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CHARACTERIZATION OF THE GENERALIZED TOP-CHOICE ASSUMPTION (SMITH) SET

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ABSTRACT. In this paper, I give a characterization of the Generalized Top-Choice Assumption set of a binary relation in terms of choice from minimal negative consistent superrelations. This result provides a characterization of Schwartz's set in tournaments.

JEL classification: D60, D71.

Key words: Negative Consistency, Generalized Top-Choice Assumption (Smith) set, Generalized Optimal-Choice Axiom (Schwartz) set.

1. INTRODUCTION

A fundamental problem of rational choice theory is to determine whether a choice is optimal relative to some preference relation or not. So a rational agent who knows his preference relation, chooses the maximal elements according to this relation, in every feasible set presented for choice. In other words, the optimal choice set of a choice process consists of the maximal alternatives in the feasible set according to the viewpoint of a binary relation. However, when will the set of optimal choices be non-empty? If the feasible set is finite and the binary relation is acyclic, then the set of optimal choices is always non-empty; When the optimal choice set is empty¹, the crucial question which has been arisen, is what to count as a choice. That is, what sets of alternatives may be considered as reasonable solutions? To answer this question, several methods (solution theories) for constructing non-empty choice sets have been proposed. Such a solution is the *Generalized Top-Choice Assumption set (GETCHA set)*, introduced by Schwartz in [4], which is a generalization of the optimal choice set. Smith in [5] introduces a generalization of Condorcet Criterion that is satisfied when pairwise election are based on simple majority choices. He uses the notion of *dominant set*, that is, any candidate in this set is collectively preferred to any candidate not into this set. But Smith does not discuss the idea of a smallest dominant set. Fishburn in [3] narrows Smith's generalization of the Condorcet Criterion to the smallest dominant set and calls it *Smith's Condorcet Principle*. Schwartz in [4] discusses the Smith's Condorcet Principle as a possible standard for optimal collective choice and he call it *GETCHA*.

¹This problem is common in the analysis of pairwise majority voting, in the choice of a winning sport team, in the aggregation of multiple choice criteria, in committee selection, in the choice under uncertainty, etc.

The \mathcal{GETCHA} set (*Smith set*)² is the choice set from a given set specified by the \mathcal{GETCHA} condition. To address the absence of maximal elements, Schwartz [4] gives another general solution for constructing non-empty choice sets which is called *Generalized Optimal-Choice Axiom* (\mathcal{GOCHA}). The choice set from a given set specified by the \mathcal{GOCHA} condition is the union of minimal sets which each one of them has the following property: no alternative outside this set is preferable to an alternative inside it.

In this paper, I show that an alternative belongs to the \mathcal{GETCHA} set of a binary relation if and only if it is maximal for a minimal negative consistent superrelation. This result provides a characterization of the \mathcal{GOCHA} set (*Schwartz set*)³ in tournaments.

2. NOTATION AND DEFINITIONS

Let X be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary relation on X . We will say that R is a subrelation of R' , and R' a superrelation of R , denoted $R \subseteq R'$, when for all $x, y \in X$, xRy implies $xR'y$. The *complement* of R is denoted by R^c , that is for all $x, y \in X$, $R^c = \{(x, y) | (x, y) \notin R\}$. We sometimes abbreviate $(x, y) \in R$ as xRy . For any $x \in X$, $Rx = \{y \in X | yRx\}$ and $xR = \{y \in X | xRy\}$ denote respectively the *upper contour set* and *lower contour set* of R at x . The *asymmetric part* $P(R)$ of R is given by: $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$. $\mathcal{M}(R)$ denote the elements of X that are *R-maximal* in X , i.e., $\mathcal{M}(R) = \{x \in X | \text{for all } y \in X, yRx \text{ implies } xRy\}$. We say that R is *transitive* if for all $x, y, z \in X$, $(x, z) \in R$ and $(z, y) \in R$ implies that $(x, y) \in R$. The *transitive closure* of R is denoted by \bar{R} , that is for all $x, y \in X$, $(x, y) \in \bar{R}$ if there exist $k \in \mathbb{N}$ and $x_0, \dots, x_k \in X$ such that $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, \dots, K\}$ and $x_k = y$. A subset $Y \subseteq X$ is an *R-cycle* if, for all $x, y \in Y$, we have $(x, y) \in \bar{R}$ and $(y, x) \in \bar{R}$. We say that R is *acyclic* if there does not exist an *R-cycle*. Suzumura [6] provides the following definition, which generalize the notions of transitivity and acyclicity: The binary relation R is *consistent*, if for all $x, y \in X$, for all $k \in \mathbb{N}$, and for all $x_0, x_1, \dots, x_k \in X$, if $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, \dots, K\}$ and $x_k = y$, then $(y, x) \notin P(R)$. If in the definition of consistent binary relation above we replace R with R^c , we get the notion of a *negative consistent* binary relation. As binary relations are subsets of $X \times X$, they are naturally partially ordered by set-inclusion. A *chain*, denoted \mathcal{C} , is a class of relations such that $B, B' \in \mathcal{C}$ implies $B \subseteq B'$ or $B' \subseteq B$. A class \mathcal{B} of relations is *closed downward* if, for all chains \mathcal{C} in \mathcal{B} , $\bigcap \{B | B \in \mathcal{C}\} \in \mathcal{B}$.

Let Ω be a family of non-empty subsets of X that represents the different feasible sets presented for choice. A choice function is a mapping that assigns to each choice situation a subset of it:

²The Smith set also appears in the literature as *weak top cycle*.

³The Smith set is also sometimes confused with the Schwartz set because in tournaments (asymmetric and complete binary relations) both sets coincide.

$C : \Omega \rightarrow X$ such that for all $A \in \Omega$, $C(A) \subseteq A$.

The traditional choice-theoretic approach takes behavior as rational if it can be explained as the outcome of maximizing a binary relation R . In this direction, best choices can be expressed as the maximization of the individuals's preferences over a set of alternatives. That is, for every $A \in \Omega$, $C(A) = \mathcal{M}(R/A)$ ($\mathcal{M}(R/A)$ denote the elements of X that are R -maximal in A). An $A \in \Omega$ is R -undominated iff for no $x \in A$ there is a $y \in X \setminus A$ such that yRx . An R -undominated set is *minimal* if none of its proper subsets has this property. The set A is R -dominant if and only if xRy for each $x \in A$ and each $y \in X \setminus A$. An R -dominant set is *minimal* if none of its proper subsets is an R -dominant subset of X . To deal with the case where the set of maximal choices $C(A)$ is empty, Schwartz [4, Definition in page 141] has been proposed the following general solutions:

Generalized Top Optimal-Choice Axiom (GETCHA): For each $A \in \Omega$, $C(A)$ is equivalent to the minimum R -dominated subsets of A .

Generalized Optimal-Choice Axiom (GOCHA): For each $A \in \Omega$, $C(A)$ is equivalent to the union of minimum R -undominated subsets of A . The *GETCHA(R)* set (resp. *GOCHA(R)* set) is the choice set from a given set specified by the *GETCHA* (resp. *GOCHA*) condition according to R .

3. MAIN RESULT

The main result in this paper establishes a binary characterization of the choices generated by negative consistent superrelations. Let \mathcal{R}_N denote the negative consistent superrelations of R , i.e., $\mathcal{R}_N = \{R \subseteq R_N \mid R_N \text{ is negative consistent}\}$, and let \mathcal{R}_{N^*} denote the elements of \mathcal{R}_N that are minimal with respect to set inclusion.

Zorn's Lemma: If every chain of a partially ordered set has a lower bound, then E has a minimal element.

In order to prove the main result of this paper, we need the following proposition which is a simplification of the dual version of the definition of Duggan [1, Definition 4] for consistent binary relations.

Proposition 1. A binary relation R is negative consistent if and only if $P(R) \subseteq P((\overline{R^c})^c)$.

Proof. Suppose that R fulfills the definition of negative transitivity and for $x, y \in X$, $(x, y) \in P(R)$. By way of contradiction, we assume that $(x, y) \notin P((\overline{R^c})^c)$. We have two cases: either $(x, y) \notin (\overline{R^c})^c$, or $(x, y) \in (\overline{R^c})^c$ and $(y, x) \in (\overline{R^c})^c$. In the first case we have $(x, y) \in \overline{R^c}$, that is, there exists a natural number n and alternatives $x_1, x_2, \dots, x_n \in X$ such that

$$x = x_1 R^c x_2 \dots x_{n-1} R^c x_n = y.$$

Thus, negative consistency yields $(y, x) \notin P(R^c)$ which contradicts the hypothesis that $(x, y) \in P(R)$. For the second case, where $(x, y) \notin \overline{R^c}$ and $(y, x) \notin \overline{R^c}$, it follows that $(x, y) \in I(R)$ which leads to a contradiction too.

To see the converse, first suppose $P(R) \subseteq P(\overline{(R^c)^c})$ and take n and $x_1, x_2, \dots, x_n \in X$ such that

$$x = x_1 R^c x_2 \dots x_{n-1} R^c x_n = y.$$

Thus, $(x, y) \in \overline{R^c}$, implying $(x, y) \notin (\overline{R^c})^c$, and by supposition $(x, y) \notin P(R)$. Therefore, $(y, x) \notin P(R^c)$, as required. \square

The following proposition is the dual of the Proposition 5 in [1].

Proposition 2. The class of all negative consistent binary relations is closed downward.

Proof. Let \mathcal{B} be the class of all negative consistent binary relations. To prove that \mathcal{B} is closed downward, take a chain \mathcal{C} in \mathcal{B} , let $C = \bigcap_{B_i \in \mathcal{C}} B_i$, and take

$(x, y) \in P(C)$. We prove that $(x, y) \in P(\overline{(C^c)^c})$. We proceed by the way of contradiction, suppose that $(x, y) \notin P(\overline{(C^c)^c})$, then there are two cases to consider: (i) $(x, y) \notin \overline{(C^c)^c}$; (ii) $(x, y) \in \overline{(C^c)^c}$ and $(y, x) \in (C^c)^c$. In the first case, if $(x, y) \notin \overline{(C^c)^c}$, then, $(x, y) \in C^c$. Thus, there exist $x_0, x_1, \dots, x_K \in X$ such that

$$x = x_0, (x_{k-1}, x_k) \in C^c \text{ for all } k \in \{0, \dots, K\} \text{ and } x_K = y.$$

But then, for each $k \in \{1, \dots, K\}$, there is a $B_k \in \mathcal{C}$ such that $(x_{k-1}, x_k) \in B_k^c$. Since \mathcal{C} is a chain, $\tilde{\mathcal{B}} = \{B_k | k = 1, 2, \dots, K\}$ contains a relation, B_λ , minimum with respect to set-inclusion. Hence,

$$x = x_0 B_\lambda^c x_1 \dots x_{K-1} B_\lambda^c x_K = y.$$

On the other hand, since $(x, y) \in P(C)$, there is $B_\mu \in \mathcal{C}$ such that $(x, y) \in P(B_\nu)$ for each $B_\nu \subseteq B_\mu$. We have the following two subcases to consider: (i_a) $B_\lambda \subseteq B_\mu$; (i_b) $B_\mu \subseteq B_\lambda$. For (i_a), we have

$$x = x_0 B_\lambda^c x_1 \dots x_{K-1} B_\lambda^c x_K = y \text{ and } (x, y) \in P(B_\lambda).$$

Since B_λ is negative consistent, it must be that $(y, x) \notin P(B_\lambda^c)$. Hence, $(y, x) \in B_\lambda$ or $(x, y) \in I(B_\lambda)$, contradicting $(x, y) \in P(B_\lambda)$. Now consider the subcase (i_b). Since $B_\mu \subseteq B_\lambda$, we have

$$x = x_0 B_\mu^c x_1 \dots x_{K-1} B_\mu^c x_K = y \text{ and } (x, y) \in P(B_\mu).$$

This is a contradiction as well.

We come now to the second case, that of $(x, y) \in \overline{(C^c)^c}$ and $(y, x) \in (C^c)^c$. In this case, we have $(x, y) \in I(C)$ which contradicts that $(x, y) \in P(B_\mu)$. \square

The next two propositions are used in the proof of Theorem 5 below. The proof of the Proposition 3 uses the technique of Lemma 1 in [2]

Proposition 3. Let R be a binary relation on X . For each $x \in X$, there exists a negative consistent superrelation $R_{C(x)} \supseteq R$ such that $\overline{R_{C(x)}^c} x = \overline{R^c} x \setminus \{x\}$.

Proof. Let us define $Y = \overline{R^c}x \cup \{x\}$. Denote by \mathcal{R} be the set of negative consistent superrelations $R_N \subseteq X \times X$ of R which satisfies the following property (c):

- (c) For each $z, y \in X$, if $(z, y) \notin R_N$, then $x = y$ or $(y, x) \in \overline{R_N^c}$.

Since $X \times X$ lies in \mathcal{R} , this set is non-empty. Let \mathcal{C} be a chain in \mathcal{R} , and let $\mathcal{D} = \bigcap \mathcal{C}$. Since the class of negative consistent relations is closed downward (Proposition 2), \mathcal{D} is negative consistent. Moreover, \mathcal{D} satisfies the condition (c). To see that, take any $s, t \in X$ such that $(t, s) \notin \mathcal{D}$, so there exists $R_N \in \mathcal{C}$ such that $(t, s) \notin R_N$. Hence, $s = x$ or $(s, x) \in \overline{R_N^c} \subseteq \overline{\mathcal{D}^c}$. Therefore, by Zorn's lemma, \mathcal{R} has an element, say $R_{C(x)}$, that is minimal with respect to set inclusion. To prove that $\overline{R_{C(x)}^c}x = Y \setminus \{x\}$, it suffices to show that $\Lambda = (Y \setminus \{x\}) \setminus \overline{R_{C(x)}^c}x = \emptyset$. Now suppose to the contrary that there exists a point $y \in \Lambda$. Then, there exists a natural number n and alternatives $y_1, y_2, \dots, y_n \in X$ such that

$$y = y_1 R^c y_2 \dots y_{n-1} R^c y_n = x \quad \text{and} \quad (y, x) \notin \overline{R_{C(x)}^c}$$

Since $x \neq y$, we may assume that the elements y_1, y_2, \dots, y_n are distinct. Now define

$$Q_N = R_{C(x)} \setminus \{(y_1, y_2), \dots, (y_{n-1}, y_n)\}.$$

Then, we have $R \subseteq Q_N \subset R_{C(x)}$. The first inclusion is easy: For each $k \in \{1, \dots, n-1\}$, $(y_k, y_{k+1}) \notin R$. For the second inclusion, it suffices to show that there is at most one $k \in \{1, \dots, n-1\}$ such that $(y_k, y_{k+1}) \in R_{C(x)}$. Indeed, if for each $k \in \{1, \dots, n-1\}$ we let $(y_k, y_{k+1}) \notin R_{C(x)}$, we obtain $(y, x) \in \overline{R_{C(x)}^c}$, a contradiction. Furthermore, Q_N satisfies the condition (c). Indeed, assume that $s, t \in X$ are such that $(t, s) \notin Q_N$. There are two cases to consider: (i) $(t, s) \notin R_{C(x)}$; (ii) $(t, s) = (y_k, y_{k+1})$ for some $i \in \{1, \dots, n-1\}$. In the first case, by construction we have $x = s$ or $(s, x) \in \overline{R_{C(x)}^c} \subset \overline{Q_N^c}$. In the second case, there must exist $k \in \{1, \dots, n-1\}$ such that $s = y_{k+1}$. If $k = n-1$, then $s = y_n = x$. Otherwise, $s = y_{k^*+1}$ for some $k^* \in \{1, \dots, n-2\}$. Since $(y_{k^*}, y_{k^*+1}) \in Q_N^c, \dots, (y_{n-1}, y_n) = (y_{n-1}, x) \in Q_N^c$, we conclude that $(s, x) \in \overline{Q_N^c}$. Therefore, by minimality of $R_{C(x)}$, it is clear that Q_N is not negative consistent. Thus, there exists a natural number m and alternatives $z_0, z_1, \dots, z_m \in X$ such that

$$\mu = z_0 Q_N^c z_1 \dots z_{m-1} Q_N^c z_m = \nu \quad \text{and} \quad (\nu, \mu) \in P(Q_N^c).$$

Since $R_{C(x)}$ is negative consistent and $Q_N^c = R_{C(x)}^c \cup \{(y_1, y_2), \dots, (y_{n-1}, y_n)\}$, there must exist $\kappa = 1, \dots, n-1$ and $\lambda = 0, 1, \dots, m-1$ such that $(y_\kappa, y_{\kappa+1}) = (z_\lambda, z_{\lambda+1})$. Consider the smallest κ for which there exist such $\mu, \nu, m, z_0, \dots, z_m$, and λ . We show that there is no $j \in \{1, \dots, n-1\}$ such that $(z_{\text{mod}[\lambda(m+1)+m+\lambda, m+1]}, z_\lambda) = (y_j, y_{j+1})$. We proceed by the way of contradiction. Suppose that $y_{j+1} = z_\lambda R^c z_{\lambda+1} = y_{k+1}$. Since the elements y_1, \dots, y_n are distinct, it follows that $\kappa \neq j$ and so $\kappa < j$. But then, from $y_\kappa = z_\lambda = y_{j+1}$ we conclude

that $\kappa = j + 1$ which is impossible. Thus, from $(z_{\text{mod}[\lambda(m+1)+m+\lambda, m+1]}, z_\lambda) \in Q_N^c$ we deduce that $(z_{\text{mod}[\lambda(m+1)+m+\lambda, m+1]}, z_\lambda) \in R_{C(x)}^c$. Since $R_{C(x)} \in \mathcal{R}$ and $(z_{\text{mod}[\lambda(m+1)+m+\lambda, m+1]}, z_\lambda) \notin R_{C(x)}$, we have $x = z_\lambda$ or $(z_\lambda, x) \in \overline{R_{C(x)}^c}$. Using $z_\lambda = y_\kappa \neq x$, we exclude the first case. Hence,

$$y = y_1 R^c y_2 \dots R^c y_k \overline{R_{C(x)}^c} x.$$

Now define

$$\Gamma_N = R_{C(x)} \setminus \{(y_1, y_2), \dots, (y_{\kappa-1}, y_\kappa)\}.$$

As in the proof of Q_N , we conclude that $R \subseteq \Gamma_N \subset R_{C(x)}$. Furthermore, for all $s, t \in X$, if $(t, s) \notin \Gamma_N$, then similarly to the case of the relation Q_N we can prove that $x = s$ or $(s, x) \in \overline{\Gamma_N^c}$. Thus, Γ_N satisfies the condition (c). Finally, because of choice of κ we conclude that Γ_N is negative consistent. Hence, $\Gamma_N \in \mathcal{R}$, contradicting the minimality of $R_{C(x)}$. This contradiction establish that $\Lambda = \emptyset$ and completes the proof. \square

Proposition 4. Let X be a nonempty set of alternatives and let R be a binary relation over X . Then, the $\mathcal{GETCHA}(R)$ set is equivalent to $\mathcal{M}(\overline{([P(R)]^c)^{-1}})$.

Proof. Let $x \in \mathcal{GETCHA}(R)$. We have two cases to consider: (i) For each $y \in Y$ there holds $(x, y) \in R$; (ii) There exists $y_0 \in Y$ such that $(x, y_0) \notin R$. In the first case, we have $(y, x) \notin P(R)$ which implies that $(x, y) \in ([P(R)]^c)^{-1} \subseteq \overline{([P(R)]^c)^{-1}}$. Hence, $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$. In the second case, since $(x, y_0) \notin R$, it follows that $y_0 \in \mathcal{GETCHA}$, for otherwise $(x, y_0) \in R$ which is impossible. Let $A_x = \{t \in \mathcal{GETCHA}(R) \mid (x, t) \in \overline{R^c}\}$. We have that $A_x \neq \emptyset$, because otherwise, for each $t \in \mathcal{GETCHA}(R)$, $(x, t) \notin \overline{R^c}$. But then, $(x, t) \in R$, which implies that $\{x\} \subset \mathcal{GETCHA}$ is an R -dominant subset of X , a contradiction because of the minimal character of $\mathcal{GETCHA}(R)$. Let $G = \mathcal{GETCHA}(R) \setminus A_x$. We prove that $G = \emptyset$. We proceed by the way of contradiction. Suppose that $G \neq \emptyset$. Then, for each $t \in A_x$ and each $s \in G$ we have $(t, s) \in R$ for suppose otherwise, $(t, s) \in R^c$ which implies that $(x, s) \in \overline{R^c}$ contradicting $s \in G$. Therefore, $A_x \subset \mathcal{GETCHA}(R)$ is an R -dominant subset of X , again a contradiction. Hence, $A_x = \mathcal{GETCHA}(R)$. Since, $y_0 \in \mathcal{GETCHA}(R)$ we conclude that $(x, y_0) \in \overline{R^c}$. Similarly, we can prove that $(y_0, x) \in \overline{R^c}$. Hence, since $R^c \subseteq [P(R)]^c$ we conclude that x and y_0 belong to a $([P(R)]^c)^{-1}$ -cycle. On the other hand, for each $y \in Y \setminus \mathcal{GETCHA}(R)$, as in the case (i), we deduce that $(x, y) \in \overline{([P(R)]^c)^{-1}}$. Hence in any case we have $(y, x) \notin P(\overline{([P(R)]^c)^{-1}})$ which implies that $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$.

To prove the converse, take any $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$. We show that $x \in \mathcal{GETCHA}(R)$. We will consider two cases:

Case 1: For each $y \in X$ there holds $(y, x) \notin \overline{([P(R)]^c)^{-1}}$. In this case we have $(x, y) \in P(R) \subseteq R$. Hence, x is an R -dominant element of X which implies that $\mathcal{GETCHA}(R) = \{x\}$.

Case 2. There exists $y \in X$ such that $(x, y) \in \overline{([P(R)]^c)^{-1}}$ and $(y, x) \in \overline{([P(R)]^c)^{-1}}$. In this case, x belongs to a $[P(R)]^c$ -cycle. Let $\mathcal{C}(x)$ be a $[P(R)]^c$ -cycle containing x that is maximal in the sense that it is not a proper subset of any other $[P(R)]^c$ -cycle. We prove that $\mathcal{C}(x) = \mathcal{GETCHA}$. Suppose on the contrary, that $(t, z) \notin R$ for some $t \in \mathcal{C}(x)$ and $z \in X \setminus \mathcal{C}(x)$; to deduce a contradiction. It follows that $(t, z) \in [P(R)]^c$ which implies that $(x, z) \in \overline{[P(R)]^c}$. Hence, $(z, x) \in \overline{([P(R)]^c)^{-1}}$. Since $(z, x) \notin P(\overline{([P(R)]^c)^{-1}})$ we conclude that $(x, z) \in \overline{([P(R)]^c)^{-1}}$. Hence, $\mathcal{C}(x) \cup \{z\}$ is a $[P(R)]^c$ -cycle, a contradiction. \square

The next result shows the connection between the $\mathcal{GETCHA}(R)$ set and the choice sets generated from negative consistent superrelations.

Theorem 5. Let X be a nonempty set of alternatives and let R be a binary relation over X . Then, the $\mathcal{GETCHA}(R)$ set is equivalent to the union of maximal elements of all minimal negative consistent superrelations of R .

Proof. Let $R_{N^*} \in \mathcal{R}_{N^*}$ be minimal, take any $x \in \mathcal{M}(R_{N^*})$. We prove that $x \in \mathcal{GETCHA}(R)$. Suppose to the contrary that $x \notin \mathcal{GETCHA}(R)$, then by Proposition 4 there exists $y \in X$ such that $(y, x) \in P(\overline{([P(R)]^c)^{-1}})$. It follows that $(x, y) \notin \overline{([P(R)]^c)^{-1}}$ which implies that $(y, x) \in P(R)$. Hence, $(y, x) \in R \subseteq R_{N^*}$. Therefore by $(y, x) \notin P(R_{N^*})$ we conclude that $(x, y) \in R_{N^*}$. Let us define $R_{N^{**}} = R_{N^*} \setminus (x, y)$. Since $(x, y) \notin R$, we conclude that $R \subseteq R_{N^{**}} \subset R_{N^*}$ and $R_{N^{**}}$ is non negative consistent (the assumption that $R_{N^{**}}$ is negative consistent contradicts to the fact that R_{N^*} is minimal with respect to set-inclusion). Hence, there exist $s, t \in X$, $\lambda \in \mathbb{N}$, and $z_0, z_1, \dots, z_\Lambda \in X$ such that $s = z_0$, $(z_{\lambda-1}, z_\lambda) \in R_{N^{**}}^c$ for all $\lambda \in \{1, \dots, \Lambda\}$, $z_\Lambda = t$ and $(t, s) \in P(R_{N^{**}}^c) \subseteq R_{N^{**}}^c$. Since R_{N^*} is negative consistent and $R_{N^{**}}^c = R_{N^*}^c \cup \{(x, y)\}$, there must exist $\lambda_0 \in \{1, \dots, \Lambda\}$ such that $(z_{\lambda_0-1}, z_{\lambda_0}) = (x, y)$ and for all $\lambda \in \{1, \dots, \Lambda\}$ with $\lambda \neq \lambda_0$, $(z_{\lambda-1}, z_\lambda) \in R_{N^{**}}$ if and only if $(z_{\lambda-1}, z_\lambda) \in R_{N^*}$. It then follows that $(z_{\lambda_0}, z_{\lambda_0-1}) \in \overline{R_{N^*}^c}$. Therefore, $(y, x) \in \overline{R_{N^*}^c} \subset \overline{R_{N^*}^c} \subset \overline{[P(R)]^c}$. But then, $(x, y) \in \overline{([P(R)]^c)^{-1}}$ contradicting $(y, x) \in P(\overline{([P(R)]^c)^{-1}})$. This contradiction confirms the claim.

To prove the converse, take any $x \in \mathcal{GETCHA}(R)$. We show that there exists $R_{N^*} \in \mathcal{R}_N$ such that $x \in \mathcal{M}(R_{N^*})$. Let $R_{C(x)}$ be as in Proposition 3. First, observe that x is $R_{C(x)}$ -maximal in X . Indeed, suppose to the contrary that there exists $y \in X$ such that $(y, x) \in P(R_{C(x)}) \subseteq P(\overline{(R_{C(x)}^c)^c})$. Since $(x, y) \notin R_{C(x)} \supseteq R$, it follows that $y \in \mathcal{GETCHA}(R)$, for otherwise $(x, y) \subseteq R \subseteq R_{C(x)}$ which is impossible. From $x \in \mathcal{GETCHA}(R)$ by using the proof of Proposition 4 we conclude that $(y, x) \in \overline{R^c}$. Therefore, $y \in \overline{R^c}x \setminus \{x\} = \overline{R_{C(x)}^c}x$, contradicting $(y, x) \in P(\overline{(R_{C(x)}^c)^c}) \subseteq \overline{(R_{C(x)}^c)^c}$. Hence, x is $R_{C(x)}$ -maximal in X . If $R_{C(x)}$ is minimal with respect to set-inclusion in X , then the proof is over. Otherwise, there exists at least one negative consistent superrelation Q such that $R \subseteq Q \subset R_{C(x)}$. Let \mathcal{Q} be the set of negative consistent superrelations Q satisfying the latter condition. Let \mathcal{C} be a chain in \mathcal{Q} , and let $\mathcal{D} = \bigcap \mathcal{C}$. Evidently, $R \subseteq$

$\mathcal{D} \subset R_{C(x)}$. Since the class of negative consistent relations is closed downward (proposition 2), \mathcal{D} is negative consistent. Therefore, by Zorn's lemma, \mathcal{Q} has an element, say \tilde{Q} , that is minimal with respect to set inclusion. We prove that x is \tilde{Q} -maximal. We proceed by way of contradiction. Let $y \in X$ such that $(y, x) \in P(\tilde{Q})$. Since $x \in \mathcal{GETCHA}(R)$ and $(x, y) \notin R$, as above, we conclude that $y \in \overline{R^c}x \setminus \{x\} = \overline{R_{C(x)}^c}x \subset \overline{\tilde{Q}^c}x$, contradicting $(y, x) \in P(\tilde{Q}) \subseteq P((\overline{\tilde{Q}^c})^c) \subseteq (\overline{\tilde{Q}^c})^c$. The proof is over. \square

REFERENCES

- [1] Duggan, J. (1999). A general extension theorem for binary relations, *Journal of Economic Theory*, **86**, 1-16.
- [2] Duggan, J. (2007). A systematic approach to the construction of non-empty Choice sets, *Social Choice and Welfare*, **28**, 491-506.
- [3] Fishburn, P., (1977). Condorcet Social Choice Functions, *SIAM Journal on Applied Mathematics*, Vol. **33**, No. 3, pp. 469-489.
- [4] Schwartz, T., *The logic of Collective Choice*, New York: Columbia University Press.
- [5] Smith, J. (1973). Aggregation of Preferences with Variable Electorate, *Econometrica*, Vol. **41**, No. 6, pp. 1027-1041.
- [6] Suzumura, K. (1976). Remarks on the theory of collective. *Economica*, **43**, 381-390.