected size of innovations. Since the expected size of innovations is increasing in κ, (23) implies when innovations are bigger on average, they occur less frequently on the steady-state equilibrium path.

We can also solve for the steady-state share of labor employed in R&D activities. From (19), the R&D share is $\frac{a(1-\kappa)n}{B}x$. Substituting into this expression using (22) yields:

$$\left(\frac{L_I}{L}\right)^* = \frac{\frac{n(1-\kappa)}{B}\frac{\kappa\chi[\sigma(\kappa+1)-1]}{(\sigma+\kappa-1)(\sigma-1)}}{(1-s_R)\left[\rho+\frac{1-B}{B}n\right]+\frac{n(1-\kappa)}{B}\frac{\kappa\chi[\sigma(\kappa+1)-1]}{(\sigma+\kappa-1)(\sigma-1)}}. \tag{24}$$

The steady-state R&D employment share is an increasing function of the R&D subsidy rate s_R and asymptotically approaches 1 as s_R approaches 1.

Finally, we solve for the steady-state rate of economic growth g^*. Substituting the individual demand function (4) into (2) and simplifying using (5), we obtain:

$$u(t) = \left[\int_0^1 q\left(j_\omega,\omega,t\right)^{\frac{1}{\sigma}} \cdot d\left(j_\omega,\omega,t\right)^{\frac{\sigma-1}{\sigma}} \; \mathrm{d}\omega\right]^{\frac{\sigma}{\sigma-1}} = \frac{c(t)}{P(t)}.$$

Since static consumer utility $u(t)$ is proportional to consumer expenditure $c(t)$, static consumer utility is a measure of real consumption and it is appropriate to use the growth rate of static utility as our measure of economic growth. Now $\frac{\dot{P}(t)}{P(t)} = \frac{1}{1-\sigma}\frac{\dot{Q}(t)}{Q(t)} = \frac{1}{1-\sigma}\frac{\kappa}{1-\kappa}\iota = \frac{n}{(1-\sigma)(1-\phi)}$ since the distribution of prices does not change over time. Thus the steady-state rate of economic growth is:

$$g^* = \frac{\dot{u}(t)}{u(t)} = -\frac{\dot{P}(t)}{P(t)} = \frac{n}{(\sigma-1)(1-\phi)}. \tag{25}$$

The steady-state rate of economic growth g^* is increasing in the population growth rate n, is increasing in the strength of R&D spillovers ϕ and is decreasing in the elasticity of substitution between products σ. Note that the R&D subsidy rate s_R does not appear in (25) and hence this is a semi-endogenous growth model.

The following Proposition summarizes the steady-state properties of the model:

Proposition 1 *The model has a unique symmetric balanced-growth equilibrium where consumer expenditure c, relative R&D difficulty x, the innovation rate in each industry ι, the R&D employment share L_I/L, and the rate of economic growth g are all constant over time and given by (21), (22), (23), (24) and (25), respectively.*

Ha and Howitt (2007) have recently called into question semi-endogenous growth models, arguing that fully-endogenous growth models have better empirical support. However, their analysis has an important limitation. They implicitly assume that convergence to steady-state is fast, so that with 50 years of data, one can just focus on the steady-state implications of growth models. This assumption is called into question in Steger (2003). He calibrates the Segerstrom (1998) semi-endogenous growth model using US data and finds that convergence to steady-state is slow: it takes almost 40 years to go half the distance to the steady-state. With such slow convergence, we think that future tests of semi-endogenous growth theory should take into account the transition path implications of the theory.[9]

5 Welfare Properties

The previous section has analyzed the equilibrium properties of the model. In this section we study the model's welfare properties. For the sake of exposition, in what follows we only report the main results of the welfare analysis and leaving to Appendix A.3 the detailed calculations. We assume that the social planner can only intervene by subsidizing or taxing R&D and solve for the R&D subsidy/tax policy that maximizes the discounted utility of the representative household.[10]

The main results of the welfare analysis can be summarized as follows: (i) the optimal R&D intensity is the same as (23) and (ii) the optimal share of labor in R&D is

$$\left(\frac{L_I}{L}\right)^s = \frac{n\kappa}{n\kappa + B\rho(\sigma - 1)}. \tag{26}$$

As in Segerstrom (1998) and Li (2003), even though the optimal innovation rate coincides with the equilibrium innovation rate (23), the government can in general improve welfare by intervening in the economy. By comparing (24) to (26), we get that the steady-state equilibrium outcome is optimal if and only if

$$1 - s_R = \Lambda, \quad \Lambda \equiv \frac{\rho\chi\,(1-\kappa)\left[\sigma(\kappa+1) - 1\right]}{(\sigma + \kappa - 1)\left[\rho + \frac{1-B}{B}n\right]}, \tag{27}$$

which implies that the optimal R&D policy is

$$\begin{array}{lcl} \text{tax R\&D} & \Longleftrightarrow & \Lambda > 1 \\ \text{subsidize R\&D} & \Longleftrightarrow & \Lambda < 1 \end{array}$$

The optimal R&D policy can be either a tax or a subsidy depending on the parameters of the model. Because Λ is a rather complicated function of the model's parameter values, we study the model's properties numerically to gain further insights about its welfare implications.

5.1 Numerical Results

In our computer simulations, we use the following benchmark parameter values: $\rho = 0.07$, $n = 0.014$, $\kappa = 0.2132$, $\sigma = 2$, $a = 1$, $\mu = 0$ and $\phi = \psi = 0.3$.

The subjective discount rate ρ is chosen to get a steady-state equilibrium interest rate matching its long-run average value in the data. The real interest rate in the model is both the risk-free interest rate as well as a measure of the average real return on the stock market. In the simulations

[9] We also think that Ha and Howitt (2007) use inappropriate data. In most of their paper (except for section 6), they use aggregate R&D expenditure data, which includes defense and space R&D. We see no reason to expect that semi-endogenous growth models would do a good job of explaining movements in defense and space R&D. In these models, all R&D decisions are made by profit-maximizing firms who sell their products to consumers, not the public sector.

[10] The optimal R&D subsidy/tax policy that we solve for does not lead to the first-best outcome since the markup of price over marginal cost differs across firms. To achieve the first-best outcome, the markup of price over marginal cost should be the same for all products. But that does not hold in equilibrium with heterogenous firms.

we set $\rho = 0.07$, consistent with the 7 percent average real return on the US stock market over the past century as calculated by Mehra and Prescott (1985).

The population growth rate $n = 0.014$ is chosen to match the world population growth rate. According to the World Development Indicators (World Bank, 2003), the average annual rate of population growth in the world between 1990 and 2000 was around 1.4 percent.

The Pareto distribution parameter κ and the elasticity of substitution parameter σ are chosen to generate markups of price over marginal cost consistent with the data. In the model, since marginal cost equals one for each firm, the markup of price over marginal cost for each firm is the firm's price, so the average markup is the expected price Ep charged by industry leaders. Using (7) and (10), the average markup in the economy is

$$Ep \equiv \int_1^\infty p(\lambda)g(\lambda)\ \mathrm{d}\lambda \quad = \quad \frac{\left(\frac{\sigma}{\sigma-1}\right)^{1+\frac{1-\sigma}{\kappa}} - 1}{\frac{\kappa}{\sigma-1} - 1} + \left(\frac{\sigma}{\sigma-1}\right)^{1+\frac{1-\sigma}{\kappa}}$$

and the highest markup in the economy is

$$p_{max} \equiv \frac{\sigma}{\sigma-1}.$$

The average markup of price over marginal cost has been estimated as ranging between 1.05 and 1.4 [see Basu (1996) and Norrbin (1993)]. We calibrate the model so the average markup is in the middle of this range, $Ep = 1.25$ or 25 percent, and the highest markup is $p_{max} = 2.0$ or 100 percent. These properties imply that $\kappa = 0.2132$ and $\sigma = 2$.

In the model, the R&D productivity parameter a plays no role in determining either the equilibrium share of labor in R&D (24) or the optimal R&D subsidy rate (27), so we set $a = 1$. Likewise, since the parameter values ψ and μ only influence equilibrium outcomes through their combined effect on $\phi \equiv \psi - \mu(1-\kappa)/\kappa$, we set $\mu = 0$ so the parameter $\phi = \psi$ fully determines the size of R&D spillovers. Finally, we chose ϕ to guarantee that the steady-state rate of economic growth is 2 percent, which is consistent with the average US GDP per capita growth rate from 1950 to 1994 reported in Jones (2005). Given $n = 0.014$, $\sigma = 2$ and $g = 0.02$, $g = \frac{n}{(\sigma-1)(1-\phi)}$ implies that the net-spillover parameter ϕ is 0.3.

The numerical results with the benchmark parameter values are reported in Table 1. The left

Table 1: The Benchmark Case

	Equilibrium	Optimal
s_R	0	0.56
g	0.02	0.02
Ep	1.25	1.25
x	1.50	3.01
c	1.04	0.91
L_I/L	0.11	0.22
$1/I$	13.5	13.5
$\hat{u}$	1.86	4.41

column shows different endogenous variables that we solve for, the middle column shows the steady-state equilibrium results when R&D is not subsidized and the right column shows the steady-state equilibrium results when R&D is optimally subsidized.

The main result from Table 1 is that it is optimal to heavily subsidize R&D. In the benchmark case where the steady-state rate of economic growth is 2 percent ($g = 0.02$) and the average markup is 25 percent ($Ep = 1.25$), the optimal R&D subsidy rate is 56 percent ($s_R = 0.56$). Given this property, it is not surprising that optimally subsidizing R&D leads to a big increase in relative R&D difficulty (x increases from 1.50 to 3.01), a big increase in the share of labor employed in R&D activities (L_I/L increases from 0.11 to 0.22), and a fall in consumer expenditure on goods (c decreases from 1.04 to 0.91).

Instead of reporting the steady-state innovation rate I in Table 1, we report its inverse $1/I$ since this provides more useful information. The inverse $1/I$ represents the average time duration between innovations or the expected duration of profits for industry leaders. With the benchmark parameter values, this expected duration of profits is a reasonable 13.5 years (both before and after R&D is optimally subsidized).

We also report information about how much consumers gain from optimal policies being implemented. It could be that implementing the optimal R&D subsidy policy is associated with just a small increase in real consumption, even though the optimal R&D subsidy rate is high. Comparing steady-state equilibrium paths before and after R&D is optimally subsidized, we show in Appendix A.3 that the percentage increase in real consumption is the same as the percentage increase in subutility $\hat{u}$. Optimally subsidizing R&D leads to a 137 percent increase in real consumption ($\hat{u}$ increases from 1.86 to 4.41). Thus in the benchmark case, not only is it optimal to heavily subsidize R&D but doing so has a big effect in increasing steady-state real consumption.

The question naturally arises, how robust is the finding that it is optimal to heavily subsidize R&D? Can we find plausible parameter values for which it is optimal to tax R&D?

As a first robustness check, we study what happens when the steady-state rate of economic growth is 0.5 instead of 2 percent ($g = 0.005$ instead of $g = 0.02$). To obtain this lower rate of economic growth (remember $g = \frac{n}{(\sigma-1)(1-\phi)}$), we lower the R&D spillover parameter ϕ from 0.3 to -1.8, hold all other parameters fixed at their benchmark values. With negative instead of positive R&D spillovers, the case for taxing R&D becomes stronger.

The results are reported in Table 2. Even when the rate of economic growth is low (only half

Table 2: The Low Growth Case

	Equilibrium	Optimal
s_R	0	0.18
g	0.005	0.005
Ep	1.25	1.25
x	3.00	3.61
c	1.11	1.10
L_I/L	0.055	0.067
$1/I$	54.2	54.2
$\hat{u}$	1.64	1.74

a percent per year), we get that it is still optimal to subsidize R&D, although the subsidy rate is lower (18 percent instead of 56 percent). Optimally subsidizing R&D leads to an increase in the share of labor employed in R&D activities by 22 percent (L_I/L increases from 0.055 to 0.067) and an increase in steady-state real consumption by 6 percent ($\hat{u}$ increases from 1.64 to 1.74). On the negative side, the expected duration of profits for industry leaders has now jumped up from 13.5 to 54.2 years, but this is an unavoidable implication of assuming a low rate of economic growth.

To get that it is optimal to tax R&D subsidy, we need to assume a very low rate of economic growth. Table 3 illustrates what happens when we keep on lowering ϕ and the corresponding economic growth rate g (holding all other parameters fixed at their benchmark values). In this

Table 3: Very Low Growth

ϕ	-1.8	-3	-6.2	-7	-10
$(s_R)^s$	0.18	0.10	0.000	-0.015	-0.052
g	0.005	0.003	0.002	0.0018	0.0013
$(L_I/L)^0$	0.055	0.043	0.0027	0.025	0.019
$(L_I/L)^s$	0.067	0.047	0.0027	0.024	0.018
$1/I$	54.2	77.4	138	154	213

table, we report the optimal R&D subsidy rate $(s_R)^s$, the R&D share when the subsidy rate is zero $(L_I/L)^0$ and the R&D share when the subsidy rate is optimal $(L_I/L)^s$. The second column represents the same information as in Table 2. What is striking is that the optimal R&D subsidy rate does not become negative (a R&D tax) until ϕ is below -6,2 and the economic growth rate is below 0.2 percent. Then the expected duration of profits exceeds 138 years.

As another robustness check, we study what happens when we vary the Pareto distribution parameter κ holding all other parameters fixed at their benchmark values. Increasing κ increases the average size of innovations and average markup of price over marginal cost in the economy. The

Table 4: Varying κ: The Benchmark Case

κ	0.05	0.1	0.3	0.5	0.7	0.9
$(s_R)^s$	0.85	0.73	0.49	0.43	0.42	0.43
g	0.02	0.02	0.02	0.02	0.02	0.02
$(L_I/L)^0$	0.04	0.07	0.13	0.14	0.14	0.14
$(L_I/L)^s$	0.22	0.22	0.22	0.22	0.22	0.22
$1/I$	2.63	5.55	21.4	50.0	116	450

results are presented in Table 4. As shown, there is a U-shaped relationship between the Pareto parameter κ and the optimal R&D subsidy rate $(s_R)^s$: as κ increases, $(s_R)^s$ first falls and then rises. Furthermore, the bottom of the U-shape occurs around $\kappa = 0.7$ and an optimal R&D subsidy rate of 42 percent. We never get close to the case of R&D taxes being optimal.

As a final robustness check, we study what happens when we vary the Pareto distribution parameter κ holding all other parameters fixed at their Table 2 values: "The Low Growth Case."

Table 5: Varying κ: The Low Growth Case

κ	0.05	0.1	0.3	0.5	0.7	0.9
$(s_R)^s$	0.55	0.35	0.15	0.17	0.22	0.28
g	0.005	0.005	0.005	0.005	0.005	0.005
$(L_I/L)^0$	0.03	0.04	0.06	0.06	0.05	0.05
$(L_I/L)^s$	0.07	0.07	0.07	0.07	0.07	0.07
$1/I$	10.5	22.2	85.7	200	467	1800

These results are presented in Table 5. Again, there is a U-shaped relationship between the Pareto parameter κ and the optimal R&D subsidy rate $(s_R)^s$: as κ increases, $(s_R)^s$ first falls and then rises. This time, the bottom of the U-shape occurs around $\kappa = 0.3$ and an optimal R&D subsidy rate of 15 percent. Even in this low growth case with $g = 0.005$, we never get close to R&D taxes being optimal.

The model has a striking property that sheds light on why it is so hard to find cases where taxing R&D is optimal. First note that $\left(\frac{L_I}{L}\right)^s = \frac{n\kappa}{n\kappa + B\rho(\sigma-1)}$ implies that $1/\left(\frac{L_I}{L}\right)^s = 1 + \frac{B\rho(\sigma-1)}{n\kappa}$. Then substituting into this expression using $g = \frac{n\kappa}{(\sigma-1)B}$, we obtain

$$\left(\frac{L_I}{L}\right)^s = \frac{g}{\rho + g}. \tag{28}$$

The optimal share of labor devoted to R&D $\left(\frac{L_I}{L}\right)^s$ only depends on the economic growth rate g and the interest rate parameter ρ. In the benchmark case where the interest rate is 7 percent ($\rho = 0.07$) and the economic growth rate is 2 percent ($g = 0.02$), the optimal R&D share is 22 percent. In "the low growth case" where the interest rate is 7 percent ($\rho = 0.07$) and the economic growth rate is 0.5 percent ($g = 0.02$), the optimal R&D share is 9 percent. Plugging possible values into (28), one quickly realizes that there do not exist plausible cases where the optimal R&D share is small and that is what one needs if taxing R&D is going to be optimal.

5.2 Discussion

To provide the economic intuition behind our numerical results, we focus on the familiar external effects associated with R&D discussed in Segerstrom (1998). First, every time a firm innovates, consumers benefit from being able to purchase higher quality products and firms do not take this into account in their profit-maximizing calculations. This *consumer surplus* effect represents one reason why firms may under-invest in R&D from a social perspective. Second, every time a firm innovates, it drives another firm out of business and firms do not take the losses of other firms into account in their profit-maximizing calculations. This *business stealing* effect represents one reason why firms may over-invest in R&D from a social perspective. Finally, there are the *intertemporal R&D spillover* effects associated with research failures and successes. For example, if $\phi < 0$, then R&D investment today raises the costs of innovating in the future and this represents an additional reason why firms may over-invest in R&D from a social perspective.

In the previous section, we found that the optimal R&D subsidy rate $(s_R)^s$ decreases when the R&D spillover parameter ϕ falls and the optimal R&D tax case only occurs when ϕ is sufficiently negative (Table 3). We can understand these findings by thinking about the above-mentioned external effects. When ϕ falls, there is no change in the size of the positive consumer surplus effect. But a fall in ϕ does contribute to lowering the innovation rate ι, makes the expected duration of each monopoly position longer (see the $1/\iota$ ratio in Table 3). Therefore, when an innovation occurs, the loss experienced by the owners of the displaced firm becomes larger when ϕ falls and there is a larger negative business stealing effect. Also a fall in ϕ directly lowers the positive intertemporal R&D spillover effect if $\phi > 0$ and raises the negative intertemporal R&D spillover effect if $\phi < 0$. Both the increase in the size of the business stealing effect and the "more negative" intertemporal R&D spillover effect contribute to reducing the optimal R&D subsidy rate $(s_R)^s$ and when these two external effects become sufficiently negative, then it becomes optimal to tax R&D. Since a low value of ϕ is associated with a low innovation rate ι, the optimal R&D tax case only occurs when the rate of economic growth g is relatively low.

6 Conclusions

In quality-ladder growth models, it is typically assumed that innovation size is constant and identical across industries. This assumption contrasts with the empirical evidence showing that innovation size is far from being identical across industries and that the innovation size distribution is skewed toward the low value side.

In this paper, we develop a quality-ladder growth model without scale effects that is consistent with the empirical evidence. In particular, we formalize the idea that there is an increasing difficulty to realize innovations of greater value by assuming that the size of quality improvements is the result of a random draw from a Pareto distribution. Our model extends the class of quality-ladder growth models to encompass firm heterogeneity: both the prices charged and the quantities produced vary across firms and depend on the type of innovation achieved.

In the earlier literature, Grossman and Helpman (1991), Segerstrom (1998) and Li (2003) solve their quality-ladder growth models analytically and find that whether it is optimal to subsidize or tax R&D depends on the size of innovations. In this paper, we go further by also solving our model numerically and this allows us to assess how important is the optimal R&D tax case. We find that the decentralized economy almost always under-invests in R&D relative to what is socially optimal. The only exceptions arise when the steady-state rate of economic growth is very low. As under-investment in R&D holds for a wide range of parameter values, our analysis provides support for a public policy of subsidizing R&D activities.

Appendix

Solving the integrals in this Appendix involves lengthy calculations. These calculations are presented in full in a "Guide to the Appendix" that is available from the authors upon request.

A.1 Calculation of the expected profit flow earned by an industry leader

In this section, we determine the expected value of the uncertain profit stream earned by an industry leader that produces the quality product $j_\omega + 1$ in industry ω at time t, that is:

$$\begin{aligned} \pi^e(j_\omega + 1, \omega, t) \;\; = \;\; & \int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \pi_I\left(j_\omega + 1, \omega, t\right) g\left(\lambda\right) \, \mathrm{d}\lambda + \\ & + \int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} \pi_D\left(j_\omega + 1, \omega, t\right) g\left(\lambda\right) \, \mathrm{d}\lambda. \end{aligned}$$

By substituting the flow of profits for incremental and drastic innovations, the expected profit flow $\pi^e(j_\omega + 1, \omega, t)$ can be written as:

$$\pi^e(j_\omega + 1, \omega, t) = \tfrac{q(j_\omega,\omega,t)c(t)L(t)}{P(t)^{1-\sigma}} \left[\underbrace{\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda \left(\lambda^{\frac{1}{\sigma-1}} - 1\right) \lambda^{-\frac{\sigma}{\sigma-1}} g\left(\lambda\right) \, \mathrm{d}\lambda +}_{(a)} \right.$$

$$\left. + \underbrace{\int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} \sigma^{-\sigma} \left(\sigma - 1\right)^{\sigma-1} \lambda g\left(\lambda\right) \, \mathrm{d}\lambda}_{(b)} \right]. \tag{A1.1}$$

First, we calculate the two integrals (a) and (b) of Eq. (A1.1). As concerns the first integral, we notice that the term $\lambda \left(\lambda^{\frac{1}{\sigma-1}} - 1\right) \lambda^{-\frac{\sigma}{\sigma-1}}$ can be easily written as $\left(1 - \lambda^{\frac{1}{1-\sigma}}\right)$. As the probability density function $g(\lambda)$ is Pareto and is equal to $\frac{1}{\kappa}\lambda^{-\frac{1+\kappa}{\kappa}}$, integral (a) can be rearranged as:

$$\frac{1}{\kappa} \int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \left(1 - \lambda^{\frac{1}{1-\sigma}}\right) \lambda^{-\frac{1+\kappa}{\kappa}} \, \mathrm{d}\lambda = \frac{\kappa\sigma - \left(\frac{\sigma}{\sigma-1}\right)^{\frac{1-\sigma}{\kappa}} \left[\sigma\left(\kappa + 1\right) - 1\right]}{\left(\kappa + \sigma - 1\right)\sigma}. \tag{A1.2}$$

Let us now focus on integral (b). Using the Pareto density function $g(\lambda)$ and solving, we get:

$$\frac{\sigma^{-\sigma}\left(\sigma - 1\right)^{\sigma-1}}{\kappa} \int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} \lambda^{-\frac{1}{\kappa}} \, \mathrm{d}\lambda = \frac{(\sigma-1)^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{\frac{(\kappa-1)(\sigma-1)}{\kappa}}}{\sigma^{\sigma}(1-\kappa)}. \tag{A1.3}$$

Summing the two expressions in Eqs. (A1.2) and (A1.3), it yields:

$$\frac{\kappa}{(\sigma+\kappa-1)}\left[1+\frac{\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{\sigma-1}{\kappa}}}{1-\kappa}\right]. \tag{A1.4}$$

Next, we calculate $P(t)^{1-\sigma}$ which is at the denominator of the ratio that is outside the square brackets of Eq. (A1.1). By using Eq. (5), this term can be written as:

$$\begin{aligned} P(t)^{1-\sigma} \quad \equiv \quad & \int_0^1 \left[\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \frac{q(j_{\omega'},\omega',t)}{\lambda} g\left(\lambda\right)\ \mathrm{d}\lambda + \right. \\ & \left. + \int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} q\left(j_{\omega'},\omega',t\right) g\left(\lambda\right)\ \mathrm{d}\lambda\right]\ \mathrm{d}\omega', \end{aligned}$$

which can be further rearranged as:

$$P(t)^{1-\sigma} \equiv \int_0^1 q\left(j_{\omega'},\omega',t\right)\left[\underbrace{\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \frac{g(\lambda)}{\lambda}\ \mathrm{d}\lambda}_{(c)}\right]\ \mathrm{d}\omega' +$$

$$+\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \int_0^1 q\left(j_{\omega'},\omega',t\right)\left[\underbrace{\int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} g\left(\lambda\right)\ \mathrm{d}\lambda}_{(d)}\right]\ \mathrm{d}\omega'. \tag{A1.5}$$

Firstly, we consider integral (c). As before, replacing the Pareto density function $g(\lambda)$ and solving, we get:

$$\frac{1}{\kappa}\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda^{-1-\frac{1+\kappa}{\kappa}}\ \mathrm{d}\lambda = \frac{1}{(1+\kappa)}\left[1-\left(\frac{\sigma}{\sigma-1}\right)^{\frac{(\kappa+1)(1-\sigma)}{\kappa}}\right]. \tag{A1.6}$$

Secondly, we calculate integral (d). Once again, using the Pareto density function $g(\lambda)$, this integral boils down to:

$$\frac{1}{\kappa}\int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} \lambda^{-\frac{1+\kappa}{\kappa}}\ \mathrm{d}\lambda = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{1-\sigma}{\kappa}}. \tag{A1.7}$$

Plugging Eqs. (A1.6) and (A1.7) into (A1.5), the term $P(t)^{1-\sigma}$ can be rewritten as:

$$\begin{aligned} P(t)^{1-\sigma} \quad \equiv \quad & \int_0^1 q\left(j_{\omega'},\omega',t\right)\left\{\frac{1}{(1+\kappa)}\left[1-\left(\frac{\sigma}{\sigma-1}\right)^{\frac{(\kappa+1)(1-\sigma)}{\kappa}}\right]\right\}\ \mathrm{d}\omega' + \\ & +\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\int_0^1 q\left(j_{\omega'},\omega',t\right)\left[\left(\frac{\sigma}{\sigma-1}\right)^{\frac{1-\sigma}{\kappa}}\right]\ \mathrm{d}\omega'. \end{aligned}$$

The latter can be rearranged as:

$$
\begin{aligned}
P(t)^{1-\sigma} \equiv & \frac{1}{(1+\kappa)}\left[1-\left(\frac{\sigma}{\sigma-1}\right)^{\frac{(1+\kappa)(1-\sigma)}{\kappa}}\right] \underbrace{\int_0^1 q\left(j_{\omega'}, \omega', t\right) \mathrm{d}\omega'}_{=Q(t)} + \\
& +\left(\frac{\sigma}{\sigma-1}\right)^{\frac{(\kappa+1)(1-\sigma)}{\kappa}} \underbrace{\int_0^1 q\left(j_{\omega'}, \omega', t\right) \mathrm{d}\omega'}_{=Q(t)}.
\end{aligned}
$$

Collecting $Q(t)$ and simplifying terms, we get:

$$
P(t)^{1-\sigma} \equiv Q(t)\left[\frac{1+\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{(\sigma-1)(1+\kappa)}{\kappa}}}{1+\kappa}\right]. \tag{A1.8}
$$

Finally, using Eqs. (A1.4) and (A1.8), we can write the expected profit stream as:

$$
\pi^e(j_\omega+1, \omega, t)=\frac{q\left(j_\omega, \omega, t\right) c(t) L(t)}{Q(t)} \cdot \frac{\kappa(1+\kappa) \chi}{(\sigma+\kappa-1)}.
$$

with χ being equal to $\left(1+\frac{\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{(\sigma-1)}{\kappa}}}{1-\kappa}\right) / \left[1+\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{(\sigma-1)(1+\kappa)}{\kappa}}\right]$.

A.2 Labour in the manufacturing sector

In this section, we determine the amount of employment in the manufacturing sector, that is:

$$
\begin{aligned}
L_M(t) & = \int_0^1 D(\omega, t) \mathrm{d}\omega \\
& = \int_0^1\left[\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} D(\omega, t) g(\lambda) \mathrm{d}\lambda+\int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} D(\omega, t) g(\lambda) \mathrm{d}\lambda\right] \mathrm{d}\omega.
\end{aligned}
$$

By using Eqs. (4) and (7) to substitute for $D(\omega, t)$, the previous expression becomes:

$$
\begin{aligned}
L_M(t)= & \underbrace{\frac{c(t) L(t) \int_0^1 q(j_\omega, \omega, t) \mathrm{d}\omega}{\int_0^1 q(j_{\omega'}, \omega', t) p(j_{\omega'}, \omega', t)^{1-\sigma} \mathrm{d}\omega'}}_{(e)}\left[\underbrace{\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda^{-\frac{\sigma}{\sigma-1}} g(\lambda) \mathrm{d}\lambda+}_{(f)}\right. \\
& \left.+\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \underbrace{\int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^{\infty} g(\lambda) \mathrm{d}\lambda}_{(g)}\right].
\end{aligned} \tag{A2.1}
$$

We first consider the ratio (e) outside the square brackets in Eq. (A2.1). The denominator is $P(t)^{1-\sigma} \equiv \int_0^1 q(j_{\omega'}, \omega', t)\, p(j_{\omega'}, \omega', t)^{1-\sigma}\, \mathrm{d}\omega'$, that is equal to (A1.8); in the numerator, $\int_0^1 q(j_\omega, \omega, t)\, \mathrm{d}\omega$ can be replaced by $Q(t)$. Simplifying terms, the ratio (e) boils down to:

$$\frac{c(t)L(t)\int_0^1 q(j_\omega,\omega,t)\ \mathrm{d}\omega}{\int_0^1 q(j_{\omega'},\omega',t)p(j_{\omega'},\omega',t)^{1-\sigma}\ \mathrm{d}\omega'} = \frac{(1+\kappa)\, c(t)\, L(t)}{1+\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{(1+\kappa)(\sigma-1)}{\kappa}}}. \tag{A2.2}$$

Then, we focus on the two integrals inside the square brackets of Eq. (A2.1). Integral (g) is the same of integral (d) of formula (A1.5); the solution is given by Eq. (A1.7). Integral (f) reads as:

$$\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda^{-\frac{\sigma}{\sigma-1}}\ g(\lambda)\ \mathrm{d}\lambda.$$

Using the Pareto density function $g(\lambda)$ and solving the integral, we get:

$$\frac{1}{\kappa}\int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda^{-\frac{\sigma}{\sigma-1}-\frac{\kappa+1}{\kappa}}\ \mathrm{d}\lambda = \frac{\sigma-1}{\sigma(\kappa+1)-1}\left[1-\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{\sigma(\kappa+1)-1}{\kappa}}\right]. \tag{A2.3}$$

By using Eqs. (A2.2), (A1.7) and (A2.3) into Eq. (A2.1), it yields:

$$\begin{aligned} L_M(t) \quad = \quad & \frac{(1+\kappa)\, c(t)\, L(t)}{1+\kappa\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{(1+\kappa)(\sigma-1)}{\kappa}}}\left\{\frac{\sigma-1}{\sigma(\kappa+1)-1}\left[1-\left(\frac{\sigma}{\sigma-1}\right)^{-\frac{\sigma(\kappa+1)-1}{\kappa}}\right]+\right. \\ & \left. +\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{\frac{1-\sigma}{\kappa}}\right\}. \end{aligned}$$

which, after a bit of algebra, can be simplified as:

$$L_M(t) = \frac{(\sigma-1)(1+\kappa)}{\sigma(\kappa+1)-1} c(t)\, L(t)\,.$$

A.3 The welfare analysis

First note that the price index $P(t) \equiv \left[\int_0^1 q(j_{\omega'}, \omega', t)\, p(j_{\omega'}, \omega', t)^{1-\sigma}\, d\omega'\right]^{1/(1-\sigma)}$ can be written as $P(t) = \left[Q(t)\cdot Ep^{1-\sigma}\right]^{1/(1-\sigma)}$ where $Ep^{1-\sigma}$ is the expected value of $p^{1-\sigma}$ and p is the price charged by each industry leader firm. This follows because q equals one for each product at time $t=0$, the innovation rate is the same in each industry and the distribution of prices does not change over time. Since $u(t) = c(t)/P(t)$,

$$\ln u(t) = \ln c(t) + \frac{1}{\sigma-1}\ln Q(t) + \frac{1}{\sigma-1}\ln Ep^{1-\sigma}.$$

In any balanced growth equilibrium where consumer expenditure c is constant over time, consumer utility grows solely because the average quality of products $Q(t)$ increases over time.

Second, we derive the state equation relevant to the social planner. Since $x(t) \equiv Q(t)^{1-\phi}/L(t)$, $\frac{\dot{x}(t)}{x(t)} = (1-\phi)\frac{\dot{Q}(t)}{Q(t)} - \frac{\dot{L}(t)}{L(t)} = (1-\phi)\frac{\kappa}{1-\kappa}\iota(t) - n$, the relevant state equation is

$$\dot{x}(t) = \left[\frac{B}{1-\kappa}\iota(t) - n\right] x(t).$$

At time $t = 0$, $Q(0) = 1$ and $L(0) = 1$ imply that the relevant initial condition is $x(0) = 1$.

Third, we derive a relationship between the state variable $x(t)$ and average quality $Q(t)$. Since $x(t) \equiv Q(t)^{1-\phi}/L(t) = e^{(1-\phi)\kappa\Phi(t)/(1-\kappa)}/e^{nt}$, $\ln x(t) = B\Phi(t)/(1-\kappa) - nt$ and using this equation we obtain $\ln Q(t) = \frac{\kappa\Phi(t)}{1-\kappa} = \frac{\kappa}{1-\kappa}\left[\ln x(t) + nt\right]\frac{1-\kappa}{B}$ or

$$\ln Q(t) = \frac{\kappa}{B}\left[\ln x(t) + nt\right].$$

Using (16), we also obtain a relationship between the state variable $x(t)$, the innovation rate $\iota(t)$ and consumer expenditure $c(t)$:

$$\ln\left[1 - a\iota\left(t\right)x\left(t\right)\right] = \ln\frac{\left(\sigma-1\right)\left(1+\kappa\right)}{\sigma\left(1+\kappa\right)-1} + \ln c\left(t\right).$$

We are now ready to state the social planner's problem. We want to maximize discounted utility $U = \int_0^\infty e^{-(\rho-n)t}\ln u\left(t\right)dt$ subject to the state equation, and using the above-derived equations, we obtain

$$\ln u(t) = \ln\left[1 - a\iota\left(t\right)x\left(t\right)\right] - \ln\frac{\left(\sigma-1\right)\left(1+\kappa\right)}{\sigma\left(1+\kappa\right)-1} + \frac{\kappa}{(\sigma-1)B}\left[\ln x(t) + nt\right] + \frac{1}{\sigma-1}\ln Ep^{1-\sigma}.$$

Since the terms $\ln\frac{(\sigma-1)(1+\kappa)}{\sigma(1+\kappa)-1}$, $\frac{\kappa nt}{(\sigma-1)B}$ and $\frac{1}{\sigma-1}\ln Ep^{1-\sigma}$ play no role in determining the optimal control, they can be ignored and the social planner's optimal control problem can be written more simply as

$$\max_{\iota(t)}\left\{\int_0^\infty e^{-(\rho-n)t}\left[\ln\left[1 - a\iota\left(t\right)x\left(t\right)\right] + \frac{\kappa}{(\sigma-1)B}\ln x(t)\right]dt\right\}$$

$$\text{subject to}\qquad \dot{x}(t) = \left[\tfrac{B}{1-\kappa}\iota(t) - n\right]x(t), \quad x\left(0\right) = 1.$$

The current value Hamiltonian function for this problem is given by

$$H \equiv \ln\left[1 - a\iota\left(t\right)x\left(t\right)\right] + \frac{\kappa}{(\sigma-1)B}\ln x(t) + \zeta(t)\left[\tfrac{B}{1-\kappa}\iota(t) - n\right]x(t),$$

where $\zeta(t)$ is the costate variable associated to the state variable $x(t)$. Pontryagin's maximum principle yields the first order conditions:

$$\frac{\partial H}{\partial\iota} = \left[-\frac{a}{1-a\iota(t)x(t)} + \zeta(t)\frac{B}{1-\kappa}\right]x(t) = 0, \tag{A3.1}$$

$$\dot{\zeta}(t) - (\rho - n)\,\zeta(t) = -\frac{\partial H}{\partial x} = -\frac{\kappa}{(\sigma-1)B}\frac{1}{x(t)} - \zeta(t)\left[\frac{B}{1-\kappa}\iota(t) - n\right] + \frac{a\iota(t)}{1-a\iota(t)x(t)}, \qquad \text{(A3.2)}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t}\zeta(t)x(t) = 0.$$

Solving (A3.1) for $\iota(t)$ yields $\iota(t) = \frac{1}{ax(t)} - \frac{1-\kappa}{\zeta(t)Bx(t)}$ and substituting this into both (A3.2) and the state equation, we obtain the following system of non-linear differential equations:

$$\dot{\zeta}(t) = \rho\zeta(t) - \frac{\kappa}{(\sigma-1)Bx(t)},$$

$$\dot{x}(t) = \frac{B}{a\,(1-\kappa)} - \frac{1}{\zeta(t)} - nx(t).$$

The curve $\dot{x} = 0$ is upward sloping in (x,ζ) space, the curve $\dot{\zeta} = 0$ is downward sloping in (x,ζ) space, and they have a unique intersection in the positive orthant given by

$$\zeta^s = \frac{a\,(1-\kappa)\,\{n\kappa + B\rho(\sigma-1)\}}{B^2\rho\,(\sigma-1)}$$

and

$$x^s = \frac{\kappa B}{a\,(1-\kappa)\,\{n\kappa + B\rho(\sigma-1)\}},$$

where the superscript s denotes the steady-state value of the variable. It is easy to verify that the system of differential equations is saddle-path stable. Jumping onto the saddle-path at time $t = 0$ and staying on it forever is a candidate solution to the social planner's problem that satisfies the first order conditions (A3.1) and (A3.2) as well as the transversality condition. Since the maximized Hamiltonian is strictly concave in x and all the sufficiency conditions for infinite-horizon autonomous optimal control problems are satisfied by this candidate solution, we conclude that it is the unique solution to the social planner's problem (see Theorem 7.14 in Acemoglu (2008)).

Focusing on the steady-state properties of this solution, we can solve for the optimal steady-state innovation rate ι^s. Setting $\dot{x} = 0$ yields $\iota^s = \frac{n(1-\kappa)}{B}$, which coincides with the equilibrium steady-state innovation rate in Eq. (23). We can also solve for the optimal steady-state share of labor employed in R&D activities $(L_I/L)^s$. Eq. (16) implies that $(L_I/L)^s = a\iota^s x^s$. Substituting into this expression for ι^s and x^s yields

$$\left(\frac{L_I}{L}\right)^s = \frac{n\kappa}{n\kappa + B\rho(\sigma-1)}.$$

Comparing the optimal R&D share $(L_I/L)^s$ with the equilibrium R&D share $(L_I/L)^*$ given by Eq. (24), they only coincide if

$$1 - s_R = \Lambda, \qquad \Lambda \equiv \frac{\rho\chi\,(1-\kappa)\,[\sigma(\kappa+1)-1]}{(\sigma+\kappa-1)\left[\rho + \frac{1-B}{B}n\right]}$$

In steady-state equilibrium, it is optimal to subsidize R&D if $\Lambda < 1$ and it is optimal to tax R&D if $\Lambda > 1$. Which case occurs depends on the parameters of the model.

Finally, we present some results used in the numerical analysis.

First, we calculate the expected price in the economy Ep. Using (7) and (10),

$$\begin{aligned} Ep \equiv \int_1^\infty p(\lambda)g(\lambda)\ \mathrm{d}\lambda &= \int_1^{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}} \lambda^{\frac{1}{\sigma-1}} g(\lambda)\ \mathrm{d}\lambda + \int_{\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1}}^\infty \frac{\sigma}{\sigma-1} g(\lambda)\ \mathrm{d}\lambda \\ &= \frac{\left(\frac{\sigma}{\sigma-1}\right)^{1+\frac{1-\sigma}{\kappa}} - 1}{\frac{\kappa}{\sigma-1} - 1} + \left(\frac{\sigma}{\sigma-1}\right)^{1+\frac{1-\sigma}{\kappa}}. \end{aligned}$$

Second, we calculate what static utility equals for the representative consumer in steady-state equilibrium. From before $u(t) = c(t)/P(t)$ and in steady-state equilibrium, consumer expenditure $c(t)$ takes on the constant value c^*. The price index satisfies $P(t) = \left[Q(t) \cdot Ep^{1-\sigma}\right]^{1/(1-\sigma)}$ and in steady-state equilibrium, since $x(t) \equiv Q(t)^{1-\phi}/L(t)$ takes on the constant value x^*, $x^* e^{nt} = Q(t)^{1-\phi}$. It follows that $Q(t) = (x^*)^{1/(1-\phi)} e^{nt/(1-\phi)}$ and $P(t) = \left[(x^*)^{1/(1-\phi)} e^{nt/(1-\phi)} \cdot Ep^{1-\sigma}\right]^{1/(1-\sigma)}$. Thus static utility in steady-state equilibrium is given by

$$u(t) = c^*(x^*)^{\frac{1}{(1-\phi)(\sigma-1)}} \left[Ep^{1-\sigma}\right]^{\frac{1}{\sigma-1}} e^{g^* t}.$$

Third, we compare static utility when R&D is not subsidized $u_0(t)$ to static utility when R&D is optimally subsidized $u_s(t)$. This yields

$$\frac{u_s(t)}{u_0(t)} = \frac{c_s^*(x_s^*)^{\frac{1}{(1-\phi)(\sigma-1)}} \left[Ep^{1-\sigma}\right]^{\frac{1}{\sigma-1}} e^{g^* t}}{c_0^*(x_0^*)^{\frac{1}{(1-\phi)(\sigma-1)}} \left[Ep^{1-\sigma}\right]^{\frac{1}{\sigma-1}} e^{g^* t}} = \frac{c_s^*(x_s^*)^{\frac{1}{(1-\phi)(\sigma-1)}}}{c_0^*(x_0^*)^{\frac{1}{(1-\phi)(\sigma-1)}}}$$

since subsidizing R&D does not affect the distribution of prices or the steady-state rate of economic growth g^*. Comparing steady-state equilibrium paths before and after R&D is optimally subsidized, the ratio $\frac{u_s(t)}{u_0(t)}$ reveals the percentage increase in real consumption from optimally subsidizing R&D. In the presentation of the numerical results, we report the subutility value $\hat{u} \equiv c^*(x^*)^{\frac{1}{(1-\phi)(\sigma-1)}}$ associated with some steady-state outcomes since this information allows the reader to calculate $\frac{u_s(t)}{u_0(t)}$.

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Figure 1: Averages number of claims made by technological fields for the U.S. economy (source: Hall *et al.*, 2001).

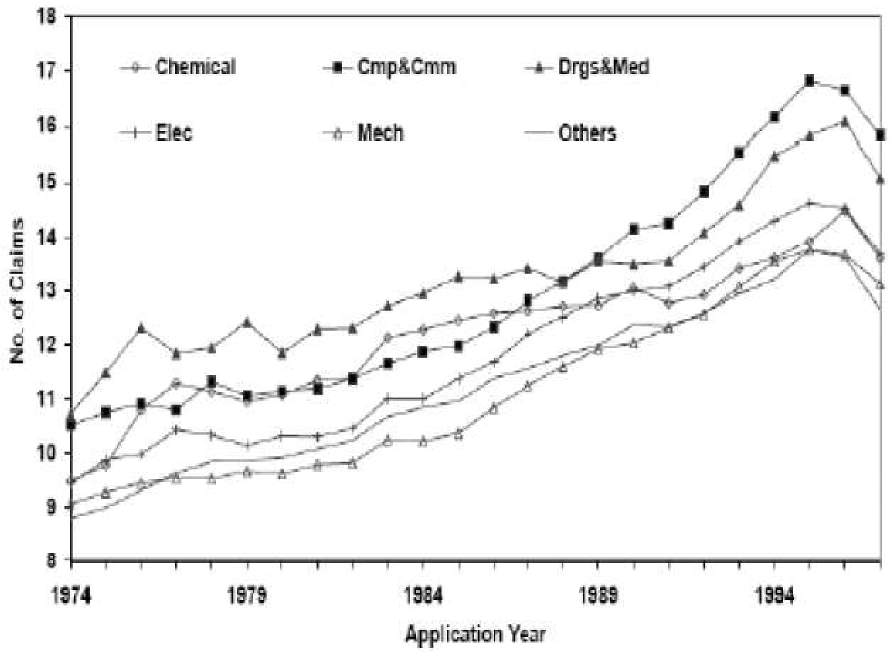

Figure 2: Estimated citation lag distribution per technology field (source: Hall *et al.*, 2000).