Options valuation.

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Abstract. This paper deals with the option-pricing problem. In the first part of the paper we study in details the discrete setting of the option-pricing problem usually referred to as the binomial scheme. We highlight basic differences between the old and the new approaches. The main qualitative distinction of the new pricing approach from either binomial or Black Scholes's is that it represents the option price as a stochastic process. This stochastic interpretation can not give straightforward advantage for an investor due to stochastic setting of the pricing problem. The new approach explicitly states that the options price is more risky than represented by binomial scheme or Black Scholes theory.

To highlight the difference between stochastic and deterministic option price definitions note that if a deterministic value is interpreted as a perfect or fair price we can comment that the stochastic interpretation provides this number or any other with the probability that real world option value at maturity will be bellow chosen number. This probability is a pricing risk of the option. Thus with an investor's motivation of the option pricing the stochastic approach gives information about the risk taking. The investor analyzing option price and corresponding risk makes a decision to purchase the option or not.

Continuous setting will be considered in the second part of the paper following [1]. A significant conclusion can be drawn from the new approach. It is shown that either binomial or Black-Scholes solutions of the option pricing problem have serious drawbacks. In particular, the binomial scheme establishes the unique price for a stock that takes two values and strike price K, $S_d < K < S_u$. According the binomial scheme this 'fair' price does not depends on real probabilities. Thus two options with that promise fixed income at maturity with probability close to 1 or 0 do have the same price. This of course does not have any sense. From this follows that there is no sense in using either neutral probabilities or 'neutral world' in options applications for valuation interest rates or credit derivatives either theoretically or numerically.

Recall that Black Scholes' approach was introduced in [2] and then later the binomial scheme was published [3]. Here we first represent discrete scheme. In several examples we discuss two-period plain vanilla option valuation. Note that the scheme can be applied for arbitrary states of a security over one step market. Then we extend the discrete scheme over an application to exotic option-pricing referred to as a compound option. The compound option in Black Scholes setting was first studied in [4] and then in [5,6].

Key words: alternative option pricing, exotics, binomial scheme, continuous time, Black Scholes equation, stochastic price.

Discrete time option valuation.

To achieve proper perspectives in studying the option pricing let us begin by reviewing a discrete form and then we comment the risk-neutral probability construction. In two periods economy let us denote the current date by 0 and the maturity date by *1*. The payoff to the European call option is defined as

$$\$ \, max \, (\, S \, (\, 1 \,) - K \, , \, 0 \,)$$

where constant K is a strike price. To begin with option pricing, we first specify the meaning of an "option price ".
When underlying security is supposed to be stochastic two questions may arise concerning the binomial scheme. It should be clear that the method of an option price should not depend on either values of the underline security nor the distribution of underlying security. What do depend on these parameters are the option values which have to be calculated.
Let us specify an option pricing framework. Suppose a security can only move up or down; the price is then designated as S_u and S_d respectively. If the security price goes up the call option has a value C_u; if it goes down, the call option has a value C_d . The value of the call option if the security rises is $C_u = max \{ \; S_u - K \, , \, 0 \, \}$, and if the security falls, the value is $C_d = max \{ \; S_d - K \, , \, 0 \, \}$. Suppose the investor is going to construct a hedge position such as the payoff B stays the same no matter which way the security moves. The initial position would be to hold the security, plus h units of the call option:

$$B_o = S_o \; + \; h \, C$$

The hedge ratio, h is chosen so that the ending payoff, or value B is the same no matter which way the security price moves. We establish the relationship that allows to find a unique value of h that will give a fixed payoff. Thus, the ending payoff is

$$B \; = \; S_u \; + h \, C_u \;\; = \; S_d \; + h \, C_d \tag{1.1}$$

The hedge position creates a less risky payoff. Solving the equation for h , the value:

$$\mathrm{h} \; = \; - \, [\, S_u \; - \; S_d \;] \, [C_u \; - \; C_d \;]^{\, -1} \tag{1.2}$$

One other important thing should be noted about the hedge position. Because the ending payoff is fixed, or certain, it must be related to the annualized riskless rate r and maturity T. This is the present value of the ending payoff. B should be equal to the investment made to construct it:

$$B_o \; = \; (\; 1 \; + \; r \, t \,)^{-1} B$$

Thus,

$$C = (1 + r t)^{-1} [\pi C_u + (1 - \pi) C_d] \qquad (1.3)$$

where $0 \le \pi \le 1$ and

$$\pi = [S_o (1 + r t) - S_d] [S_u - S_d]^{-1} \qquad (1.4)$$

If the option is worth less than this amount, the investor could make a greater return than on the risk free rate. In the case where the interest rate is continuously compounded the value $(1 + r t)$ should be replaced by $exp\ r t$. The expression (1.3) is what is referred to as risk-neutral pricing and probabilities π and $(1 - \pi)$ are often referred to as risk-neutral probabilities or equivalent martingale probabilities. We examine the option pricing formula (1.3) from the two points of view. Let's recall option pricing problem solution. We introduce a construction of the European call option problem using the Black-Scholes approach.
Let $w (t)$ be one-dimensional Wiener process and stock price $S (t)$ is the solution to the equation

$$d S (t) = \mu S (t) d t + \sigma S (t) d w(t) \qquad (1.5)$$

The European Call option on security $S (t)$ over a given interval $[t , T]$ is an agreement of buying shares at $\$ max (S (T; t, x) - K , 0)$ at maturity T, which is also known as the expiration date. Constant K is a pre-established strike price. The European Put option gives the buyer the right to sell shares at $\$ max (S (T; t, x) - K , 0)$. Here $S (T; t , x)$ is the solution of (1.1), such that $S (t; t, x) = x$. The act of making this transaction is referred to as exercising an option. Black and Scholes gave the definition of the European call option price. By definition it is a nonrandom function that is equal to the value of the portfolio containing a certain number of stocks and bonds. In B&S interpretation, a bond is a financial instrument that is governed by equation

$$d B (t) = r B (t) d t \qquad (1.6)$$

where r is a known constant interest rate.
In the simple deterministic example below we show that popular derivatives pricing models, including Black Scholes' lead to arbitrage opportunity and therefore can not be used. Then we show that the mathematical derivation of the Black Scholes equation is also incorrect. The solution of the Black Scholes equation (BSE)

$$\partial_t C (t , x) + r x \partial_x C (t , x) + ½ \sigma^2 x^2 \partial_{xx} C (t , x) = 0 \qquad (1.7)$$

with the boundary condition

$$C(T, x) = max(x - K, 0)$$

represents the value of the European call option contract on a common stock, which price is governed by the equation (1.5). Using probabilistic representation of the solution for the parabolic Cauchy problem, the Black Scholes equation solution can be written in the form

$$\mathrm{C}(t, x) = E\, exp - r(T - t)\, max(\eta(T; t, x) - K, 0) \qquad (1.8)$$

where $\eta(l; t, x)$, $l > t$ is the solution of the Ito equation

$$d\eta(l) = r\eta(l)\, dl + \sigma\, \eta(l)\, dw(l)$$

such that $\eta(t) = x$.

It seems instructive first to examine correspondence of the Black-Scholes formula (1.8) to the binomial scheme represented above. The time values are $t = 0, T = 1$. Then continuous random variable $w(1) - w(0)$ should be replaced by a random variable δ that assumes only two possible values. Let S_o be the value of the stock at time 0 and S_u, S_d are the stock values at time 1. In conforming with the stock model (1.5) we put

$$S_1 = S_o + \mu S_o + \sigma S_o \delta$$

Assume that the random variable $S(1)$ takes values S_u, S_d with probabilities p_u, $p_d = 1 - p_u$; the mean and the variance of the random variable δ is 0 and 1 respectively. S_o is a nonrandom then

$$E S_1 = S_u p_u + S_d p_d = S_o + \mu S_o$$

Solving equation for μ we obtain

$$\mu = [S_u p_u + S_d p_d - S_o] S_o^{-1}$$

From the other hand of the equation

$$E[S_1 - E S_1]^2 = (\sigma S_o)^2$$

has a unique solution for σ. Let $r \geq 0$ be a risk free interest rate. Then risk-neutral security price is

$$\eta_1 = S_o + r S_o + \sigma S_o \delta$$

Then the values of the option price given by the formulae (1.3), (1.8) are different.
By construction, the risk-neutral probabilities are independent of the real world probabilities. Therefore they hold their values when the real probability of the states S_u (S_d) tends to be 0 or 1. One can easily discover that in this case the state S_u or S_d can be eliminated from consideration by making the real probability of the state, say S_d

equal to 0. The straightforward conclusion is that the binomial method of the option valuation (1.3) does not make sense.

We scrutinize the Black-Scholes design and show that their option pricing solution admits arbitrage opportunity. This arbitrage arises when one compares rates of return on an option to an underlying security.

Let $S(0) = \$10$, $S(1) = \$16$, and $K = \$15$, $r = 0$. Applying formula (1.8) we can see that the Black-Scholes option price is

$$C(t, x) = E \exp - r(T - t) \max(\eta(T; t, x) - K, 0) = \max(10 - 15, 0) = 0$$

Indeed, the deterministic stock price implies that $\sigma = 0$. On the other hand we can establish the option price by comparing the rates of return on two investment opportunities. These opportunities are stock or option investments. Thus,

$$S(1) / S(0) \chi\{ S(1) > K \} = C(1, S(1)) / C(0, S(0)) \qquad (1.9)$$

or

$$16 / 10 = (16 - 15) / C(0, S(0))$$

Therefore,

$$C(0, S(0)) = \$0.625$$

This price differs from Black - Scholes's price. In this basic example the Black and Scholes' solution provides a different rate of return on stock and its option and therefore could not be accepted as a definition of the option price problem.

Here we highlight the main difference among the two option pricing approaches. Assume that a call option is at-the-money, the current security and strike prices are equal to \$75. When security moves it will either go up to $S_u = \$100$ or down to $S_d = \$80$. The risk-free rate of interest is not involved in calculations and should be assumed to be equal to $r = 0$. It can also be a chosen arbitrary if an investor follows a binomial scheme. The binomial approach first establishes the hedge ratio h. It can be found solving the equation

$$S_u - h C_u = S_d - h C_d$$

Thus $S_u - hC_u = S_d - hC_d$, where

$$C_u = \max\{ S_u - K, 0 \} = 25, \qquad C_d = \max\{ S_d - K, 0 \} = 5$$

Solving the equation for h, we get $h = 1$. As far as $r = 0$ then the future value of the portfolio $S_u - hC_u = S_d - hC_d = \$75$ should be equal to its present value at date 0. Therefore, $75 - C = 75$ and $C = \$0$. This example of the binomial scheme is a benchmark of an option valuation and it was represented in any college handbook. On the other hand in this particular framework it is possible to establish the option price that perfectly replicates the stock return. Indeed, let us introduce the elementary events

$$\omega_u = \{ S(1) = 100 \}, \qquad \omega_d = \{ S(1) = 80 \}$$

Then the equation

$$S(1) / S(0) = C(1) / C(0)$$

results that $C(0) = C(0, \omega)$, where $\omega \in \{ \omega_u, \omega_d \}$ and

$$C(0, \omega) = \begin{cases} 18.75, & \text{when } \omega = \omega_u \\ 4.6875, & \text{when } \omega = \omega_d \end{cases}$$

The binomial scheme states that the call option price is 0, and therefore there is no reason to buy it. The new interpretation of the option price shows that investing in call option results the positive profit and will be beneficial regardless of the stock value at expiration. The drawback of the option price represented by binomial scheme is quite evident.

Remarkably, the discount factor has not been involved. The risk free interest rate generated by the Treasury bond does have an effect on derivatives pricing, but not in the way as it was performed in the Black-Scholes theory.

Let us consider the rolling dice example that can serve as an example of the stochastic stock price. Let $t = 0$ be the initial time and $t = 1$ an expiration date of the option. The set $\{ 1, 2..., 6 \}$ represents the set of all the possible values of the stock. The probability of the event $\{ S(1) = x ; x = 1, 2, ... 6 \}$ does not depend on x and is equal to $1/6$. We are trying to avoid some technical difficulties that will arise in the case when the payoff at maturity admits the value 0.

The option price is a random variable which values are defined such that its rate of return replicates the return of the underlying security. Therefore the current option price is a function depending on the stock values at expiration. Note that if the option payoff admits the value 0 that is if the probability of the event $\{ S(T = 1) = 0 \}$ is strictly positive then the current value of the option is assumed to be equal to 0.

Letting $K = \$0.8$, $S(0) = \$2$ and applying equation (1.9) we arrive at the option price $C(0, 2) = C(0, S(0) = 2)$

Table1.1 Two time periods economy $t = 0$, $T = 1$, $S(0) = 2$, $K = \$\,0.8$

$S(1)$	**1**	**2**	**3**	**4**	**5**	**6**
$C(1, S(1))$	0.2	1.2	2.2	3.2	4.2	5.2
$C(0, 2)$	0.4	1.2	1.47	1.6	1.68	1.73

Each entry in the third row of the table has the same probability of $1/6$ as the correspondent value S (1). The calculation represented in the Table 1 is quite simple and cover the case when the option's return can perfectly replicates the stock return. In a general case, when the possibility to replicate stock return by the option perfectly is impossible, we suppose that $K = \$\,2.5$. The payoff at the maturity is $C(1, S(1)) =$

$max\ (S (1) - K ,\ 0)$ and then the correspondent call option price can be calculated on the equal rate of return basis as follow

Table 1.2 Two time periods economy $t = 0, T = 1, S (0) = 2, K = \$ 2.5$

$S (1)$	**1**	**2**	**3**	**4**	**5**	**6**
$C (1 , S (1))$	0	0	0.5	1.5	2.5	3.5
$C (0 , 2)$	0	0	1/3	3/4	1	7/6

One can see that in this example the option payoff admits value 0. It occurs when stock price at maturity is equal to 1 or 2. The probability of such event equal 1/3. Since the option price in this case can not perfectly replicate the stock return we introduce a portfolio that does perfectly replicate underlying equity. Let P be a portfolio value which value at date $t = 0$ is

$$P (0 , S (0)) = S (0) \chi \{ S (1) \le 2.5 \} + C (0 , S (0)) \chi \{ S (1) > 2.5 \} \qquad (1.10)$$

At maturity date $T = 1$ the value of the portfolio is

$$P (1 , S (1)) = S (1) \chi \{ S (1) < 3 \} + C (1 , S (1)) \chi \{ S (1) \ge 3 \}$$

and therefore the portfolio's rate of return coincides with the stock return. Indeed,

$$P (1 , S (1)) / P (0 , S (0)) = \chi \{ S (1) < 3 \} [S (1) / S (0)] +$$

$$+ \chi \{ S (1) \ge 3 \} [C (1 , S (1)) / C (0 , S (0))] =$$

$$= \chi \{ S (1) < 3 \} [S (1) / S (0)] + \chi \{ S (1) \ge 3 \} [S (1) / S (0)] =$$

$$= S (1) / S (0)$$

Thus stock and the established portfolio offer the same rate of return for an arbitrary outcome associated with the stock value at maturity. Note that the portfolio at date $t = 0$ contains random portions $\chi \{ S (1) < 3 \}$ and $\chi \{ S (1) \ge 3 \}$ of stocks and the call options respectively and replacing these random variables by its estimates such as their probabilities involves the risk. This is the risk that the owner of the portfolio meets when the portfolio return is below the return of underlying security.

The next important problem of the option pricing is the uniqueness of the option price. This issue one should not mix with an amount an investor agrees to pay for the option. Note that an investor can also name it option price. By paying a certain premium for the option the investor can specify the return accepted in the deal and the risk associated with the event that the return occurs below than it is initially specified.

To justify the uniqueness of the option price definition in stochastic environment consider two steps of economy t = 0, 1. Assume that the stock at maturity T = 1 holds two values $S_d < S_u$ with probabilities p_d, p_u respectively. Assume for instance that $K < S_d$. Then the call option price is a random variable that has admitted two values

$$C_d = \frac{S_0}{S_d} (S_d - K) \ , \ C_u = \frac{S_0}{S_u} (S_u - K)$$

Note that option price is monotone decreasing function of the stock price. Let a one who does not accept stochastic option price definition wish to represent deterministic pricing solution. Here we need to recall the risk neutral pricing model arguments that had been used above. In addition to the above comments note that any non-random option price can not reconstruct underlying stock values. Therefore in the transition from the real world events describing stock values and deterministic option value some significant information is lost. On the other hand the stochastic portfolio with random portions of the stocks and options represents the same information as the underlying security itself.
Now let us consider the most important real-world problem. How much an investor should pay for an option. There are two sides of the problem. Let an investor pays \$M for the call option. How describe this investment position and how to compare two option prices to chose optimal price. Note that the term 'optimal' should be clarified.
Return to the dice example which solution is presented in the Table 1.2. Let the investor thinks that the price of $C_{\#} (0, 2) = \$1/2$ is the fair price for the call option. Then the average loss of this choice is

$$1/6 [2 \times (- 1/2) + (1/3 - 1/2) + (3/4 - 1/2) + (7/6 - 1/2)] = - 1/24$$

We can also calculate other statistical characteristics of the price $C_{\#} (0, 2)$. Thus any option price implies the certain risk characteristics that represent risk exposure of the investor's choice. This exposure can be expressed either in profit / loss terms or as return form. For instance using the above example the return representation of the risk means the calculation of the probability

$$P \{ C (1, S (1)) / C_{\#} (0, 2) < \delta \}$$

for any $\delta > 0$. The other problem that we outlined above is how to find optimal option value $C_{opt} (0, 2)$. Note that meaning of the term optimal is not unique. One example of the optimal choice can be defined as follow. For given $0 < \alpha, \delta < 1$ the optimal choice is the minimum number $C_{\alpha\delta} (0, 2)$ such that

$$P \{ C (1, S (1)) / C_{\alpha\delta} (0, 2) > \alpha \} > 1 - \delta$$

Note that in the dice option problem the random variable $C (1, S (1))$ admits exactly 6 different values and therefore the distribution of the random return
$C (1, S (1)) / C_{\alpha\delta} (0, 2)$ is a stepwise function.
On the other hand it is possible to introduce another fair price. That is the expected value of the $C (0, 2)$. Let us find this expected value for the dice problem. The values $C (0, 2)$ are given in the Table 2. Therefore $C_{avg} (0, 2) = E\, C (0, 2) = 1/6 [1/3 + 3/4 + 7/6] =$
$= \$3/8$. In contrast with the price $C_{\alpha\delta} (0, 2)$ which was constructed based on given risk characteristics the value $C_{avg} (0, 2)$ specifies probability $1 - \delta$ for any given α.

Now, let us focus on the pricing problem over three time periods. This is more complex problem that highlights new factors that impact on the pricing.

Denote 0, 1, $T = 2$ the current, intermediate, and maturity moments of time. Let us consider the example in which rolling dice is a model of the stochastic security. Assume that events $\{ S(j) = q_j \}$ for $j = 1, 2$ are mutually independent, $q_j = 1, 2, \dots 6$ and put $K = \$2.5$, and $S(0) = \$2$. Applying equation (1.9) over the interval [0 , 2] we see that the call option price holds the same values as above in Table 1.2. Therefore the perfect portfolio should be taking in the form (1.10). This follows from the fact that cost of borrowing money is assumed to be equal 0. Now let us look at the intermediate moment of time $t = 1$. At this moment the stock price q takes values from 1 to 6. The probability of the event $\{ S(1) = q \}$, $q = 1, 2, \dots 6$ is independent on the value $S(0)$, $S(2)$ and equal to 1 / 6. Thus the option price at time 0 is "double" random. It depends on two random variables $S(1)$, $S(2)$ that form the trajectory of the security price over the interval [1 , 2]. We examine the possibility to exercise an option at a date 1_l when the stock offers upper return than at maturity. This early exercise possibility suggests a higher option premium. Next table represents the option price when early exercise does not possible.

Table 1.3 Three periods economy $t = 0, 1, 2$; $T = 2$, $S(t_0) =$ \$2, K = \$2.5

$S(2)$	1	2	3	4	5	6
$C(0, 2)$	0	0	1/3	3/4	1	7/6

The Table 1.4 the stock values enclosed in the first column and the first row. The k×j-th entry in the Table 4 represents values of the call option $C(1_l, S(1))$ when $S(1) = k$ and $S(2) = j$. The probability of the such entry is 1/36 for any k and j.

Table 1.4 Three time periods economy $t = 0, 1$, $T = 2$, $S(0) =$ \$2, K = \$ 2.5

S (1) \ S (2)	1	2	3	4	5	6
1	0	0	1/6	3/8	1/2	7/12
2	0	0	1/3	3/4	1	7/6
3	0	0	1/2	9/8	3/2	7/4
4	0	0	2/3	3/2	2	7/3
5	0	0	5/6	15/8	5/2	35/12
6	0	0	1	9/4	3	7/2

Let at date t = 0 an investor assumes that S (1) = 5. The return on stock over interval [0, 1] is equal to 2.5 per dollar and therefore the option's return is the same. Indeed, the option's return over [0 , 1] is $C(1, 5) / C(0, 2)$ and therefore

Table 1.5

$S(2)$	1	2	3	4	5	6
$C(1, 5)$	0	0	5/6	15/8	5/2	35/12
$C(0, 2)$	0	0	1/3	3/4	1	7/6
$C(1, 5)/C(0, 2)$	0	0	5/2	5/2	5/2	5/2

From Table 1.5 it follows that with the dice-security example we do not have a chance to adjust option price by exercising it earlier than maturity. This unimproved setting stems from the fact that the stock price model is a random process with independent values at each dates t = 1,2 and with the same uniform distribution.
Owner of the portfolio that replicates stock return should restructure the portfolio at future moments. To specify these changes we first form a portfolio that perfectly replicates stock at time $t = 1$. Thus, if $S(1) = 5$ then portfolio's value at $t = 1$ is

$$P(1,5) = 5\chi\{S(2) \le 2.5\} + C(1,5)\chi\{S(2) > 2.5\} \tag{1.11}$$

In order to perform the portfolio reconstruction from the state $P(0, 2)$ to the state $P(1,5)$ one needs to solve the equation

$$P(0,2) + \mathrm{X} = P(1,5)$$

or

$$\mathrm{X} = P(1,5) - P(0,2) = 3\chi\{S(2) \le 2.5\} +$$

$$+ [C(1,5) - C(0,2)]\,\chi\{S(2) > 2.5\}$$

The value of X represents growth in value of the portfolio at time $t = 1$. One can remark that the increment of the option price $[C(1,5) - C(0,2)]$ is a random with values of

Table 1.6

$C(1,5) - C(0,2)$	0	0	1/2	9/8	3/2	7/4

As far as each value of the random variable $C(1,5) - C(0,2)$, depends only on the value of $S(2)$ then the probability of the event is 1/6. Thus, expected expenses on reconstruction the hedge portfolio over [0 , 1] is

$$E[P(1,5) - P(0,2)] = 3P\{S(2) \le 2.5\} +$$

$$+ E[C(1,5) - C(0,2)]\,\chi\{S(2) > 2.5\} = 5\ 7/8$$

It is also of interest to the investor to display the difference between European and American types of the options. Considering three periods in economy we are able to compare the portfolio profitability for both option types. Let us consider the American option evaluation using the example that was represented above. Recall the prevailing opinion that in the case where the underlying asset pays no dividends, it is never optimal to exercise an American call option early. The equal rate of return rule for the economy with more than two periods can be studied in the similar form as for the two-periods European counterpart

$$S(t)/S(0)\chi\{S(2) > K\} = C(t, S(t))/C(0, S(0)) \tag{1.12}$$

$t = 1, 2$. The only difference between two options are the values of the options at intermediate time $t = 1$ that effects the option price at date 0. The scheme of the European valuation along with the perfect replication portfolio was established by the formula (11) and presented numerically by the Table 1.5. American option can be exercised at any date t and its payoff is

$$max\ \{\ S(t) - K,\ 0\ \}$$

Denote $C_A(t, S(t); T)$ the American option price at t when the stock price is $S(t)$ and maturity date T. Then

$$\begin{aligned} C_A(0, S(0); T=2) = \ & C_E(0, S(0); T=1))\, \chi\{S(1) \geq S(2)\} \times \\ & \times \chi\{S(2) > K\} + C_E(0, S(0); T=2)\, \chi\{S(2) > S(1)\}\, \chi\{S(1) > K\} \end{aligned} \tag{1.13}$$

Note, for example, that when the event { S (1) ≥ S (2) } is true then the rate of return on stock over the period [0, 1] is higher than the stock return over the [0, 2]. By the option price construction this conclusion also remains true for the option return. The formula (1.13) can be easily extended on the n-steps economy.
There exists a class of contingent claims in which underlying securities are either options or other types of derivative instruments. Recall that options European or American or their combinations of the four basic option strategies: long call, long put, short call, short put are referred to as plain vanilla options. Any option that is not a regular plain vanilla is called *exotics*.

Let's consider a model example in which the underlying instrument is an option. This class is also called *compound* or split free options. Possible specifications are a call on a call or a put, and put on a call or a put options. Let $C(t, S(t)) = C(t, S(t); T, K)$ denote the value of a plain-vanilla call option at date t with maturity T and strike price K written on a security $S(\cdot)$. First consider option on a call option. The buyer of a compound call-call option at date t has the right to buy underlying call option with maturity date T at a fixed premium $C_c(T_c)$ on a fixed date $T_c > t$ in the future.
We denote the price of the compound call on call option at date t with maturity T_c, $t \leq T_c < T$ with strike price K_c by

$$C_c(t, S(t)) = C_c(t, S(t); T_c, K_c; T, K))$$

Then payoff of the compound call-call option with strike price K_c and maturity T_c is a delivery of a plain call option with a strike price K and an expiration date T. Note that the lifetime of the compound option is $[t, T_c]$. Thus

$$\begin{aligned} C_c(T_c) &= C_c(T_c, C(T_c, S(T_c); T, K); T_c, K_c) = \\ &= \max\{ C(T_c, S(T_c); T, K) - K_c, 0 \}. \end{aligned}$$

Here we perform a discrete scheme to illustrate numerical calculations needed to establish the price of this exotic instrument. Let $K_c = 0.6$, $T_c = 1$, $K = 3$, $T = 2$. The data in the table

Table 1.7

t = 0	t = 1	t = 2
		S(2) = 6 p(5, 6) = 1/4
	S(1) = 5 p(4, 5) = 2/3	S(2) = 5 p(2, 5) = 1/8
S(0) = 4		S(2) = 4 p(2, 4) = 1/4
	S(1) = 2 p(4, 2) = 1/3	S(2) = 3 p(5, 3) = 3/4
		S(2) = 1 p(2, 1) = 5/8

represent evolution of the underlying security price over three periods of time t = 0, 1, and 2, along with the correspondent transition probabilities. From the table one can see that the security price at time 0 is \$4, then at time 1 it can be either \$5 or \$2 with probabilities 2/3 and 1/3 respectively. At the event S (1) = 5 security can go either to the state \$6 with probability 1/4 or to the state \$3 with probability ¾. If at date t = 1, S (1) = \$2 then security can move either to \$5, \$4, \$1 with transition probabilities 1/8, 1/4, 5/8 respectively. All transitions from one state to the other are assumed to be mutually independent The problem is to establish the value of the compound call-call at date t = 0 given $K_c = 0.6$, $T_c = 1$; K = 3, T = 2. We start evaluation from the date $T_c = 1$. Denote C (1; 5, 6) = C (t = 1; S (1) = 5, S (2) = 6) the value of the plain call option at time 1, given S (1) = 5, S (2) = 6. Note that the unique opportunity to avoid arbitrage is to put C(1; 5, 6) = \$2.5. Indeed, underlying security return over the interval [1, 2] is then equal to 1.2. Therefore the option return can replicate the security return for this particular event. Note that there is no other way to prescribe an option's value. Indeed, the option pricing does not depend on distribution and one can admit that the probability of the event { S (1) = 5, S (2) = 6 } is close to 1 and therefore the joint chance of other states can be neglected. Then from the equation (6 – 3) / C (1; 5, 6) = 6/5 follows the established price. The same way of calculations bring us to the prices:

C (1; 5, 6) = 2.5, C (1; 2, 5) = 0.8,
C (1; 2, 4) = 0.5, C (1; 5, 3) = C (1; 2, 1) = 0.

Thus,

$$C(1;\cdot) = \begin{cases} 2.5, & \text{when } \cdot = \{S(0)=4, S(1)=5, S(2)=6\} \\ 0.8, & \text{when } \cdot = \{S(0)=4, S(1)=2, S(2)=5\} \\ 0.5, & \text{when } \cdot = \{S(0)=4, S(1)=2, S(2)=4\} \\ 0, & \text{otherwise} \end{cases} \quad (1.14)$$

The random variable C (1; ·) represents a price of the underlying plain call option at date t = 1. Note that from (14) follows that

$$C(0;\cdot) = \begin{cases} 2, & \text{when } \cdot = \{S(0)=4, S(1)=5, S(2)=6\} \\ 1.6, & \text{when } \cdot = \{S(0)=4, S(1)=2, S(2)=5\} \\ 1, & \text{when } \cdot = \{S(0)=4, S(1)=2, S(2)=4\} \\ 0, & \text{otherwise} \end{cases}$$

Note that E C (0; ·) = 2/6 + 1.6 /24 + 1/12 = 5.8/12, E C (1; ·) = 2.5/6 + 0.8/24 + + 1/24 = 5.9/12, E C (2, ·) = 3/6 + 2/24 + 1/12 = 8/12 = 2/3. This calculation performs the fact that optimal average strategy does suggest to buy the plain option for

$5.9/12 = $0.492 at $t = 1$ and exercise it at $T = 2$ because $E\,C(2, \cdot) > E\,C(1, \cdot)$. The return on expected option values over $[1, 2]$ is $[E\,C(2, \cdot)] / [E\,C(1, \cdot)]$. That is about $1.356 at $t = 2$ per $1 investment at $t = 1$.
The direct analysis based on the precise option pricing solution shows that average return on call option is

$$E[\chi\{S(2) > 3\} \times C(2; \cdot)/C(1; \cdot)] = [3/2.5]/6 + [2/0.8]/24 + [1/0.5]/12 = 5.25/12.$$

That suggests less than $1 return at $t = 2$ per $1 investment at $t = 1$. This shows that the option price will not increase in average over $[1, 2]$.Nevertheless the option price at $t = 1$ that bellow than $2/3 offers increase in average value of the investment. In particular the price $5.9/12 at $t = 1$ looks reasonable enough. The investor's risk associated with the event when investment return billow than 1 coincides with the probability of the event to lose all capital is

$$\text{Prob}\{C(2, S(2)) < 5.9/12\} = P\{S(2) < 4\} = 7/24$$

That is about 29%.
Now let us turn back to the compound option valuation. The date $T_c = 1$ is the expiration date of the compound option. The buyer of the compound call option has the right to buy underlying plain vanilla option for $0.6. The compound payoff received at the date $T_c = 1$ is

$$\$\max\{C(1, S(1); T = 2, K = 3) - 0.6, 0\} \tag{1.15}$$

Therefore at the date $T_c = 1$ the buyer has the right but not an obligation to pay the strike price of $0.6 and in return to get plain call option with strike $K = 3$ and expiration date $T = 2$. Let us return to the recovery of the compound premium. From (1.15) it follows that compound payoff is

$$C_c(1, C(T_c, S(T_c); T, K); T, K_c) = \begin{cases} 2.5 - 0.6 = 1.9, & p = 2/3 \times 1/4 = 1/6 \\ 0.8 - 0.6 = 0.2, & p = 1/3 \times 1/8 = 1/24 \\ 0, & p = 19/24 \end{cases}$$

Using values $C_c(1, S(1))$ we enable to establish the compound option price at date $t = 0$. Indeed, the stock return over the interval $[0, 1]$ is either 5/4 or 2/4 with the probabilities 2/3 and 1/3 respectively. If the payoff to the compound call at $T_c = 1$ is equal to 0 then the value of the compound call at date 0 is obviously equal to 0. At the event when payoff is positive the return on stock and on the compound option should be equal. Otherwise as we mentioned above one can assume that the probability of a particular stock value that exceeds the strike price is as close to 1 that we can ignore the all other values. In this case we actually arrive at deterministic stock that uniquely establishes option price by replicating stock return. Note that continuous distributed stock can also perfectly be replicated by the option. This is the case when the lowest stock value is higher then option's strike price.
One can see that the investor pays compound premium at date 0 than pays strike price K_c at date 1 and in return he or she obtains the plain vanilla option that cost $C(1, S(1)) = C(1, S(1); 2, 3)$. Thus

$$S(1)/S(0)\ \chi\{C(1,S(1)) > 0.6\} = C_c(1,S(1))/\ C_c(0,C(0,S(1))$$

If the underlying option exists on [0 , 1] then by definition

$$S(1)/S(0)\ \chi\{S(2)) > 3\} = C(1,S(1);2,3)/C(0,S(0);2,3))$$

and stock return on the left hand side can be replaced by the correspondent option return. The compound option premium at t = 0 in both cases is equal. Thus,

$$C_c(0,4) = \begin{cases} 4/5 \times 1.9 = \$1.52, & p = 1/6 \\ 4/2 \times 0.2 = \$0.4, & p = 1/24 \\ 0, & p = 19/24 \end{cases}$$

Note that the average value and the standard deviation of the compound call-call option are

$$E\,C_c(0,4) = 1.52 \times 1/6 + 0.4 \times 1/24 = \$0.27$$
$$\text{standard deviation } C_c(0,4) = \$0.319$$

For the practical use assume that an investor pays compound premium, say \$0.5 at the date 0 and the compound strike of \$0.6 at the date t = 1. How to represent the buyer's risk? Analyzing the situation we see that the only event that suggests profit is the event associated with the trajectory $\cdot = \{S(0) = 4, S(1) = 5, S(2) = 6\}$. Indeed, the investor pays compound premium \$0.5 at date 0 and then compound strike of \$0.6 at date 1 that is \$1.1. In return the investor receives the assets which price at date 0 is \$1.52. Therefore the positive balance is \$0.42. The probability of such event is 1/6. Any other possible outcome leads to the loss. Thus, the compound option risk on the investment associates with the return that is below than 1 and therefore it coincides with the event

$$\{C(1,S(1))/1.1 < 1\}$$

The probability of such an event is $1 - P\{C_c(1,S(1)) \geq 1.1\} = 5/6$. The average loss of exercising a compound option price is

$$\text{Expected losses} = E[C_c(0,4) - 1.1]\,\chi\{C(1,S(1)) < 1.1\} =$$

$$= [(0.4 - 1.1) \times 1/24 - 1.1 \times 19/24] = -\$0.9$$

Expected profit is

$$E[C_c(0,4) - 1.1]\,\chi\{C(1,S(1)) > 1.1\} = 0.42 \times 1/6 = \$0.07$$

An important investment characteristic over [t, T] can serve a ratio

$$\wp(t,T) = \text{Expected profit/ Expected losses} \times 100\% \tag{1.16}$$

We call it risk-effectiveness. Then the risk-effectiveness of the compound call-call investment is $\wp(0, 1) = 7.78\%$, that might be too low to attract an investor.
Let us consider next specification of the compound option. In this case assume that underlying of the compound option is a put option. Assume for simplicity that the security data remains the same as in the Table 7. Then underlying plain vanilla put payoff at its expiration T = 2 by definition is $P(2, S(2)) = \max\{K - S(2), 0\}$. Therefore

$$P(2, S(2)) = \begin{cases} 0, & \text{when } S(2) = 6, 5, 4, 3 \\ 2, & \text{when } S(2) = 1 \end{cases}$$

From the equation

$$\frac{P(2, S(2))}{P(1, S(1))} = \frac{S(2)}{S(1)} \chi\{ S(2) < K \}$$

it follows that

$$P(1, S(1, \cdot)) = \begin{cases} 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 6 \} \\ 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 5 \} \\ 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 4 \} \\ 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 3 \} \\ 4, & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 1 \} \end{cases}$$

The compound call-put option payoff at $T_c = 1$ is then can be written as

$$C_p(1, S(1)) = = \max\{ P(1, S(1)) - K_p, 0 \}$$

where K_p is assumed to be equal to $K_c = 0.6$. Solving the equation

$$\frac{C_p(1, S(1))}{C_p(0, S(0))} = \frac{S(1)}{S(0)} \chi\{ P(1, S(1)) > K_p \}$$

for $C_p(0, S(0))$ bring us to the compound call-put premium

$$C_p(0, 4) = \begin{cases} 13.6, & p = 5/24 \\ 0, & p = 19/24 \end{cases}$$

The expected value of the compound call-put option price is then $EC_p(0, 4) = \$2.83$. If an investor pays a premium of \$2 at date 0 and the compound strike price of \$0.6 at date 1, then the profit exists only when the outcome is $\{ S(0) = 4, S(1) = 2, S(2) = 1 \}$. Thus the investment of \$2.6 might only bring \$13,6 and therefore result in \$11 of the pure profit. The probability of this event is 5/24. The event $\{ P(1, S(1)) < K_p \}$ represents the risk associated with this investment. Expected losses, profit, and the profitability (1.16) are

$$2.6 \times 19/24 = \$2.06\,, \qquad 11 \times 5/24 = \$2.29\,, \qquad \wp(0, 1) = 111.17\%$$

correspondingly. This data for the long position looks much attractive for the investor than for the call-call option.

We have studied two types of compound options, call-call and call-put. Now let us take a look at two other types: put-call and put-put options. For the put-call option we again use the same data given in the Table 1.7. Put option payoff at date T_c is

$$P_c(T_c, S(T_c)) = \max\{K_c - C(T_c, S(T_c); T, K), 0\}$$

Bearing in mind values of the call option $C(1, S(1); 2, 3)$ given above in (1.15) we see that

$$P_c(1, C(1, S(1))) = \begin{cases} 0.6\,, & \text{when} \cdot = \{S(0) = 4, S(1) = 5, S(2) = 3\} \\ 0.6\,, & \text{when} \cdot = \{S(0) = 4, S(1) = 2, S(2) = 1\} \\ 0.1\,, & \text{when} \cdot = \{S(0) = 4, S(1) = 2, S(2) = 4\} \\ 0\,, & \text{otherwise} \end{cases}$$

where

$\text{Prob}\{S(0) = 4, S(1) = 5, S(2) = 3\} = 0.5$,
$\text{Prob}\{S(0) = 4, S(1) = 2, S(2) = 1\} = 5/24$,
$\text{Prob}\{S(0) = 4, S(1) = 2, S(2) = 4\} = 1/12$,
$\text{Prob}\{P_c(1, S(1)) = 0\} = 5/24$

In calculating these probabilities we used the assumption that all transitions are mutually independent. Then taking into an account the equation

$$\frac{P_c(1, S(1))}{P_c(0, S(0))} = \frac{S(1)}{S(0)} \chi\{C(1, S(1)) < K_c\}$$

it follows that the put-call compound option price $P_c(0, 4)$ is a random variable with distribution

values	0	0.2	0.48	1.2
probabilities	5/24	1/12	0.5	5/24

We would like to point out an interesting moment in the above calculations. One might note (14) that plain vanilla call option has the same price \$0.6 for two different ways at time 1. $S(0) = 4$, and $S(1)$ can be either 5 or 2 then return on the stock is different though compound put payoff along two paths holds the same value

$$P_c\{t = 1; S(1) = 5, S(2) = 3\} = P_c\{t = 1; S(1) = 2, S(2) = 1\} = 0.6$$

The explanation of this phenomenon is that the price C (1, S (1); 2, K) depends on σ-algebra F_{12} events generated by the values of S (·) occurred over the time interval [1, 2]. Therefore when the price C (1 , *) admits the same value for two different ω is F_{12} – measurable. On the other hand compound put pricing depends on the values S (·) over time interval [0, 1]. The distinction in stock return on [0, 1] results in a compound put pricing.

The expected value of the put-call compound option is about \$0.51. An investor who purchases the put-call option for \$0.3 at t = 0 and pays the strike price \$0.6 at date t = 1 purchases put-call option contract for the total of \$0.9. The chance that the price of the underlying call option at date t = 2 will be bellow than \$0.9 is

$$P\{ C(2, S(2)) \le 0.9 \} = P\{ S(2) - 3 \le 0.9 \} = P\{ S(1) = 5, S(2) = 3 \} +$$

$$+ P\{ S(1) = 2, S(2) = 1 \} = 1/2 + 5/24 = 17/24.$$

and the average value of the losses are $3 \times 1/2 + 1 \times 5/24 = \$1.71$.

The last compound option type is put-put option. By applying the same pricing methods for P (1, S (1)) we see that the put-put payoff at $T_p = 1$

$$P_p(1, S(1)) = \max\{ K_p - P(1, S(1); T, K), 0 \}$$

is a random variable, where $K_p =$ \$0.6, K = \$3, T = 2. Then using (1.16) and that $P_p(1, P(1, S(1)) = \max\{ 0.6 - P(1, S(1)), 0 \}$ we see that

$$P_p(1, P(1, S(1, \cdot)) = \begin{cases} 0{,}6 & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 6 \} \\ 0{,}6 & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 5 \} \\ 0{,}6 & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 4 \} \\ 0{,}6 & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 3 \} \\ 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 1 \} \end{cases}$$

Applying equation

$$\frac{P_p(1, S(1))}{P_p(0, S(0))} = \frac{S(1)}{S(0)} \chi\{ P(1, S(1)) < K_p \}$$

we figure out that the compound put-put price is a random variable

$$P_p(0, 4) = \begin{cases} 0{,}48 & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 6 \} \\ 1{,}2 & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 5 \} \\ 1.2 & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 4 \} \\ 0{,}48 & \text{when } \cdot = \{ S(0) = 4, S(1) = 5, S(2) = 3 \} \\ 0, & \text{when } \cdot = \{ S(0) = 4, S(1) = 2, S(2) = 1 \} \end{cases}$$

with the distribution

values	0	0.48	1.2
probabilities	5/24	2/3	1/8

The expected value of put-put compound option is \$0.47. An investor who managed to buy put-put option say for \$0.21 and paid a strike price of \$0.6 at date $t = 1$ holds the risk 5/24 to lose complete investment.

Exotic options.

In the early 1980's, exchanges in Amsterdam, Montreal, Philadelphia, and Chicago were traded in standardized foreign currency options. Now, currency options are available to many currencies. The Security Exchange Commission regulates the options exchanges in the US. In addition to the exchanges there is an over-the-counter market where the currency options are offered by the commercial banks and brokerage firms. Unlike the currency options traded on an exchange, these currency options are tailored to the specific firm's interests. The number of units, strike price desired, and expiration date can be chosen by the clients.

In this paper we discuss several types of standard and non-standard options contracts. The standard options that can be reduced to the European and American options types are sometimes referred to as plain vanilla options. The non-standard options types are commonly called exotics and they usually divide into two main classes path-depended and path-independent. We provide a survey of some popular exotic options and simultaneously introduce their valuation.

Recall some of the main definitions. A call (put) option is a contract between two parties: buyer and a seller. It gives the holder the right to buy (sell) an asset at a stated price (strike price, exercise price) on (European) or before (American) predetermined date, called maturity (expiration date). The current option price is called option price or premium.

Let $S(T; t, x)$ be a price of an underlying asset at a future date T, $T > t$ so that $S(t; t, x) = x$. Denote $C(t, x)$ [$P(t, x)$] the call [put] option price at date t when $S(t) = x$. Formally a plain vanilla option contract is defined by its expiration and its payoff at the expiration date. Next we will use T as a date of expiration. Then call and put values at expiration are defined as follows:

$$C(T, S(T)) = \max\{ S(T; t, x) - K, 0 \}$$
$$P(T, S(T)) = \max\{ K - S(T; t, x), 0 \} \tag{2.1}$$

where t is a current date and x is the price of an underlying asset at t.

The problem of the option pricing is to determine the call (put) option price at any moment of time t before the expiration date T. It is clear that the option price does not cost anything when payoff at maturity is equal to 0, and for any particular value Ξ of the asset at expiration there is a unique value of the option contract that replicates underlying stock return. The possibility that the underlying stock in the future admits different values implies stochastic setting of the problem. We simplify the scheme of the pricing problem. Assume that an option can only be exercised at maturity T. Such setting can be interpreted as a two step economy. Next, we shall indicate the price adjustment for the continuous time and discuss the possibility to exercise the option prior to the expiration date.

Let us make a comment regarding a binomial scheme that is widely used as a simplified model representing an option valuation. The main drawback of the binomial benchmark pricing is that this method prescribes the same option price that does not depend on real world states probabilities. To illustrate this inaccuracy let us consider an example. Lets say the asset price today is \$2, strike price \$3, and at the expiration date tomorrow the asset price is either \$5 or \$1. Then the binomial scheme represents the constant call option price that is independent upon distribution. Thus, the option price remains the same, constant, whether the probability of the event {asset price = \$5} is 0.0001 or 0.9999. One can see that such a distribution with a high reliability can be interpreted as deterministic. In the first case the call option payoff is 0 and therefore option price must be 0. In the second case the positive payoff suggests positive call option price. This remark should suggest a critical revision of the modern derivatives pricing theory.

We begin with a new framework of the option pricing. The pricing problem implies the answer to the question what price should an investor pay for the option contract today given a particular payoff at the expiration?

Let an asset price today be $S(t) =$ \$29 and strike K = \$29. If the price at maturity is $S(T) = Q >$ \$29, then what is the call option price today? Note that for a particular value Q at maturity there is a unique value of the option that represents an equal rate of return on stock and the option. Other possible pricing will lead us to an estimate of the option price that will be introduced bellow.

Consider the table in which the first line represents specified asset values at maturity. The values are chosen arbitrary only to illustrate the method used for calculations. The second line of the table shows the values of the call option payoff at date T, and the third line represents current option price (premium). We use the method that suggests the same rate of return on asset and the specified valuable option

$$\frac{C(T, S(T))}{C(t, x)} = \frac{S(T)}{x}\chi(S(T) > K) \qquad (2.2)$$

Table 2.1 $x = S(t) =$ \$29, K = \$29

S (T)	28	28.5	29	29.5	30	30.5
C (T, S (T))	0	0	0	0.5	1	1.5
C (t , x)	0	0	0	0.49	0.97	1.43

Though these calculations are simple algebra the only problem that makes a difference between market participants is the distribution that an investor prescribes to the values of the underlying asset. Recall that the widely popular assumption in continuous time investment sciences is that the stock price is log-normal. The accurate investigation of this assumption does not affect the problems that we are studying in this paper. Recall

that the log-normal assumption has been commonly applied without statistically testing it. Therefore the corresponding models and parameter estimates are implied. This approach is quite popular in the investment sciences.
We begin with the option valuation. Applying (2.2), it follows that

$$C(T,S(T)) = \frac{x}{S(T;t,x)} \max\{ S(T;t,x) - K,\ 0 \}$$

$$P(T,S(T)) = \frac{x}{S(T;t,x)} \max\{ K - S(T;t,x),\ 0 \} \tag{2.3}$$

We see that the option price is a random function. It is positive for the event when payoff of the option is positive and equal to 0 when payoff is 0. When option price is positive then the rate of return on an underlying asset and its option is the same. The possibility that the option's payoff may be equal to 0 represents the fact that the option is a more risky financial instrument than the underlying security.

Let us consider now of a standard (plain vanilla) credit option contract. This contract's underlying are the exchange rates between two currencies. Though the setting of the problem is quite similar to the above constructions, some peculiarities need to be specified. Let K be a strike price measured in \$ / £ and q (t) denotes the exchange rate at time t measured in the same units. That is £1 = \$ q (t) and therefore a £1 can be interpreted as an asset that can be sold or bought on the \$-market. All contracts are settled by delivery of the underlying currency. By definition, the contract payoff at maturity T is N max { Q (T) – K , 0 }, where N denotes a contract's size. For instance, the size of a British pound call option contract traded on PLHX is N = £ 31,250. Equation (2.2) now can be rewritten in the form of:

$$\frac{C(T,\ q(T))}{C(t,\ q(t))} = \frac{q(T)}{q(t)} \chi\{\ q(T) > K\ \} \tag{2.4}$$

Then the \$-value of the call option contract at date t is

$$C(t,\ q(t)) = \$N \frac{q(t)}{q(T)} \max\{\ q(T) - K\ ,\ 0\ \} \tag{2.5}$$

Formula (2.5) holds regardless; whether the currency exchange rate is stochastic or deterministic. For instance, let N = £ 31,250, K = \$/ £ 1.50,
q (T) = \$/ £1.55. Then the payoff at maturity T is equal to

£ 31,250 \$/ £ (1.55 – 1.5) = \$ 1,562.5

Now, let us consider the two periods in the currency exchange market where
q (t) = K = \$/ £1.50 and

$$q_u = \$1.55, \qquad p_u = 0.25$$

$$q(T) = \{ \quad q_d = \$1.48 \quad p_d = 0.75$$

The problem is to determine the call option price at the initial date t. For simplicity we define the option price related to the size N = £100 and this value could be easy replaced by an actual contract size to represent the real option's value. Applying formula (2.5) it follows;

$$C(t, q(t)) = \begin{cases} \$4.84 & p_u = 0.25 \\ 0 & p_d = 0.75 \end{cases}$$

The average and standard deviation of the option price are 1.21, and 1.815 respectively. The return on exchange rate is a random variable with a given distribution

$$q(T) / q(t) = \begin{cases} 1.0333, & p_u = 0.25 \\ 0.9867, & p_d = 0.75 \end{cases}$$

for which the mean and the standard deviation are 0.9983 and 0.309 respectively. The return on a call option is equal to 1.0333 with a probability of 0.25, and 0 with a probability of 0.75. Note that the positive value of the option return coincides with the correspondent value of the return on an underlying rate of exchange. There might be a speculator who wishes to make a profit, agrees to receive 1.02% return on his investment. Then the only event that suggests this return is the event associated with the future rate q_u. Therefore the maximum price that the investor might pay for the option is \$4.90 = = $(q_u - K) / 1.02$. The probability of the event is 0.25 that may not be enough to accept such a deal. Indeed the average rate of return would be only $1.02 \times 0.25 = 0.255$. If the investor accepts 2% premium on the expected return then the upper bound of the option price is $0.25 \times 4.90 = \$1.225$. This is one simple illustration of the discrete method of the option valuation. On the other hand an investor may ask a reasonable price for the option. Ignoring real additional expenses the investor can consider the option price (\$/ £)Ξ that suggests equal expected return on an option and underlying exchange rates. This setting leads to the equation

$$p_u \times [(q_u - K) / \Xi] = p_u \times [q_u / q(t)] + p_d \times [q_d / q(t)]$$

The solution of the equation is Ξ = (\$/ £) 0.1269.

We now apply this method to a more complex problem that also involves an intermediate moment of time with more than 2 states at expiration. Assume that the value of 100 British pounds over three dates 0,1,2 are given by

Table 2.2

t = 0	t = 1	t = 2
		q(2) = 186 p(185, 186) = 1/4
	q(1) = 185, p(180, 185) = 2/3	q(2) = 182 p(178, 182) = 1/8
q(0) = 180		q(2) = 181 p(178, 181) = 1/4
	q(1) = 178, p(180, 178) = 1/3	q(2) = 179 p(185, 179) = 3/4
		q(2) = 176 p(178, 176) = 5/8

where p (a, b) represents transition probability from the state ‘a’ to state ‘b’. Assume that all transitions are mutually independent.
Let us consider European call option with the strike price K = ($/ £)180. We begin option price construction moving backward in time. Let us first consider the span period [1, 2]. Applying the method we used above it is easy to see that

$$C(1, 185) = \begin{cases} 5.968, & p(180, 185, 186) = 1/6 \\ 0, & p(180, 185, 179) = 1/2 \end{cases}$$

and

$$C(1, 178) = \begin{cases} 1.956, & p(180, 178, 182) = 1/24 \\ 0.983, & p(180, 178, 181) = 1/12 \\ 0, & p(180, 178, 176) = 5/24 \end{cases}$$

Then

$$C(0, 180) = \begin{cases} 5.807, & p(180, 185, 186) = 1/6 \\ 1.978, & p(180, 178, 182) = 1/24 \\ 0.994, & p(180, 178, 181) = 1/12 \\ 0, & p(\{180, 185, 179\} \cup \{180, 178, 176\}) = \\ & = 2/3 \times 3/4 + 1/3 \times 5/8 = 17/24 \end{cases}$$

Here we denote the probability of the path {q (0) = a, q (1) = b, q (2) = c } as p (a, b, c) and {a} ∪ {b} the union of two events ‘a’ and ‘b’. We summarize the results of these calculations with the help of Table 2.3

Table 2.3

C(0, 180)	**C (1, ω)**	**C (2, ω)**	**p (ω)**
5.807	5.968	6	1/6
1.978	1.956	2	1/24
0.994	0.983	1	1/12
0	0	0	17/24

The probabilities in the fourth column are related to the events in each cell in the row.
Now let us investigate feasible investor’s pricing strategy. The average return on the exchange rate over [0, 1] is E q (1) / q (0) = 1.015. The investor might be interested in

finding the option price that suggests the average return of 1.015. This expected price is the solution of the equation

$$E\,C(1, \omega) / 1.015 = \$1.116.$$

Then if the investor pays \$1.116 for the option, then the risk of this position is the probability

$$P\{C(1, \omega) < 1.116\} = P[\{C(1, \omega) = 0.983\} \cup \{C(1, \omega) = 0\}] = 1/12 + 17/24 = 19/24$$

This high risk results from nonsymmetrical distribution of the stochastic exchange rate. Note that in this example one can reach an arbitrary high average return on an option by choosing the option price that is sufficiently small. But the risk of any price will not be less than 17/24. We can use the data provided by Table 2.2; to determine put option price. Let us consider an European put option with the strike price of K = 182. Repeating the previous steps used for the call option valuation we arrive at

$$P(1, 185) = \begin{cases} 0, & p(180, 185, 186) = 1/6 \\ (185/179) \times 3 = 3.1001, & p(180, 185, 179) = 1/2 \end{cases}$$

and

$$P(1, 178) = \begin{cases} 0, & p(180, 178, 182) = 1/24 \\ (178/181) \times 1 = 0.9834, & p(180, 178, 181) = 1/12 \\ (178/181) \times 6 = 6.0682, & p(180, 178, 176) = 5/24 \end{cases}$$

Then

$$P(0, 180) = \begin{cases} 6.1364, & p(180, 178, 176) = 5/24 \\ 3.0168, & p(180, 185, 179) = 1/2 \\ 0.9945 & p\{180, 178, 181\}) = 1/12 \\ 0, & p\{180, 185, 186\} \cup \{180, 178, 182\}) = 5/24 \end{cases}$$

The mean and the standard deviation of the put premium are 2.8697, 2.0598 correspondingly. Let the investor pay \$1 premium for the put option. Then the risk to receive less funds than invested at expiration is 7/24. This risk is associated with the events $\{q(2) = 186, 182, \text{ or } 181\}$. If an investor decides to pay \$4 then the risk is $P\{q(2) = 186, 182, 181, 179\} = 19/24$.

Now let us consider the nonstandard derivative contracts. Exotic option is generic name that refers to variations of the basic options. Options are referred to as being path-independent if their payoff does not depend on the path during the life of the option. First let us examine some exotic option contracts.

Cash-or- nothing call or put options are defined by their payoff at maturity as

$$C_{cn}(T, q(T)) = X\ \chi\{q(T) > K\} \tag{2.6}$$

$$P_{cn}\,(T, q(T)) = X\ \chi\{q(T) < K\}$$

where X is a predetermined constant and q (t) is interpreted as the spot exchange rate in dollars per unit of foreign currency at time t, t $\leq$ T. Note, that in contrast to the continuous payoff of the standard option (2.1) the cash-or-nothing options have discontinuous payoff. In contemporary financial books the cash-or nothing options are also known as digital or binary options. In this case the constant X usually assumed to be equal to 1. The valuation of options contracts follows the formula

$$C_{cn}(t, q(t)) = \$(t)\ N\ \frac{q(t)}{q(T)}\ X\ \chi\{\ q(T) > K\ \}$$

$$P_{cn}(t, q(t)) = \$(t)\ \ N\ \frac{q(t)}{q(T)}\ X\ \chi\{\ q(T) < K\ \} \tag{2.7}$$

where N is the contract size expressed in foreign currency, K is the strike price, q (T) is the currency exchange rate at date T. Let us use a numeric example. Assume that the underlying security data is given by the Table 2.2, N = X = 1. Then using the same algebra one arrives at the table

Table 2.4

C_{cn} (0, 180)	C_{cn} (1, ω)	C_{cn} (2, ω)	p (ω)
0.9677	0.9946	1	1/6
0.989	0.978	1	1/24
0.9945	0.9834	1	1/12
0	0	0	17/24

Each row in the Tables 2.3 or 2.4 is the path of call option for some fixed elementary event ω_0 = { q (0, ω_0), q (1, ω_0), q (2, ω_0) }, and therefore for the fixed ω_0 the option's rates of return coincide with the correspondent rates of return of the underlying exchange rate. This remark is valid for the other exotic options of the European type.

Assets-or- nothing call or put option's payoff at maturity is defined as

$$C_{an}\,(T, q(T)) = q(T)\ \chi\{q(T) > K\}$$

$$P_{an}\,(T, q(T)) = q(T)\ \chi\{q(T) < K\}$$

The pricing formulas can be found from the equation (2.4). Then

$$C_{an}\,(t, q(t)) = q(t)\ \chi\{q(T) > K\}$$

$$P_{an}\,(t, q(t)) = q(t)\ \chi\{q(T) < K\}$$

Gap options are those contacts for which call payoff is defined to be

$$C_g(T, q(T)) = (q(T) - R)\ \chi\{q(T) > K\}$$

where $K > R$. The value of the contracts can be represented by the (2.7) where $X = q(T) - R$. If $K < R$ then payoff can be represented as

$$(q(T) - R)\ \chi\{q(T) > K\} = (q(T) - R)\ \chi\{K < q(T) \leq R\} +$$

$$+ (q(T) - R)\ \chi\{q(T) > R\}$$

Note that if the buyer of the Gap call option still has the right but not an obligation to exercise option, it is clear that for a particular exchange rate such that $K < q(T) \leq R$ the investor will not exercise it. Indeed, why would the buyer pay $(K + R) in order to receive less then invested? Thus the payoff amount is actually reduced to the standard option

$$(q(T) - R)\ \chi\{q(T) > K\} = (q(T) - R)\ \chi\{q(T) > R\}$$

The gap-put payoff is

$$P_g(T, q(T)) = (R - q(T))\ \chi\{q(T) < K\}$$

where $K < R$. Then the gap-put price can be performed by the formula (2.7) where $X = R - q(T)$.

Paylater options are defined by their payoff as follows

$$\begin{aligned} C_{pl}(T, q(T)) &= (q(T) - K - C_{pl}(t, q(t)))\ \chi\{q(T) > K\} \\ P_{pl}(T, q(T)) &= (K - q(T) - P_{pl}(t, q(t)))\ \chi\{q(T) < K\} \end{aligned} \qquad (2.8)$$

where $C_{pl}(t, q(t))$, $P_{pl}(t, q(t))$ are premiums to the options specified at date t and paid only on the exercise of the options. These are up-front payments paid at date t. One might think that paylater payoff can be negative. We show that it is impossible under the interpretation of the option price that is introduced in this paper. To produce the valuation of the problem let us look at benchmark formula (2.5). The solution to this equation is

$$C_{pl}(t, q(t)) = \frac{q(t)}{q(T)} C_{pl}(T, q(T))$$

The put problem solution can be represented in the similar form. Bearing in mind that the payoff of the paylater option depends on its premium, we arrive at the equation

$$C_{pl}(t, q(t)) = \frac{q(t)}{q(T)} [q(T) - K - C_{pl}(t, q(t))]\ \chi\{q(T) > K\} \qquad (2.9)$$

Solving the equation for C_{pl} (t , q (T)) we get the call paylater option solution

$$C_{pl}(t, q(t)) = \frac{q(t)}{q(t)\chi\{q(T) > K\} + q(T)} [q(T) -$$

$$- K]\chi\{q(T) > K\} = \frac{q(t)}{q(t) + q(T)} [q(T) - K]\chi\{q(T) > K\} \tag{2.10'}$$

Similarly,

$$P_{pl}(t, q(t)) = \frac{q(t)}{q(t) + q(T)} [K - q(T)]\chi\{q(T) > K\} \tag{2.10''}$$

The solution of the paylater option problem in Black-Scholes' setting can be found for example in [7]. Their approach to the solution construction is different, therefore we make some remark to their design. In [7] the payoff of the call option was decomposed into sum of two terms

$$[q(T) - K] \chi(q(T) > K) - X_c \chi(q(T) > K)$$

where X_c is an unknown constant. The first term then was interpreted as the ordinary option payoff. Though the value X_c was considered as a constant, the second term was interpreted as the binary option payoff with the option premium X_c. Bearing in mind this interpretation the value $X_c + K$ was interpreted as the paylater option premium. It is easy to see that the correctness of this decomposition is doubtful as far as X_c relates to paylater option and not to the cash-or-nothing option. Furthermore, X_c is not a constant it by payoff definition is an unknown function in t.
In the Table 2.5 we enclose the valuation of the paylater call option, when underlying is the value of foreign currency unit. Its value is given in Table 2.2.

Table 2.5

C_{pl} (0, 180)	**C_{pl} (1, ω)**	**C_{pl} (2, ω)**	**p (ω)**
0.2.911	2.9919	3.089	1/6
0.9944	0.9889	1.0056	1/24
0.4978	0.4958	0.5022	1/12
0	0	0	17/24

Indeed, applying the formula (2.10′) we see that

$$C(1, 185) = \begin{cases} [185 / (185 + 186)] (186 - 180) = 2.9919, & p = 1/4 \times 2/3 = 1/6 \\ 0, & p = 3/4 \times 2/3 = 1/2, \end{cases}$$

$$C(1, 178) = \begin{cases} [178/(178+182)](182-180) = 0.9889, & p = 1/8 \times 1/3 = 1/24 \\ [178/(178+181)](181-180) = 0.4958, & p = 1/4 \times 1/3 = 1/12 \\ 0, & p = 5/8 \times 1/3 = 5/24, \end{cases}$$

$$C(1, q(1)) = \begin{cases} C(1, 185) \\ \\ C(1, 178) \end{cases} = \begin{cases} 2.9919, & p = 1/6 \\ 0.9889, & p = 1/24 \\ 0.4958, & p = 1/12 \\ 0, & p = 17/24, \end{cases}$$

$$C(0, q(0)) = \begin{cases} 2.911, & p = 1/6 \\ 0.9944, & p = 1/24 \\ 0.4978, & p = 1/12 \\ 0, & p = 17/24, \end{cases}$$

$$C(2, q(2)) = [q(2) - K - C(0, 180)]\,\chi\{q(2) > K\} = \begin{cases} 186 - 180 - 2.911 = 3.089 \\ 182 - 180 - 0.9944 = 1.0056 \\ 181 - 180 - 0.4978 = 0.5022 \\ 0 \end{cases}$$

Note, that for the simplicity we omitted index 'pl' that specifies paylater option. One might see that the risk characteristics of the paylater call option as well as other exotics call option with the same strike price that have been introduced above, coincide with the correspondent risk characteristics of the standard European option with the same strike price. All these options offered the same return even though their premiums and payoffs are different.

<u>A collar contract</u> payoff at maturity T is defined as

$$I(T) = \min\{\max\{q(T), K_1\}, K_2\}.$$

Note, that this payoff can be rewritten in the form

$$I(T) = K_1\,\chi\{q(T) \le K_1\} + q(T)\,\chi\{q(T) \le (K_1, K_2]\} + + K_2\,\chi\{q(T) > K_2\} \qquad (2.11)$$

Below, we introduce the standard arguments that perform the valuation idea. Using an identity

$$\chi\{q(T) \le K\} = 1 - \chi\{q(T) > K\}$$

one can see that

$$\chi\{q(T) \in (K_1, K_2]\} = \chi\{q(T) > K_1\} - \chi\{q(T) > K_2\}$$

Therefore,

$$I(T) = K_1 - K_1 \chi\{ q(T) > K_1 \} + q(T) \chi\{ q(T) > K_1 \} -$$

$$- q(T) \chi\{ q(T) > K_2 \} + K_2 \chi\{ q(T) > K_2 \} = K_1 +$$

$$+ [q(T) - K_1] \chi\{ q(T) > K_1 \} - [q(T) - K_2] \chi\{ q(T) > K_2 \}$$

The right hand side of this equality is equal to a portfolio holding $\$K_1$ cash, long European call with the strike price K_1 , and short European call with the strike price K_2. This decomposition of the $I(T)$ is not a unique representation. Indeed, it can be easily checked that

$$I(T) = K_1 + K_2 - q(T) + [q(T) - K_1] \chi\{ q(T) > K_1 \} -$$

$$- [K_2 - q(T)] \chi\{ q(T) < K_2 \}$$

Thus, collar payoff is equivalent now to the value of the portfolio that contains $\$(K_1 + K_2)$ cash, short stock, long European call, and short European put. The price of a collar contract at any time prior to the expiration coincides with the price of the portfolio. We introduce the direct evaluation of the collar contract. From formula (2.11) it follows that collar payoff is a collection of three hypothetical financial instruments with payoffs

$$I_1(T) = K_1 \chi\{ q(T) \le K_1 \}$$

$$I_2(T) = q(T) \chi\{ q(T) \in (K_1 , K_2] \}$$

$$I_3(T) = K_2 \chi\{ q(T) > K_2 \}$$

maturity T. Then the collar contract price at date t is $I(t) = I_1(t) + I_2(t) + I_3(t)$, where

$$I_1(t) = \frac{K_1 \, q(t)}{q(T)} \chi\{ q(T) \le K_1 \}$$

$$I_2(t) = q(t) \chi\{ q(T) \in (K_1 , K_2] \}$$

$$I_3(t) = \frac{K_2 \, q(t)}{q(T)} \chi\{ q(T) > K_2 \}$$

A chooser or **as-you-like option** is the next exotic option type.
A holder of this option can choose whether the option is a call or a put after a specified period of time. Consider a chooser option that matures at moment T_{ch} , the maturity of

the underlying call and put denote T_c , T_p respectively. Here $\min(T_c , T_p) > T_{ch}$. Thus, the values of underlying call and put at date T_{ch} are $C(T_{ch}, q(T_{ch}); T_c, K_c)$, $P(T_{ch}, q(T_{ch}); T_p, K_p)$; where $q(t)$ is the call and a put deliverable security, and K with subscript c and p as correspondent strike prices. To start evaluation of the chooser option it is necessary to establish payoff to the option. The payoff to the chooser option is

$$co(T_{ch}, q(T_{ch})) = \max\{ C(T_{ch}, q(T_{ch}); T_c, K_c),\ P(T_{ch}, q(T_{ch}); T_p, K_p) \}$$

Then

$$co(T_{ch}, q(T_{ch})) = C(T_{ch}, q(T_{ch}); T_c, K_c) \times$$

$$\times\ \chi\{ C(T_{ch}, q(T_{ch}); T_c, K_c) \geq P(T_{ch}, q(T_{ch}); T_p, K_p) \} + P(T_{ch}, q(T_{ch}); T_p, K_p) \} \times$$

$$\times\ \chi\{ C(T_{ch}, q(T_{ch}); T_c, K_c) < P(T_{ch}, q(T_{ch}); T_p, K_p) \}$$

Note that:

$$C(T_{ch}, q(T_{ch}); T_c, K_c))\, \chi\{ C(T_{ch}, q(T_{ch}); T_c, K_c) \geq P(T_{ch}, q(T_{ch}); T_p, K_p) \} =$$

$$= C(T_{ch}, q(T_{ch}); T_c, K_c))\, \chi\{ q(T_c) > K_c \},$$

$$P(T_{ch}, q(T_{ch}); T_p, K_p))\, \chi\{ C(T_{ch}, q(T_{ch}); T_c, K_c) < P(T_{ch}, q(T_{ch}); T_p, K_p) \} =$$

$$= P(T_{ch}, q(T_{ch}); T_p, K_p))\, \chi\{ q(T_p) < K_p, q(T_c) < K_c \}$$

Therefore,

$$\frac{co(T_{ch}, q(T_{ch}))}{co(t, q(t))} = \frac{C(T_{ch}, q(T_{ch}))}{co(t, q(t))}\, \chi\{ q(T_c) > K_c \} +$$

$$+\ \frac{P(T_{ch}, q(T_{ch}))}{co(t, q(t))}\, \chi\{ q(T_c) < K_c \}, q(T_p) < K_p) \} = \qquad (2.12)$$

$$= \frac{q(T_{ch})}{q(t)} [\, \chi\{ q(T_c) > K_c \} + \chi\{ q(T_c) < K_c \}, q(T_p) < K_p) \}]$$

The solution of the equation (2.12) can be written in the form

$$co(t) = \frac{q(t)}{q(T_{ch})} [\, C(T_{ch}, q(T_{ch}); T_c, K_c)\, \chi \{ q(T_c) > K_c \} +$$

$$+ \; P(T_{ch}, q(T_{ch}); T_p, K_p)\, \chi \{ q(T_c) \le K_c ,\, q(T_p) < K_p \}]\} \; =$$

$$= \frac{q(t)}{q(T_c)} [\, q(T_c) - K_c \,]\, \chi \{ q(T_c) > K_c \} +$$

$$+ \frac{q(t)}{q(T_p)} [\, K_c - q(T_c) \,]\, \chi \{ q(T_c) \le K_c ,\, q(T_p) < K_p \}$$

An interesting point is that a chooser option has some peculiarities that distinguish it from other derivatives. The payoff of the chooser option does not specify the strike price and therefore this contract can not be viewed as call or put options. More realistically this type of derivatives could be named as a forward-choice option contract. That is why it may be reasonable to perform its valuation.

Let the underlying asset be the pound value described in Table 2.2. Let a call option have the strike price K_c = \$180 and expiration at $T_c = 2$. The values of the call option is given in Table 2.3, and the put option with the strike price K_p = \$183 expired at $T_p = 1$. Assume that the chooser option expiration date $T_{ch} = T_p = 1$. Then the payoff of the chooser option at date 1 is $co(1) = \max \{ C(1, q(1); 2, 180),\; P(1, q(1); 1, 183) \}$ and equals to

$$co(1) = \begin{cases} 5, & P\{ q(1) = 178 \} = 1/3 \\ 2.9919, & P\{ q(1) = 185,\; q(2) = 186 \} = 1/6 \\ 0, & P\{ q(1) = 185,\; q(2) = 179 \} = 1/2 \end{cases}$$

Then the values of the chooser option at date 0 could be found by solving the equation

$$co(0) = \frac{q(0)}{q(1)}\, co(1)$$

or

$$co(0) = \begin{cases} 5.0562, & \text{with probability } 1/3 \\ 2.911, & \text{with probability } 1/6 \\ 0, & \text{with probability } 1/2. \end{cases}$$

A **cliquet** or **ratchet** option is a series of at the money options, with periodic settlement, resetting the strike price at the current price level, at which the option locks in the difference between the old and new strike prices and pays that difference out as the profit. This profit might be paid out at each reset date or could be accumulated until maturity. Thus a cliquet option can be thought as a series of options that settles periodically and resets the strike price at the spot level. Let us introduce a n-years cliquet option with k-resets annually. Let $t_{j,i}$ be reset moments of time, j = 0, 1, … , n ; I = 0, 1, …k – 1 and T a maturity. The amount that will accumulate at maturity over the time interval [t , T] is

$$\sum_{j=0}^{n} \sum_{i=0}^{k-1} \max\{ q(t_{j,i+1}) - q(t_{j,i}), 0 \}$$

Here $t = t_o$. This formula covers the case when an option writer pays out reset differences periodically. Let us denote the underlying of the cliquet option q (s) = = q (s ; t , x), s $\geq$ t and C (t , x ; T , K) the value of the European call option at date t with strike price K and expiration date T. Then,

$$\frac{\max\{ q(t_{j,i+1}) - q(t_{j,i}), 0 \}}{C(t_{j,i}, q(t_{j,i}); t_{j,i+1}, q(t_{j,i}))} = \frac{q(t_{j,i+1})}{q(t_{j,i})} \chi\{ q(t_{j,i+1}) > q(t_{j,i}) \}$$

Therefore the pricing equation for the cliquet option call can be taking in the form

$$\frac{\sum_{j=0}^{n} \sum_{i=0}^{k-1} \max\{ q(t_{j,\, i+1}) - q(t_{j,\, i}), 0 \}}{C_q(t, x)} =$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{k-1} \chi\{ q(t_{j,\, i+1}; t, x) > q(t_{j,\, i}; t, x) \} \frac{q(t_{j,\, i+1}; t, x)}{q(t_{j,\, i}; t, x)}$$

Then the solution to this equation is

$$C_q(t, x) = \frac{\sum_{j=0}^{n} \sum_{i=0}^{k-1} \max\{ q(t_{j,\, i+1}; t, x) - q(t_{j,\, i}; t, x), 0 \}}{\sum_{j=0}^{n} \sum_{i=0}^{k-1} \dfrac{q(t_{j,\, i+1}; t, x)}{q(t_{j,\, i}; t, x)}}$$

Let us represent the value of the cliquet call option based on the information in Table 2.2. The cliquet call option is a random variable, which can be calculated using the above formula. For example the value of the option along the path ω = {180, 185, 186} is equal to [(185 – 180) + (186 – 185)] / [(185/180) + (186 / 185)] = \$2.9494. The probability of such event is p (ω) = 1/6. Next option value is equal to [185 – 180] / [(185 / 180)]

= \$4.8596 and has been realized along the path $\omega = \{180, 185, 179\}$. Using the same type of calculations we arrive at

$$
C_q(0, 180) = \begin{cases} 2.9494, & p(\{180, 185, 186\}) = 1/6 \\ 4.8596, & p(\{180, 185, 179\}) = 1/6 \\ 3.9121, & p(\{180, 178, 182\}) = 1/24 \\ 2.9503, & p(\{180, 178, 181\}) = 1/12 \\ 0, & p(\{180, 178, 176\}) = 5/24 \end{cases}
$$

It is possible to introduce the cliquet option put by defining correspondent stream payments at reset dates. Then payoff at maturity can be expressed by

$$
\sum_{j=0}^{n} \sum_{i=0}^{k-1} \max\{\ q(t_{j,\,i};\,t,\,x) - q(t_{j,\,i+1};\,t,\,x)\ ,\,0\,\}
$$

The value of the cliquet put option at date t is

$$
P_q(t,\,x) = \frac{\displaystyle\sum_{j=0}^{n} \sum_{i=0}^{k-1} \max\{\ q(t_{j,\,i};\,t,\,x) - q(t_{j,\,i+1};\,t,\,x)\ ,\,0\,\}}{\displaystyle\sum_{j=0}^{n} \sum_{i=0}^{k-1} \frac{q(t_{j,\,i+1};\,t,\,x)}{q(t_{j,\,i};\,t,\,x)}}
$$

The price formulas for the call and put options assume that the payoffs to the holder take place immediately at the reset dates. The value of the cliquet put option with the underlying is given by Table 2.2 is

$$
P_q(0, 180) = \begin{cases} 0, & p(\{180, 185, 186\}) = 1/6 \\ 5.8054, & p(\{180, 185, 179\}) = 1/6 \\ 2.0225, & p(\{180, 178, 182\}) = 1/24 \\ 2.0225, & p(\{180, 178, 181\}) = 1/12 \\ 1.9777, & p(\{180, 178, 176\}) = 5/24 \end{cases}
$$

Couple options are almost similar to the cliquet options. Cliquet option payout to the holder could take place either at prespecified reset dates or at maturity. The only distinction between couple and cliquet is that couple options at reset dates switch its value to the smaller of the current spot level and the initial strike price. If payments are delivered to the holder at reset dates then the cash flow generated by the couple call option is

$$
\sum_{j=0}^{N-1} \max\{\ q(t_{j+1}) \ -\ \min[\ q(t_j),\ K\ ]\ ,\,0\,\}
$$

Here, $t = t_0 < t_1 < \ldots < t_N = T$ are reset dates, $\min[\,q(t_j), K\,]$ is reset strike price. The price of the call and put couple options is

$$C_{cp}(t, x) = \frac{\sum_{j=0}^{N-1} \max\{\, q(t_{j+1}; t, x) - \min[\, q(t_j; t, x),\, K\,],\, 0 \,\}}{\sum_{j=0}^{N-1} \frac{q(t_{j+1}; t, x)}{q(t_j; t, x)}}$$

$$P_{cp}(t, x) = \frac{\sum_{j=0}^{N-1} \max\{\, \max[\, q(t_j; t, x),\, K\,] - q(t_{j+1}; t, x),\, 0 \,\}}{\sum_{j=0}^{N-1} \frac{q(t_{j+1}; t, x)}{q(t_j; t, x)}}$$

Based on information in Table 2.2 we see that the value of the couple call option K = $182 can be written as follows:

Table 2.6

C_{cp} (0, 180)	ω	p (ω)
4.4241	{ 180, 185, 186 }	1/6
4.8596	{180, 185, 179}	1/2
3.9121	{180, 178, 182}	1/24
2.9503	{180, 178, 181}	1/12
0	{180, 178, 176}	5/24

Only the first value is larger than the correspondent value of the cliquet option. Other values of the couple and cliquet options remain the same.

A **ladder** option payoff is somewhat similar to a cliquet option with an exception that the gains are locked in when the asset price breaks through a certain predetermined rung. The strike price is then intermittently reset. The ladder option is also known as a ratchet or lock-in option.

Let us consider the ladder option on exchange rates q (s) , s $\geq$ t. An investor buys the ladder option with a strike price $Q = Q_o$. Thus, a ladder start with the height Q and goes upwards in the step interval of $\varepsilon > 0$ until the maximum rung of Q_N, $Q_j = Q + j\varepsilon$, j = 0, 1, … , N. At maturity T the buyer of the ladder call would receive a payoff

$$\sum_{j=0}^{N-1} \max \{ Q_j - Q \ , \ q(T) - Q \} \chi \{ Q_j < \max_{t \le t \le T} q(t) \le Q_{j+1} \} +$$

$$+ [q(T) - Q] \chi \{ \max_{t \le t \le T} q(t) > Q_N \}$$

We see that this payoff reflects the possibility to maximum value of the underlying price over lifetime of the option. In order to construct a solution of the ladder call option we used an equation

$$\frac{callpayoff\,(T)}{C_{lad}\,(t, q(t))} = \max_{t \le l \le T} \frac{q(l)}{q(t)} \chi \{ \max_{\le l \le T} q(l) > Q \}$$

and therefore,

$$C_{lad}\,(t, q(t)) = \max_{t \le t \le T} \frac{q(t)}{q(l)} \{ [q(T) - Q] \chi \{ \max_{t \le t \le T} q(t) > Q_N \} +$$

$$+ \sum_{n=0}^{N-1} \max \{ Q_n - Q \ , \ q(T) - Q \} \chi \{ Q_n < \max_{t \le t \le T} q(t) \le Q_{n+1} \} \}$$

Let us consider valuation of the ladder call option based on exchange rate data given in Table 2.2. Putting Q = \$1.78, Q_1 = \$1.80, Q_2 = \$182, Q_3 = \$1.84 we see for example that

C_{lad} (0, 180; ω_1) = [180 / 185] × (186 – 178) ×

× χ { max [180, 185, 186] ≥ 184 } = 7.7838, ω_1 = { 180, 185, 186 }

C_{lad} (0, 180; ω_2) = [180 / 176] × max [182 – 178 , 176 – 178] ×

× χ { 178 ≤ max [180, 180, 179] < 184 } = 1.0056, ω_2 = { 180, 178, 176 }.

This type of calculation lead us to the representation of the option price as follows:

Table 2.7

C_{cp} (0, 180)	ω	p (ω)
7.7838	{ 180, 185, 186 }	1/6
1.0056	{180, 185, 179}	1/2
4.0449	{180, 178, 182}	1/24
4.0449	{180, 178, 181}	1/12
4.0909	{180, 178, 176}	5/24

Note for example that other modification of the ladder option can be introduced by performing another payoff

$$\sum_{n=0}^{N-1} (Q_n - Q)\chi\{ Q_n < \max_{t \le t \le T} q(t) \le Q_{n+1} \} + (Q_N - Q)\chi\{ \max_{t \le t \le T} q(t) > Q_N \}$$

In this particular case, the payoff is really similar to the ladder and assumes a finite number of values 0, Q_1 - Q, … , Q_N – Q with the probabilities

Values	0	Q_1 - Q	Q_2 - Q	Q_N - Q
Probabilities	P_0	P_1	P_2	P_N

where $P_j = P\{ Q_j < \max q(t) \le Q_{j+1} \}$, j = 0, 1, … , N - 1, $P_N = P\{ \max q(t) \ge Q_N \}$.
The purchaser of the ladder put will receive payoff at maturity of

$$\sum_{j=0}^{K} \max \{ Q - Q_{-j} , Q - q(T) \}\chi \{ Q_{-j+1} < \min_{t \le t \le T} q(t) \le Q_{-j} \} +$$

$$+ [Q - q(T)] \chi \{ \min_{t \le t \le T} q(l) \le Q_{-K} \}$$

Here, $Q_{-k} < Q_{-k+1} < \ldots < Q_{-1} < Q$ is a rung sequence. The pricing equation for the ladder put is

$$\frac{putpayoff(T)}{P_{lad}(t, q(t))} = \min_{t \le l \le T} \frac{q(l)}{q(t)} \chi \{ \min_{\le l \le T} q(l) < Q \}$$

The value of the put ladder option is

$$P_{lad}(t, q(t)) = \max_{t \le l \le T} \frac{q(l)}{q(t)} \{ [Q - q(T)] \chi \{ \min_{t \le l \le T} q(l) \le Q_{-K} \} +$$

$$+ \sum_{j=0}^{K} \max \{ Q - Q_{-j} , Q - q(T) \}\chi \{ Q_{-j+1} < \min_{t \le l \le T} q(l) \le Q_{-j} \} \}$$

Extendible options have become popular over a recent time for a volatile underlying. There are two types of extendible options: holder’s and writer’s extendible. A holder’s extendible option is the option that can be extendible by the holder of the option at maturity T_e. An additional premium is required to do so. The holder of an extendible option on call or put receives at maturity T_e a choice to get an ordinary call option payoff or by paying a predetermine premium \$d to the writer at time T_e , to get the call option with an extended maturity. This means that the payoff at T_e is

$$C_{eh}(T_e, q(T_e)) = \max\{ q(T_e) - Q , C(T_e, q(T_e); K, T) - d, 0 \}$$

$$P_{eh}(T_e, q(T_e)) = \max\{ Q - q(T_e) , P(T_e, q(T_e); K, T) - d, 0 \}$$

Here C (t , x ; K , T) , P (t , x ; K , T) are values of the European call or put options at date t , x = q (t) , K is the strike price, T is the European call and put options expiration date T_e < T, and d is an additional premium paid by the holder for having the opportunity to apply the extendable option's feature. This is somewhat a more complex derivative instrument than introduced above. This complexity is bounded with the fact that a few new factors are involved to the problem. A valuation equation can be represented in the form

$$\frac{\max\{\, q(T_e) - Q\ ,\ C(T_e\,, q(T_e);\, K\,, T) - d\ ,\ 0\,\}}{C_{eh}(t\,,\, x)} =$$

$$= \frac{q(T_e\,;\, t\,,\, x)}{x}\ \chi\,[\{q(T_e\,;\, t\,,\, x) > Q\} \cup \{C(T_e\,, q(T_e);\, K\,, T) > d\ \}]$$

Note that the indicator in the right hand side of the equality contains a union of two events.

$$\chi\,[\{q(T_e\,;\, t\,,\, x) > Q\} \cup \{C(T_e\,, q(T_e);\, K\,, T) > d\ \}] = \chi\,\{q(T_e\,;\, t\,,\, x) > Q\} +$$

$$+ \chi\,\{C(T_e\,, q(T_e);\, K\,, T) > d\ \} - \chi\,\{q(T_e\,;\, t\,,\, x) > Q\}\chi\{C(T_e\,, q(T_e);\, K\,, T) > d\ \}$$

Then,

$$\chi\,\{C(T_e\,, q(T_e);\, K\,, T) > d\ \} = \chi\,\{\ \frac{q(T_e)}{q(T)}\max\,[\ q(T\,;\, T_e\,, q(T_e)) - K\,, 0\,] > d\ \}$$

Taking this into account and solving a call option price equation we arrive at the premium formula

$$C_{eh}(t\,,\, x)\ =\ \frac{\max\{\ q(T_e) - Q\ ,\ C(T_e\,, q(T_e);\, K\,, T) - d\ ,\ 0\,\}}{q(T_e\,;\, t\,,\, 1)}\ \chi\,\{q(T_e\,;\, t\,,\, x) > Q\} +$$

$$+ \chi\,\{C(T_e\,, q(T_e);\, K\,, T) > d\ \} - \chi\,\{q(T_e\,;\, t\,,\, x) > Q\ \}\chi\{C(T_e\,, q(T_e);\, K\,, T) > d\ \} =$$

$$= [\,q(T_e) - Q\,]\chi\,\{q(T_e) > Q\}\chi\ \{q(T_e\,;\, t\,,\, x) - \frac{\max\ \{\ q(T_e\,;\, t\,,\, x)\times q(T\,;\, T_e\,, 1) - K\ , 0\,\}}{q(T\,;\, T_e\,, 1)}\ >$$

$$> Q - d\ \} + [\ C(T_e\,, q(T_e);\, K\,, T) - d\ ]\ \chi\,\{\ \frac{q(T_e)}{q(T)}\max\,[\ q(T\,;\, T_e\,, q(T_e)) - K\,, 0] > d\ \}\times$$

$$\times\,\chi\,\{\ q(T_e\,;\, t\,,\, x) - \ \frac{q(T_e)}{q(T)}\max\,[\ q(T\,;\, T_e\,, q(T_e)) - K\,, 0\,] >\ Q - d\ \}$$

Another reciprocal problem can arise here. Given the option price derive the extended premium d. Actually, this inverse problem is more difficult than the initial one, and is not any less important because this extendible premium is the contract agreement provision. The formula for the holder extendible put option can be performed in the similar way

$$P_{eh}(t, x) = [\,Q - q(T_e)\,]\chi\{\,Q > q(T_e; t, x)\,\}\times$$

$$\times\chi\,\{q(T_e; t, x) + \frac{\max\{\,K - q(T_e; t, x)\times q(T; T_e, 1)\,,\,0\,\}}{S(T; T_e, 1)} < Q + d\,\} +$$

$$+[\,P(T_e, q(T_e); K, T) - d\,]\,\chi\,\{\,\frac{\max\{K - q(T_e; t, x)\times q(T; T_e, 1), 0\}}{S(T; T_e, 1)} > d\,\}\times$$

$$\times\chi\{\,q(T_e; t, x) + \frac{\max\{K - q(T_e; t, x)\times q(T; T_e, 1), 0\}}{q(T; T_e, 1)} > Q + d\,\}$$

Writer extendible option allows the seller of the option to extend the option with zero cost. The writer can extend the option at the maturity T_e if the option is out-of-money. Recall that an option call or put is out-of-money if its value is equal to 0, this option is immediately exercised. Thus the payoffs at maturity T_e of the writer extendible calls and puts respectively are

$$C_{ew}(T_e, q(T_e)) = \min\{\,q(T_e) - Q\,,\; C(T_e, q(T_e); T, K)\,\} =$$

$$= [\,q(T_e) - Q\,]\,\chi\{q(T_e) > Q\} + C(T_e, q(T_e); T, K)\chi\{\,q(T_e) \le Q\,\},$$

$$P_{ew}(T_e, q(T_e)) = \min\{\,Q - q(T_e)\,,\; C(T_e, q(T_e); T, K)\,\} =$$

$$= [\,Q - q(T_e)]\,\chi\{q(T_e) \le Q\} + P(T_e, q(T_e); T, K)\,\chi\{\,q(T_e) > Q\}$$

These bring us to valuation formulas

$$C_{ew}(t, x) = \frac{[\,q(T_e) - Q\,]\,\chi\{q(T_e) > Q\} + C(T_e, q(T_e); T, K)\chi\{\,q(T_e) \le Q\,\}}{S(\,T_e; t, 1\,)}$$

$$P_{ew}(t, x) = \frac{[\,Q - q(T_e)\,]\,\chi\{q(T_e) \le Q\} + P(T_e, q(T_e); T, K)\chi\{\,q(T_e) > Q\,\}}{q(\,T_e; t, 1\,)}$$

where C (t , q (t) ; T , K), P (t , q (t) ; T , K) are European call and put option prices respectively. Note that European underlying options can also be replaced by American options.

<u>The Extreme or Reverse Extreme exotic options</u> were introduced in 1996. These options payoff at maturity T is determined by the difference between the maximum values on compliment subintervals constituted the lifetime of an asset. If $t < T_0 < T$ then the payoffs to the call option at maturity for the extreme and inverse extreme options are

$$\max\{ \max_{t \le v \le T_0} q(v) - \max_{T_0 \le u \le T} q(u),\ 0 \}$$

$$\max\{ \max_{T_0 \le u \le T} q(u) - \max_{t \le v \le T_0} q(v),\ 0 \}$$

The payoffs to the put option at maturity for the extreme and inverse extreme options are

$$\min\{ \min_{t \le v \le T_0} q(v) - \min_{T_0 \le u \le T} q(u),\ 0 \}$$

$$\min\{ \min_{T_0 \le u \le T} q(u) - \min_{t \le v \le T_0} q(v),\ 0 \}$$

We would say that associations of these derivative contracts with a well-known classical interpretation of the call and put options are not very obvious. The pricing formulae to these derivative contracts can be written in the form

$$C_{ex}(t, x) = \frac{x \max\{ \max_{t \le v \le T_0} q(v) - \max_{T_0 \le u \le T} q(u),\ 0 \}}{q(T; t, x)}$$

$$C_{ex}(t, x) = \frac{x \max\{ \max_{T_0 \le u \le T} q(u) - \max_{t \le v \le T_0} q(v),\ 0 \}}{q(T; t, x)}$$

respectively. The put option formulae can be obtained from above formulae by replacing max operations by its min counterpart.

Let us represent the valuation of the extreme call option using the second form. Here $t = 0$, $T_0 = 1$, $T = 2$. Then the payoff to the option on exchange rate given in Table 2.2 is

$$\max\{ \max(185, 186) - \max(180, 185),\ 0 \} = 1,$$
$$\max\{ \max(185, 179) - \max(180, 185),\ 0 \} = 0,$$
$$\max\{ \max(178, 182) - \max(180, 178),\ 0 \} = 2,$$
$$\max\{ \max(178, 181) - \max(180, 178),\ 0 \} = 1,$$
$$\max\{ \max(178, 176) - \max(180, 178),\ 0 \} = 0$$

Though there are no ambiguities in these calculations, it probably makes sense for the discrete schemes to exclude from the payoff differentials of two maximums the common point of time. This will lead us to the different payoff and therefore a different premium. Having payoff values it is easy to present an option premium.

Table 2.8

C_{eh} (0, 180)	ω	p (ω)
0.9677	{ 180, 185, 186 }	1/6
0	{180, 185, 179}	1/2
1.978	{180, 178, 182}	1/24
0.9945	{180, 178, 181}	1/12
0	{180, 178, 176}	5/24

Other path-dependent option class is **Lookback** options. Note that the extreme exotic option introduced above sometimes is considered as a subclass of lookback options, called extrema lookback options. Two primary forms of the options exist based on strike price definition. First form is defined as lookback options with a fixed strike price. The payoffs of the call and put options are

$$\max \left\{ \max_{t \le u \le T} q(u) - K , 0 \right\}$$

$$\max \left\{ K - \min_{t \le u \le T} q(u) , 0 \right\}$$

respectively. Hence, the values of lookback options can be given as:

$$C_{lx}(t,x) = \frac{x \max \left\{ \max_{t \le u \le T} q(u;t,x) - K , 0 \right\}}{q(T;t,x)} \chi\left(\max_{t \le u \le T} q(u;t,x) > K \right)$$

$$P_{lx}(T,x) = \frac{x \max \left\{ K - \min_{t \le u \le T} q(u;t,x) , 0 \right\}}{q(T;t,x)} \chi\left(\min_{t \le u \le T} q(u;t,x) < K \right)$$

The premium on a lookback option on exchange rate given in Table 2.2 is

Table 2.9 K = 180

C_{lx} (0, 180)	P_{lx} (0, 180)	p (ω)	ω
5.8065	0	1/6	{ 180, 185, 186 }
5.0279	1.0056	1/2	{180, 185, 179}
1.978	1.978	1/24	{180, 178, 182}
0.9945	1.989	1/12	{180, 178, 181}
0	4.0909	5/24	{180, 178, 176}

The lookback options with floating strike price were introduced in 1979, these can be settled in cash or asset in contrast with the fixed strike options in which cash settlement is only admitted. The payoff of the lookback call and put options with floating strike price are

$$C_{lf}\ (T, q(T)) = \max\ \{\ q(T) - \min_{t \le u \le T} q(u), 0\ \}$$

$$P_{lf}\ (T, q(T)) = \max\ \{\ \max_{t \le u \le T} q(u) - q(T)\ , 0\ \}$$

An attractive feature of the lookback options with floating strike price is that they are never out-of-the-money. The generalization on American options is straightforward. The formulae representing current options price are

$$C_{lf}\ (t, x) = \frac{x \max\ \{\ q(T; t, x) - \min_{t \le u \le T} q(u; t, x), 0\ \}}{q(T; t, x)}$$

$$P_{lf}\ (t, x) = \frac{x \max\ \{\ \max_{t \le u \le T} q(u; t, x) - q(T; t, x)\ , 0\ \}}{q(T; t, x)}$$

Asian options are quite popular exotic options. Underlying of Asian options is the average price of an asset. One can see that Asian underlying has lower volatility than the asset itself. There are three main subclasses of the Asian options in which underlying are arithmetic, geometric, or weighted averages. A further specification is that the introduced underlying can be used either as a security or as a strike price. Therefore, the payoffs for Asian call options can be represented as

$$C_{as}(T) = \max\{\ \frac{1}{n}\sum_{j=0}^{n} q(t_j) - K\ ,\ 0\ )$$

$$C_{asa}(T) = \max\{\ \ q(T) - \frac{1}{n}\sum_{j=0}^{n} q(t_j)\ ,\ 0\ )$$

and Asian put payoffs for average underlying and average strike price are

$$\max\ \{\ \ K - \frac{1}{n}\sum_{j=0}^{n} q(t_j)\ ,\ 0\ )$$

$$\max\ \{\ \frac{1}{n}\sum_{j=0}^{n} q(t_j) - q(T)\ ,\ 0\ )$$

The American style of the Asian options is also available for trade.
The pricing formulas are

$$C_{as}(t) = \frac{x \max\{ \frac{1}{n} \sum_{j=0}^{n} q(t_j) - K, 0 \}}{q(T; t, x)} \quad ; \quad P_{as}(t) = \frac{x \max\{ K - \frac{1}{n} \sum_{j=0}^{n} q(t_j), 0 \}}{q(T; t, 1)}$$

$$C_{asa}(t) = \frac{x \max\{ q(T) - \frac{1}{n} \sum_{j=0}^{n} q(t_j), 0 \}}{q(T; t, x)} \quad ; \quad P_{asa}(t) = \frac{x \max\{ \frac{1}{n} \sum_{j=0}^{n} q(t_j) - q(T), 0 \}}{q(T; t, x)}$$

where $q(t_j) = q(t_j; t, x)$, $j = 0, 1, \ldots, n$. For the Asian options that involve the geometric or weighted averages to obtain valuation formulae one needs to replace arithmetic average in the above formulae by their geometric or weighted average counterparts. Calculations of the Asian call options based on data given in Table 2.2 result in

Table 2.10

$C_{as}(0)$	$C_{asa}(0)$	p (ω)	ω
3.5484	2.2581	1/6	{ 180, 185, 186 }
1.3408	0	1/2	{180, 185, 179}
0	1.978	1/24	{180, 178, 182}
0	1.326	1/12	{180, 178, 181}
0	0	5/24	{180, 178, 176}

<u>Compound option</u>.

In the first part we introduced an example that illustrates compound option valuation. Here we present general compound option formulas. There exists a class of derivatives in which the underlying securities are options or other types of contingent claims. Let an underlying instrument is an option. This class is called compound or split free options. Possible specifications are a call on a call or a put, and put on a call or a put. Let $C(t, q(t)) = C(t, q(t); T, K)$ be the value of an option at date t with the maturity T and the strike price K written on security q (*). First consider option on call option. We denote the compound call option price at date t with maturity t ″ $T_c < T$ with strike price K_c as

$$C_c(t, q(t)) = C_c(t, q(t); T_c, K_c; T, K)).$$

Then,

$$C_c(T_c, q(T_c); T_c, K_c; T, K)) = \max\{ C_c(T_c, q(T_c); T, K) - K_c, 0 \}$$

Having defined the compound payoff we can provide the valuation of the call on call option. The equal rate of return on compound call and underlying option given that the underlying option price is positive yields the equation

$$\frac{C_c(T_c, q(T_c); T_c, K_c; T, K)}{C_c(t, q(t); T_c, K_c; T, K)} =$$

$$= \frac{C(T_c, q(T_c); T, K)}{C(t, q(t); T, K)} \chi\{C(T_c, q(T_c); T, K) > K_c\} =$$

$$= \frac{q(T_c)}{q(t)} \chi\{q(T) > K\} \chi\{C(T_c, q(T_c); T, K) > K_c\}$$

Substituting corresponding terms in the equation and then solving,

$$C_c(t, q(t); T_c, K_c; T, K) =$$

$$= \frac{q(t)}{q(T_c)} [C(T_c, q(T_c); T, K) - K_c] \chi\{q(T) > K\} \chi\{C(T_c, q(T_c); T, K) > K_c\} =$$

$$= q(t) [\frac{q(T) - K}{q(T)} - \frac{K_c}{q(T_c)}] \chi\{q(T) > K\} \chi\{\frac{q(T) - K}{q(T)} > \frac{K_c}{q(T_c)}\}$$

Note that the price of the compound option depends on the underlying security price $q(t)$ at three future dates $t < T_c < T$.
Let us consider other types of compound option. The pricing equation for the compound put on European call option is

$$\frac{P_c(T_c, q(T_c); T_c, K_c; T, K)}{P_c(t, q(t); T_c, K_c; T, K)} = \frac{K_c - C(T_c, q(T_c); T, K)}{P_c(t, q(t); T_c, K_c; T, K)} \times$$

$$\times \chi\{C(T_c, q(T_c); T, K) < K_c\} = \frac{C(T_c, q(T_c); T, K)}{C(t, q(t); T, K)} \chi\{C(T_c, q(T_c); T, K) < K_c\} =$$

$$= \frac{q(T_c)}{q(t)} \chi\{q(T) > K\} \chi\{C(T_c, q(T_c); T, K) < K_c\}$$

Solving this equation for the current moment of time leads to the formula

$$P_c(t, q(t); T_c, K_c; T, K) =$$

$$= q(t)\left[\frac{K_c}{q(T_c)} - \frac{q(T) - K}{q(T)}\right]\chi\{q(T) > K\}\chi\left\{\frac{K_c}{q(T_c)} > \frac{q(T) - K}{q(T)}\right\}$$

The pricing formulae for compound call or put written on European put option can be obtained in a similar way

$$C_p(t, q(t); T_c, K_c; T, K) =$$

$$= q(t)\left[\frac{K_c}{q(T_c)} - \frac{q(T) - K}{q(T)}\right]\chi\{q(T) < K\}\chi\left\{\frac{K - q(T)}{q(T)} > \frac{K_c}{q(T_c)}\right\}$$

$$P_p(t, q(t); T_c, K_c; T, K) =$$

$$= q(t)\left[\frac{K_c}{q(T_c)} - \frac{q(T) - K}{q(T)}\right]\chi\{q(T) < K\}\chi\left\{\frac{K_c}{q(T_c)} > \frac{K - q(T)}{q(T)}\right\}$$

Let us consider a derivative contract that admits a choice between two or more foreign bonds at a future moment of time. This type of the contract sometimes called options on **maximum or minimum of several risky assets** or **rainbow options**. Rainbow options get their name from the fact that there more than one exchange rate. Assume that at maturity T a holder of the contract has the right to choose a bond: domestic or foreign. It is clear that the face value of the bond contracts can not be arbitrary. There are two possibilities to formalize setting of the problem. First, we can assume that at initial moment t, the size of two contracts is the same. Let q (t) be the spot exchange rate between two currencies. That means that

1 unit of foreign currency (at date t) = q (t) units of the domestic currency (at date t)

This is referred to as a direct quotation. The indirect quotation that shows number of units of the foreign currency per domestic is the reciprocal value 1/q (t). Here we ignore the bid-ask spread differential. The value of a 0-defualt and 0-coupon bond is either domestic or foreign at time t with the face value of 1, and is defined by the relationship

1 unit currency (at date T) = B (t , T) 1 unit currency (at date t)

This equation can also be interpreted as a relationship between future and current values of the currency. For example the domestic currency is US dollars and foreign is GB pounds. Then the payoff at maturity T can be established in the form

$$\max\{ \$1(T),\ £1(T)/q(t) \} = \max\{ 1,\ q(T)/q(t) \}\ \$1(T) \qquad (2.13)$$

Other possibility is to use the forward exchange rate. In this case, at date t we suppose that the values of two contracts look equal. The payoff to this contract is

$$\max\{ 1,\ q(T)/q(T;t,q(t)) \}\ \$1(T) \qquad (2.14)$$

where $q(T;t,q(t))$, $T \geq t$ is the forward exchange rate established at date t, quoted on $T - t$ period given that $q(t;t,q(t)) = q(t)$. The price of the contracts is their present values of the payoff

$$\max\left\{ \frac{1}{B_\$(t,T)},\ \frac{q(T)}{B_\$(t,T)q(t)} \right\} \$1(t)$$

$$\max\left\{ \frac{1}{B_\$(t,T)},\ \frac{q(T;t,q(t))}{B_\$(t,T)q(t)} \right\} \$1(t)$$

for (2.13) and (2.14) payoffs respectively. The generalization of the currency-rainbow option on three or more underlying currencies is straightforward. Indeed we will use $B_i(t,T)$, $i = 0, 1, \ldots n$ to denotes bonds values at date t with maturity T. Index 0 is arranged for US Treasury bond. Denote $q_i(t)$ the direct quotation of i-th currency with respect to domestic. If the payoff to the option at maturity is chosen as

$$\max\{ 1_0(T),\ 1_1(T)/q_1(t),\ \ldots,\ 1_n(T)/q_n(t) \}$$

Then the price of $(n+1)$-rainbow option at time t is

$$\max\left\{ \frac{1}{B_0(t,T)},\ \frac{q_1(T)}{B_1(t,T)q_1(t)},\ldots,\ \frac{q_n(T)}{B_n(t,T)q_n(t)} \right\} \$1(t)$$

Let us consider the case when n assets and cash are involved in payoff. We will see that the pricing formulas for the contract that deals with the maximum or minimum of several stocks differ from the one presented above. It follows from the fact that foreign exchange market instruments can be compared if they have the same currency format and therefore the exchange rates play a significant role in valuations.
Let $S_1(t), \ldots, S_n(t)$ be a price of n – assets. Assume that the payoff at maturity T is given as:

$$\max\{ S_1(T),\ \ldots,\ S_n(T),\ K \}$$

where a constant $K \geq 0$ is a cash. The reasonable price for the contract at time t is the price that suggests the maximum possible return among underlying instruments. Therefore,

$$\frac{\max\{ S_1(T), .., S_n(T), K \}}{rainbow(t)} = \max\{ \frac{S_1(T)}{S_1(t)}, ..., \frac{S_n(T)}{S_n(t)}, \frac{K}{present\ value\ of\ K} \}$$

where *rainbow (t)* is the contract price at date t. The fractions on the right hand side can be simplified. Indeed, putting $S_i(T) = S_i(T; t, S_i(t))$ and having noted the linear dependence of $S_i(T)$ on initial data one can see that the right hand side of the equation above can be rewritten in the form

$$\max\{ S\Im(T;t,1), \ ..., \ S_n(T;t,1), B(t,T) \}$$

The linear dependence of an asset on the initial data follows from the fact that N assets price N S (t) over an arbitrary period of time is governed by the same low as the single asset time N , that is $N \times [S(T;t,S(t))] = S(T;t,NS(t))$, and this observation does not depend on a format in which the asset price is established. Hence,

$$rainbow(t) = \frac{\max\{ S_1(T;t,S_1(t)), ..., S_n(T;t,S_n(t)), K \}}{\max\{ S_1(T;t,1), ..., S_n(T;t,1), B(t,T) \}} =$$

$$= \frac{K + \max\{ \max[S_1(T;t,S_1(t)), ..., S_n(T;t,S_n(t))] - K, 0 \}}{B(t,T) + \max\{ \max[S_1(T;t,1), ..., S_n(T;t,1)] - B(t,T), 0 \}}$$

The rainbow with minimum of n risky assets is similar to the best of n assets and changing the operation from max to min one can easily represent the correspondent formulas.

Other type of the exotic derivatives is a **spread** option. The payoff for European calls and puts at maturity T with the strike price K can be written respectively as

$$C_{sd}(T, S_1(T), S_2(T)) = \max\{ S_1(T) - S_2(T) - K, 0 \}$$

$$P_{sd}(T, S_1(T), S_2(T)) = \max\{ K - S_1(T) + S_2(T), 0 \}$$

The generalization on n-assets in long position and m-assets in short is obvious. The price of the call on spread can be expressed by an equation

$$\frac{C_{cd}(T, S_1(T), S_2(T))}{C_{cd}(t, x, y)} = \frac{S_1(T) - S_2(T)}{x - y} \chi\{ S_1(T) - S_2(T) > K \}$$

from which it follows, that

$$C_{cd}(t, x, y) =$$

$$= \max\{ S_1(T) - S_2(T) - K, 0 \} \frac{x - y}{S_1(T) - S_2(T)} \chi \{ S_1(T) - S_2(T) > K \}$$

The price of the put can be written in analogous form

$$P_{cd}(t, x, y) =$$

$$= \max\{ K - S_1(T) + S_2(T), 0 \} \frac{x - y}{S_1(T) - S_2(T)} \chi \{ S_1(T) - S_2(T) < K \}$$

Now, we will study a popular type of exotics options called a **barrier option**. This family of options is path-dependent because the value of the option at maturity depends on the path of the spot exchange rate over the lifetime of the option. The value of the barrier option is specified by an event whether or not the underlying spot exchange rate crosses a given barrier. There are two different ways of intersections that can be regarded as 'in' and 'out' and two types of the level 'up' or 'down' with respect to the initial value of the spot rate. A double barrier option is a barrier option with two 'up' and 'down' barriers. The down-and-out (knock-out) option specifies a low barrier. If the spot exchange rate breaches this barrier during the life of the option then the option payoff is equal to 0. In some cases a rebate can also be provided if the barrier is crossed. Denote d as a barrier level, K a strike price, d < K. The payoff to the down-and-out call option is defined as

$$C_{do}(T, q(T)) = \max\{ q(T) - K, 0 \} \chi(\min q(l) > d, l \in [t, T])$$

Let τ_d denote the first moment when process $q(l)$, $l \geq t$ attains the level d. Then

$$\chi(\min q(l) > b, l \in [t, T]) = \chi(\tau_d > T)$$

with probability 1. The payoff to the down-and-out put option at maturity T is defined

$$P_{do}(T, q(T)) = \max\{ K - q(T), 0 \} \chi(\tau_d > T)$$

The down-and-in (knock-in) call option exercise price at maturity T is

$$C_{di}(T, q(T)) = \max\{ q(T) - K, 0 \} \chi(\tau_d \leq T)$$

And the down-and-in (knock-in) put option exercise price is defined as

$$P_{di}(T, q(T)) = \max\{ K - q(T), 0 \} \chi(\tau_d \leq T)$$

Assume that exchange rate data is defined by the Table 2.2 and K = 180, d = 178. Then,

Table 2.12 K = 180, d = 178

C_{do} (0, ω)	C_{do} (1, ω)	C_{do} (2, ω)	ω	p(ω)
5.8064	5.9677	6	{180, 185, 186}	1/6
0	0	0	{180, 185, 179}	1/2
0	0	0	{180, 178, 182}	1/24
0	0	0	{180, 178, 181}	1/12
0	0	0	{180, 178, 176}	5/24

Table 2.13 K = 180, d = 178

P_{do} (0, ω)	P_{do} (1, ω)	P_{do} (2, ω)	ω	p(ω)
0	0	0	{180, 185, 186}	1/6
1.0056	1.0335	1	{180, 185, 179}	1/2
0	0	0	{180, 178, 182}	1/24
0	0	0	{180, 178, 181}	1/12
0	0	0	{180, 178, 176}	5/24

Table 2.14 K = 180, d = 178

C_{di} (0, ω)	C_{di} (1, ω)	C_{di} (2, ω)	ω	p(ω)
0	0	0	{180, 185, 186}	1/6
0	0	0	{180, 185, 179}	1/2
1.978	1.956	2	{180, 178, 182}	1/24
0.9945	0.9834	1	{180, 178, 181}	1/12
0	0	0	{180, 178, 176}	5/24

Table 2.15 K = 180, d = 178

P_{di} (0, ω)	P_{di} (1, ω)	P_{di} (2, ω)	ω	p(ω)
0	0	0	{180, 185, 186}	1/6
0	0	0	{180, 185, 179}	1/2
0	0	0	{180, 178, 182}	1/24
0	0	0	{180, 178, 181}	1/12
4.0909	4.0455	4	{180, 178, 176}	5/24

Let us present a risk analysis for the investment in down-and-in call option. The average option price at date 0 is 0.16529 times the size of a contract. For example, the contract size is 1000 units of foreign currency (£). Thus, the mean value of one contract of down-and-in call option costs $165.29. Assume that an investor wishes to compare the two scenarios in which the prices are a $100 or $200 that represents 0.1 or 0.2 per £. If the option price is 0.1 then the chance to exercise the option at expiration is when next union of two events are realized {180, 178, 182} ∪ {180, 178, 181}. The probability of such an event is 1/24 + 1/12 = 1/8. The expected rate of return is equal to

$$(1 / 24) \times [1.978 - 0.1] / 0.1 + (1 / 12) \times [0.9945 - 0.1] / 0.1 = 1.5279$$

If the option price is 0.2 then the expected rate of return is about

$$(1 / 24) \times [1.978 - 0.2] / 0.2 + (1 / 12) \times [0.9945 - 0.2] / 0.2 = 0.7015$$

The break even mean price is the solution to the equation

$$(1 / 24) \times [1.978 - x] / x + (1 / 12) \times [0.9945 - x] / x = 1$$

That is x = 0.14693 or $146.93 per contract. The mean value analysis does not cover the risk exposure. For the given options price, risk is a random variable $\Re$ that represents possible rate of return values x_i , i = 1, 2, … along with correspondent distribution function $\Phi (x_i) = P (\Re < x_i)$. In case when the down-and-in call option price is 0.1 per £, then

$$\Re = \begin{cases} 0, & P (\Re < 0.9945) = 21/24 \\ 0.9945 & P (0.9945 \le \Re < 1.978) = 1/12 \\ 1.978 & P (1.978 \le \Re) = 1/24 \end{cases}$$

One can note that this rate of return risk performance can be easily converted into currency form as it was presented in the earlier examples. On the other hand, the rate of return performance uses one parameter rate of return, and the currency risk form uses two parameters; current and future spot exchange rates that may be inconvenient.

Now let us look at the next type of barrier option in which 'up' barrier is specified. If the spot exchange rate goes above the 'up' barrier then an up-and-out option ceases to exist. A rebate that should be specified at initiation of the contract may also be provided as the barrier is crossed.

The payoff to the up-and-out call/put options at maturity T is defined as

$$C_{uo} (T, q (T)) = \max \{ q (T) - K , 0 \}\} \chi (\tau_u > T)$$
$$P_{uo} (T, q (T)) = \max \{ K - q (T) , 0 \}\} \chi (\tau_u > T)$$

respectively. Note that,

$$\chi (\tau_u > T) = \chi (\max q (l) < u , l \in [t , T])$$

The payoff to the up-and-in call, put options at maturity T are defined as

$$C_{ui} (T, q (T)) = \max \{ q (T) - K , 0 \}\} \chi (\tau_u \le T)$$
$$P_{ui} (T, q (T)) = \max \{ K - q (T) , 0 \}\} \chi (\tau_u \le T)$$

Assuming that the underlying exchange rate given in Table 2 and putting K = 180, u = 185 we have

Table 2.16

$C_{uo}(0, \omega)$	$C_{uo}(1, \omega)$	$C_{uo}(2, \omega)$	ω	$p(\omega)$
0	0	0	{180, 185, 186}	1/6
0	0	0	{180, 185, 179}	1/2
1.978	1.956	2	{180, 178, 182}	1/24
0.9945	0.9834	1	{180, 178, 181}	1/12
0	0	0	{180, 178, 176}	5/24

Analogously, we see that

Table 2.17

$P_{uo}(0, \omega)$	$P_{uo}(1, \omega)$	$P_{uo}(2, \omega)$	ω	$p(\omega)$
0	0	0	{180, 185, 186}	1/6
1.0056	1.0335	1	{180, 185, 179}	1/2
0	0	0	{180, 178, 182}	1/24
0	0	0	{180, 178, 181}	1/12
4.0909	4.0455	4	{180, 178, 176}	5/24

The valuation of the up-and-in barrier option is similar to the represented above.
Let us consider a double barrier pricing scheme for the case when $K = 180$, $u = 185$, $d = 178$. The payoff to the double-out barrier call and put options at maturity may be defined as

$$C_{dbo}(T, q(T)) = \max\{q(T) - K, 0\}\,\chi\{d < \min q(l),\ \max q(l) < u,\ l \in [t, T]\}$$
$$P_{dbo}(T, q(T)) = \max\{K - q(T), 0\}\,\chi\{d < \min q(l),\ \max q(l) < u,\ l \in [t, T]\}$$

The payoff to the double-in barrier call and put options at maturity is

$$C_{dbi}(T, q(T)) = \max\{q(T) - K, 0\}\,\chi\{d \geq \min q(l),\ \max q(l) \geq u,\ l \in [t, T]\}$$
$$P_{dbi}(T, q(T)) = \max\{K - q(T), 0\}\,\chi\{d \geq \min q(l),\ \max q(l) \geq u,\ l \in [t, T]\}$$

The scheme of calculations of a double barrier options is similar to the one developed above and therefore we can omit it.

Continuous time option pricing.

Now we discuss the continuous time pricing problem. Historically there are only two approaches to the problem in continuous time setting. We call them Samuelson-Merton, Black-Scholes, and binomial scheme. We begin with the approach that had been developed by P.A. Samuelson and R.C. Merton at the end of 60's. The main attention they paid to the warrant pricing that theoretically is very close but does not seem completely coincides with the option pricing. The final version of their studies was represented in [8, 9].
As a price definition of the warrant they admitted a statement: "…under what conditions will everyone be willing to hold a warrant, giving the right to buy a share of the common

stock for an exercise price of \$1 per share at any time in the next n-periods, and at the same time be willing to hold the stock and cash? "

In their paper (1965) it was postulated that the warrant must have a specified gain per dollar not less than the expected return per dollar invested in the common stock representing in this scheme underlying security. Denote $Y(t, n)$ the warrant price at the date t with n periods still run to maturity and let $X(t)$ be the stock price at t. Then they assumed that

$$E\{ X(t + T) / X(t) \} = \exp \alpha T$$

(S-M-assumption)

$$E\{ Y(t + T, n - T) / Y(t, n) \} = \exp \beta T$$

where $\alpha \leq \beta$ if the warrant is to be held. It was arbitrary postulated that expected percentage gains α, β are given data but then it became clear that these parameters can be derived from knowledge of the risk aversion properties of an utility function. The method that lead them to the construction of the warranty price is the induction with respect to n. For $n = 0$ we have

$$Y(t, 0) = \max\{ 0, X(t) - 1 \} = F(0, X(t))$$

It follows from (S-M-assumption) that when $n = 1$

$$\exp \beta T = E\{ Y(t + 1, 0) / Y(t, 1) \}$$

They put $X(t + 1) = X(t) Z$ and therefore

$$Y(t + 1, 0) = \max\{ 0, Z X(t) - 1 \} = F(0, Z X(t))$$

Let $X(t) = x$ and $dP(z)$ be a distribution function of the random variable Z. Given that the warrant price $Y(t, 1)$ is the nonrandom function it follows that

$$\exp \beta = E\{ F(0, Z X(t)) / Y(t, 1) \} = \int \{ F(0, z x) \, d P(z) / Y(t, 1)$$

That gives us the warrant value in the form

$$Y(t, 1) = \exp(-\beta) \int F(0, z x) \, d P(z)$$

Putting

$$F(1, x) = \exp(-\beta) \int F(0, z x) \, d P(z)$$

and then successively introducing expression $F(k, x)$ for the $F(k - 1, x)$, $k = 2, 3, \ldots$ as it shown above for $F(1, x)$ and $F(0, x)$ we arrive at the Samuelson-Merton pricing model.

Now let us examine the Black-Scholes option price construction. Recall that there exist few ways of derivation of the Black-Scholes equation. First is the original one belonging to Black and Scholes. Its modern design can be found in [10]. Then we briefly outline other construction following [11]. And one more method also probably deserve our attention is referred to as neutral martingale approach [7]. This method significantly uses binomial scheme and we will consider it later.

Let us introduce some definitions. An option contract is an agreement to buy or sell an asset at a certain future time for a predetermine price. The party that agree to buy takes a long position and other party that agree to sell takes a short position in the option contract. The option contract is settled at maturity. The holder of short position delivers the asset to the holder of long position in return for the cash amount equal to the strike price. Let $w(t)$ be one-dimensional Wiener process. Then, the stock price $S(t)$ is the solution of the equation (1.5).

Following [9] we briefly outline original method of the derivation of the Black-Scholes equation. Imagine one's wealth can be apportioned between two assets. One of these assets is a risk-free bond (1.6) with a constant interest rate r. The other one is a stock, whose price is described by a linear Ito equation (1.5) with known constant mean rate of return μ constant volatility σ. Black and Scholes denoted the value of the portfolio over the given time interval as

$$\Pi(t, x) = p(t, x)\, x + b(t, x)\, B(t) \tag{3.1}$$

where functions $p(t, x)$ and $b(t, x)$ represent the amount of stocks and bonds in portfolio at time t respectively, when the stock price $S(t) = x$. They also supposed that the value of the portfolio change along the function $S(\cdot)$ is

$$\begin{aligned} d\,\Pi(t, S(t)) &= p(t, S(t))\, dS(t) + b(t, S(t))\, dB(t) = \\ &= [\, p(t, S(t))\, \mu S(t) + b(t, S(t))\, r B(t)\,]\, dt + \\ &\quad + p(t, S(t))\, \sigma S(t)\, dw(t) \end{aligned} \tag{3.2}$$

Let $f(t, x)$ be an arbitrary nonrandom smooth function subject to constraint $S(t) = x$. Applying Ito formula we have

$$d f(t, S(t)) = \{\, \partial f(t, S(t)) / \partial t + \mu S(t)\, \partial f(t, S(t)) / \partial x +$$

$$+ \tfrac{1}{2} \sigma^2 S^2(t)\, \partial^2 f(t, x) / \partial x^2 \,\}\, dt + \sigma S(t)\, [\, \partial f(t, S(t)) / \partial x\,]\, dw(t)$$

Black and Scholes put

$$\begin{aligned} p(t, x) &= \frac{\partial f(t, x)}{\partial x} \\ b(t, x) &= \frac{1}{B(t)} \left[f(t, x) - x \frac{\partial f(t, x)}{\partial x} \right] \end{aligned} \tag{3.3}$$

Setting

$$d\,\Pi(t, S(t)) = d\,f(t, S(t)) \qquad (3.4)$$

and taking into account (3.3), (3.4) we obtain Black-Scholes equation

$$\partial f(t, x)/\partial t + rx\,\partial f(t, x)/\partial x + ½\,\sigma^2 x^2\, \partial^2 f(t, x)/\partial x^2 = r f(t, x) \qquad \text{(BSE)}$$

The solution of the Black Scholes option pricing equation with boundary condition

$$f(T, x) = \max(x - K, 0)$$

represents the value of the European call option contract (1.7) on common stock which price governed by the equation (1.5). Using probabilistic representation of a solution for the parabolic Cauchy problem the Black Scholes equation solution can be written in the form (1.8). Remarkably that in this construction the definition of the option price was not used.

We now specify the sense of Black-Scholes' the option price. This is the arbitrary nonrandom smooth function that satisfies (3.4). A substitutions (3.3) bring us to the equation (BSE). Black and Scholes called the solution of the equation the option price.

Statement: In order to obtain BSE assumption (3.1) should be omitted. Indeed, from (1.8) follows

$$\Pi(t, S(t)) = p(t, S(t))\,S(t) + b(t, S(t))\,B(t) \qquad (3.5)$$

where $p(t, x)$, $b(t, x)$ are given by (1.11), (1.12). Using the formula of integration by parts results

$$\begin{aligned} d\Pi(t, S(t)) &= p(t, S(t))\,dS(t) + b(t, S(t))\,dB(t) + \\ &+ S(t)\,dp(t, S(t)) + B(t)\,db(t, S(t)) + \qquad (3.6) \\ &+ < dp(t, S(t))\,dS(t) > \end{aligned}$$

where dp, db are stochastic differentials of the smooth nonrandom functions p, b along the $S(t)$ and using Ito formula we see that

$$< d\,p(t, S(t)), d\,S(t) > = \sigma^2\, p'_x(t, S(t))\; S^2(t)\,dt$$

The representation (3.6) is based on the exact formula of integration by parts of the function Π and it differs from (3.6). It is easy to note that the terms not equal to 0 are lost. Thus, equalities (3.5) and (3.6) are mutually exclusive when the function $S(t)$ is either random or nonrandom. This misleading situation compels reconsider the Black-Scholes conception. We are going to specify the B&S construction. This does not mean that we suggest accept the Black-Scholes price definition. We will show that there exist other substitutions that lead us to other equation and we can not choose the best. On the

other hand we can interpret set of the solutions of the equations as strategies but neither can be accepted as an option price definition.
Assume that the change in value of the portfolio is given by the formula

$$d\Pi(t, S(t)) = p(t, S(t))\, dS(t) + b(t, S(t))\, dB(t) \tag{3.7}$$

In this case we should express (3.1) in the following integral form. Note that only this integral form holds the real sense in stochastic calculus. Hence, along the paths $S(l; t, x)$ we have

$$\Pi(t, x) = \Pi(T, S(T; t, x)) - \int_t^T [\, p(l, S(l; t, x))\, \mu\, S(l; t, x) + $$
$$+ b(l, S(l; t, x))\, r\, B(l)\,]\, dl - \int_t^T [\, p(l, S(l; t, x))\, \sigma\, S(l; t, x)\, dw(t) \tag{3.8}$$

where $f(t, x)$ is undefined option price and therefore it still an arbitrary deterministic function. By definition the value $f(T, x) = \max\{S(T) - K, 0\}$. Let us introduce another example of the choice of the function (3.2) that lead to other equation that can be used as another option price definition. Our goal to present conditions when 'perfect replication' idea expressed by identity

$$\Pi(t, x) = f(t, x)$$

holds for all $t < T$. This is new option definition. To achieve this goal we set $\Pi(T, x) = f(T, x)$ and then in order to eliminate the risk term assume

$$p(t, x) = \frac{\partial f(t, x)}{\partial x}$$

Comparing the integrand expressions in the ordinary integrals in (3.8) we arrive at identity

$$\mu x \frac{\partial f(t, x)}{\partial x} + b(t, x) r B(t) = \frac{\partial f(t, x)}{\partial t} + \mu x \frac{\partial f(t, x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2}$$

or

$$b(t, x) = \frac{1}{r B(t)} \left[\frac{\partial f(t, x)}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2} \right] \tag{3.9}$$

where functions b and f are still unknown. If $b(t, x)$ is chosen as (3.3) then substituting it in the left hand side of (3.9), we arrived at the (BSE). Other choice of the $b(t, x)$ bring us to the different equation that can be establish as an option price definition. This fact again confirms the logical rule that definitions should be given before studying their properties. Selecting

$$b(t, x) = \frac{1}{B(t)} [f(t, x) - \frac{\theta x}{\mu} \frac{\partial f(t, x)}{\partial x}]$$

we obtain a new valuation equation depending on parameter $\theta \geq 0$

$$\frac{\partial f_\theta(t, x)}{\partial t} + \theta \frac{r x}{\mu} \frac{\partial f_\theta(t, x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f_\theta(t, x)}{\partial x^2} = r f_\theta(t, x)$$

with the same boundary condition $max(x - K, 0)$ at $t = T$. The solution of this equation represents definition of the option price for each θ. In particular when $\theta = \mu$ we arrive at Black-Scholes substitution. Other choice $\theta = 0$ leads to the other solution $g(t, x)$.
Note that in our case it is easy to check that for all (t, x), $g(t, x) \leq f(t, x)$.
Indeed, using probabilistic representation one can write down that

$$f(t, x) = \exp -r(T - t) E f_T(\eta(T; t, x))$$

$$g(t, x) = \exp -r(T - t) E f_T(\zeta(T; t, x))$$

where

$$\eta(l; t, x) = x + \int_t^l r \eta(q) dq + \int_t^l \sigma \eta(q) dw(q)$$

$$\zeta(l; t, x) = x + \int_t^l \sigma \zeta(q) dw(q) \qquad , \qquad l \leq T$$

Applying comparison theorem it follows that $\zeta(l; t, x) \leq \eta(l; t, x)$ with probability 1. This inequality along with the probabilistic representation of the functions f, g proves the statement.
Assume now that two investors are going to use these option price definitions trying to find out the reasonable position for the deal. The payoff is the same in both cases. The one who intends to take long position, i.e. buying the option wish to pay $g(t, x)$ because he spends less money buying the option but the second investor at the same time is about to use value $f(t, x)$ in order to take short position, selling the option that offers him larger profit. This deviation in understanding of the option value suggests that the price concept is not logically correct.
One may attempt to follow by Black and Scholes and apply the equation

$$d\Pi\,(T,S(T)) = r\Pi\,(T,S(T))\,dt$$

where Π is given by (3.7). In order to find the function $b(t,x)$ based on (3.2), (3.3) given

$$p(t,x) = \frac{\partial f(t,x)}{\partial x}$$

one can note that we need to solve the equation

$$p(t,\,S(t))\,dS(t) + b(t,\,S(t))\,dB(t) = r\,\{\,\Pi(T,S(T)) -$$

$$-\int_t^T [\,p(l,\,S(l))\mu\,S(l) + b(l,\,S(l))rB(l)\,]\,dl \quad -$$

$$-\int_t^T p(l,\,S(l))\;\sigma\,S(l)\,dw(l)\,\}dt$$

The solution of this equation can not be expressed in compact form given by Black Scholes.

Now let us introduce other approach to the option price definition that also have to lead us to BSE. Here, we follow [11] where that approach was introduced. Let Δ be a portfolio which value is equal to

$$\Xi(t,x) = -f(t,x) + \frac{\partial f(t,x)}{\partial x}\,x \tag{3.10}$$

In other words the portfolio Δ contains one option in short position and $/f(t,x)//x$ stocks in long position. In conformity with BS construction it was assumed that change in value of the portfolio over the time is given by

$$d\,\Xi(t,\,S(t)) = -\,d\,f(t,\,S(t)) + \frac{\partial f(t,S(t))}{\partial x}\,dS(t) \tag{3.11}$$

In order to derive the equation for option price it was set

$$d\,\Xi(t,\,S(t)) = r\,\Xi(t,\,S(t))\,dt \tag{3.12}$$

where r is given interest rate. In order to define equation for the function $f(t,x)$ it was recommended first to substitute values $\Delta(t,x)$, $d\Delta(t,S(t))$ from right hand sides of (3.10) and (3.11) into (3.11) and then apply Ito formula for the stochastic differential $df(t,S(t))$. That brings us to (BSE). The boundary condition

$$f(T, x) = \max(x - K, 0)$$

directly follows from definition of the function $f(t, x)$. Taking into account formula integration by parts one can easy justify that formula (3.10) and (3.11) are mutually exclusive. Therefore, they could not be substitute in (3.5) simultaneously.

One should remark that some researchers discussed mathematical arguments used by Black and Scholes. They attempted to make original derivation more accurate. In [12] were introduced arguments following by the [13]. Here we represent the referred results with some technical adjustments. We apply the common time direction. Let $f(t, x)$ be the Black-Scholes equation solution. Define stochastic processes

$$V(t) = f(t, S(t)), \quad p(t) = \frac{\partial f(t, S(t))}{\partial x},$$

$$q(t) = \frac{V(t) - p(t)S(t)}{B(t)} = [f(t, S(t)) - S(t)\frac{\partial f(t, S(t))}{\partial x}]\frac{1}{B(t)}$$

The components p, q can be interpreted as a number of stocks and bonds correspondingly of the hypothetical portfolio. It easy to see identity

$$V(t) = p(t)S(t) + q(t)B(t)$$

The timing condition for the portfolio follows from the fact that value of the portfolio at maturity coincides with the payoff of the option. Therefore

$$V(T) = \max(S(T) - K, 0)$$

Then applying Ito formula and taking into account that $f(t, x)$ is BSE solution , we obtain

$$dV(t) = d\, f(t, S(t)) = p(t)\, dS(t) + q(t)\, dB(t)$$

It seems that one of the main problem of the Black and Scholes derivation is corrected. On the other hand we can note that this construction proves only the statement.
Statement. Let $f(t, x)$ be the solution of the deterministic equation with partial derivatives (BSE). Then along the $S(t)$ we have representation (a 'self-financing strategy')

$$dV(t) = p(t)\, dS(t) + q(t)\, dB(t)$$

One may see that this statement does not have any relation with option price definition. Indeed, it was prior assumed that the solution of the (BSE) is the option price that of course was not proved by Black and Scholes.
Remarkably that the seeming progress in elimination of the contradiction comes from the assumption that $f(t, x)$ is a solution of the Black Scholes equation and therefore the term in (3.6) that was not equal to 0 now becomes equal to 0.

Now if $f(t,x)$ is an arbitrary smooth nonrandom function then Ito formula confirms the self-financing portfolio

$$df(t,S(t)) = p(t,S(t))\,dS(t) + b(t,S(t))\,dB(t)$$

where

$$p(t,x) = \partial f(t,x)/\partial x\ ,$$

$$b(t,x) = [\,r\,B(t)\,] \,/\, \{\,\partial f(t,x)/\partial t + \tfrac{1}{2}\sigma^2 x^2 \partial^2 f(t,x)/\partial x^2\,\}$$

without any definition of the option price. Follow interpretation of the option price one can use any relationship between functions p (t , x) and b (t , x) to produce own definition and Black Scholes approach could not specifies which one is better.

Let us compare Black-Scholes and Samuelson-Merton models that introduced at the beginning of this section. Though these models are somewhat similar it may interesting to emphasize their distinctions. To investigate distinctions we assume that the stock price in Samuelson-Merton model is the solution of the Ito equation

$$dX(t) = \mu X(t)\,dt + \sigma X(t)\,dw(t) \tag{3.13}$$

where w (t) be one-dimensional Wiener process. The solution of the equation (3.31) can be written in the form

$$X(t+T) = X(t)\exp\{\,(\mu - \sigma^2/2)\,T + \sigma[\,w(t+T) - w(t)\,]\,\}$$

Letting $T = 1$ we can see that the distributions of the random variable Z, where $Z = X(t+1)/X(t)$ coincide with the distributions

$$\exp\{\,(\mu - \sigma^2/2) + \sigma\, w(1)\,\}$$

and therefore

$$dP(z) = \exp\{\,(\mu - \sigma^2/2) + \sigma z\,\}\exp\{-\ z^2/2\,\}\,dz$$

Now the formula for $Y(t,1)$ can be rewritten as

$$Y(t,1) = \exp(-\beta)\int \max(0, zx - 1)\exp\{\,(\mu - \sigma^2/2 + \sigma z) - z^2/2\,\}\,dz \tag{3.14}$$

It was remarked by R. Merton [9] that Black and Scholes claim that one reason why Samuelson-Merton did not arrive at their formula was because they did not consider the same model of assets. As far as risk-free bond was not involve in Samuelson-Merton model one can assume that the formula (3.12) or (3.14) can be obtained as a particular case of the Black and Scholes' pricing formula when the risk-free interest rate equal to 0. Thus Samuelson-Merton omitted this factor as insignificant. Our point of view is that

these two models are different in sense that it is not clear what assumptions one can make in order to obtain (3.12) from Black-Scholes formula.
Now we recall the idea used above for the discrete option valuations, which leads to the unique pricing of contingent claims.
We show hoe Black-Scholes 'perfect replication' should be modified. We show that the derivative price for all possible outcomes which suggest strictly positive payoffs can be defined in the way that provides equal return on derivative and its underline security. Applying the investment principle to two alternative investment opportunities one notes that these opportunities should offer the same rate of return on the stock and the derivative over the predetermine time interval. Formally that means that

$$\frac{f(T, S(T))}{f(t, S(t))} = \frac{S(T)}{S(t)} \chi\{ S(T) > K \}$$

where χ {A} is indicator of an event A. Therefore

$$f(t, S(t)) = f(T, S(T)) \frac{S(t)}{S(T)} \chi\{ S(T) > K \} \qquad (3.15)$$

Note that indicator in the right hand side can be omitted whereas the function $f(T, S)$ equal to 0 when $S \le K$. Let $S(l; t, x)$, $l \ge t$ be a random function such that $S(t; t, x) = x$. The equality (3.15) contains the function $S(T; t, x)$ in denominator and it can be proof that $S(l; t, x) > 0$, $t \le 1 \le T$ with probability 1. Thus (3.15) can be rewritten in the form

$$f(t, x) = f(T, S(T; t, x)) S^{-1}(T; t, 1) \qquad (3.16)$$

Formulae (3.15), (3.16) determines option price on the stock and holds sense regardless of the whether the stock price $S(T; t, x)$ is random or nonrandom function. Assuming that $S(l; t, x)$ solution of (1.5) one can see that by definition the option price should be the random function measurable with respect to the family of σ - algebras

$$F^{t} = F_{[t,T]} = \sigma\{ w(t_1) - w(t_2) \;\; ; \;\; t_1, t_2 \in [t, T] \}, \;\; t \in [0, T]$$

Note, that we have not supposed that there exist risk free financial instrument and therefore no discount factor involves in (3.16). If we assume that then the equation representing equal rate of return principle should be rewritten in the form

$$\frac{\exp\{-r(T-t)\} f(T, S(T))}{f(t, S(t))} = \frac{\exp\{-r(T-t)\} S(T)}{S(t)} \chi\{ S(T) > K \}$$

and also bring us to the same equation (3.16). It may be important now to comment the uniqueness and stochastic nature of the option price. As far as a nonrandom function is a particular case of the random functions we assume that exist other random or nonrandom function that one may call option price. If for some event ω the alternative value is a

number, say A, that is not equal to $f(t, x) = f(t, x, \omega)$ then we consider deterministic example with the values of stock $S(T) = S(T; t, x, \omega)$. If this value is larger than K and $A \neq f(t, x)$ then it is easy to note arbitrage opportunity. On the other hand if $S(T) \leq K$ then no one wish to pay for the worthless option but this will be profitable opportunity for a writer. This argument justifies stochastic environment and uniqueness of the option price definition. Here we also remark that as far as the given definition is unique then the possibility of the perfect replication of a portfolio containing option, asset and cash to any financial instrument in the market is only technical result that can not contradict or eliminate construction of the option price. Now we can consider an investor's benefits of the stochastic price definition. At date t the investor of course can not pay a stochastic price and it is up to him to suppose what is the fair option price. Any option's value established by the writer of the contract or the value that the investor willing to pay for the option calculated on the same base using equal rate of return principal. The only difference between two market participants is what value at maturity of underlying security they subjectively prescribe at date t and what is probability of such event. Admitting hypothetical distribution of an approximation of the security price at maturity in the form $\Sigma S_j \chi \{ S_j \leq S(t) < S_{j+1} \}$ and thinking that maximum probability events $\{\omega: S_j \leq S(t) < S_{j+1}\}$ occurs for the particular number j_o the investor would specify a number from the interval $[S_j, S_{j+1}]$. Chosen number uniquely determines the option price B_j at date t. Along with the price the investor can calculate risk which by definition is associated with the probability of the event $\{\omega: f(t, x, \omega) > B_j\}$. Thus the risk of the option price construction is associated with factors. Assume that the stock price is a random process with the given distribution. Then the most probable value admitted by the investor at date T could be smaller value at maturity T. If at expiration the stock price distribution is exactly the same as it was assumed the option price is also random. Therefore the most probable option price given distribution of the S (T) can be violated. Thirdly the stock's distribution is only hypothetical and all calculated probabilities are only estimates of the real probability assuming that the assumption that real probabilities has less risk than all enumerated factors.
Our primary goal to study statistical properties of the function $f(t, x)$. First we introduce stochastic system which will specify dynamic characteristics of the option price function $f(t, x)$. Using Markov property of the function $S(l; t, x)$ follows

$$f(t - \Delta t, x) = f(T, S(T; t, q))_{q = S(t; t - \Delta t, x)} \; S^{-1}(T; t, p)_{p = S(t; t - \Delta t, 1)}$$

This identity suggests to introduce an auxiliary function

$$F(t, x, y) = f(T, S(T; t, x)) \; S^{-1}(T; t, y) \tag{3.17}$$

Then the derivative price *f* can be interpreted as a contraction of the *F*,

$$f(t, x) = F(t, x, 1).$$

This remark is essential for the next study. The function $f(t, x)$ does not hold the Markov property over one dimensional coordinate space but it actually does over two dimensional coordinate space. Thus, we should first study analytic properties of the function $F(t, x, y)$ and then putting $y = 1$ we can specify them for the option value f. Consider the problem of finding moments of the derivatives price. Assume that $f(T, x)$ is a continuous function in x having growth not faster than a polynomial function. Then expectation of the derivative price $m(t, x, y)$ is a classical solution of the backward Cauchy problem

$$\frac{\partial m(t, x, y)}{\partial t} + \mu (x\nabla_x + y\nabla_y) m(t, x, y) +$$

$$+\frac{\sigma^2}{2}(x\nabla_x + y\nabla_y)^2 m(t, x, y) = 0 ; \qquad t, x \geq 0 , \quad y > 0 \tag{3.18}$$

with boundary condition $m(T, x, y) = f(T, x)/y$.
Thus the function $m(t, x) = m(t, x, 1) = E\, f(t, x)$ is the mean of the option price. Indeed, if the function $f(T, x)$ two times continuously differentiable in x then the proof of the theorem follows from standard stochastic calculus results [16,17]. Based on option contract definition this function always contains irregular point $x = K$ so that we could not apply general results. We will prove smoothness of the function $m(t, x, y)$ directly using its analytic representation. With help of Ito formula one can verify that the function $S(T; t, x)$ have normal distribution with mean

$$\ln x + (\mu - \frac{\sigma^2}{2})(T - t)$$

And standard deviation σ ∉ (T – t). Hence

$$P\{ S(T; t, x) < x \} = \Phi \{ \frac{\ln q - \ln x - (\mu - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{(T - t)}} \}$$

where

$$\Phi(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y} \exp - \frac{q^2}{2}\, dq$$

The density of the process $S(T; t, x)$ is

$$\rho(t, x; T, y) = [2\pi (T - t)\sigma^2 y^2]^{-\frac{1}{2}} \exp - \frac{[\ln \frac{y}{x} - (\mu - \frac{\sigma^2}{2})(T - t)]^2}{2\sigma^2 (T - t)}$$

Having the density formula one can to find probability that the option price is not zero

$$P\{ f(t,x) > 0 \} = P\{ S(T;t,x) > K \} =$$
$$= 1 - \Phi\left(\frac{\ln K - \ln x - (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \right)$$

Taking into account (3.16) note

$$m(t, x, y) = E\, f(T, S(T;t,x))\, S^{-1}(T;t,y) = \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \frac{f[T, \ln x - (\mu - \frac{\sigma^2}{2})(T-t) + \sigma q]}{y \exp[(\mu - \frac{\sigma^2}{2})(T-t) + \sigma q]} (2\pi)^{-\frac{1}{2}} \exp - \frac{q^2}{2}\, dq$$

By definition f (T , x) = max (x – K , 0) and therefore

$$m(t, x, y) = \sqrt{\frac{T-t}{2\pi}}\ \frac{x}{y}\left[1 - \Phi\left(\frac{\ln K - \ln x - (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \right)\right]$$

We can check directly that partial derivatives of the first and the second orders with respect to x and y are continuous functions for any $t < T$. Here we present formulas only for the first order derivatives in x , y because expressions for higher order derivatives look pretty cumbersome. Thus,

$$\frac{\partial}{\partial x} m(t, x, y) = \frac{1}{x} m(t, x, y) +$$
$$+ \frac{\exp - \{ [\sigma\sqrt{2(T-t)}]^{-1} \ln\frac{K}{x} - (\mu - \frac{\sigma^2}{2})(T-t) \}^2}{\sigma\sqrt{(T-t)}})]$$

$$\frac{\partial}{\partial y} m(t, x, y) = -\frac{1}{y} m(t, x, y)$$

Using the same method one can verify that the moment of the n-th order

$$m_n(t, x, y) = E F^n(t, x, y), \quad n = 2, 3, \ldots$$

is the solution of the equation (34) with boundary condition

$$m_n(T, x, y) = \left[\frac{f(T, x)}{y}\right]^n, \quad n = 2, 3, \ldots$$

In particular, standard deviations of the derivative price can be computed by the formula

$$[\, m_2(t, x, 1) - m(t, x, 1)\,]^{\frac{1}{2}}$$

Assuming that the function F (t , x , y) is 2-times continuously differentiable in x , y we can proof that this function is a solution of stochastic partial differential equation. Detailed investigation in that field can be found in [15, 16]. The correspondent techniques used are more complicated than standard stochastic calculus.
Statement. Function $F(t, x, y)$ is a solution of the backward Cauchy problem

$$F(t, x, y) = \frac{f(T, x)}{y} + \int_t^T [\, \mu\,(x \nabla_x + y \nabla_y) + \frac{\sigma^2}{2}(x \nabla_x + y \nabla_y)^2\,]\, F(l, x, y)\, dl - \int_t^T \sigma\,(x\nabla_x + y \nabla_y)\, F(l, x, y)\, d\overleftarrow{w}(l) \tag{3.20}$$

where stochastic integral at right hand side of (3.20) is Ito backward stochastic integral. This equation can be rewritten in differential form

$$\frac{\partial F(t, x, y)}{\partial t} + [\mu\,(x \nabla_x + y \nabla_y) + \frac{\sigma^2}{2}(x \nabla_x + y \nabla_y)^2\,]\, F(l, x, y) + \sigma\,(x\nabla_x + y \nabla_y)\, F(l, x, y)\, \dot{w}(l), \qquad F(T, x, y) = \frac{f(T, x)}{y}$$

Here is the short draft of the proof. Detailed techniques used in the proof can be found in [16]. Set

$$F(t, x, y) - F(T, x, y) = \sum_k F(t_{k-1}, x, y) - F(t_k, x, y) \tag{3.21}$$

where

$$t = t_1 < t_2 < \ldots\ldots < t_N = T$$

is a partition of the interval [0 , T]. Applying Taylor's formula note

$$F(t - \Delta t, x, y) - F(t, x, y) = F(t, S(t; t - \Delta t, x), S(t; t - \Delta t, y)) -$$

$$- F(t, x, y) = [\mu\,(x\nabla_x + y\nabla_y) + \frac{\sigma^2}{2}(x\nabla_x + y\nabla_y)^2]\, F(t, x, y)\Delta t +$$

$$+ [\sigma\,(x\nabla_x + y\nabla_y)\, F(t, x, y)\; \Delta w(t) + \varepsilon((\Delta t)^2)]$$

where P. lim • (h) / h = 0. Putting there

$$t = t_k\,,\quad \Delta t = t_k - t_{k-1},\quad \Delta w(t) = w(t_k) - w(t_{k-1})$$

and taking into account that function $F(t, x, y)$ is measurable with respect to family σ- algebras σ{ $\Delta w(l),\ l \le t$ } and substitute these into the formula (3.21) and observing that

$$\Delta w(t) = -[\,w(t_{k-1}) - w(t_k)\,]$$

we find out that right hand side of (3.21) tends to (3.20) when partition becomes finer.

As far as the option pricing model can not perfectly replicates security we need to establish a portfolio that can do it. From (3.16) ensues that

$$\frac{f(T, S(T; t, x))}{f(t, x)} + \frac{S(T; t, x)}{x}\,\chi\{S(T; t, x) \le K\} = \frac{S(T; t, x)}{x} \qquad (3.22)$$

The first term on the left is strictly positive when { ω: S (T ; t , x) > K }. Introduce a portfolio which value at date *t* is

$$\Pi(t, x) = x\,\chi\{S(T; t, x) \le K\} + f(t, x)\,\chi\{S(T; t, x) > K\}$$

At the expiration date the value of the portfolio equal to

$$\Pi(T, S(T)) = S(T)\,\chi\{S(T; t, x) \le K\} + f(T, S(T))\,\chi\{S(T; t, x) > K\}$$

Thus the return of the portfolio identical to the return on stock

$$\frac{\Pi(T, S(T))}{\Pi(t, x)} = \frac{\Pi(T, S(T))}{\Pi(t, x)}[\,\chi\{S(T; t, x) \le K\} + \chi\{S(T; t, x) > K\}] =$$

$$= \frac{S(T)}{x}\,\chi\{S(T; t, x) \le K\} + \frac{f(T, S(T))}{f(t, x)} = \frac{S(T; t, x)}{x}$$

The portfolio shows how to include the option into market and avoid arbitrage opportunity. We can replace the theoretical portfolio Π by its approximation

$$\Xi(t, x) = x P\{ S(T; t, x) \le K \} + f(t, x) P\{ S(T; t, x) > K \}$$

The amount of stocks and options in the portfolio Ξ can be easy calculated using the explicit distribution function of the random process S (T ; t , x) given above. Some additional risk of replacing portfolio Π by the Ξ can be described by the event

$$\frac{\Xi(T, S(T))}{\Xi(t, S(t))} < \frac{\Pi(T, S(T))}{\Pi(t, S(t))}$$

In this case the rate of return of the portfolio Ξ will be lower than the rate of Π and therefore could not perfectly replicate stock rate that leads to the change of the option price. In more realistic option market there is a set of strike prices. Denote

$$K_1 < K_2 < .... < K_N$$

an ordered sequence of the strike prices. Applying formula (3.22) for each strike price one can justify that

$$\frac{f(T, S(T; t, x); K_j)}{f(t, x; K_j)} - \frac{f(T, S(T; t, x); K_{j+1})}{f(t, x; K_{j+1})} =$$
$$= \frac{S(T; t, x)}{x} \chi\{ K_j \le S(T; t, x) < K_{j+1} \}$$

Summing up over all values of j we arrive at identity

$$\sum_{j=1}^{N-1} \frac{f(T, S(T; t, x); K_j)}{f(t, x; K_j)} - \frac{f(T, S(T; t, x); K_{j+1})}{f(t, x; K_{j+1})} =$$
$$= \frac{S(T; t, x)}{x} \chi\{ K_1 \le S(T; t, x) < K_N \}$$

To reach perfect replication of the security rate we need to rewrite this formula

$$\frac{S(T; t, x)}{x} \chi\{ K_1 \le S(T; t, x) \} + \sum_{j=1}^{N-1} [\frac{f(T, S(T; t, x); K_j)}{f(t, x; K_j)} -$$
$$- \frac{f(T, S(T; t, x); K_{j+1})}{f(t, x; K_{j+1})}] + \frac{f(T, S(T; t, x); K_N)}{f(t, x; K_N)} = \frac{S(T; t, x)}{x}$$

Let us introduce the portfolio

$$\Pi(t, x) = x\chi\{ S(T) \le K_1\} + \sum_{j=1}^{N-1}[f(t, x; K_j) -$$

$$- f(t, x; K_{j+1})]\chi\{ K_{j+1} \le S(T) < K_{j+1}\} + f(t, x; K_N)\chi\{ S(T) > K_N\}$$

The portfolio $\Pi(t, x)$ combines random portion

$$\chi\{ S(T) \le K_1 \}$$

of stock, the portion

$$\chi\{ K_j \le S(T) < K_{j+1} \}$$

invested in the vertical call bullspread

$$K_j \,/\, K_{j+1} \quad j = 1, 2, ..., N$$

and the portion

$$\chi\{ S(T) > K_N \}$$

invested in the option with the highest strike price.
Now we will study a model that represents an effect of the bond. Until now we ignore bond as a security that can change option valuation. Nevertheless bond is an important factor of the market. We will show how to adjust the given option price definition. A generalization of the equal rate of return principle for the market with bond and stock can be taking in a form

$$\frac{f(T, S(T; t, x))}{f(t, x)} = \max\left\{ \frac{S(T; t, x)}{x}, \frac{B(T)}{B(t)} \right\}$$

Then the option price at time t is

$$f(t, x) = f(T, S(T; t, x))\left[\max\left\{ \frac{S(T; t, x)}{x}, \frac{B(T)}{B(t)} \right\} \right]^{-1}$$

Then the call option price can be rewritten

$$f(t,x) = \max(\ S(T;t,x) - K\ ,\ 0\ ) \max\{\ \exp - r(T - t)\ ,\ S^{-1}(T;t,x)\ \} =$$

$$= \max(\ S(T;t,x) - K\ ,\ 0\ )\{\ \exp - r(T - t)\,\chi(\ S(T;t,x) \geq \exp - r(T - t)) +$$

$$+ S^{-1}(T;t,x)\,\chi(\ S(T;t,x) < \exp - r(T - t))\}$$

Using this formula one can write down analytic expression for the n-th moment of the call option price

$$m_n(t\ ,x\ ,y) = E\ f^{\,n}(\ t,x\ ) =$$

$$= \int_K^{\infty}(q - K)^n\ [\exp - r(T - t)\,\chi(\frac{q}{x} \geq \exp\ r(T - t)) +$$

$$+ \frac{x}{q}\,\chi(\frac{q}{x} < \exp\ r(T - t))\,\Phi(\frac{\ln q - \ln x - (\mu - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{(T - t)}})\ dq$$

n = 1, 2, … . The perfect portfolio now should contain the certain portions of options, stocks, and bonds. The structure of the portfolio at time t is

$$\overline{\Pi}(t,x) = f(t,x)\,\chi\{\ S(T) > K\ \} + x\,\chi\{\ x\frac{B(T)}{B(t)} < S(T) \leq K\ \} +$$

$$+\ B(t)\,\chi\{\ S(T) \leq x\frac{B(T)}{B(t)}\ \}$$

Assume, for simplicity that

$$\frac{B(T)}{B(t)} < \frac{K}{x}$$

though the general cases can also be considered. Note that as it was shown above that

$$\frac{\overline{\Pi}(T, S(T))}{\overline{\Pi}(t,x)} = \max\{\ \frac{S(T)}{x}\ ,\ \frac{B(T)}{B(t)}\ \}$$

Inasmuch as

$$\chi\{\ S(T) > K\ \} + \chi\{\ x\frac{B(T)}{B(t)} < S(T) \leq K\ \} + \chi\{\ S(T) \leq x\frac{B(T)}{B(t)}\ \} = 1$$

then each term in the left hand side can be interpreted as a random portion of shares in the portfolio.
Here we make a few remarks to some important for applications results widely used in the modern option theory.
Remark 1. The study of arbitrage in option pricing gave birth to the theory well known as put-call parity. Let us represent call-put parity based on given definitions. From the definition of option price follows that

$$c(t,x) = \frac{x}{S(T)}\,[\,S(T) - K\,]\,\chi\{\,S(T) > K\,\}$$

$$p(t,x) = \frac{x}{S(T)}\,[\,S(T) - K\,]\,\chi\{\,S(T) \le K\,\}$$

Therefore the corrected put-call parity can be expressed by the formula

$$p(t,x) - c(t,x) = \frac{x}{S(T)}\,[\,K - S(T)]$$

Note that this relationship does not depend on a security model.
Remark 2. Here we comment the numeric method, which was applied for the calculation of the Black-Scholes call option price. This approach to the problem is quite important and used in many books.
To calculate the theoretical value of a call option applying Black-Scholes option model we need to know the following: the stock price, the option striking price, the time until expiration, the riskless interest rate. The stock price $S(t)$ is the solution of the equation

$$d\ln \mathrm{S}(t) = (\mu - \frac{\sigma^2}{2})\,dt + \sigma\, d\,w(t)$$

(1.5) and has a log-normal distribution. Indeed applying Ito formula to the function $S(t)$ we have

Therefore the difference [$\ln S(T) - \ln x$] is normally distributed with mean and variance

$$(\mu - \frac{\sigma^2}{2})(T-t) \quad and \quad \sigma^2 (T-t)$$

correspondingly. Parameters μ and σ are unknown and should be estimated from data given by observation. The parameter μ is the expected rate of return. In the deterministic setting it follows that

$$\frac{d\ln S(t)}{S(t)} = d\ln S(t)$$

and therefore instantaneous rate of return equal to the differential of natural logarithm. In stochastic setting it is wrong and we will proof it bellow.
We first look at the explanation given in [11, p.231]. Applying Ito formula for the natural logarithm function they noticed that the expected continuously compounded rate of return over the interval [t , T] is

$$(\mu - \frac{\sigma^2}{2}) (T - t)$$

Then in previous chapter μ was interpreted as the expected value of the rate of return in any short time interval. To illustrate how it can be the example was considered. We follow by the book and consider the example. Suppose that the following is the sequence of the actual returns per annum realized on a stock in five consecutive years, measured using annual compounding: 15% , 20% , 30% , 20% , 25% . The arithmetic mean of the returns is calculated by taking the sum of returns and divided by 5 give us 14%. However, investors would earn less than 14% if they left money invested in the stock for five years. The actual average return per a year earned by the investor compounded annually is

$$[(1+0.15)(1+0.2)(1+0.3)(1+0.2)(1+0.25)]^{\frac{1}{5}} - 1 = 0.124$$

or 12.4% per annum. This example indicates that we should expect estimate 14% to be greater than estimate 12.4% and the difference between

$$\mu \text{ and } \mu - \frac{\sigma^2}{2}$$

how that this is the similar case. The conclusion is the expected rate of return in infinitesimally short period of time is μ and the expected continuously compounded rate of return is

$$\mu - \frac{\sigma^2}{2}$$

These arguments show that the term expected return is ambiguous. It can refer either to

$$\mu \text{ and } \mu - \frac{\sigma^2}{2}$$

and unless otherwise stated we will use it to refer to μ throughout the book.
Now we will explain the paradox. Let us recall the algebraic inequality

$$(\prod_{j=1}^{N} a_j)^{\frac{1}{N}} \le \frac{1}{N} \sum_{j=1}^{N} a_j$$

valid for non negative numbers. Letting

$$a_j = 1 + r_j \ , \quad r_j > -1$$

we will get the proof the phenomenon. Also this algebraic inequality does not have any connections to the stochastic structure of the Ito equation where parameter σ is involved. From the general stochastic calculus follows that if $\sigma \neq 0$ then

$$\frac{d\,S(t)}{S(t)} \neq d \ln S(t)$$

and therefore *d ln S (t)* an not be referred to as expected rate of return and there are no ambiguous with the stock rate of return.

Now we introduce more general construction of the option price problem. Let $B (t , T^*)$ be the bond price at time t , T^* stands for maturity, and the face value of such bond is \$1, i.e. $B (T^* , T^*) = \$1$. We also assume that $T > T^*$, where T denotes option maturity. Next idea provides a new more secure method of derivatives pricing.

Assume that an investor decides to buy 'covered' call. A covered call means a portfolio with long call option and long underlying stock. In such case it is reasonable to find the option price which guarantee the portfolio the same rate of return as the stock. This price can be found by solving equation

$$\frac{f(T, S(T)) + q\,S(T)}{f(t, x) + q\,x} = \frac{S(T)}{x}$$

where q is a constant. Remarkably that this equation makes sense for any elementary event not only when $S (T) > K$.

Consider a contract which contain one call option and q , $q = 0, 1, 2, \ldots$ shares of bond. The payoff of such contract is $max (X - K , 0) + q\,B (t , T^*)$, where X is the stock price at time T , and K is the option strike price. Our goal is to find the fair price of such derivatives contract. The equal rate of return on the derivatives contract and stock lead us to the equation

$$\frac{G(T, S(T; t, x)) + q\ B(T, T^*)}{G(t, x) + q\ B(t, T)} = \frac{S(T; t, x))}{x} \tag{3.23}$$

Here, G denotes the option price in the derivatives contract. Note that the numerator in left-hand side does not equal to 0. Therefore the denominator is also strictly positive. Thus, if the bond price is the solution of the equation (1.6) with boundary condition

$B(T,T) = \$1$ then the call option price is that

$$G(t,x) = \frac{x}{S(T;t,x)} \max(S(T;t,x) - K, 0) +$$

$$+ \exp - r(T^* - T)\left[\frac{x}{S(T;t,x)} - \exp - r(T-t)\right]$$

Remarkably, that $G(t, x)$ can be positive or negative signs nevertheless the value of the derivatives contract is always positive.
More general model comes into the existence if the rate of return of the derivatives contract we used above considered against the stock portfolio consisting from the stock and p shares of bonds. Here p, $p > 0$,can be equal or not to q. In this case equal rate of return on two investment opportunities can be expressed by the equation

$$\frac{G(T, S(T;t,x)) + q\, B(T, T^*)}{G(t,x) + q\, B(t,T)} = \frac{S(T;t,x)) + p\, B(T,T^*)}{x + p\, B(t,T^*)}$$

Then the value G (t , x) of the call option in the such contract is

$$G(t,x) = \frac{x + p\, B(t, T^*)}{S(T;t,x) + p\, B(t, T^*)} \max(S(T;t,x) - K, 0) +$$

$$+ \left[\frac{x\, B(T, T^*) - B(t, T^*)\, S(T;t,x)}{S(T;t,x) + p\, B(t, T^*)}\right]$$

Thus we show the way in which variety of option price strategies can be produce. Every solution has particular properties depending on the parameters p and q. These parameters may also be functions depending on time. This, in turn, gives possibility more closely to meet a particular investor's interest.
Now we introduce the other fundamental option type called American option. American call option gives a buyer the right to exercise it at any moment of time within its lifetime. American option is very popular and traded on numerous exchanges all over the world. The American option feature is often embedded in various financial instruments. For instance, US T-bonds , corporate bonds often contain provision similar to American call option. Convertible bonds can be converted into common stocks after a given period of time, implying that the option is initially European and then American. Recall well-known strategy stems from the American option-pricing model on no dividend common stock. It does not recommend exercising an American call option early. This can be interpreted that by waiting an investor saves the interest that he would lost by paying strike price earlier than maturity. For put option on no dividends stock it may be optimal to exercise early because the potential gains are bounded by the strike price.
Here we develop another model of American option pricing based on idea of equal rate of

return represented above for European option. It is obvious that the base of the known strategy ensues from Black and Scholes interpretation and it is irrelevant to our construction.

Let $S(l)$, $l \in [t, T]$ be the stock price governed by the equation (1.5). The American call option on stock is the right to buy shares at

$$f_{ac}(l, S(l; t, x)) = \$\max\{S(l; t, x) - K, 0\}$$

at any moment at $l \in [t, T]$ of the lifetime of the option. American put option is aright to sell shares at

$$f_{ap}(l, S(l; t, x)) = \$\max\{K - S(l; t, x), 0\}$$

at any moment $l \in [t, T]$ within its lifetime. The American option pricing problem is to determine fair option price at time t. The advantage of American type with respect to its European counterpart is that the American option can be exercised at any moment $l \in [t, T]$ that in turn suggests higher rate of return than European. Therefore American option price should be greater. We will show how these intuitive arguments can be formally expressed by mathematical formulas.

The main pricing principle that suggests equal rate of return on option and underlying stock in case when option payoff is greater than zero makes it possible to reach maximum theoretical return during the option's lifetime. At any $l \in [t, T]$ the return on stock is

$$S(l; t, 1) = \frac{S(l; t, x)}{x}$$

Thus maximum return on stock can be written as

$$\bar{S} = \max_{t \le l \le T} S(l; t, 1)$$

Denote random time $\tau(\omega)$ by the formula

$$\bar{S} = S(\tau(\omega); t, 1)$$

It can be proved that $\tau(\omega)$ is a random variable and this random variable is not a Markov moment of time because the realization of the event $\{\omega : \tau(\omega) > u\}$ can not be justified based on observations of the stock price until the moment u. Thus it is impossible to apply powerful theory of Markov processes to get optimal return on stock. Nevertheless having distribution of the $\tau(\omega)$ we can estimate risk that predetermine return will be reached over the lifetime of the stock. In this scheme the first passage time of the chosen level of return can be considered as a reasonable approximation of the $\tau(\omega)$. First find an explicit formula for the distribution function of the random variable $\bar{S}$. Without loss of generality we put $t = 0$. The solution of the equation (1.5) with the initial condition $S(0) = 1$ can be written in the form

$$S(l;0,1) = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) l + \sigma\, w(l)\right]$$

Then

$$P(\overline{S} > x) = P\{\max_{t \le l \le T} \exp[(\mu - \frac{\sigma^2}{2})l + \sigma\, w(l)] > x\} =$$

$$= P\{\max_{t \le l \le T} [(\mu - \frac{\sigma^2}{2})l + \sigma\, w(l)] > \ln x\}$$

Applying Girsanov's theorem to the right hand side of this equality we arrive at

$$P(\overline{S} > x) = \bar{E}\,\chi\{\max_{t \le l \le T} \overline{w}(l) > \frac{\ln x}{\sigma}\} \exp\left[\frac{(\mu - \frac{\sigma^2}{2})}{\sigma}\overline{w}(T) - \right.$$

$$\left. - \frac{(\mu - \frac{\sigma^2}{2})^2}{2\sigma^2} T\right\} = \exp\left[-\frac{(\mu - \frac{\sigma^2}{2})^2}{2\sigma^2} T\right] \times \tag{3.24}$$

$$\times\, \bar{E}\,\chi\{\max_{t \le l \le T} \overline{w}(l) > \frac{\ln x}{\sigma}\} \exp\left[\frac{(\mu - \frac{\sigma^2}{2})}{\sigma}\overline{w}(T)\right]$$

where $\overline{w}$, $\overline{E}$ are a Wiener process and the expectation with respect to the probability measure

$$\overline{P}(d\omega) = \exp\left\{\frac{(\mu - \frac{\sigma^2}{2})}{\sigma}\overline{w}(T) - \frac{(\mu - \frac{\sigma^2}{2})^2}{2\sigma^2} T\right\} P(d\omega)$$

Next, for brevity, we wll omit upper 'bar' symbol in (3.24). To calculate mathematical expectation in the right hand side of the equality (3.24) we need the auxiliary theorem.

Theorem [16] (Bachelier ***)***. For any $q > 0$,

$$P\{\max_{t \le l \le T} w(l) > q,\ \ w(T) \in [c, d]\} = \int_{c \vee q}^{d \vee q} \rho(y)\, dy + \int_{(2q-d)\vee q}^{(2q-c)\vee q} \rho(y)\, dy \tag{3.25}$$

where $d\ 2\, q = max\ (d, q)$, and

$$\rho(y) = \frac{1}{\sqrt{2\pi T}} \exp - \frac{y^2}{2T}$$

This formula we use to perform the density with respect to variable *d*. Putting $d = c + \Delta c$ where $\Delta c > 0$ is infinitesimally small the first integral in the right will be equal to

$$\int_{c \vee q}^{(c+\Delta c)\vee q} \rho(y)\, dy = \{ \int_{c}^{c+\Delta c} \rho(y)\, dy \;, \; when \; c \geq q$$

$$and \;\; 0\,, \quad when \;\; c < q \} = \chi\{ c \geq q \} \int_{c}^{c+\Delta c} \rho(y)\, dy$$

Here, $\Delta c > 0$ was chosen such small that if $c < q$ then $c + \Delta c < q$. The second integral on the right hand side of (3.25) we then represent as

$$\int_{(2q-c-\Delta c)\vee q}^{(2q-c)\vee q} \rho(y)\, dy = \{ \int_{2q-c-\Delta c}^{2q-c} \rho(y)\, dy \;, \; when \; 2q - c \geq q$$

$$and \;\; 0\,, \quad when \;\; 2q - c < q \} = \chi\{ c < q \} \int_{2q-c-\Delta c}^{2q-c} \rho(y)\, dy$$

With the help of these calculations the formula (3.25) can be rewritten as

$$P\{ \max_{t \leq l \leq T} w(l) > q\,, \;\; w(T) \in [\, c, \; c + \Delta c) \; \} =$$

$$= E\,[\, \chi\{ c \geq q \}\rho(c) + \chi\{ c < q \}\rho(2q - c)\,]\,\Delta c + o(\Delta c) \tag{3.26}$$

where $\lim\; o(\Delta c)/\Delta c = 0$. Now we will apply formula (3.25) to compute the expectation in the right of the (3.25). Denote

$$\lambda = \frac{(\mu - \frac{\sigma^2}{2})}{\sigma} \;, \quad Q = \exp - \frac{\lambda^2 T}{2}.$$

Then

$$P(\overline{S} > x) = Q\int_{-\infty}^{+\infty} \exp \lambda c \, [\, \chi\{c \geq \frac{\ln x}{\sigma}\}\rho(c) + \chi\{c < \frac{\ln x}{\sigma}\}\rho(\frac{2\ln x}{\sigma} - c)\,]\, dc =$$

$$= Q\int_{\frac{\ln x}{\sigma}}^{+\infty} \rho(c)\exp \lambda c \, dc + \int_{-\infty}^{\frac{\ln x}{\sigma}} \rho(\frac{2\ln x}{\sigma} - c)\exp \lambda c \, dc = Q\{ I_1 + I_2 \}$$

Let us calculate integrals in the right hand side.

$$I_1 = \frac{1}{\sqrt{2\pi T}} \int_{\frac{\ln x}{\sigma}}^{+\infty} \exp(\lambda c - \frac{c^2}{2T})\, dc = \frac{1}{\sqrt{2\pi T}} \exp \frac{\lambda^2 T}{2} \times$$

$$\times \int_{\frac{\ln x}{\sigma}}^{+\infty} \exp -\frac{1}{2}(\frac{c}{\sqrt{T}} - \lambda\sqrt{T})^2 \, dc = \frac{1}{\sqrt{2\pi}} Q^{-1} \int_{\frac{\ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T}}^{+\infty} \exp -\frac{c^2}{2}\, dc =$$

$$= Q^{-1}(1 - \Phi(\frac{\ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T}))$$

where $\Phi(y)$ is the standard Gaussian distribution function.
Then

$$I_2 = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{\ln x}{\sigma}} \exp\{\lambda c - \frac{1}{2T}(c - \frac{2\ln x}{\sigma})^2\}\, dc =$$

$$= \exp\{-\frac{2\ln^2 x}{\sigma^2 T} + \frac{T}{2}(\lambda + \frac{2\ln x}{\sigma T})^2\} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{\ln x}{\sigma}} \exp -\frac{1}{2}[\frac{c}{\sqrt{T}} -$$

$$-\sqrt{T}(\lambda + \frac{2\ln x}{\sigma T})]^2 dc = Q^{-1} x^{\frac{2\lambda}{\sigma}} \Phi[\frac{\ln x}{\sigma\sqrt{T}} - \sqrt{T}(\lambda + \frac{2\ln x}{\sigma T})]$$

Thus we arrive at the formula

$$P\{ \max_{0<l<T} S(l;0,1) > x \} = 1 - \Phi\left(\frac{\ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T} \right) +$$

$$+ \quad x^{\frac{2\lambda}{\sigma}} \Phi\left(- \frac{\ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T} \right) \tag{3.27}$$

where S (1 ; 0, 1), l ∈ [0 , T] is the solution of the equation (1.5) such that S (0 ; 0, 1) = 1. Using (3.27) it is easy to compute statistical characteristics of the random variable $\overline{S}$. Then the chance that the stock price touches the barrier L, L > K over time interval [0 , T] is

$$P\{ \max_{0<l<T} S(l;0,x) > L \} = 1 - \Phi\left(\frac{\ln L - \ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T} \right) +$$

$$+ \left(\frac{L}{x}\right)^{\frac{2\lambda}{\sigma}} \Phi\left(- \frac{\ln L - \ln x}{\sigma\sqrt{T}} - \lambda\sqrt{T} \right)$$

As it was indicated the random time τ (ω) is the optimal exercise time of the American option and τ (ω) is not the Markov stopping time. Therefore an investor who wishes to receive maximum return need to collect the stock quotes over lifetime span [0 , T]. This circumstance makes it difficult the practical realization of the maximum return strategy. On the other hand an investor can establish reasonable barrier ν, ν > K such that if stock crosses this barrier then the investor exercises the option. Others strategies may also be reasonable. Let ν be the chosen barrier and

$$p_\nu = P\{ \overline{S} > \nu)$$

be the correspondent risk the investor willing to hold while the stock return hits level ν. Note that this risk-probability can be also expressed by the formula

$$p_\nu = P\{ \max_{l<T} S(l;0,x) > \nu\, x)$$

Denote $\tau = \tau_x(\nu x)$ the first passage time of the barrier ν *x* by the random process *S (l ; 0 , x)*. Then one can note that $\tau = \tau_1(\nu)$. The equal rate of return principle applied to the Markov stopping time yields

$$\frac{f(T \wedge \tau, S(T \wedge \tau; 0, x))}{f(0, \mathrm{x})} = \frac{S(T \wedge \tau; 0, x)}{x} \chi\{ S(T \wedge \tau; 0, x) > K \} \tag{3.28}$$

where $c \wedge d = min(c, d)$. By definition we also put $f(t, x) = 0$ for

$$\omega \in \{ \omega : S(T \wedge \tau ; 0, x) \le K \}$$

Then the option price at initial time $t = 0$ is

$$f(0, x) = S^{-1}(T \wedge \tau)\, f(T \wedge \tau,\ S(T \wedge \tau))\, \chi\{ S(T \wedge \tau) > K \} =$$

$$= [\, \nu^{-1}\, \chi(\tau \le T) + S^{-1}(T)\ \ \chi(\tau > T)\,] \times \tag{3.29}$$

$$\times \max \{[(\nu - K)\, \chi\ (\tau \le T) + (S(T) - K)\chi(\tau > T)\ ], 0\}\, \chi\{ S(T \wedge \tau) > K\}$$

The formula (3.29) represents American-barrier call option pricing solution in analytic form and this solution for the given level ν can be treated as an approximation general American option pricing problem. The investor's portfolio

$$\Pi_{ac}(t, x) = x\, \chi\{ S(T \wedge \tau) \le K \} + f(t, x)\, \chi\{ S(T \wedge \tau) > K \}$$

perfectly replicates stock rate return.

The modern stochastic calculus provides elegant way to describe statistical properties of the first passage time moments. Let $\delta > 0$ be a small number and introduce the first passage time

$$\tau(\delta, \nu) = \min\{ t : S(l ; 0, x) \notin (\delta, \nu)\}, \qquad \textit{and} \qquad \tau(\delta, \nu) = +\infty ,$$

if the set in the braces is empty. The important condition that should be fulfilled is that diffusion coefficient must be nondegenerated. Though the diffusion coefficient of the equation (1.5) is equal to 0 when x = 0 nevertheless bearing in mind Lemma we can conclude that this level is not accessible by the solution of the equation (1.5).

Theorem [17]. The first and the second moments $\{ u(x), v(x) \}$ of the random moment $\tau(\delta, \nu)$ are the solution of the system

$$L(x)\, u(x) = -1$$

$$L(x)\, v(x) = -2\, u(x), \qquad x \in (\delta, \nu) \quad , \tag{3.30}$$

$$L(x) = \frac{(\sigma x)^2}{2} \frac{d^2}{d x^2} + \mu x \frac{d}{d x}$$

with 0-boundary conditions.

Corollary. The solution of the system (3.30) depends on parameter δ. Using analytical form of the solution of the system it is possible to prove that there exist limit of the solution when δ tends to zero and therefore we can obtain the statistical characteristics of the moments $\tau_1(\nu)$. This theorem helps to estimate average waiting time until exercising the derivative instrument.

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