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ABSTRACT

This paper studies the sequential sampling scheme as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In sequential sampling, the signal is sampled at intervals of ΔT, $\Delta T+\Delta\tau$, $\Delta T+2\Delta\tau$, $\Delta T+3\Delta\tau$, ...; where $\Delta\tau < \Delta T$ and $\Delta\tau$ may be selected as desirable. Sequential sampling is, however, analyzed and it is proved that when the ratio $\Delta T/\Delta\tau$ is an integral number, the associated spectral estimates give a Nyquist frequency $\frac{1}{2\Delta\tau}$. This sampling scheme can, therefore, be employed to yield a required cut- off frequency.

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INTRODUCTION

Some data acquisition systems have a minimum allowable sampling interval and do not provide a desired sampling period less than a minimum allowable value. This may be due to some restrictions set by the measuring instrument that has to be used [1-7].

Let the minimum allowable sampling time be ΔT; if the uniform sampling scheme is employed then, the Nyquist or cut-off frequency is known [8] to be given as:

$$f_c = \frac{1}{2\Delta T} \tag{1}$$

This would mean that if frequencies higher than f_c are present, aliasing will occur. Otherwise, the signal would have to be filtered so that only frequencies below f_c are passed and, therefore, the spectral analysis will be restricted [8].

The sequential sampling scheme can, however, be employed to obtain an autocorrelation function with estimates $\Delta\tau$ apart, where $\Delta\tau < \Delta T$, with the exception of coefficients lying inside the range $R(0) \rightarrow R(\Delta T)$. In this sampling scheme, the signal would be sampled at intervals of:

$\Delta T, \Delta T+\Delta\tau, \Delta T+2\Delta\tau, \Delta T+3\Delta\tau, \ldots$

From the sampled signal, an autocorrelation function can be obtained with the coefficients:

$R(0), R(\Delta T), R(\Delta T+\Delta\tau), R(\Delta T+2\Delta\tau), \ldots$

While ΔT is restricted, the value of $\Delta\tau$ may be chosen as desirable. It will be proved, in this paper, that the sequential sampling can give an increased cut-off frequency as:

$$f_{cs} = \frac{1}{2\Delta\tau} \tag{2}$$

The sequential sampling can, therefore, be employed to overcome aliasing and the restrictions of spectral analysis, by selecting a sufficiently small value for $\Delta\tau$.

THE CUT OFF FREQUENCY IN THE SEQUENTIAL SAMPLING

The cut- off frequency provided by the sequential sampling scheme is considered in this section. The analysis employs the impulse representation of a continuous signal as an approach to discretization [9-14].

In the sequential sampling, the signal is sampled at intervals of:

$\Delta T, \Delta T+\Delta\tau, \Delta T+2\Delta\tau, \Delta T+3\Delta\tau, \ldots$

The sampling instants are, therefore, given by:

$$t_i = 0, \quad \Delta T, 2\Delta T+\Delta\tau, 3\Delta T+3\Delta\tau, 4\Delta T+6\Delta\tau, \ldots \tag{3}$$

This can be written as:

$$t_i = i\Delta T + [\sum_{r=0}^{i} r - i]\Delta\tau\,, \quad i = 0,1,2,3,\ldots \qquad (4)$$

Since, $\sum_{r=0}^{i} r = \frac{i}{2}(i+1)$, then equation (4) gives:

$$t_i = i\Delta T + \frac{i}{2}(i-1)\Delta\tau \qquad (5)$$

When a continuous signal x(t) is sampled, the sample values $x(t_i)$ are acquired. A discrete autocorrelation function, with coefficients $R(\tau_j)$, may be obtained from the discrete signal, by contributions of the products $x(t_i)x(t_{i+j})$. Equation (5) can be used to give the time delay τ_j as:

$$\tau_j = t_{i+j} - t_i = \Delta T + [\frac{j}{2}(j-1) + ij + (j-1)\mu]\Delta\tau \qquad (6)$$

i= o, 1, 2, ...

j= o, 1, 2, ...

where μ is a constant given by:

$$\mu = \Delta T/\Delta\tau \qquad (7)$$

It is seen from equation (6) that for j=0, the time delay is zero and for j=1, the time delay is $\Delta T+i\Delta\tau$ (where i=0, 1, 2,3, ...). An autocorrelation function is, therefore, obtainable at discrete values of the time delay given as:

$$\tau_n = o,\ \Delta T+n\Delta\tau n= 0, 1, 2, 3, \ldots \qquad (8)$$

If the ratio μ is an integral number, then higher values of j would also provide more contributions to the autocorrelation estimates at the above time delays τ_n. This is because j(j-1)/2 is always even, and any value of j would hence add a multiple of $\Delta\tau$ to ΔT.

The discrete autocorrelation function may be represented as:

$$R^*(\tau) = \Delta\tau.R(\tau)\delta_b(\tau) \qquad (9)$$

where $R(\tau)$ is the continuous autocorrelation function and $\delta_b(\tau)$ is the following form of the delta comb:

$$\delta_b(\tau) = \delta(\tau) + \delta(\tau-\Delta T) + \delta(\tau-\Delta T-\Delta\tau) + \delta(\tau-\Delta T-2\Delta\tau) + L \qquad (10)$$

It is established [9-10] that the Fourier transform of equation (10) can be written as:

$$\Delta_b(\omega) = 1 + e^{-j\omega\Delta T}\sum_0^{\infty} e^{-j\omega n\Delta\tau} \qquad (11)$$

which by manipulation [9-10] can be re-written as:

$$\Delta_b(\omega) = \frac{1 - e^{-j\omega\Delta\tau} + e^{-j\omega\mu\Delta\tau}}{1 - e^{-j\omega\Delta\tau}} \qquad (12)$$

where substitution has also been made for ΔT from equation (7).

The Fourier transformation of $R^*(\tau)$ gives the spectral density $S^*(\omega)$ corresponding to the sampled signal and that of $R(\tau)$ would yield the spectral density $S(\omega)$ of the original continuous signal. The approach adopted for the Fourier transformation of equation (9) is based on the convolution and residue theorems [9]. By evaluating the residue terms [9] and using the convolution property [9], for substitution into equation (9), the Fourier transform of this equation ca be obtained as:

$$S^*(\omega) = \sum_{-\infty}^{\infty} e^{j2\pi n\mu} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi/_{\Delta\tau} \qquad (13)$$

However, if the ratio $\mu=(\Delta T/\Delta\tau)$ is an integral number then,

$$e^{j2\pi n\mu} = 1$$

noting that n is also an integer. Substituting this into equation (13) gives:

$$S^*(\omega) = \sum_{-\infty}^{\infty} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi/_{\Delta\tau} \qquad (14)$$

Now, consider the periodicity of $S^*(\omega)$; this can also be examined by applying the corresponding methods [9-14]. Using equations (9), (10), (11) and the rules established for discrete Fourier transformation [9-10], it can be written:

$$\Delta\tau[R(0) + e^{-j\omega\Delta T}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] = \Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \quad (15)$$

and then for an integer m:

$$\Delta\tau[R(0) + e^{-j(\omega+2m\omega_{cs})\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j(\omega+2m\omega_{cs})n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}e^{-j2m\omega_{cs}\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2m\omega_{cs}n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2mn\pi}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \qquad (16)$$

since m and n are integers. If μ is also an integral number, this would reduce to:

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}]$$

from which it follows that:

$$S^*(\omega + 2m\omega_{cs}) = S^*(\omega) \qquad (17)$$

This is the mathematical statement for $S^*(\omega)$ to be periodic with period $2\omega_{cs}$. Otherwise, if μ is not an integral number,

$$S^*(\omega + 2m\omega_{cs}) \neq S^*(\omega) \qquad (18)$$

and the requirement for periodicity is not met.

It is, therefore, seen that when the ratio $\mu(=\Delta T/\Delta\tau)$ is an integral number, the periodic pattern, conforming with the Nyquist theorem,[9-14] is obtained. That is, the sequential sampling gives a cut-off frequency $\omega_{cs} = \pi/_{\Delta\tau}$ or $f_{cs} = \frac{1}{2\Delta\tau}$. On the contrary, when μ is not a whole number, $S*(\omega)$ is related to the true spectral density by equation (13); it includes a complex term and is not periodic.

CONCLUSIONS

This paper has considered the sequential sampling scheme, as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In the sequential sampling, the signal is sampled at intervals of $\Delta T, \Delta T + \Delta\tau, \Delta T + 2\Delta\tau, \Delta T + 3\Delta\tau, \ldots$; where $\Delta\tau$ p ΔT and may be selected as desirable.

The sequential sampling was considered analytically and it was proved that, when the ratio $\Delta T / \Delta\tau$ is an integral number, the corresponding spectral estimates give a cut- off frequency of $\frac{1}{2\Delta\tau}$. On the contrary, when the ratio is not a whole number, the associated spectrum of the sequentially sampled data was found to comprise a complex term in its relation to the true spectrum and would not be periodic in terms of the cut-off frequency.

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