Evolution of Coalition Structures under Uncertainty

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Abstract

In Hart and Kurz (1983), stability and formation of coalition structures has been investigated in a noncooperative framework in which the strategy of each player is the coalition he wishes to join. However, given a strategy profile, the coalition structure formed is not unequivocally determined. In order to solve this problem, they proposed two rules of coalition structure formation: the γ and the δ models.

In this paper we look at evolutionary games arising from the γ model for situations in which each player can choose mixed strategies and has vague expectations about the formation rule of the coalitions in which is not involved; players determine at every instant their strategies and we study how, for every player, subjective beliefs on the set of coalition structures evolve coherently to the strategic choices. Coherency is regarded as a viability constraint for the differential inclusions describing the evolutionary game. Therefore, we investigate viability properties of the constraints and characterize velocities of pairs belief/strategies which guarantee that coherency of beliefs is always satisfied. Finally, among many coherent belief revisions (evolutions), we investigate those characterized by minimal change and provide existence results.

Keywords: Coalition formation; coherent beliefs; differential inclusions; viability theory; minimal change belief revision.

1 Introduction

As recognized by von Neumann and Morgenstern (1944) in their seminal work, the problem of coalition formation plays a central role in game theory. A significative topic in the theory of coalition formation is the investigation of mechanisms and processes which determine how individuals choose to partition themselves into mutually exclusive and exhaustive coalitions, that is, into a coalition structure. It has been shown that in many situations

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the payoffs of the players belonging to a coalition S depend also on the way players outside S are organized in coalitions and, as a consequence, the strategic choices of the agents depend on the whole coalition structure (see for example Greenberg (2002)). In the context of classical TU characteristic form games, such a point of view has been taken into account, for instance, by defining "power indexes" deriving from the coalition structure formed or by considering "characteristic functions" depending on the whole coalition structure (see also Owen (1977), Hart and Kurz (1983) or Myerson (1978)).

Stability of coalition structures has been analyzed via concepts of equilibrium in associated strategic form games. The key feature for this approach is that the strategy set of each player j is the set of all subgroups of players containing j and his choice represents the coalition he wishes to join. However, given a strategy profile (i.e. a coalition for each player), the coalition structure formed is not unequivocally determined. So, different rules of coalition structure formation can be considered, namely, functions associating to every strategy profile a coalition structure. In Hart and Kurz (1983) the following rules are proposed: the so called model γ in which a coalition S forms if and only if all its members have chosen it; the other players become singletons. The model δ in which a coalition S forms if and only if it is a maximal coalition in which all its members have chosen the same coalition T (which might be different from S); the other players become singletons. Note that, given a strategy profile, the γ and the δ rules determine whether each coalition forms or not and *consequently* determine a unique coalition structure.

A fundamental assumption in the Hart and Kurz model is that each player j makes his choice knowing not only the strategies of every other player, but also the formation rule of coalitions in which is not involved. However, in many situations it happens that the formation of a coalition is the outcome of private communication within the members of the coalition (see Moreno and Wooders (1996) and references therein). Hence, differently from the previous literature, in this paper, we consider the case in which each player has vague expectations about the choices of his opponents corresponding to the coalitions in which is not involved and about the formation rule of these coalitions. Moreover, in this paper, we are interested in the evolutionary games arising from the static model of coalition formation. More precisely, we consider the situation in which players determine at every instant the set of players they wish to join and then we study how the coalition structure evolves according to these strategic choices. To this purpose the classical concept of coalition as a subset of players, also called crisp coalition, seems to be not well suited. In fact, such concept implicitly presupposes that a group of players signs a contract (either they cooperate or not), the realization of which requires the involved players to cooperate with a full commitment. In a dynamical setting, it seems natural to assume that a player is not asked to sign a contract at any instant, but rather to announce the set of players he wishes to join. According to this point of view, Konishi and Ray (2003) state that uncertainty enters into the coalition formation process in a natural way: whenever a player has more than one reasonable move he might randomize among them. So we assume that, at any instant, each player j can randomize his choice by playing a mixed strategy, that is, a probability distribution on the set of all coalitions containing j, called *mixed coalition* (De Marco and Romaniello (2006)).

Therefore, in our approach, each player j, knowing only the components of the strategies of the other players corresponding to the coalitions S containing j, can only infer, via a *mixed rule* of coalition formation, the subjective probabilities that coalitions containing j will eventually form and, *consequently*, subjective *"coalition structure beliefs"*, *cs beliefs* for short, (that is, probability distribution on the set of coalition structures). However, this approach embodies two fundamental problems:

i) Even if player j knows the components of the strategies of the other players corresponding to the coalitions S containing j, there exist many rules which assign to every mixed strategy profile the probabilities that coalitions containing j will eventually form and which generalize the γ or the δ rules.

ii) Given a probability assignment on coalitions containing j corresponding to a mixed strategy profile under one of the possible generalizations of the γ or of the δ rules then, the coalition structure belief is not unequivocally determined. In other words, there might exist many cs beliefs which are consistent (in terms of laws of probability) with the probability assignment.

In this paper we focus only on the γ model and, to tackle the first question, we introduce the so called *mixed* γ model, which gives back the pure γ model whenever agents play only pure strategies and which is the natural probabilistic extension of the pure γ model since the probability of each coalition S is calculated as the product of the probabilities given by every player in S to this coalition.

The second question is unsolvable in a sense. In fact, if one interprets the probability of a coalition S as the probability of the event "coalition structures containing S", then, there might exist multiple coalition structure beliefs which are *coherent* in the sense of de Finetti (1931) with this probability assignment (roughly speaking, such that the total probability theorem is satisfied for the probability of every event/coalition). Therefore, such multiplicity problem do not allow for an unambiguous and well-defined decision mechanism of each player. For instance, the multiplicity of coherent coalition structure beliefs for player j implies multiplicity of von Neumann - Morgestern expected utilities to player j, given the mixed strategy profile. We will show below that it is possible to deal with these difficulties by selecting a mechanism of *revision* of prior beliefs in a dynamical environment. In particular, we consider the coalition structure beliefs updating problem of the generic player j and state the condition that coalition structures beliefs be consistent (in terms of de Finetti's coherency) with his subjective probabilities that coalition containing j will eventually form, at all instants, as a viability constraint. Then, we give characterizations for continuous evolutions in both the players' strategies and (at least one) corresponding subjective coherent belief, by applying the main viability theorem for differential inclusions.

More precisely, we consider evolutionary games in which players act on the velocities of the strategies which are regarded as decisions (controls) and used by the players to govern the evolution of coherent coalition structure beliefs. For every player j, the evolution of the strategies determine, through the mixed γ model, an unique evolution of the player j subjective probability assignment on coalitions containing j. Moreover, among every possible evolution of player j subjective coalition structure belief we consider only those which are coherent with the given probability assignments at every instant. The coherency condition is regarded as a viability constraint and we characterize the so called *regulation map* (Aubin (1991)) which gives the velocities which guarantee viable evolution of couples coalition structure belief/mixed strategy profile starting from every point in the viability constraint's domain. Moreover, starting from such characterizations, we exhibit some paradoxes for differentiable evolutions of pairs coalition structure belief/mixed strategy profile whenever they start from or arrive at a pure cs belief.

Finally, given feedback (state dependent) controls of the players, the evolution of the beliefs can be regarded as a problem of *probabilistic belief revision*. Namely, the classical question in belief revision theory is the following: Suppose one holds a certain belief about the states of the world and at a given moment something that contradicts these belief is observed. How should the belief be revised? Of course, different approaches might be considered; we focus on the idea of minimal change revision (see Schulte (2002) and Perea (2007)). In fact, in belief revision theory, it is a generally accepted idea that if one observes an event that contradicts the previous belief, then the new belief about the world should explain the event just observed, and should be "as close as possible" to the previous ones. The intuition behind this principle is that previous belief should change only as far as necessary. In our case, belief revision works in continuous time and revised beliefs explain observations at every instant through the coherency conditions, since observations are in terms of probability assignments on coalitions rather than events; moreover, the idea of minimal change is translated in terms of revision with minimal velocity. Therefore we provide an existence theorem for evolutions of cs belief of minimal velocity in the mixed γ.

As a final remark we recall that, to describe uncertainty in cooperative games, a different approach is the use of the concept of fuzzy coalition (Aubin (1974, 1981) in which each player is characterized by his participation rate. However this concept is more suited to describe stability of the grand coalition rather than coalition structures formation. Moreover differential cooperative games were firstly introduced by Filar and Petrosjan (2000) and then developed for fuzzy coalitions by Aubin (2002, 2003) in the framework of characteristic form games.

2 Mixed strategies and coherent coalition structure beliefs

2.1 Preliminaries

Let $I = \{1, \dots, n\}$ be the set of players. Then the *coalition structures set* (cs set, for short) is the set $\mathscr{B}$ of all partitions of I, that is

$$\mathcal{B} \in \mathscr{B} \iff \bigcup_{S \in \mathcal{B}} S = I, \; S \cap T = \emptyset \quad \forall S, T \in \mathcal{B}$$

In Hart and Kurz (1983), the strategy set of each agent i is $\Sigma_i = \{S \subseteq I \mid i \in S\}$ so that a strategy profile is the n-tuple $(S_1, \dots, S_n) \in \prod_{i \in I} \Sigma_i$ and the strategy S_i is the set of players that player i wishes to join. The total number of partitions of a set with n elements is given by the *Bell number*, denoted with $B(n)$, which can be defined by the following recursive equation

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(k). \tag{1}$$

Moreover $B(n) = \sum_{k=0}^{n} S(n,k)$ where $S(n,k)$ are the Stirling numbers of the second kind which are a doubly indexed sequence of natural numbers, each element representing the number of ways to partition a set of n objects into k groups.

On the other hand, it is well known that the number $C(n)$ of nonempty coalitions of a set with n elements is given by

$$C(n) = \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1.$$

Therefore, since the number of strategies of each player is equal to the number of nonempty coalitions in a set with $n-1$ players plus 1 (corresponding to the strategy *singleton*), we get $|\Sigma_i| = 2^{n-1}$ and call $|\Sigma_i| = k$.

The γ model proposed in Hart and Kurz (1983) for coalition structures formation can be represented by the function $h^\gamma : \prod_{i \in I} \Sigma_i \to \mathscr{B}$ defined by:

$$T \in h^\gamma(S_1, \dots, S_n) \iff T = S_j \text{ for all } j \in T \quad \text{or} \quad T = \{l\} \text{ for some } l \in I$$

Remark 2.1: A relation between the number of coalition structures and the number of not empty coalition is given by the following

Lemma 2.2: *If $B(n) > C(n)$ then $B(n+1) > C(n+1)$.*

Proof. Trivially

$$C(n+1) = \sum_{k=1}^{n+1} \binom{n+1}{k} = 2^{n+1} - 1 = 2(2^n - 1) + 1 = 2C(n) + 1$$

Consider

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k),$$

by Pascal's rule

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

so

$$B(n+1) = \binom{n}{0} B(0) + \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} + \binom{n-1}{k} \right] B(k) + \binom{n}{n} B(n)$$

since

$$\sum_{k=1}^{n-1}\binom{n-1}{k-1}B(k) > \sum_{k=1}^{n-1}\binom{n-1}{k-1} = C(n-1)\ ,\ \sum_{k=1}^{n-1}\binom{n-1}{k}B(k) = B(n) - \binom{n-1}{0}B(0)$$

then

$$B(n+1) > B(0)+C(n-1)+B(n)-B(0)+B(n) = 2B(n)+C(n-1) > 2C(n)+1 = C(n+1)$$

□

Since, it can easily calculated that $B(5) = 52 > 31 = C(5)$, then $B(n) > C(n)$ for all $n \geq 5$.

2.2 Mixed strategies and coalition formation rules

Differently from Hart and Kurz (1983), we assume that every player i is allowed to choose a mixed strategy, called mixed coalition. Denote with $\mathcal{S}_i = \{S \subseteq I \mid i \in S\}$ then, a mixed coalition is a vector of probabilities $m_i = (m_{i,S})_{S\in\mathcal{S}_i}$ such that $m_{i,S} \geq 0$ for every $S \in \mathcal{S}_i$ and $\sum_{S\in\mathcal{S}_i} m_{i,S} = 1$. The set of mixed strategies of player i is denoted with $\Delta_i \subset \mathbb{R}^k$, with $k = |\Sigma_i|$.

As stated in the Introduction, we consider the situation in which the generic player j observes only the probability $\mu_{j,S}$ that coalition S will eventually form, for every $S \in \mathcal{S}_j$. Following the idea in the γ and the δ models, each $\mu_{j,S}$ should be given by a function $\lambda_{j,S} : \prod_{i=1}^n \Delta_i \to [0,1]$, that is, $\mu_{j,S} = \lambda_{j,S}\big((m_i)_{i\in I}\big)$. Each function $\lambda_{j,S}$ represents the subjective coalition S formation rule to player j.

The mixed γ model

There are different ways to generalize the γ model to the case of mixed strategies, that is there are different set of functions $(\lambda_{j,S})_{S\in\mathcal{S}_j}$ which extend the γ model. The mixed γ model

Among the possible extensions to the mixed strategy case of the γ model we consider the following:

$$\text{mixed } \gamma \text{ model} := \begin{cases} \mu_{j,S} = \lambda_{j,S}(m_1,\ldots,m_n) = \prod_{i\in S} m_{i,S} \quad \forall S \in \mathcal{S}_j, |S| \geq 2 \\ \\ \mu_{j,\{j\}} = \lambda_{j,\{j\}}(m_1,\ldots,m_n) = 1 - \sum_{S\in\mathcal{S}_j,\ |S|\geq 2} \mu_{j,S} \end{cases} \tag{2}$$

In this case $\mu_{j,S} = \prod_{i\in S} m_{i,S}$ translates the idea that player j evaluates the probability of coalition S as the product of the probabilities announced by players in S and the probabilities left are assigned to the singleton. Observe that in the mixed γ model, the functions $\lambda_{j,S}$ are multiaffine and therefore continuously differentiable. Finally, note that whenever players choose only pure strategies the mixed γ model is equivalent to the (pure) γ model.

2.3 Coherent beliefs

A *coalition structure belief* (*cs belief* for short) is a probability distribution on $\mathscr{B}$, that is a vector of probabilities $\varrho = (\varrho_{\mathcal{B}})_{\mathcal{B}\in\mathscr{B}}$ such that $\varrho_{\mathcal{B}} \geq 0$ for all $\mathcal{B} \in \mathscr{B}$ and $\sum_{\mathcal{B}\in\mathscr{B}} \varrho_{\mathcal{B}} = 1$. Denote the set of all of cs beliefs with $\Delta_{\mathscr{B}} \subset \mathbb{R}^b$ with $b = B(n) = |\mathscr{B}|$.

It is obvious that a coalition S can be interpreted as an event in the set of all coalition structures $\mathscr{B}$, more precisely as the event $E_S = \{\mathcal{B} \in \mathscr{B} \mid S \in \mathcal{B}\}$ for every $S \subseteq I$, so that the probability $\mu_{j,S}$ can be regarded as $\mu_{j,S} = prob\{\mathcal{B} \in \mathscr{B} \mid S \in \mathcal{B}\}$ for every $S \subseteq I$.
Therefore, as stated in the Introduction, the generic player j considers feasible only those coalition structure beliefs which are coherent in the sense of de Finetti (1931) with his subjective probability assignment on the event/coalition S for every $S \in \mathcal{S}_j$. This is equivalent to say that, for every event/coalition $S \in \mathcal{S}_j$ the total probability theorem should be satisfied, that is, a cs belief must satisfy the following *coherency constraint*:

$$\sum_{\mathcal{B}\ni S} \varrho_{\mathcal{B}} = \mu_{j,S} \quad \forall S \in \mathcal{S}_j \tag{3}$$

REMARK 2.3: Usually, Probability Theory works with probability measures on σ-algebras and needs the specification of probabilities of all the events in the σ-algebra. There are, however, situations in which one can be interested in working with partial assignments of probability. In such cases the collection of all events for which probabilities are known (or believed to be something) need not have any algebraic structure (for example, do not form a σ-algebra). In such cases, one would like to know if there is a probability space $(\Omega, \Sigma, \mathbb{P})$ such that Σ contains all events of interest to us and $\mathbb{P}$ assigns the same probabilities to these as we believe them to be. In other words, one would like to know if the probability assignment is *coherent* in the sense of de Finetti (1931).
In our model the probability assignments to player j are the probabilities $\mu_{j,S}$ to the event/coalition S. The probability distributions on $\mathscr{B}$ are the cs-beliefs ϱ and constraints (3) determine coherency, that is, if constraints (3) are satisfied then the cs belief assigns to the coalitions the same probabilities as player j believes them to be in light of his strategies and the corresponding mixed coalition rule $(\lambda_{j,S})_{S\in\mathcal{S}_j}$.

Existence

The system of equations in constraints (3) define a linear system in the unknowns $(\varrho_{\mathcal{B}})_{\mathcal{B}\in\mathscr{B}}$ where the number of unknowns is greater than the number of equations. The next proposition gives sufficient conditions for the existence of coherent cs beliefs:

PROPOSITION 2.4: *For every probability assignment $(\mu_{j,S})_{S\in\mathcal{S}_j}$ satisfying the following condition*

$$\sum_{S\in\mathcal{S}_j} \mu_{j,S} = 1 \tag{4}$$

there exists at least a cs belief satisfying the coherency constraints (3).

Proof. For every coalition S with at least two players, let $\mathcal{B}_S$ be the coalition structure defined by $\mathcal{B}_S = \{S, (\{l\})_{l\notin S}\}$. Obviously $S \neq T \iff \mathcal{B}_S \neq \mathcal{B}_T$. Given the assignment

$(\mu_{j,S})_{S\in\mathcal{S}_j}$ satisfying (4), let $\mathcal{B}'$ be the coalition structure having only singletons as elements and $\widehat{\varrho}$ a cs belief defined by

$$\widehat{\varrho}_{\mathcal{B}_S} = \mu_{j,S} \quad \forall S \in \mathcal{S}_j,\ |S| \geq 2; \quad \widehat{\varrho}_{\mathcal{B}'} = 1 - \sum_{S\in\mathcal{S}_j, |S|\geq 2} \mu_{j,S}; \quad \widehat{\varrho}_{\mathcal{B}} = 0 \text{ otherwise.}$$

It results that

$$\sum_{\mathcal{B}\ni S} \widehat{\varrho}_B = \widehat{\varrho}_{\mathcal{B}_S} \quad \text{for all } S \in \mathcal{S}_j,\ |S| \geq 2$$

so, being $\widehat{\varrho}_{\mathcal{B}_S} = \mu_{j,S}$, the coherency constraints for coalitions with at least two players is satisfied.
Moreover, the only coalition structures containing $\{j\}$ which have positive probability in the cs belief $\widehat{\varrho}$ is $\mathcal{B}'$. Therefore, from the assumption (4), it results that

$$\mu_{j,\{j\}} = 1 - \sum_{S\in\mathcal{S}_j,\ |S|\geq 2} \mu_{j,S} = \widehat{\varrho}_{\mathcal{B}'} = \sum_{\mathcal{B}\ni\{j\}} \widehat{\varrho}_{\mathcal{B}}.$$

Hence the coherency constraint is for coalition $\{j\}$ is satisfied. Since $\widehat{\varrho}$ is a probability distribution on $\mathscr{B}$ then the assertion follows. $\square$

Multiplicity

In the next examples we show that multiple coalition structure beliefs might be supported by the same probability assignment $(\mu_{j,S})_{S\in\mathcal{S}_j}$.

EXAMPLE 2.5: Consider a 3 player game and the following strategies:

$$\begin{cases} i)\ m_{1,\{1,2,3\}} = 1/2,\ m_{1,\{1,2\}} = 1/4,\ m_{1,\{1,3\}} = 0,\ m_{1,\{1\}} = 1/4 \\ ii)\ m_{2,\{1,2,3\}} = 1/3,\ m_{2,\{1,2\}} = 0,\ m_{2,\{2,3\}} = 0,\ m_{2,\{2\}} = 2/3 \\ iii)\ m_{3,\{1,2,3\}} = 1,\ m_{3,S} = 0 \text{ otherwise} \end{cases} \tag{5}$$

Consider player 1 and calculate $\mu_{1,S}$ for all $S \in \mathcal{S}_1$ as in the γ model, we obtain the following

$$\begin{cases} \mu_{1,\{1,2,3\}} = 1/6, \\ \mu_{1,\{1,2\}} = 0,\ \mu_{1,\{1,3\}} = 0, \\ \mu_{1,\{1\}} = 5/6 \end{cases}$$

then it easily follows that coherent cs beliefs must satisfy the following coherency conditions

$$\begin{cases} 1)\ \mu_{1,\{1,2\}} = \varrho_{\{\{1,2\},\{3\}\}} = 0,\ \mu_{1,\{1,3\}} = \varrho_{\{\{1,3\},\{2\}\}} = 0 \\ 2)\ \mu_{1,\{1,2,3\}} = \varrho_{\{\{1,2,3\}\}} = 1/6 \\ 3)\ 5/6 = \mu_{\{1\}} = \varrho_{\{\{1\},\{2\},\{3\}\}} + \varrho_{\{\{2,3\},\{1\}\}} \end{cases}$$

and therefore we get from 3) we get infinite solutions.

3 Evolution of coherent coalition structure beliefs

Now we introduce the evolutionary games arising from the γ and δ models of coalition formation. We consider the situation in which players determine at every instant the set of players they wish to join, more precisely players act on the velocities of the strategies which are regarded as controls. Then, fixed a generic player j, we study how his subjective coalition structure beliefs might evolve, governed by Nature, according to these strategic choices, that is, coherently with his subjective probability assignment on coalitions determined by the strategies.

For every player $i \in I$, $U_i : \Delta_{\mathscr{B}} \times \prod_{l=1}^{n} \Delta_l \rightsquigarrow \mathbb{R}^k$ is the set-valued map of *feasible controls* of player i; the set valued map of the a-priori feasible dynamics of cs beliefs is $H : \prod_{j=1}^{n} \Delta_j \times \Delta_{\mathscr{B}} \rightsquigarrow \mathbb{R}^b$, with $b = B(n) = |\mathscr{B}|$. Note that H could be constant and given, for instance, by the entire space $\mathbb{R}^b$ or by a closed ball with radius η and center 0, $B(0, \eta) \subset \mathbb{R}^b$. Moreover, the evolution of pairs cs belief/mixed strategy profile $(\varrho(t), m(t))$ should satisfy the following simplex constraints

$$\left\{ \begin{array}{l} \varrho_{\mathcal{B}} \geq 0 \quad \forall \mathcal{B} \in \mathscr{B} \\ \sum_{\mathcal{B} \in \mathscr{B}} \varrho_{\mathcal{B}} = 1 \end{array} \right. ; \quad \left\{ \begin{array}{l} m_{i,S} \geq 0 \quad \forall \in \mathcal{S}_i \\ \sum_{S \in \mathcal{S}_i} m_{i,S} = 1 \end{array} \right. \quad \text{for all } i \in I \tag{6}$$

and the coherency constraints of player j, which can be rewritten as

$$\sum_{\mathcal{B} \ni S} \varrho_{\mathcal{B}} = \lambda_{j,S}(m_1, \ldots, m_n) \quad \forall S \in \mathcal{S}_j.$$

Summarizing, evolutions of coherent pairs coalition structure belief/mixed strategies should be solutions of the following dynamical system (i.e. absolutely continuous functions satisfying the following system for almost all t):

$$\left\{ \begin{array}{l} \varrho'_{\mathcal{B}}(t) = h_{\mathcal{B}}(t) \quad \forall \mathcal{B} \in \mathscr{B} \\ m'_i(t) = u_i(t), \quad \forall i \in I \\ u_i(t) \in U_i(m(t), \varrho(t)) \quad \forall i \in I \\ (h_{\mathcal{B}})_{\mathcal{B} \in \mathscr{B}} \in H(m(t)), \varrho(t)) \end{array} \right. \tag{7}$$

under the viability constraints K_j given by:

$$(\varrho, m) \in K_j \iff \left\{ \begin{array}{l} i) \varrho_{\mathcal{B}} \geq 0 \; \forall \mathcal{B} \in \mathscr{B} \\ ii) \sum_{\mathcal{B} \in \mathscr{B}} \varrho_{\mathcal{B}} - 1 = 0 \\ iii) m_{i,S} \geq 0 \; \forall S \in \mathcal{S}_i, \text{ and} \forall i \in I \\ iv) \sum_{S \in \mathcal{S}_i} m_{i,S} - 1 = 0 \quad \forall i \in I \\ v) \chi_S(\varrho, m) = \sum_{\mathcal{B} \ni S} \varrho_{\mathcal{B}} - \lambda_{j,S}(m_1, \ldots, m_n) = 0 \quad \forall S \in \mathcal{S}_j \end{array} \right. \tag{8}$$

Note that the control $u_i(t)$ is the vector $u_i(t) = \big(u_{i,S}(t)\big)_{S \ni i}$, where each component $u_{i,S}(t)$ governs the velocity of $m_{i,S}$.

Of course, there is no a-priori reason why a solution $(\varrho(t), m(t))$ of the system (7) should be *viable* in the constraints (8) for every $t \in [0, +\infty[$, that is, satisfy the constraints (8) for every $t \in [0, +\infty[$. So we are interested to characterize velocities $\big(h(t), u(t)\big) = \Big(\big(h_{\mathcal{B}}(t)\big)_{\mathcal{B} \in \mathscr{B}}, \big(u_i(t)\big)_{i \in I}\Big)$ such that the corresponding solutions are viable. To this purpose we will apply the main viability theorems for control systems as stated in Aubin (1991).

The viability theorem

Now we recall some classical definitions and state the main viability theorem for the previous system. The constraints set K_j is said to be *viable* under the control system (7) if for every point $(\varrho_0, m_0) \in K_j$ there exists at least one solution $(\varrho(\cdot), m(\cdot))$ starting at (ϱ_0, m_0) and governed by (7) such that $(\varrho(t), m(t)) \in K_j$ for all $t \geq 0$.

We recall that if $K \subset \mathbb{R}^q$ and $y \in K$, a direction $v \in \mathbb{R}^q$ belongs to the *contingent cone* $T_K(y)$ if there exist a sequence $\varepsilon_n > 0$ and $v_n \in \mathbb{R}^q$ converging to 0 and v respectively, such that:

$$y + \varepsilon_n v_n \in K \quad \forall n \in \mathbb{N}.$$

To simplify notations denote $\mathcal{M}(\varrho, m) = H(\varrho, m) \times \left[\prod_{i=1}^n U_i(\varrho, m)\right]$, then the Viability Theorem (Aubin, 1991, 1997) for system (7) under the constraints (8) reads:

THEOREM 3.1: *Assume that the set-valued maps H and U_i, with $i \in I$, are Marchaud, that is with closed graphs, not empty, compact and convex images for every point in the domain and bounded by linear growth, that is, there exists $c, \psi_1, \ldots, \psi_n > 0$ such that, for all $(\varrho, m) \in K_j$:*

$$\sup_{y \in H(\varrho, m)} \|y\| \leq c(\|(\varrho, m)\| + 1)$$

and

$$\sup_{z_i \in U_i(\varrho, m)} \|y\| \leq \psi_i(\|(\varrho, m)\| + 1) \quad \forall i$$

Then, K_j is viable under the control system (7) if and only if for every $(\varrho, m) \in K_j$, the images of the regulation map $R_{K_j} : K_j \rightsquigarrow \mathbb{R}^b \times \prod_{j=1}^n \mathbb{R}^k$ are not empty, where R_{K_j} is defined by:

$$R_{K_j}(\varrho, m) = \left\{(h, u) \in \mathcal{M}(\varrho, m) \mid (h, u) \in T_{K_j}(\varrho, m)\right\}. \tag{9}$$

Moreover, every evolution $(\varrho(\cdot), m(\cdot))$ viable in K_j is regulated by (it is a solution of) the system:

$$\begin{cases} \varrho'_{\mathcal{B}}(t) = h_{\mathcal{B}}(t) \quad \forall \mathcal{B} \in \mathscr{B} \\ m'_i(t) = u_i(t), \quad \forall i \in I \\ (u(t), h(t)) \in R_{K_j}(\varrho(t), m(t)). \end{cases} \tag{10}$$

This previous theorem provides existence conditions and characterization of continuous evolutions in both the players' strategies and corresponding coherent beliefs to player j for a general class of set valued maps of feasible controls (satisfying classical regularity assumptions). This approach obviously might include, as particular cases, set valued maps of feasible controls related, for instance, to myopic optimization criteria (such as best reply dynamics). However, at this point, the definition of suitable preference relations over strategy profiles is not straightforward since we deal with ambiguous probabilities or expected payoffs (Ellsberg (1961)) arising from the multiplicity of beliefs for a given strategy profile and therefore deserves a future accurate analysis.

Conditional viability and belief revision

Let $(\varrho, m, u) \rightsquigarrow \mathcal{A}(\varrho, m, u)$ be the set valued map defined by

$$\mathcal{A}(\varrho, m, u) = \big\{ h \in H(\varrho, m) \mid (h, u) \in R_{K_j}(\varrho, m) \big\} \tag{11}$$

and $u(\varrho, m)$ a profile of feedback controls of the players. Consider a solution $(\widehat{\varrho}(t), \widehat{m}(t))$ of the following differential inclusion:

$$\begin{cases} \varrho'(t) \in \mathcal{A}\Big(\varrho(t), m(t), u\big(\varrho(t), m(t)\big)\Big) \\ m' = u(\varrho(t), m(t)) \end{cases} \tag{12}$$

then it follows that the evolution of the cs belief $\widehat{\varrho}(t)$ satisfies the coherency constraints at every instant t given the probability assignment $\lambda_{j,S}(\widehat{m}(t))$. Therefore, it can be regarded as a revision, in continuous time, of player j subjective probabilistic belief *conditioned* (in terms of the coherency constraints) by the evolution, in continuous time, of the assignment of probability on coalitions in $\mathcal{S}_j$ which, on the other hand, is determined by the evolution of the mixed strategy profile $\widehat{m}(t)$. Of course, there are no a-priori reasons why system (12) should admit a solution. Existence could be guaranteed, for instance, by the lower semicontinuity of the set valued map $\mathcal{A}$ (see, for example, the proof of Theorem 9.2.4 in Aubin (1997)) which, however, is not assured in general even when the model satisfies the hypothesis of the main Viability Theorem 3.1. Finally, observe that, even if system (12) admits a solution, then it could be not unique. Therefore it could be reasonable to refine the set of feasible solutions of system (12) by restricting the set valued map $\mathcal{A}$. In Section 4 we will apply this procedure to select belief revision of minimal velocity.

4 The regulation map in the mixed γ model

We give the formula for the regulation map $R_{K_j}(\varrho, m)$ of system (7) under the constraints (8) in the mixed γ and δ models. To this purpose, denote with $\nabla(\varrho_{\mathcal{B}})$, $\nabla\left(\sum_{\mathcal{B}\in\mathscr{B}} \varrho_{\mathcal{B}}\right)$, $\nabla(m_{i,S})$, $\nabla\left(\sum_{S\ni i} m_{i,S}\right)$, $\nabla\left(\sum_{\mathcal{B}\ni S} \varrho_{\mathcal{B}}\right)$ and $\nabla\left(\prod_{i\in S} m_{i,S}\right)$ the gradients of the functions with respect to the variables (ϱ, m). Moreover recall that a set $C \subseteq \mathbb{R}^n$ is said to be it regular in a point $\overline{x} \in C$ (also called *sleek*, see Aubin and Frankowska (1990)) if the set valued map $x \rightsquigarrow T_C(\cdot)$ is lower semicontinuous in $\overline{x}$. If the set C is regular in $\overline{x}$, then the *normal cone* of C in $\overline{x}$ $N_C(\overline{x})$ is given by set of *regular normal vectors*, i.e.:

$$N_C(\overline{x}) = \{\omega \mid \langle \omega, x - \overline{x}\rangle \le o(\|x - \overline{x}\|) \quad \forall x \in C\}.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *lower subdifferentially regular* in a point $\overline{x}$ if the epigraph

$$epi\ f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \ge f(x)\}$$

is a regular set in $(\overline{x}, f(\overline{x}))$. Moreover, f is said to be *upper subdifferentially regular* in x if $-f$ is lower subdifferentially regular in x.

The characterization theorem

Let $L_{\mathcal{B}}(\varrho, m)$ (resp. $L_{i,S}(\varrho, m)$) be the functions defined by $L_{\mathcal{B}}(\varrho, m) = 1$ if $\varrho_{\mathcal{B}} = 0$ and $L_{\mathcal{B}}(\varrho, m) = 0$ otherwise (resp. $L_{i,S}(\varrho, m) = 1$ if $m_{i,S} = 0$ and $L_{i,S}(\varrho, m) = 0$ otherwise).

PROPOSITION 4.1: *Assume that each function λ_S is defined as in the mixed γ model. Let $(\varphi_{\mathcal{B}})_{\mathcal{B}\in\mathscr{B}}$, $(\eta_{i,S})_{i\in I, S\ni i}$ be nonnegative real numbers and ζ, $(\beta_i)_{i\in I}$, $(\alpha_S)_{S\in\mathcal{S}_j,\,|S|\geq 2}$, θ be real numbers, if the following transversality condition holds:*

$$\begin{cases} i)\ \varphi_{\mathcal{B}} L_{\mathcal{B}}(\varrho, m) + \zeta + \sum_{S\in\mathcal{B}\cap\mathcal{S}_j,\,|S|\geq 2} \alpha_S = 0, \quad \forall \mathcal{B}\in\mathscr{B},\ \textit{such that } \{j\}\notin\mathcal{B} \\ ii)\ \varphi_{\mathcal{B}} L_{\mathcal{B}}(\varrho, m) + \zeta + \sum_{S\in\mathcal{B}\cap\mathcal{S}_j,\,|S|\geq 2} \alpha_S + \theta = 0, \quad \forall \mathcal{B}\in\mathscr{B},\ \textit{such that } \{j\}\in\mathcal{B} \\ iii)\ \eta_{i,S} L_{i,S}(\varrho, m) + \beta_i = 0, \quad \forall (i,S)\ \textit{such that } S\notin\mathcal{S}_j,\ i\in S \\ iv)\ \eta_{i,S} L_{i,S}(\varrho, m) + \beta_i - \alpha_S\left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right) + \theta\left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right) = 0, \\ \quad \forall (i,S)\ \textit{such that } S\in\mathcal{S}_j,\ i\in S \end{cases}$$
$$\Downarrow$$
$$\varphi_{\mathcal{B}} = \zeta = \theta = \alpha_S = \eta_{i,S} = \beta_i = 0 \quad \text{for all } \mathcal{B}, S, i \tag{13}$$

Then $(h, u)\in\mathcal{M}(\varrho, m)$ belongs to $R_{K_j}(\varrho, m)$ if and only if:

$$\begin{cases} i) & h_{\mathcal{B}} \geq 0 \quad \text{whenever } \varrho_{\mathcal{B}} = 0 \\ ii) & \sum_{\mathcal{B}\in\mathscr{B}} h_{\mathcal{B}} = 0 \\ iii) & u_{i,S} \geq 0 \quad \text{whenever } m_{i,S} = 0 \\ iv) & \sum_{S\in\mathcal{S}_i} u_{i,S} = 0, \quad \forall i\in I \\ v) & \sum_{\mathcal{B}\ni S} h_{\mathcal{B}} - \sum_{i\in S} u_{i,S}\left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right) = 0, \quad \forall S\in\mathcal{S}_j, |S|\geq 2 \\ vi) & \sum_{S\notin\mathcal{S}_j}\left[\sum_{i\in S} u_{i,S}\left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right)\right] + \sum_{\mathcal{B}\ni\{j\}} h_{\mathcal{B}} = 0. \end{cases} \tag{14}$$

For the proof of the previous proposition the following Lemma is needed:

LEMMA 4.2: *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ with $i\in J_1$ and $g_l : \mathbb{R}^n\to\mathbb{R}$ with $l\in J_2$ be continuously differentiable functions and let K be the set defined by:*

$$K = \left\{ x\in\mathbb{R}^n \;\middle|\; \begin{array}{l} f_i(x) = 0, \quad \forall i\in J_1 \\ \text{and} \\ g_l(x)\leq 0, \quad \forall l\in J_2 \end{array} \right\}$$

Let $H(\overline{x})\subseteq J_2$ denote the set of active constraints in a point $\overline{x}$, that is $l\in H(\overline{x}) \iff g_l(\overline{x}) = 0$. Assume that $\nabla f_i(\overline{x})\neq 0$ for all $i\in J_1$ and $\nabla g_l(\overline{x})\neq 0$ for all $l\in H(\overline{x})$ and that the following transversality condition holds: given $v_i\in\mathbb{R}$ for all $i\in J_1$ and $q_l\in\mathbb{R}_+$ for all $l\in H(\overline{x})$ then

$$\sum_{i\in J_1} v_i\nabla f_i(\overline{x}) + \sum_{l\in H(\overline{x})} q_l\nabla g_l(\overline{x}) = 0 \implies v_i = 0\ \forall i\in J_1 \text{ and } q_l = 0\ \forall l\in H(\overline{x}). \tag{15}$$

Then

$$w\in T_K(\overline{x}) \iff \begin{cases} \langle\nabla f_i(\overline{x}), w\rangle = 0 \quad \forall i\in J_1 \\ \text{and} \\ \langle\nabla g_l(\overline{x}), w\rangle \leq 0 \quad \forall l\in H(\overline{x}) \end{cases} \tag{16}$$

Proof. Consider the sets $E_i = \{x \in \mathbb{R}^n \mid f_i(x) = 0\}$ $\forall i \in J_1$ and $F_l = \{x \in \mathbb{R}^n \mid g_l(x) \leq 0\}$ $\forall l \in J_2$. From the assumptions it follows that E_i and F_l are regular sets in every point and

$$T_{E_i}(\overline{x}) = \{w \mid \langle \nabla f_i(\overline{x}), w\rangle = 0\} \quad \forall i \in J_1, \quad T_{F_l}(\overline{x}) = \{w \mid \langle \nabla g_l(\overline{x}), w\rangle \leq 0\} \quad \forall l \in H(\overline{x}).$$

The normal cones are

$$N_{E_i}(\overline{x}) = \{w \nabla f_i(\overline{x}) \mid w \in \mathbb{R}\} \quad \forall i \in J_1, \quad N_{F_l}(\overline{x}) = \{w \nabla g_l(\overline{x}) \mid w \in \mathbb{R}_+\} \quad \forall l \in H(\overline{x}).$$

So, in light of (15) in the assumptions, from Proposition 6.42 in Rockafellar and Wets (1998) it follows that

$$T_K(\overline{x}) = \left(\bigcap_{i \in J_1} T_{E_i}(\overline{x})\right) \cap \left(\bigcap_{l \in H(\overline{x})} T_{F_l}(\overline{x})\right). \tag{17}$$

Hence, we get the assertion. $\square$

Proof of Proposition 4.1. Note that the functions defining the constraints in (8) are continuously differentiable. Since the functions in equations *v)* in (8) are defined as in the mixed γ model (2), then they can be rewritten as

$$\sum_{\mathcal{B} \ni S} \varrho_{\mathcal{B}} - \prod_{i \in S} m_{i,S} = 0 \quad \text{for all } S \in \mathcal{S}_j, \ |S| \geq 2$$

and

$$\sum_{S \in \mathcal{S}_j, |S| \geq 2} \left[\prod_{i \in S} m_{i,S}\right] + \sum_{\mathcal{B} \ni \{j\}} \varrho_{\mathcal{B}} - 1 = 0.$$

One first computes the gradients with respect to the variables (ϱ, m); $\nabla(\varrho_{\mathcal{B}})$ is a vector with 1 as the entry corresponding to partition $\mathcal{B}$ and 0 elsewhere; $\nabla\left(\sum_{\mathcal{B} \in \mathscr{B}} \varrho_{\mathcal{B}}\right)$ is a vector with 1 as the first $B(n)$ entries (associated to all the partitions $\mathcal{B}$) and 0 elsewhere; $\nabla(m_{i,S})$ is a vector with 1 as the entry associated to the pair (i, S) and 0 elsewhere; $\nabla\left(\sum_{S \in \mathcal{S}_i} m_{i,S}\right)$ is a vector with 1 as the entries associated to the pair (i, S) for all $S \in \mathcal{S}_i$ and 0 elsewhere. Then one considers the constraints *v)* in (8). Let S contain at least two players and consider the entries of the gradient $\nabla\left(\sum_{\mathcal{B} \ni S} \varrho_{\mathcal{B}} - \prod_{i \in S} m_{i,S}\right)$; among the first $B(n)$ components (those associated to the partitions) one finds 1 corresponding to the partitions containing S and 0 elsewhere; for the remaining entries, $\prod_{l \in S\setminus\{i\}} m_{l,S}$ appears for the entries corresponding to the pairs (i, S), for every $i \in S$, while the remaining entries are equal to 0. Finally consider the entries of the gradient $\nabla\left(\sum_{S \in \mathcal{S}_j, |S| \geq 2}\left[\prod_{i \in S} m_{i,S}\right] + \sum_{\mathcal{B} \ni \{j\}} \varrho_{\mathcal{B}} - 1\right)$; among the first $B(n)$ components (those associated to the partitions) one finds 1 corresponding to the partitions containing $\{j\}$ and 0 elsewhere; for the remaining entries, $\prod_{l \in S\setminus\{i\}} m_{l,S}$ appears for the entries corresponding to the pairs (i, S), for all $i \in S$ and all $S \in \mathcal{S}_j$ with $|S| \geq 2$, while the entries are equal to 0 elsewhere. So the gradients are different from 0 and

$$
\begin{array}{ll}
i) \left\langle \nabla \left(\varrho_{\mathcal{B}}\right), (h,u)\right\rangle = h_{\mathcal{B}}, \ \forall \mathcal{B} \in \mathscr{B} & \\
ii) \left\langle \nabla \left(\sum_{\mathcal{B}\in\mathscr{B}} \varrho_{\mathcal{B}}\right), (h,u)\right\rangle = \sum_{\mathcal{B}\in\mathscr{B}} h_{\mathcal{B}} & \\
iii) \left\langle \nabla \left(m_{i,S}\right), (h,u)\right\rangle = u_{i,S}, \ \forall S \subseteq I \text{ with } S \ni i, \ \forall i \in I & \\
iv) \left\langle \nabla \left(\sum_{S\in\mathcal{S}_i} m_{i,S}\right), (h,u)\right\rangle = \sum_{S\in\mathcal{S}_i} u_{i,S}, \quad \forall i \in I & \\
v) \left\langle \nabla \left(\sum_{\mathcal{B}\ni S} \varrho_{\mathcal{B}} - \prod_{i\in S} m_{i,S}\right), (h,u)\right\rangle = \sum_{\mathcal{B}\ni S} h_{\mathcal{B}} - \sum_{i\in S} u_{i,S} \left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right) & (18) \\
\forall S \in \mathcal{S}_j, \ |S| \geq 2 & \\
vi) \left\langle \nabla \left(\sum_{S\in\mathcal{S}_j,\, |S|\geq 2} \left[\prod_{i\in S} m_{i,S}\right] + \sum_{\mathcal{B}\ni\{j\}} \varrho_{\mathcal{B}} - 1\right), (h,u)\right\rangle = & \\
= \sum_{S\in\mathcal{S}_j,\, |S|\geq 2} \left[\sum_{i\in S} u_{i,S} \left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right)\right] + \sum_{\mathcal{B}\ni\{j\}} h_{\mathcal{B}} &
\end{array}
$$

Condition (13) in the assumptions guarantees that

$$
\left\{
\begin{array}{c}
\sum_{\mathcal{B}\in\mathscr{B}} \Big(\varphi_{\mathcal{B}} \left[\nabla \left(\varrho_{\mathcal{B}}\right)\right] L_{\mathcal{B}}(\varrho, m)\Big) + \zeta \left[\nabla \left(\sum_{\mathcal{B}\in\mathscr{B}} \varrho_{\mathcal{B}}\right)\right] \\
+ \sum_{i\in I} \sum_{S\ni i} \Big(\eta_{i,S} \left[\nabla \left(m_{i,S}\right)\right] L_{i,S}(\varrho, m)\Big) \\
+ \sum_{i\in I} \left(\beta_i \left[\nabla \left(\sum_{S\in\mathcal{S}_i} m_{i,S}\right)\right]\right) + \sum_{S\in\mathcal{S}_j,\, |S|\geq 2} \Big(\alpha_S \left[\nabla \left(\sum_{\mathcal{B}\ni S} \varrho_{\mathcal{B}} - \prod_{i\in S} m_{i,S}\right)\right]\Big) \\
+ \theta \left[\nabla \left(\sum_{S\in\mathcal{S}_j, |S|\geq 2} \left[\prod_{i\in S} m_{i,S}\right] + \sum_{\mathcal{B}\ni\{j\}} \varrho_{\mathcal{B}} - 1\right)\right] = 0 \\
\Downarrow \\
\varphi_{\mathcal{B}} = \zeta = \beta_i = \eta_{i,S} = \alpha_S = \theta = 0 \quad \text{for all } i, S, \mathcal{B},
\end{array}
\right.
\tag{19}
$$

which implies that condition (15) in Lemma 4.2 holds true.
Hence $(h,u) \in T_{K_j}(\varrho, m)$ if and only if

$$
\left\{
\begin{array}{ll}
i) & h_{\mathcal{B}} \geq 0 \quad \text{whenever } \varrho_{\mathcal{B}} = 0 \\
ii) & \sum_{\mathcal{B}\in\mathscr{B}} h_{\mathcal{B}} = 0 \\
iii) & u_{i,S} \geq 0 \quad \text{whenever } m_{i,S} = 0 \\
iv) & \sum_{S\in\mathcal{S}_i} u_{i,S} = 0, \quad \forall i \in I \\
v) & \sum_{\mathcal{B}\ni S} h_{\mathcal{B}} - \sum_{i\in S} u_{i,S} \left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right) = 0 \quad \forall S \in \mathcal{S}_j, \ |S| \geq 2 \\
vi) & \sum_{S\in\mathcal{S}_j,\, |S|\geq 2} \left[\sum_{i\in S} u_{i,S} \left(\prod_{l\in S\setminus\{i\}} m_{l,S}\right)\right] + \sum_{\mathcal{B}\ni\{j\}} h_{\mathcal{B}} = 0
\end{array}
\right.
\tag{20}
$$

Hence, the assertion follows. □

A Paradox in the mixed γ model

PROPOSITION 4.3: *If the assumption of Proposition 4.1 are satisfied and if $(\varrho, m) \in K_j$ is such that $\varrho_{\mathcal{B}'} = 1$ and $\varrho_{\mathcal{B}} = 0$ for all $\mathcal{B} \neq \mathcal{B}'$. Then*

$$
(h,u) \in R_{K_j}(\varrho, m) \implies h_{\mathcal{B}} = 0 \ \forall \mathcal{B} \in \mathscr{B} \text{ such that } \exists S \in \mathcal{B} \cap \mathcal{S}_j \text{ with } S \notin \mathcal{B}' \text{ and } |S| \geq 2
$$

Proof. Let $(h,u) \in R_{K_j}(\varrho, m)$ and consider a coalition $S \in \mathcal{S}_j$ such that $S \notin \mathcal{B}'$ and $|S| \geq 2$. Being $m_{i,S} = 0$ for all $i \in S$, we have:

$$\sum_{i \in S} u_{i,S} \left(\prod_{l \in S \setminus \{i\}} m_{l,S} \right) = 0.$$

Therefore, in light of condition *v)* in (14), it follows that $\sum_{\mathcal{B} \ni S} h_{\mathcal{B}} = 0$.
However, since $S \notin \mathcal{B}'$ and $\varrho_{\mathcal{B}} = 0$ for all $\mathcal{B} \neq \mathcal{B}'$, from *i)* in (14) it follows that $h_{\mathcal{B}} \geq 0$ for all $\mathcal{B} \ni S$, and so $h_{\mathcal{B}} = 0$ for all $\mathcal{B} \ni S$.

□

REMARK 4.4: The *"only if"* part in Proposition 4.1 does not require the transversality assumption (13) in Proposition 4.1. In fact, from Proposition 6.42 in Rockafellar and Wets (1998), it follows that the contingent cone to the intersection of sets is a subset of the intersection of the contingent cones, while only for the converse statement the transversality conditions is required. Therefore Proposition 4.3 can be extended also when condition (13) is not satisfied.

Proposition 4.3 can be also interpreted as follows: whenever at a given time the cs belief is pure, that is players are partitioned in coalitions with probability 1, then any differentiable deviation of a player from his pure strategy has the only effect of increasing his probability to stay alone. In other words, even if two or more players jointly deviate from a pure coalition in order to form a new one, then feasible cs beliefs evolve in such a way that the probability of this new coalition coalition remains 0. We will better illustrate the previous paradox in the following example:

EXAMPLE 4.5: Let $I = \{1, 2, \ldots, 5\}$ be the set of players and consider (ϱ, m) such that $\varrho_{\mathcal{B}'} = 1$, with $\mathcal{B}' = \big\{\{1, 2, \ldots, 5\}\big\}$, and $\varrho_{\mathcal{B}} = 0$ for all $\mathcal{B} \neq \mathcal{B}'$. Of course this implies that, for all $i \in I$, $m_{i,\{1,2,\ldots,5\}} = 1$ and $m_{i,S} = 0$ otherwise. Consider the following controls of the players:

$$\begin{cases} u_{1,\{1,3\}} = -u_{1,\{1,2,\ldots,5\}} > 0, \ u_{3,\{1,3\}} = -u_{3,\{1,2,\ldots,5\}} > 0 \\ u_{i,\{2,4,5\}} = -u_{i,\{1,2,\ldots,5\}} > 0, \ \forall i = 2, 4, 5 \\ u_{i,S} = 0 \text{ otherwise} \end{cases} \tag{21}$$

Let h be velocities of coalition structure beliefs such that (h, u) belongs to the regulation map. From *(v)* and *(ii)* in (14)

$$h_{\mathcal{B}'} = \sum_{i=1}^{5} u_{i,\{1,2,\ldots,5\}} < 0, \quad \sum_{\mathcal{B} \in \mathscr{B}} h_{\mathcal{B}} = 0.$$

Consider the evolution of beliefs of player 1. In light of Proposition 4.3 $h_{\mathcal{B}} = 0$ for all $\mathcal{B} \neq \mathcal{B}'$ such that $\exists S \in \mathcal{B} \cap \mathcal{S}_j$ with $S \notin \mathcal{B}'$ and $|S| \geq 2$. In particular $h_{\mathcal{B}} = 0$ for all $\mathcal{B}$ containing $\{1, 3\}$. Therefore, let $\overline{\mathscr{B}} = \{\mathcal{B} \in \mathscr{B} \mid \{1\} \in \mathcal{B}\}$, then

$$h_{\mathcal{B}'} + \sum_{\mathcal{B} \in \overline{\mathscr{B}}} h_{\mathcal{B}} = 0 \implies \sum_{\mathcal{B} \in \overline{\mathscr{B}}} h_{\mathcal{B}} > 0$$

This means that even if players' deviations are somehow in the direction of coalition structure $\{\{1,3\},\{2,4,5\}\}$, the beliefs evolve only in the direction of the coalition structure in which player 1 stays alone.

5 Minimal Change of Beliefs

As stated in Section 2, given a profile of feedback controls of the players $u(\varrho, m) = (u_i(\varrho, m))_{i\in I}$, a solution $(\widehat{\varrho}(t), \widehat{m}(t))$ of the differential inclusion (12) provides revision, in continuous time, of cs belief *conditioned* (in terms of the coherency constraints) by the control $u(\varrho, m)$ (therefore by the corresponding evolution of the mixed strategy profile $\widehat{m}(t)$). However some evolution of cs beliefs might show inconsistencies. Consider the following example:

Example 5.1: Assume we are in the mixed γ model. Fixed $(\widehat{\varrho}, \widehat{m})$, let $\widehat{u}$ a profile of feedback controls such that $\widehat{u}(\widehat{\varrho}, \widehat{m}) = 0$. Consider velocities of the cs belief $\widehat{h}$ satisfying

$$\begin{cases} \widehat{h}_{\{\{1,2,3\},\{4,5\}\}}(\widehat{\varrho}, \widehat{m}) = \widehat{h}_{\{\{1\},\{2\},\{3\},\{4\},\{5\}\}}(\widehat{\varrho}, \widehat{m}) = 1 \\ \widehat{h}_{\{\{1,2,3\},\{4\},\{5\}\}}(\widehat{\varrho}, \widehat{m}) = \widehat{h}_{\{\{1\},\{2\},\{3\},\{4,5\}\}}(\widehat{\varrho}, \widehat{m}) = -1 \\ \widehat{h}_{\mathcal{B}}(\widehat{\varrho}, \widehat{m}) = 0 \quad \text{otherwise} \end{cases} .$$

It follows that for every coalition $S \in \mathcal{S}_j$, conditions *v), vi)* in (14) are satisfied so that $\widehat{h}$ belongs to $\mathcal{A}\big(\widehat{\varrho}, \widehat{m}, \widehat{u}(\widehat{\varrho}, \widehat{m})\big)$. However notice that in this case velocities lead to a change of the coalition structure even if players are not changing their strategies. From (14), it is easy to check that $0 \in \mathcal{A}\big(\widehat{\varrho}, \widehat{m}, \widehat{u}(\widehat{\varrho}, \widehat{m})\big)$. Hence, if we follow the idea of minimal change belief revision (see, for instance, Schulte (2002) or Perea (2007)), which states that the new belief should be as similar as possible to the previous one, we expect that, whenever the mixed strategy profile reaches $\widehat{m}$ with velocity 0 then, the corresponding cs belief reaches $\widehat{\varrho}$ where it remains there in equilibrium.

The previous example shows that in order to capture the idea minimal change belief revision, we could restrict velocities to those characterized by minimal norm and then consider the corresponding solutions. More precisely:

Definition 5.2: An evolution $\varrho(t)$ is a *minimal change cs belief revision* of system (12) for a given continuous feedback control profile $(\varrho, m) \to \widetilde{u}(\varrho, m)$ if there exists an evolution of strategy profile $m(t)$ such that $(\varrho(t), m(t))$ is a solution of the following system

$$\begin{cases} \varrho'(t) = \widetilde{h}\Big(\varrho(t), m(t), \widetilde{u}\big(\varrho(t), m(t)\big)\Big) \\ m' = \widetilde{u}(\varrho(t), m(t)) \end{cases} \tag{22}$$

for a function $\widetilde{h}(\varrho, m, u(\varrho, m))$ defined by

$$\|\widetilde{h}(\varrho, m, \widetilde{u}(\varrho, m))\| = \min_{h\in\mathcal{A}(\varrho,m,\widetilde{u}(\varrho,m))} \|h\|.$$

Of course, there are no a-priori reasons why system (12) should admit minimal change cs belief revision. We give some existence results below. Note also that the concept of minimal change cs belief revision corresponds to a slight modification of the concept of *heavy solution* to a differential inclusions which has been investigated in Aubin (1991, 1997) and Aubin and Saint-Pierre (2006).

LEMMA 5.3: *If the assumptions of Theorem 3.1 are satisfied and if $\mathcal{A}(\varrho, m, \widetilde{u}(\varrho, m))$ is a lower semicontinuous set valued map with not empty and compact values for all $(\varrho, m) \in K_j$ and the feedback control $\widetilde{u}(\varrho, m)$ is continuous and bounded by linear growth in K_j, then every point in K_j is the starting point of a minimal change cs belief revision.*

Proof. Since $\mathcal{A}(\varrho, m, \widetilde{u}(\varrho, m))$ has not empty and compact values for all (ϱ, m), then the function

$$(\varrho, m) \to \pi(\varrho, m) = \min_{h \in \mathcal{A}(\varrho, m, \widetilde{u}(\varrho, m))} \|h\|$$

is well defined, moreover $\mathcal{A}$ is a lower semicontinuous set valued map so the assumptions of the Marginal Function Theorem (see for instance theorem 1.4.16 in Aubin and Frankowska (1990)) hold true, and π is also an upper semicontinuos function, that is

$$\limsup_{(\overline{\varrho}, \overline{m}) \to (\varrho, m)} \pi(\overline{\varrho}, \overline{m}) \leq \pi(\varrho, m) \quad \forall (\varrho, m) \in K_j.$$

The set valued map $(\varrho, m) \rightsquigarrow B(0, \pi(\varrho, m))$ has closed graph; in fact consider a sequence $\{(\varrho_\nu, m_\nu)\}_{\nu \in \mathbb{N}}$ converging to (ϱ, m) and a sequence $\{h_\nu\}_{\nu \in \mathbb{N}}$ converging to h with $h_\nu \in B(0, \pi(\varrho_\nu, m_\nu))$ for all ν then

$$\|h_\nu\| \leq \pi(\varrho_\nu, m_\nu) \implies \|h\| = \limsup_{\nu \to \infty} \|h_\nu\| \leq \limsup_{\nu \to \infty} \pi(\varrho_\nu, m_\nu) \leq \pi(\varrho, m)$$

therefore $h \in B(0, \pi(\varrho, m))$ and $(\varrho, m) \rightsquigarrow B(0, \pi(\varrho, m))$ has closed graph. From the assumptions, the set valued map H has closed graph so the set valued map $(\varrho, m) \rightsquigarrow \mathcal{W}(\varrho, m)$ defined by:

$$\mathcal{W}(\varrho, m) = B(0, \pi(\varrho, m)) \cap H(\varrho, m) \quad \forall (\varrho, m) \in K_j$$

has closed graph. Moreover $\mathcal{W}(\varrho, m)$ is the intersection of compact and convex sets and so it is compact and convex, for every (ϱ, m). Finally, from $\mathcal{W}(\varrho, m) \subseteq H(\varrho, m)$, it follows that $\mathcal{W}$ is bounded by linear growth.
Therefore, the system:

$$\begin{cases} \varrho'(t) = \widetilde{h}(t) \\ m'(t) = \widetilde{u}(\varrho(t), m(t)) \\ \widetilde{h}(t) \in \mathcal{W}(\varrho(t), m(t)) \end{cases} \tag{23}$$

satisfies the assumptions of Theorem 3.1 then K_j is viable under this auxiliary system. Hence, every point $(\varrho, m) \in K_j$ is the staring point of at least a solution $(\varrho(t), m(t))$ of system (22) which remains in K_j, that is, an evolution $(\varrho(t), m(t))$ such that $m(\cdot)$ is governed by $\widetilde{u}(\varrho(t), m(t))$ and $\varrho(\cdot)$ by a control $\widetilde{h}(t)$ which satisfies

$$\|\widetilde{h}(t)\| \leq \min_{h \in \mathcal{A}(\varrho(t), m(t), \widetilde{u}(\varrho(t), m(t)))} \|h\| = \pi(\varrho(t), m(t), \widetilde{u}(\varrho(t), m(t))) \tag{24}$$

for almost all $t \geq 0$. Hence, there exists a minimal change cs belief revision starting from every point in K_j. □

PROPOSITION 5.4: *If the assumptions of Theorem 3.1 are satisfied, the functions λ_S are given as in the mixed γ model and the set valued maps H, U_i for $i = 1, \ldots, n$ are also lower semicontinuous in K_j and satisfy the following:*

$$\forall (\varrho, m) \in K_j \quad \exists \eta > 0, \theta > 0 \text{ such that}$$

$$B(0, \eta) \subset \mathcal{M}(\widehat{\varrho}, \widehat{m}) - T_{K_j}(\widehat{\varrho}, \widehat{m}) \quad \forall (\widehat{\varrho}, \widehat{m}) \in B\big((\varrho, m), \theta\big) \tag{25}$$

then, given a feedback control $\widetilde{u}(\varrho, m)$ continuous and bounded by linear growth in K_j such that $\mathcal{A}\big(\cdot, \cdot, \tilde{u}(\cdot, \cdot)\big)$ has not empty values, for every initial condition in K_j there exists a minimal change cs belief revision.

Proof. Since the functions $\lambda_{j,S}$ are defined by system (2), the set K_j defined by (8) is regular (as the functions that define the constraints are differentiable or concave), then, by definition, it follows that $(\varrho, m) \rightsquigarrow T_{K_j}(\varrho, m)$ is a lower semicontinuous set valued map. We claim that the regulation map R_{K_j} is lower semicontinuos in K_j. In fact, fix $(\varrho, m) \in K_j$, $z = (h, u) \in R_{K_j}(\varrho, m)$ and a sequence (ϱ_ν, m_ν) in K_j converging to (ϱ, m). Since the set valued maps T_{K_j} and $\mathcal{M}$ are lower semicontinuous in (ϱ, m), there exists sequences $x_\nu \to z$ and $y_\nu \to z$ such that $x_\nu \in \mathcal{M}(\varrho_\nu, m_\nu)$ and $y_\nu \in T_{K_j}(\varrho_\nu, m_\nu)$ for all $\nu \in \mathbb{N}$. From the assumptions there exists $\eta > 0$ and $\overline{\nu}$ such that

$$B(0, \eta) \subset \mathcal{M}(\varrho_\nu, m_\nu) - T_{K_j}(\varrho_\nu, m_\nu) \quad \forall \nu \geq \overline{\nu}$$

Set $\|x_\nu - y_\nu\| = \varepsilon_\nu$ and $\alpha_\nu = \frac{\eta}{\eta + \varepsilon_\nu} \in]0, 1[$, it follows that $\alpha_\nu \varepsilon_\nu = (1 - \alpha_\nu)\eta$ and then

$$\alpha_\nu(x_\nu - y_\nu) \in B(0, \alpha_\nu \varepsilon_\nu) = B(0, (1 - \alpha_\nu)\eta) \subset (1 - \alpha_\nu)\big(\mathcal{M}(\varrho_\nu, m_\nu) - T_{K_j}(\varrho_\nu, m_\nu)\big)$$

Thus

$$\alpha_\nu(x_\nu - y_\nu) = (1 - \alpha_\nu)(\varphi_\nu - \psi_\nu) \quad \text{with} \quad \varphi_\nu \in \mathcal{M}(\varrho_\nu, m_\nu),\ \psi_\nu \in T_{K_j}(\varrho_\nu, m_\nu).$$

Therefore,

$$\alpha_\nu(x_\nu - y_\nu) = (1 - \alpha_\nu)(\varphi_\nu - \psi_\nu) \iff \alpha_\nu x_\nu + (1 - \alpha_\nu)\varphi_\nu = \alpha_\nu y_\nu + (1 - \alpha_\nu)\psi_\nu.$$

Moreover, since $\mathcal{M}(\varrho_\nu, m_\nu)$ and $T_{K_j}(\varrho_\nu, m_\nu)$ are convex sets

$$\alpha_\nu x_\nu + (1 - \alpha_\nu)\varphi_\nu \in \mathcal{M}(\varrho_\nu, m_\nu) \quad \text{and} \quad \alpha_\nu y_\nu + (1 - \alpha_\nu)\psi_\nu \in T_{K_j}(\varrho_\nu, m_\nu)$$

So,

$$\xi_\nu = \alpha_\nu x_\nu + (1 - \alpha_\nu)\varphi_\nu \in \mathcal{M}(\varrho_\nu, m_\nu) \cap T_{K_j}(\varrho_\nu, m_\nu) = R_{K_j}(\varrho_\nu, m_\nu).$$

Moreover $\alpha_\nu \to 1$ as $\nu \to \infty$ and then $\xi_\nu \to z$ as $\nu \to \infty$, which means that R_{K_j} is lower semicontinuous in $\varrho, m)$. Then, it easily follows that $\mathcal{A}$ is lower semicontinuous. Thus, since $\tilde{u}$ is continuous in K_j, $\mathcal{A}\big(\cdot, \cdot, \tilde{u}(\cdot, \cdot)\big)$ is lower semicontinuous in K_j. In light of (20) the images of $\mathcal{A}\big(\cdot, \cdot, \tilde{u}(\cdot, \cdot)\big)$ are convex and compact and, in light of the assumptions, non empty, then the assumptions of Lemma 5.3 are satisfied. Hence we get the assertion. □

6 Conclusion

This paper proposes an evolutionary game style model for the dynamics of coalition structure beliefs when players announce the coalition they wish to join by using a mixed strategy. These models extend the static γ models of coalition formation introduced in Hart and Kurz (1983) for situations in which each player has vague expectations about the choices of his opponents corresponding to the coalitions in which is not involved and about the formation rule of these coalitions as a consequence of private communication within the members of each coalition.
In particular, an evolutionary game is considered, where strategies and coalition structure beliefs are state variables and players act on the velocities of their strategies. Fixed a generic player j, we state the condition that his subjective coalition structures beliefs be consistent (in terms of de Finetti's coherency) with the mixed strategy choices of the players at all instant as a viability constraint and then, give characterizations for continuous evolutions in both the players' strategies and corresponding coherent belief, by applying the main viability theorem. Finally we relate the evolution of the beliefs to probabilistic belief revision; in particular, we propose to reduce the set of viable evolutions of beliefs by selecting the changes with minimal norm and provide existence results.

As a final remark, in this paper we considered a general class of set valued maps of feasible controls of the players (satisfying classical assumptions). Further research might focus on the problem of considering particular controls; for instance, related to some myopic optimization criteria (such as best reply dynamics); however, this approach is not straightforward since it requires the definition of suitable preference relations in case of ambiguous probabilities arising from the multiplicity of coherent beliefs for a given strategy profile.

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