Some critical comments on credit risk modeling.

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Abstract. In this notice we are focus on drawbacks in the basis of the credit risk modeling.

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The credit risk along with the credit derivatives is a modern area of the financial business. In recent years this field becomes the most successful innovation, which accumulates significant cash flows as well as the highest attention within financial community. In this notice we present some general comments concerning mathematical techniques used for the credit risk modeling. We will not discuss here specific forms of the credit derivatives.

Unfortunately looking at the progress in this area one might note that it has been achieved using somewhat inaccurate modeling techniques. We highlight several drawbacks related to the benchmark models that in turn imply a problem of using correspondent software.

Next example illustrates an error of the benchmark plain-vanilla option price definition using a very simple algebra.

Let S (t), t = 1, 2 be a security price at date t and K a strike price. Assume for simplicity that S(1) = K = \$2 and there are two hypothetical securities value at the date t = 2 which is also an option maturity date

$$S_1 (2 , \omega) = \begin{cases} \$4 & \text{with probabilities } 0.99 \\ \$1 & \text{with probabilities } 0.01 \end{cases}$$

$$S_2 (2 , \omega) = \begin{cases} \$4 & \text{with probabilities } 0.01 \\ \$1 & \text{with probabilities } 0.99 \end{cases}$$

The average return on the first security is equal to 98.5% and - 48.5% on the second security. The volatility in both cases is the same 0.0891. One can easily to recall that the binomial scheme suggests the same call option price C = 2/3 for either security. Thus, for investors the binomial scheme suggests the option price that does not take into account the real expected return on underlying security. Indeed one can see that the call option on the first security in 99 cases out of 100 promises positive payoff and only in 1 case a loss. With the second security the situation is opposite. Nevertheless in both scenarios binomial scheme suggests the same price. We also note that there is a significant difference in expected option returns. The expected rate of return on the call option for the first security is about 197% = [(4 – 2)*0.99 – 2/3] / (2/3) and for the second security is - 97% = [(4 – 2)*0.01 – 2/3] / (2/3) that explicitly demonstrates failing of the option pricing method. Indeed the present value of the future cash flows is different but their spot prices are equal. This observation contradicts the understanding of the price itself.

Recall that one of the basic interpretations of the spot price is the present value of the future cash flow. In stochastic setting it means in particular that if the volatility of two instruments is equal and expected returns are different then the spot prices of the two instruments should be different otherwise the price is not correct. Sometimes using the arbitrage argument as a necessary condition of the fair pricing could be helpful vehicle to identify incorrect pricing. On the other hand an incorrect pricing does not automatically leads to the arbitrage opportunity. More details could be found in [2].

There is a discrete space-time approach used for derivatives valuation. For the 0-default options it is the binomial scheme. The above example is a simple one of the binomial scheme. In a case when the chance of default could not be assumed equal to 0 the binomial approach was implemented by Jarrow and Turnbull, see for instance [7]. The core of the model is the risk neutral probabilities prescribed to the risky bond. Recall that risk neutral (martingale) probabilities were first developed for the option valuation in discrete space-time setting. In the above example we demonstrated that the risk-neutral option price construction does not have any sense it looks also reasonable to highlight its application to default studies. In the equity case presented above the security value $S(1) = 2$ is known at $t = 1$. The value of $S(2)$ is random with given probability distribution. Risk neutralization replaces real world probability distribution by so-called martingale probabilities. This transformed instrument used then for derivatives valuation only. That is if an investor ask about the security price then it is $S(t)$, $t = 1,2$ supplied by the original probability distribution. In risky bond evaluation the mistaken approach was extended. The same risk neutralization framework was applied for risky bond valuation. Follow [7] let $v(1, 2)$ denote the value at $t = 1$ a zero-coupon bond issued by the firm. The only date when the risky bond might default is assumed to be the bond's maturity $t = 2$. The value of the risky bond at $t = 2$ is a random variable. This variable admits two values 1, if no default and $\delta \in [0, 1]$ if default occurred at $t = 2$. The value δ is known as the recovery rate. Having these two value at $t = 2$ and deterministic value at $t = 1$ one can easy to construct "risk-neutral" probabilities. As it follows from the binomial scheme the real world probability distribution of the states $\{\delta, 1\}$ does not have any effect on risk neutral probabilities, i.e. the risk neutral price at $t = 1$ is the same for either risky bonds

$$\nu_1(1,2) = \begin{cases} 1 & \text{with probability } 0.001 \\ \delta & \text{with probability } 0.999 \end{cases}$$

$$\nu_2(1,2) = \begin{cases} 1 & \text{with probability } 0.999 \\ \delta & \text{with probability } 0.001 \end{cases}$$

that in turn contradict the company ratings. It would be good if the rating companies have not used risk neutralization in their business. In this comment we somewhat simplified original reduced form approach by separating two different issues. The first one is the interpretation of the default in the simplest discrete time case and the second one is the interpretation of the default with the help of Poisson random process. In this case the probability $[1 - \lambda(1)\Delta]$ is prescribed to the risk neutral probability of the 0 default scenario and $\lambda(1)\Delta$ if default occurred at t = 2. Here the time interval $\Delta = 2 - 1 = 1$. These transformations make it possible to interpret the time of default as the time of the first jump of the Poisson process. Starting from this point the well developed mathematical techniques have been adjusted for the reduced form approach to the credit derivatives valuations.

Now let us take a look at a credit derivatives structural model. The primary results were introduced in [4-6]. In these papers a new canonical distribution was used to describe losses of the large debt portfolio. This distribution plays now a key role in the structural models as well as for CreditMark product of the Moody's KMV. In the New Basel Accord the regulatory capital of a Bank should be calculated using Internal Rating Based method. Under this method the canonical distribution must be applied for regulatory capital calculations.

Follow [4] we assume that Wiener Z_i processes are equally correlated and

$$E[\Delta Z_i(t)]^2 = \Delta t, \quad E\,\Delta Z_i(t)\,\Delta Z_j(t) = \rho\,\Delta t, \quad i \neq j$$

where $\Delta Z_i(t) = Z_i(t + \Delta t) - Z_i(t)$, i = 1,2,…n.

Statement 1. [4]. The Wiener processes $Z_i(t)$ admit representation

$$Zi(t) = \sqrt{\rho}\,x(t) + \sqrt{1-\rho}\,\varepsilon_i(t) \tag{1}$$

where $x(t)$ and $\varepsilon_i(t)$, i = 1,2,…n are mutually independent Wiener processes. The idea of the proof is presented bellow follows Vasicek's private communication.

Let $U(t)$ be a Wiener process independent on the given Wiener processes $Z_i(t)$, i = 1,2,…n. Putting

$$x(t) = a\sum_{i=1}^{n} Z_i(t) + bU(t), \qquad \varepsilon_i(t) = \frac{1}{\sqrt{1-\rho}}\left(Z_i(t) - x(t)\sqrt{\rho}\right)$$

where

$$a = \frac{\sqrt{\rho}}{1 + (n-1)\rho} \quad \text{and} \quad b = \frac{\sqrt{1-\rho}}{\sqrt{1 + (n-1)\rho}}$$

one can check that x (t) and ε_i (t), i = 1,2,…n are independent Wiener processes. There are several pitfalls here that we wish highlight bellow.
Remark. It is clear that the proof of the statement 2 should be refined. Indeed the decomposition used in the statement 2 is correct when and only when the assumption regarding the Wiener process U (t) is true. If the Wiener process U (t) did not exist then stated decomposition for the given system { Z i (t) } fails. In [6] it is stated that (1) is not an assumption but a property of the equicorrelated normal distributed system. This statement does not correct and does not correspond to the real situation. Also note that all terms in (1) are depend on the number n by the construction.
Thus the main result [4,5] regarding existence of the limit distribution of the large loan portfolio is subject to the assumption that for any system of Wiener processes Z = { Z i (t), i = 1,2,…n }(n is an arbitrary number) there exist a Wiener process U(t) independent on the system Z. Here the Wiener process U(t) might depend or independent on n.

A new approach that at some degree is close to the structural model was introduced in [3]. In this paper the presentation (1) is used indirectly in order to describe joint defaults on underlying loans. Thus the problem that related to the construction of the process U (t) does not exist in such model.

We comment that construction. Let M and Z_i , i = 1, 2, …, n be independent random variables with mean 0 and variance 1. Define random variables X_i , i = 1,2,…,n with a help of equality

$$X_i = a_i M + \sqrt{1 - a_i^2}\, Z_i$$

where the constants a_i satisfy a condition: $|a_i| < 1$. Let t_i be the time of default of a i-th obligator and Q_i is the cumulative distribution function (cdf) of the random time t_i and H_i (x) is the cdf of Z_i . Then

$$P\{X_i < x \mid M\} = P\{a_i M + \sqrt{1 - a_i^2}\, Z_i < x \mid M\} =$$
$$= P\{ Z_i < \frac{x - a_i m}{\sqrt{1 - a_i^2}} \}|_{m=M} = H_i\left(\frac{x - a_i M}{\sqrt{1 - a_i^2}}\right) \tag{2}$$

Let F_i (x) denotes the cdf of the random variable X_i . Define mappings

$$x = F_i^{-1}(Q_i(t)), \qquad t = Q_i^{-1}(F(x)) \tag{3}$$

Hence x = x (t) and t = t (x). Then from (3) it follows
<u>Conclusion</u> [3]. Conditionals on M defaults are independent. Indeed

$$Q_i(t \mid M) = P\{t_i < t \mid M\} = H_i \{ \frac{F_i^{-1}(Q_i(t)) - a_i M}{\sqrt{1 - a_i^2}} \} \qquad (4)$$

This conclusion would follow from the fact that H_i are cumulative distribution functions that independent on variables Z_i.
One could see that the link between (3) and (4) does not mathematically accurate. Indeed the equality (3) deals with unconditional probabilities and left-hand side (4) relates to the conditional probabilities. It is not difficult to present an example that shows that from equality unconditional expected values of two random processes does not follows the equality their conditional expected values. Indeed the Wiener processes w (t) and the constant 0 have the same expectations equal to the 0 nevertheless the unconditional expectation of and $E\{ w(t) \mid F_t^w \} = w(t) \neq 0$ where $F_t^w = \sigma\{ w(s), s \leq t \}$. Note also that by the definition the random variables t_i and M do not have any relationship between them. Therefore the derivation [3] leading to the equality (4) is correct for either when the random variables X_i and t_i are independent or when they are dependent. Hence it follows that the construction introduced in [3] rather incorrect.

There is another approach to the credit risk modeling referred to as to the reduced form of the credit risk [1]. This approach deals with the statistical data regarding the default events ignoring structural parameters such as firm's equity and debt costs. Statistical data is used to help providing analysts by probabilities of credit events. In a simple setting the default time is associated here with the first jump of the counting process N (t). The Poisson process or its generalization can be considered as a model example of the Process N (t). If $\tau = \tau(\omega)$ is a random time associated with a default event then reduced form interprets it as

$$\tau = \min \{ t \geq 0 : N(t) = 1 \}$$

The hazard rate $\lambda(t)$ is defined with a help of conditional probability

$$P\{ \tau \leq t + \Delta t \mid \tau > t \} = \lambda(t) \Delta t + o(\Delta t) \qquad (5)$$

With the reduced form models we can make two comments regarding to approach rather than to the problems solution. The first comment is related to the notorious risk neutral world and risk neutral probabilities. As it was shown above the Black Schloes option price definition is incorrect and as a conclusion the risk neutral interpretation is irrelevant for the derivative pricing. Therefore the reduced form approach interprets the credit risk in the risk neutral environment and therefore is inappropriate. In addition recall that the risk neutral setting has been used for the modeling of the stochastic short interest rates.

The second comment relates to the drawback of the construction of the doubly stochastic hazard rates that is the essence of the reduced form models. In these models the stochastic intensity $\lambda(t)$ is assumed to be in the form

$$\lambda(t) = \Lambda(X(t)),$$
$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dw(t) \qquad (6)$$

where $\Lambda(x)$ is a nonrandom appropriate function. Our remark relates to the informal substitution of the random process $X(t)$ at the right hand side (5). Indeed it is clear that it makes sense to consider first the case when the function $X(t)$ is equal to the constant scalar or vector x. It implies in turn that the default time $\tau(\omega)$ is also a function on x, Thus on the first step of the modeling one should introduce the parameter x and explain the way how it effects on default time. On the next step we need to justify the substitution $x = X(t)$ into both sides of the equality (5). On the right hand side of (5) this substitution is correct when for example the function $\lambda(x)$ is a Borel measurable function with respect to x (i.e. B-measurable). On the left hand side of the equality (5) the corresponding substitution would be correct if the conditional expectation on the left hand side of (5) is measurable with respect to σ-algebra $\sigma\{X(t)\} \times B$. This comment shows that setting of the model should be done more accurate. Besides that it looks possible that some additional assumptions on the counting process $N(t)$ might be taking.

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