A note on the transitional behavior of the saving rate in the neo-classical growth model (the Cobb-Douglas case)

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Abstract

In this short note I clarify two features of Figure 2.3 in Barro and Sala-i-Martin [2004]. The figure, as it appeared in the first and second editions of the book, is confusing if not wrong. I hope this note will serve as a corrigendum to the figure.

Key words: Transition Dynamics; Saving Rate; Neo-classical Growth Model

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Barro and Sala-i-Martin [2004] (BS from here on) discuss the behavior of the saving rate during transition to the steady state in the neo-classical growth model in Section 2.6.4 of their book. They summarize the behavior in figure 2.3 (p.109). I reproduce the figure below.

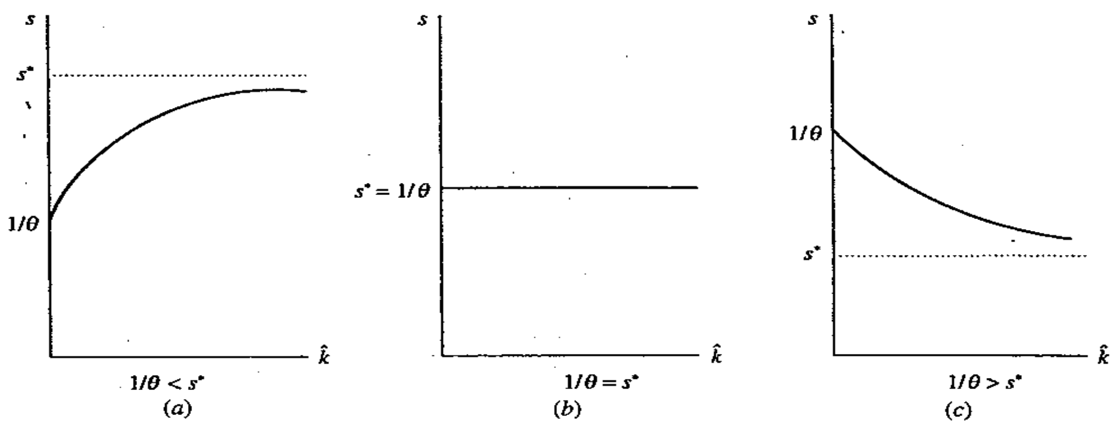

Figure 1: Phase diagram for the behavior of the saving rate (in the Cobb-Douglas case)

There are two features of this figure which are confusing. First, in each part of the figure, the curve showing the transitional behavior of the saving rate (from here on, the stable saddle path) starts from point $1/\theta$ on the y-axis. It implies that y intercept of these curves is $1/\theta$. This is confusing because when capital is zero, given the assumptions of the model, the saving rate is undefined. It is also confusing because, in parts (a) and (c) of the figure, if the system begins at point $1/\theta$ it will diverge instead of converging to the steady-state saving given by s^*.

Second, in parts (a) and (c) of the figure, it appears that as $\hat{k}$ grows large, the saving rate asymptotically approaches s^*. This is confusing too. Since a unique finite steady state exists, there is a finite $\hat{k}^*$ at which the stable saddle paths in parts (a) and (c) of the figure will cross the horizontal line representing the steady state saving rate. In what follows I attempt to clarify these confusions.[1]

I use the same notation as in BS except that I replace $\hat{c}/\hat{y}$ with $\hat{z}$ to ease the notation a little bit. With this notation, equations (2.37) and (2.38) in BS become

[1] There is another problem with the figure as it appears in the second edition of the book: the caption does not correspond to the figure. According to Robert Barro, the caption was prepared for another figure which was inadvertently deleted in production.

$$\hat{z} = \left(1 - \frac{1}{\theta}\right) + \psi \cdot \frac{\hat{k}^{1-\alpha}}{\alpha A}, \qquad (2.37)$$

$$\hat{z} = 1 - \frac{(x+n+\delta)}{A} \cdot \hat{k}^{1-\alpha}, \qquad (2.38)$$

where in (2.37), $\psi \equiv [(\delta + \rho + \theta x)/\theta - \alpha(x + n + \delta)]$. Depending on the values of model parameters, ψ could be greater than, equal to or less than zero. Hence the plot of (2.37) could be upward sloping, horizontal or downward sloping. The plot of (2.38) is always downward sloping. I plot these equations in Figure 2. The three cases correspond to three possible scenarios about ψ. The directional arrows follow from equations (2.35) and (2.36) in BS. The curves with arrows represent the stable saddle paths.

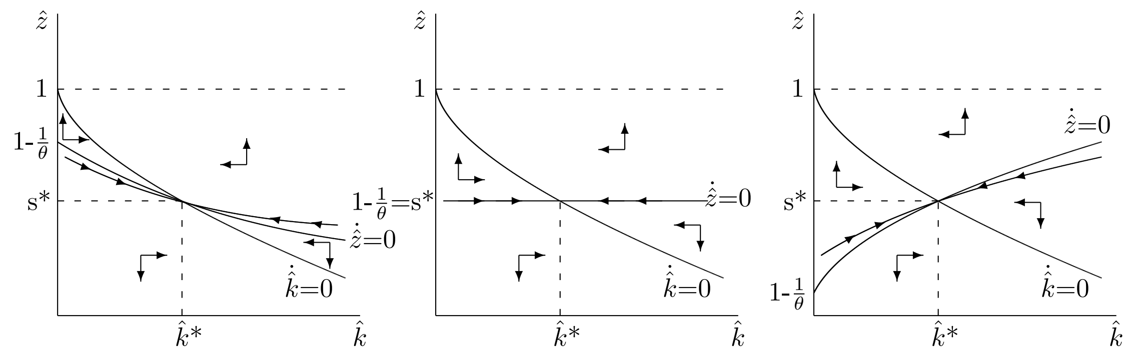

Figure 2: Stable Saddle Path in $\hat{z}$ and $\hat{k}$ space

The stable saddle path represents the equilibrium ratio $\hat{z}$ for any given $\hat{k}$. Since the saving rate is given by $1 - \hat{z}$, it is represented by the vertical distance between the stable saddle path and the horizontal dashed line at $\hat{z} = 1$ in Figure 2. I plot this vertical distance in Figure 3, which is the same as Figure 1 (i.e. Figure 2.3 in BS) except for two differences.

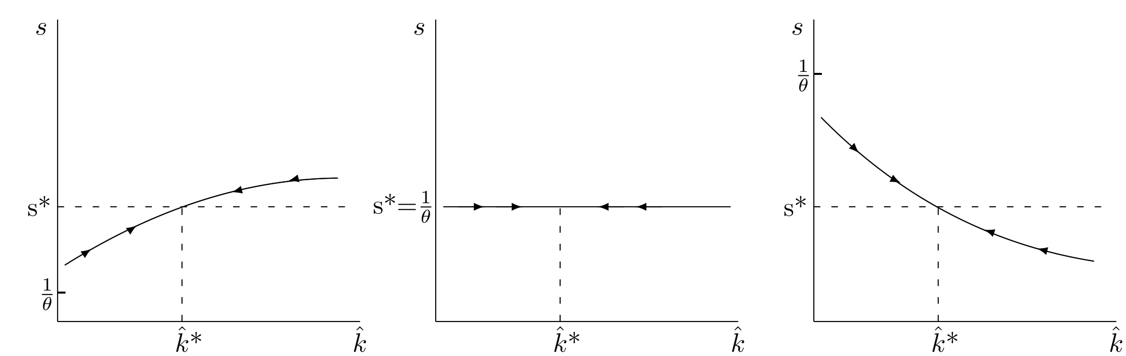

Figure 3: Saving Rate during transition to Steady State

The first difference is that these curves do not start from $1/\theta$ on y-axis. The reason is that the saving rate is undefined when $\hat{k} = 0$. This is because when capital is zero, output and consumption are both zero and their ratio $\hat{z}$ is undefined. Even if we ignore this indeterminacy of the saving rate when $\hat{k} = 0$, the stable saddle path cannot start from point $1/\theta$ on y-axis in parts (a) and (c) of the figure. To see this first consider part (a). If $s = 1/\theta$ in Figure 3(a) then $\hat{z} = 1 - 1/\theta$ in Figure 2(a). At this point, although $\dot{\hat{z}} = 0$, $\dot{\hat{k}} > 0$ and hence the system will move eastwards. When the system moves eastwards it will be above $\dot{\hat{z}} = 0$ line and diverge. The system will eventually reach point (0,1) in Figure 2(a) and the Euler's equation will be violated. Using a similar argument one can show that if the system starts at point $\hat{z} = 1 - 1/\theta$ in Figure 2(c), it will diverge to a point (not shown) where $\hat{z} = 0$ and the economy will save everything it produces. This will violate the transversality condition. Hence the only possible starting position for the stable saddle path in Figure 2(a) is somewhere below point $1 - 1/\theta$. Similarly the only possible starting position for the stable saddle path in Figure 2(c) is somewhere above the point $1 - 1/\theta$.

The second difference is that these stable saddle paths intersect the horizontal line representing s^* when $\hat{k} = \hat{k}^*$. This is clear in parts (a) and (c) of Figures 2 and 3. Hence in Figure 3(a), for example, if initial period capital stock is less than $\hat{k}^*$ the saving rate will increase during the transition. But if the initial capital stock is greater than $\hat{k}^*$ the saving rate will decrease during the transition (a possibility not shown in Figure 1).

To sum up, the transitional behavior of the saving rate in the Neo-classical growth model (in the Cobb-Douglas case) is more accurately described by Figure 3 than by Figure 1.

References

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