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16 July 2008

Online at https://mpra.ub.uni-muenchen.de/14922/
MPRA Paper No. 14922, posted 30 Apr 2009 00:30 UTC

Lévy Density Based Intensity Modeling of the Correlation Smile

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July 16, 2008, Revised: April 6, 2009

Abstract

The jump distribution for the default intensities in a reduced form framework is modeled and calibrated to provide reasonable fits to CDX.NA.IG and iTraxx Europe CDOs, to 5, 7 and 10 year maturities simultaneously. Calibration is carried out using an efficient Monte Carlo simulation algorithm suitable for both homogeneous and heterogeneous collections of credit names. The underlying jump process is found to relate closely to a maximally skewed stable Lévy process with index of stability $\alpha \sim 1.5$.

The market standard for pricing credit derivatives sensitive to default dependency is based on the Gaussian copula. As is well-known, this method is inadequate to price non-standard products. The model as such is not able to explain the correlation smile. Better models addressing these issues have been developed by various authors. Some recent work in this direction in the reduced form framework involves modeling the default intensities as in Joshi and Stacey [2005], Di Graziano and Rogers [2005], Chapovsky, Rennie and Tavares [2006], Errais, Giesecke and Goldberg [2006], Balakrishna [2007]; modeling dependency with simultaneous defaults as in Putyatin, Prieul and Maslova [2005], Balakrishna [2006], Brigo, Pallavicini and Torresetti [2006a], Hull and White [2007]; and modeling loss distributions as in Bennani [2005], Sidenius, Piterbarg and Andersen [2005], Schönbucher [2005].

The model presented here is based on pure-jump processes for the default intensities driven by a common underlying Lévy process. It involves a framework for handling such dependent processes and the associated jump distributions and an efficient Monte Carlo algorithm to generate default scenarios to price default correlation products. The jump in the default intensity of a credit name is taken to be proportional to its hazard rate from the credit curve leading to an increasing dependence of default correlation on the hazard rate in agreement with our expectations. This important feature of the model enables it to offer a reasonable explanation of the correlation smile and account for a correlation term structure. The model provides a reasonable fit to CDX.NA.IG and iTraxx Europe CDOs, simultaneously to all three maturities: 5, 7 and 10 year. The behavior $y^{-1-\alpha}$ of the Lévy density describing the jump distribution of jump size y, that is characteristic of stable Lévy processes with index of stability α, appears to be largely responsible for generating

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a correlation smile. The underlying Lévy process is found to relate closely to a maximally skewed stable Lévy process with $\alpha \sim 1.5$.

Section 1 presents the model. Section 2 discusses some analytical properties. Section 3 presents an efficient Monte Carlo algorithm applicable to both homogeneous and heterogeneous collections. Section 4 presents the simulation results obtained by calibrating the model to CDX.NA.IG and iTraxx Europe CDOs of 5, 7 and 10 year maturities. Section 5 concludes with a summary. Some explicit solutions and a series expansion are presented in Appendices A and B.

1 The Model

The model is based on the following pure jump process for the default intensity $\lambda_i(t)$ for each of the credit names $i = 1, ..., n$:

$$d\lambda_i(t) = [\phi_i(t) - \mu_i \lambda_i(t)]\, dt + \int_x h_i(x,t) dN(dx,t). \tag{1}$$

$N(dx,t)$'s are independent Poisson processes with intensities $\zeta(x,t)dx$, labeled by a variable x and associated with the intervals $(x, x+dx)$, modeling the arrival of common events causing jumps in the default intensities. If the Poisson process $N(dx,t)$ jumps up by one at time t due to the arrival of a common event, $dN(dx,t)$ causes each $\lambda_i(t)$ to jump up respectively by $h_i(x,t)$ at time t. The jump in $\lambda_i(t)$ decays at rate μ_i until the arrival of another event at a later time. Poisson intensity density $\zeta(x,t)$, assumed to be integrable over x, is factored as $\zeta(t)\nu_X(x)$ introducing an overall intensity $\zeta(t)$ and a probability density function $\nu_X(x)$ (assumed to be t-independent) describing the distribution of a random variable X. Hence, the above process can also be viewed as being driven by a compound Poisson process.

The drift term $\phi_i(t)$ can be handled by introducing $\overline{\lambda}_i(t)$ via $\phi_i(t) = d\overline{\lambda}_i(t)/dt + \mu_i \overline{\lambda}_i(t)$ so that the process for $\lambda_i(t) - \overline{\lambda}_i(t)$ is driftless in between Poisson events. $\overline{\lambda}_i(t)$ may be viewed as a firm-specific component of the default intensity. Model consistency suggests that we require $\overline{\lambda}_i(t) \geq 0$. From the perspective of $\overline{\lambda}_i(t)$, μ_i is a decay rate. It is also a mean reversion rate since, as can be noted from (1), $\lambda_i(t)$ mean reverts at rate μ_i to mean default intensity (say $\widehat{\lambda}_i(t)$, given by $\phi_i(t) = d\widehat{\lambda}_i(t)/dt + \mu_i \widehat{\lambda}_i(t) - \int_x dx \zeta(x,t) h_i(x,t)$).

Intensity models with jumps have been discussed before, usually with exponentially distributed jump sizes (see for instance Duffie, Pan and Singleton [1998]). It is known that pure jump models with such distributions are not capable of explaining the correlation smile. Process (1) is kept quite general with no such distributional assumptions. It can be viewed as the continuum version of a model involving a discrete set of common factors discussed in Balakrishna[2007]. It is hence expected to exhibit default contagion with a cluster life of the order of μ^{-1}. It admits some explicit solutions that are presented in Appendix A.

Just as in the discrete version, one may note here a certain limit of the model as $h_i(x,t)$ and μ_i tend to infinity such that $h_i(x,t)/\mu_i \to h'_i(x,t)$. Jumps in $\lambda_i(t)$ then become infinitely large but decay instantaneously and contribute to the time-integral $\Lambda_i(t) = \int_0^t ds \lambda_i(s)$ (this is better appreciated within the context of the Monte Carlo algorithm discussed later). The limiting model of simultaneous defaults is described by

$$d\Lambda_i(t) = \overline{\lambda}_i(t)dt + \int_x h'_i(x,t) dN(dx,t). \tag{2}$$

This model of simultaneous defaults has appeared in the literature under various guises. It belongs to a class of shock models related to Marshall-Olkin copula (for a discussion of shock models as applied to credit risk, see for instance Elouerkhaoui [2003], Lindskog and McNeil [2003], Brigo, Pallavicini and Torresetti [2006a]). It's version involving a discrete set of common factors is studied parametrically in Balakrishna [2006]. The model of Putyatin, Prieul and Maslova [2005] can be identified with its homogeneous version with one discrete factor. The constant jump version of Hull and White [2007] can also be identified with it for one discrete factor having a dynamical formulation of the above kind.

Coming back to process (1), note that the jumps in the default intensity are independent of the level of the default intensity. If these jumps are taken to be uniform across all the credit names in a heterogeneous collection, they will contribute a uniform amount to the hazard rates implied by the model and hence will be limited by the smallest hazard rate in the collection. To avoid this unrealistic feature, let us assume that $h_i(x,t)$ is proportional to some name-specific intensity $h_i(t)$,

$$h_i(x,t) = \mu_i y_i(x,t) h_i(t). \tag{3}$$

Heterogeneous collection can now be handled naturally with a uniform μ_i and $y_i(x,t)$ if desired. A factor of μ_i is included to make it convenient taking the $\mu_i \to \infty$ limit.

Different choices can be tried for $h_i(t)$. A straightforward one is to take $h_i(t)$ to be $\overline{\lambda}_i(t)$. A more appealing choice is to take it to be the mean default intensity. The choice considered in this the article, convenient from the point of view of calibration to individual credit curves though with a less appealing dynamical nature, is to take $h_i(t)$ to be the hazard rate from the credit curve. A more realistic model would perhaps involve jumps proportional to default intensity $\lambda_i(t)$ itself. Such a model has no available solutions but can be investigated using the Monte Carlo algorithm and appears to be capable of providing similar quality fits.

For a time-independent $y_i(x,t)$, with the i dependence suppressed, process (1) can be viewed as being driven by an underlying Lévy process $L(t)$ given by

$$dL(t) = \int_x y(x) dN(dx,t). \tag{4}$$

This corresponds to what is called a subordinator because of our restriction to non-negative jumps for the default intensity. The associated Lévy density is $\zeta\nu_Y(y) = \zeta\nu_X(x(y))\,|dx(y)/dy|$ where $x(y)$ is inverse of the relation $y(x) = y$. As already mentioned, we are here concerned with integrable Lévy densities that give rise to probability density functions $\nu_Y(y)$.

Given this framework, one could attempt to imply the jump distribution from the market data given certain plausible assumptions about its form and some distributional assumptions for the underlying x variable. One approach is to assume $y(x) \propto x$ and to model the Lévy density by modeling the distribution of X itself. This approach is based on the implicit assumption that Lévy densities, and hence the jump distributions, associated with different processes for different credit names are essentially identical to each other. It is hence both appealing and convenient to assume instead that X follows some simple distribution so that all the model complexities are embodied into the Lévy density $\zeta\nu_Y(y)$. Let us hence assume that $x \in [0,1]$ is uniformly distributed with $\nu_X(x) = 1$ and that the probability density function of $y(x)$ is of the form

$$\nu_Y(y) = a^\alpha g(a/y) y^{-1-\alpha}. \tag{5}$$

Any i and t dependencies of the parameters a and α, or of the function $g()$ itself, are suppressed for convenience. Function $g(z)$ is assumed to be regular at $z = 0$ in some positive power of z. Parameter a determines the scale of y that is suitably chosen based on one's choice for the function $g(z)$.

The behavior $y^{-1-\alpha}$ of the Lévy density is characteristic of stable Lévy processes with index of stability α. Calibration results indicate that α lies in between 1 and 2 for which this behavior leads to a divergence as $y \to 0$ in the first moment of the jump distribution. In our case, function $g(z)$, beside ensuring an integrable Lévy density, is expected to 'regularize' such divergences by going to zero sufficiently fast as $z \to \infty$. It will thus be referred to here as the regularizing function, though it may be more than so in reality. Because $\nu_Y(y)$ is a normalized density, $g(z)$ should satisfy

$$\int_0^\infty dz\ g(z) z^{\alpha-1} = 1. \tag{6}$$

Function $y(x,t)$ is now determined by matching the cumulative densities of y and x. For a uniformly distributed X and $\nu_Y(y)$ of the form (5) this becomes

$$\int_{a/y(x)}^\infty dz\ g(z) z^{\alpha-1} = x. \tag{7}$$

This determines $y(x,t)$ given the individual Lévy densities for each of the credit names.

Different choices for $g(z)$ appear to be capable of providing a reasonable fit to market prices. An easily integrable choice is

$$g(z) = \alpha, \quad z \leq 1, \text{ zero otherwise}, \quad y(x) = a(1-x)^{-1/\alpha}. \tag{8}$$

This yields a Pareto distributed Lévy density $\zeta \alpha a^\alpha y^{-1-\alpha}$, $y \geq a$, zero otherwise. Another easily integrable choice is

$$g(z) = \alpha \exp(-z^\alpha), \quad y(x) = a(-\mathrm{Ln}x)^{-1/\alpha}. \tag{9}$$

A more generic choice $g(z) = \omega[\Gamma(\alpha/\omega)]^{-1}\exp(-z^\omega)$ given some $\omega > 0$, of which the above two are special cases, solves for $y(x)$ in terms of the incomplete gamma function for which efficient numerical algorithms are available. Alternately, if integrability becomes an issue, $y(x)$ can be modeled directly to be used in the Monte Carlo algorithm. The Pareto distribution being the simplest is chosen for the article.

The fit to the five tranches turns out to be better for a slightly lower spread of index credit default swap. This appears to be due to the fact that the model as such is not able to generate sufficient spread for the super senior tranche, at least with a manageable number of Monte Carlo scenarios attempted here. One could model the relatively large jumps affecting the super senior tranche, but a shortcut seems to suffice at this point. Let us force $h(x,t)$ to ∞ for $h(x,t) > \mu b$ given some relatively large b leading to a small probability of simultaneous defaults. This corresponds to setting $y(x,t) = \infty$ for $y(x,t) > b/h(t)$ given a suitably redefined $g(z)$ for $z < ah(t)/b$. This shortcut is capable of generating significant spread for the super senior tranche to effect a better fit to the remaining five tranches.

2 Analytic Properties

The model offers some analytical results that can be useful in calibrating the model to individual credit curves. They contain integrals involving a parameter, say u, representable as Laplace transform of the density $\nu_Y(y)$,

$$\int_x dx\ \nu_X(x) e^{-uy(x)} = \int_0^\infty dy\ \nu_Y(y) e^{-uy} = \int_0^\infty dz\ g(z) z^{\alpha-1} e^{-au/z}. \tag{10}$$

For a generic regularization supplied by $g(z)$, the integral is approximated for small au in Appendix B up to second order as

$$\int_0^\infty dz\ g(z) z^{\alpha-1} e^{-au/z} \simeq 1 - g_1 au + g_\alpha (au)^\alpha - \frac{1}{2} g_2 (au)^2. \tag{11}$$

It is assumed that α lies between 1 and 2, and that $g(z) - g(0) \sim z^\omega$ as $z \to 0$ for some $\omega > 2 - \alpha$. The remainder above is of $\mathcal{O}(au)^3$ for $\omega > 3 - \alpha$, but of $\mathcal{O}(au)^{\alpha+\omega}$ otherwise. Coefficients g_1, g_α and g_2 are (prime denotes differentiation)

$$g_1 = -\frac{1}{\alpha - 1} \int_0^\infty dz\ g'(z) z^{\alpha-1}, \quad g_\alpha = g(0)\Gamma(-\alpha), \quad g_2 = -\frac{1}{2-\alpha} \int_0^\infty dz\ g'(z) z^{\alpha-2}. \tag{12}$$

For the Pareto distributed Lévy density, $g_1 = \alpha/(\alpha - 1)$, $g_\alpha = \alpha\Gamma(-\alpha)$ and $g_2 = \alpha/(2 - \alpha)$. If $y_i(x,t) = \infty$ for $y_i(x,t) > b/h_i(t)$ given some relatively large b as discussed earlier, the expansion needs to be corrected by adding

$$-\int_0^{au/b'} dz\ g(z) z^{\alpha-1} e^{-au/z} \simeq -(au)^\alpha g(0) \Gamma(-\alpha, b'), \tag{13}$$

where $b' = bu/h_i(t)$ and $\Gamma(,)$ is the incomplete gamma function[1]. Its effect is to correct g_α by a multiplicative factor $1 - \Gamma(-\alpha, b')/\Gamma(-\alpha)$. However, since b' could turn out to be dependent on t or the number of names involved, g_α is referred to in the following as given by (12) without this multiplicative factor.

To start with, to get some insight into the model, consider the Laplace exponent $\eta(u)$ of the underlying Lévy process (4) given by

$$\begin{aligned}
\eta(u) &\equiv -\frac{1}{t} \log \left\{ \mathrm{E}\left[e^{-uL(t)} \right] \right\} = \zeta \int_x dx\ \nu_X(x) \left[1 - e^{-uy(x)} \right] \\
&= \zeta \int_0^\infty dz\ g(z) z^{\alpha-1} \left[1 - e^{-au/z} \right] \simeq \zeta g_1 u - \zeta g_\alpha (au)^\alpha + \frac{1}{2} \zeta g_2 (au)^2.
\end{aligned} \tag{15}$$

Expectation E is taken over the underlying Poisson processes given the information at time zero. In expressions of this kind encountered below, parameter u turns out to be small being

[1] Efficient algorithms are available to compute both the complete and incomplete gamma functions. If they are available only for positive arguments, one can use

$$\Gamma(-\alpha) = \frac{1}{\alpha(\alpha-1)} \Gamma(2-\alpha), \quad \Gamma(-\alpha, x) = \frac{1}{\alpha(\alpha-1)} \left[\Gamma(2-\alpha, x) + (\alpha - 1 - x) x^{-\alpha} e^{-x} \right]. \tag{14}$$

proportional to a combination of one or more $h_i(t)$. The $\mathcal{O}(au)$-term just contributes to the implied hazard rate. $\mathcal{O}(au)^\alpha$-term is largely responsible for default dependency. This term is characteristic of the behavior $\nu_Y(y) \sim y^{-1-\alpha}$. It is in fact the two-sided Laplace exponent of an α-stable Lévy process maximally skewed to the right with zero mean[2]. $\mathcal{O}(au)^2$-term contributes relatively less to default dependency. Higher order terms are smaller, and negligible for small hazard rates as far as individual credit curves are concerned or not too large a number of credit names is under consideration.

Given the explicit results presented in Appendix A, one obtains for the survival probability of a credit name,

$$Q_i(t) \equiv \mathrm{E}\left\{\exp\left[-\int_0^t ds\ \lambda_i(s)\right]\right\} = \exp\left[-c_i\beta_i(t) - \int_0^t ds\ \pi_i(s,t)\right]. \tag{16}$$

Here $c_i \equiv \lambda_i(0) - \overline{\lambda}_i(0)$ arises due to contributions from events earlier to time zero and

$$\begin{aligned} \beta_i(t) &= \frac{1}{\mu_i}\left(1 - e^{-\mu_i t}\right), \\ \pi_i(s,t) &= \overline{\lambda}_i(s) + \zeta(s)\int_x dx\ \nu_X(x)\left[1 - e^{-h_i(x,s)\beta_i(t-s)}\right]. \end{aligned} \tag{17}$$

The hazard rate is given by $p_i(t) = -d\mathrm{Ln}Q_i(t)/dt$. In the following, the subscript i is dropped for clarity. If $au = ah(s)\mu\beta(t-s)$ is expected to be small, the integral can be approximated according to (11). Assuming constant parameters, this gives the following result for the time-integral $P(t) \equiv \int_0^t ds\ p(s) = -\mathrm{Ln}Q(t)$ (with $\overline{\Lambda}(t) \equiv \int_0^t ds\ \overline{\lambda}(s)$),

$$\begin{aligned} P(t) &= \overline{\Lambda}(t) + c\beta(t) + \zeta\int_0^t ds\ \int_0^\infty dz\ g(z)z^{\alpha-1}\left[1 - e^{-ah(s)\mu\beta(t-s)/z}\right] \\ &\simeq \overline{\Lambda}(t) + c\beta(t) + \zeta\int_0^t ds\,\Big\{ \\ &\qquad g_1 ah(s)\mu\beta(t-s) - g_\alpha(ah(s))^\alpha(\mu\beta(t-s))^\alpha + \frac{1}{2}g_2(ah(s))^2(\mu\beta(t-s))^2\Big\}. \end{aligned} \tag{18}$$

The hazard rate $p(t) = dP(t)/dt$ is given by

$$\begin{aligned} p(t) &\simeq \overline{\lambda}(t) + ce^{-\mu t} + \mu\zeta\int_0^t ds\ e^{-\mu(t-s)}\,\Big\{ \\ &\qquad g_1 ah(s) - \alpha g_\alpha(ah(s))^\alpha(\mu\beta(t-s))^{\alpha-1} + g_2(ah(s))^2\mu\beta(t-s)\Big\}. \end{aligned} \tag{19}$$

Correction to g_α is time-dependent since here $b' = b\mu\beta(t-s)$. For a piecewise constant $h(t)$-curve with $0 = t_0 < t_1 < \ldots$ defining the piecewise intervals, $h(t) = h_k$ in an interval (t_k, t_{k+1}) and $h_{-1} = 0$, one finds

$$\begin{aligned} p(t) &\simeq \overline{\lambda}(t) + ce^{-\mu t} + \zeta\sum_{t_k<t}\big\{g_1 a(h_k - h_{k-1})\mu\beta(t-t_k) \\ &\qquad - g_\alpha a^\alpha(h_k^\alpha - h_{k-1}^\alpha)(\mu\beta(t-t_k))^\alpha + \frac{1}{2}g_2 a^2(h_k^2 - h_{k-1}^2)(\mu\beta(t-t_k))^2\Big\}. \end{aligned} \tag{20}$$

[2] One is not able to reach a stable process via a limiting procedure wherein $a \to 0$ and $\zeta \to \infty$ keeping ζa^α fixed for a given α between 1 and 2 to suppress all higher order terms of the expansion. This is due to $\zeta g_1 a$ diverging in the limit and the model becoming inconsistent as it reaches a certain value. This is as expected since our intensity process can not handle negative jumps arising from the resulting stable process. However, it is interesting note that a model involving jumps proportional to default intensity λ_i can absorb this divergence into the decay rate μ_i to result in a mean-reverting model driven by a stable process.

Correction to g_α here involves $b' = b\mu\beta(t - t_k)$.

The above results can be useful in calibrating the model to individual credit curves, to determine $\overline{\lambda}(t)$ to be used in the Monte Carlo algorithm. Note that the integrals in (19) are in the form of convolutions which can be efficiently handled with Fast Fourier Transform techniques. For a given $p(t)$-curve, the method is straightforward if $h(t)$ is independently known, or is chosen to be $p(t)$ itself. In this case, the above results directly determine the $\overline{\lambda}(t)$-curve. If $h(t)$ is chosen to be $\overline{\lambda}(t)$, the above can be used to imply the $\overline{\lambda}(t)$-curve.

The limiting model of simultaneous defaults is interesting in its own right as it allows easier calibration to individual credit curves. Here, the result for the hazard rate takes a simpler form as $\mu \to \infty$ with $u = h(s)$, giving

$$\begin{aligned} p(t) &= \overline{\lambda}(t) + \zeta \int_0^\infty dz\ g(z) z^{\alpha-1} \left[1 - e^{-ah(t)/z}\right] \\ &\simeq \overline{\lambda}(t) + \zeta g_1 a h(t) - \zeta g_\alpha (ah(t))^\alpha + \frac{1}{2}\zeta g_2 (ah(t))^2. \end{aligned} \tag{21}$$

Correction to g_α is now t-independent since $b' = b$. The result can be used to easily calibrate the model to individual credit curves; for instance with constant parameters and a piecewise constant $p(t)$ and $h(t)$ curves, it gives piecewise constant $\overline{\lambda}(t)$ curves to be used in the Monte Carlo algorithm.

The model of simultaneous defaults admits a formulation of instantaneous default correlation. Here, the probability that two names, say i and j, default during an infinitesimal interval $(t, t + dt)$ is given by $p_{ij}(t)dt$ where

$$\begin{aligned} p_{ij}(t) &= \int_x dx\ \zeta(x,t)\{1 - \exp\left[-y_i(x,t)h_i(t)\right]\}\{1 - \exp\left[-y_j(x,t)h_j(t)\right]\}, \\ &\simeq \zeta g_\alpha \left[(a_i h_i(t) + a_j h_j(t))^\alpha - (a_i h_i(t))^\alpha - (a_j h_j(t))^\alpha\right] - \zeta g_2 a_i a_j h_i(t) h_j(t). \end{aligned} \tag{22}$$

The Lévy densities are assumed to be the same for the two names except perhaps for the scale parameter a. Given the hazard rates $p_i(t)$ and $p_j(t)$, this yields an expression for the instantaneous default correlation $\rho(t)$ which, for the choice $h_i(t) = p_i(t)$ and for a homogeneous collection of credit names, is approximately

$$\rho(t) = \frac{p_{ij}(t)}{\sqrt{p_i(t)p_j(t)}} \simeq (2^\alpha - 2)\sigma^\alpha \left(p(t)\right)^{\alpha-1} - \zeta g_2 a^2 p(t), \quad \text{where } \sigma = a\left(\zeta g_\alpha\right)^{1/\alpha}. \tag{23}$$

Thus, given constant parameters and small hazard rates, $\rho(t)$ increases with $p(t)$ in agreement with our expectation, approximately as its $(\alpha - 1)^{\text{th}}$ power, giving rise to an increasing correlation term structure for an increasing hazard rate curve. Its growth rate relative to that of the hazard rate is also reasonable. It evaluates to within an acceptable range; for instance with $\alpha = 1.5$, $a = 4.0$, $\zeta = 300$bp and $p(t) = 30$bp, it gives $\sigma \simeq 0.9$ and $\rho \simeq 3.4\%$.

Terms of order beyond $\mathcal{O}(ah)^2$ are negligible for small hazard rates as far as individual credit curves are concerned or not too large a number of credit names is involved. To the extent that they are negligible, the model of simultaneous defaults, as far as defaults are concerned, can be considered to be largely independent of the regularizing function $g(z)$. To see this for the choice $h(t) = p(t)$, assuming Lévy densities are the same for all the names, introduce, in addition to $\sigma = a\left(\zeta g_\alpha\right)^{1/\alpha}$ mentioned earlier, $\sigma_2 = a\sqrt{\zeta g_2}$ (for a positive g_2) and $\widehat{\lambda}(t) = \overline{\lambda}(t) + \zeta g_1 ah(t)$. Different regularizations calibrate to nearly the same σ, σ_2 and $\widehat{\lambda}(t)$ but to possibly different ζ, a and $\overline{\lambda}(t)$ (the g_2 contribution being relatively small, the model with a given regularization gets nearly calibrated for a range of ζ values). The only characteristics needed then of $g(z)$ are its normalization (fixed to define ζ) and its z-scale (set appropriately to define a). However, some regularizations may lead to too large a value of ζ that could make $\overline{\lambda}$ turn negative making the model inconsistent (a preferred choice would be the one that sets $\overline{\lambda}$ to a realistic firm-specific contribution if known). Similar regularization independence does not generally hold good for finite μ (a likely exception is when $h(t)$ is chosen to be the mean default intensity $\widehat{\lambda}(t)$ given by $d\widehat{\lambda}(t)/dt + \mu\widehat{\lambda}(t) - \mu\zeta g_1 ah(t) = d\overline{\lambda}(t)/dt + \mu\overline{\lambda}(t)$).

If some control over g_2 is desired, consider a $g(z)$ that is a non-increasing function of z so that $g_2 > 0$. Given this, another regularization $\widetilde{g}(z)$ can be constructed whose density can be viewed as a weighted sum of two densities: one with index α and the other with index $\alpha + \omega$. This $\widetilde{g}(z)$ for some $\omega > 2 - \alpha$ is given by

$$\begin{aligned} \widetilde{g}(z) &= \mathcal{C} g(z)(1 + \kappa z^{\omega}), \quad \text{where } \mathcal{C} = \left[1 + \kappa \int_0^{\infty} dz\ g(z) z^{\alpha+\omega-1}\right]^{-1}, \\ \widetilde{g}_2 &= \mathcal{C}\left\{-\frac{1}{2-\alpha}\int_0^{\infty} dz\ g'(z) z^{\alpha-2} + \frac{\kappa}{\alpha+\omega-2}\int_0^{\infty} dz\ g'(z) z^{\alpha+\omega-2}\right\}. \end{aligned} \tag{24}$$

It is interesting to note that there is a positive κ for which $\widetilde{g}_2$ can be identically set to zero[3]. As for ω, any value larger than $2 - \alpha$ such that the correction term of $\mathcal{O}(au)^{\alpha+\omega}$ occurs sufficiently farther away from $\mathcal{O}(au)^2$ can be chosen.

3 Monte Carlo Algorithm

The model can be handled numerically using a Monte Carlo simulation algorithm suitable for both homogeneous and heterogeneous collections. It is a modified version of the algorithm presented in Balakrishna[2007] with in the context of a discrete version of the model. It is based on the well-known observation that λ_i's are independent variables given a realization of the common Poisson processes and that $c_i(t) \equiv \lambda_i(t) - \overline{\lambda}_i(t)$ is driftless decaying at rate μ_i in between Poisson events. The algorithm reads as follows.

1. Draw n independent exponentially distributed random numbers $u_i, i = 1, ..., n$ with unit mean. For each i referring to a credit name, set the time-integral of default intensity $\Lambda_i(0)$ to zero and set $c_i(0)$ to any contribution from events earlier to time zero. Set $t_o = 0$.

2. Draw an independent exponentially distributed random number v with unit mean. Determine the next event arrival time t by solving $\int_{t_o}^{t} ds\zeta(s) = v$.

[3]This is helpful in a limiting procedure to a stable process wherein $a \to 0$ and $\zeta \to \infty$ keeping ζa^{α} fixed for a given α between 1 and 2. The limiting procedure is slow because of the $g_2(au)^2$ term in the expansion of $\eta(u)$. Besides, when α is closer to 2, the $g_2(au)^2$ term, absorbing the $1/(2-\alpha)$ divergence of g_{α}, results in the characteristic exponent containing $u^2\mathrm{Ln}u$ which is not expected of the $\alpha = 2$ stable process, namely the Brownian process. With $\widetilde{g}_2 = 0$ and no $\widetilde{g}_{\alpha}$ divergence, one obtains faster convergence to a stable process. The regularizing function is simplest in the Pareto case for $\omega \to \infty$ with $\widetilde{g}(z) = \alpha(2-\alpha)/2$ for $z \leq 1$ and zero otherwise, and a Dirac-delta function of magnitude $\alpha/2$ placed at $z = 1$. The Dirac-delta function represents a Poisson process of intensity $\zeta\alpha/2$.

3. For each i referring to a surviving credit name, update $\Lambda_i(t_o)$ to $\Lambda_i(t)$ by adding

$$F(t) = c_i(t_o)\beta_i(t - t_o) + \int_{t_o}^{t} ds\ \overline{\lambda}_i(s). \tag{25}$$

Check if $\Lambda_i(t) > u_i$ to determine whether this credit name defaults before time t. If so, determine its default time t_i by solving $\Lambda_i(t_o) + F(t_i) = u_i$.

4. If t is beyond the time horizon of interest or there are no more surviving credit names, go to step 6.

5. Draw an independent uniform random number $x \in [0, 1]$. For each i referring to a surviving credit name, determine $y_i(x,t)$ by solving (7). If $y_i(x,t) > b/h_i(t)$, set $y_i(x,t) = \infty$. Update $c_i(t_o)$ to

$$c_i(t) = c_i(t_o)e^{-\mu_i(t-t_o)} + \mu_i y_i(x,t)h_i(t). \tag{26}$$

Set $t_o = t$ and go to step 2.

6. Given the default times, price the instrument. For the next scenario, go to step 1.

7. Average all the prices thus obtained to get a price for the instrument.

In step 1, $c_i(0)$ arises due to contributions from events earlier to time zero. This would be especially relevant when one is in the middle of a default contagion. However, more study is needed to be able to use the algorithm in such situations. In our case, for a choice such as $h_i(t) = p_i(t)$, without loss of generality, it can be set to zero since its contribution $c_i(0)e^{-\mu_i t}$ can be absorbed into a redefined $\overline{\lambda}_i(t)$. In step 3, if t is beyond the time horizon of interest, time horizon can be used in place of t to avoid solving for any default times beyond.

The algorithm has a limit as h_i and μ_i tend to infinity simulating the model of simultaneous defaults described by the process (2). Expressing c_i as $\mu_i c_i'$ and letting $\mu_i \to \infty$, one observes that $c_i'(0) = 0$ in step 1, $c_i(t_o)\beta_i(t-t_o) \to c_i'(t_o)$ in step 3 and $c_i'(t) = y_i(x,t)h_i(t)$ in step 5. When solving for the default time in step 3, one could first check if $\Lambda_i(t_o)+c_i'(t_o) > u_i$ to determine whether default occurs exactly at the event arrival time t_o.

CDOs can be priced the usual way. Given a scenario of default times, one proceeds processing the defaults one by one, starting from the first up to maturity, picking up payments by the default leg, switching to the next tranche whenever a tranche gets wiped out, at the same time computing the premium legs of all the surviving tranches. Whenever a default leg pays out the loss amount, the notional of that tranche gets reduced by the same amount, and the notional of the super senior tranche gets reduced by the recovery amount so that the total notional of the CDO tranches keeps up with that of the index default swap (when the super senior is the only survivor, it gets treated like a default swap). The legs can be added across tranches to obtain those for the index default swap. They can also be reused for higher maturities within a scenario. The premium legs are computed per unit spread leaving out the computation of the prices or the par spreads to end of the simulation.

The efficiency of the algorithm is dependent on the number of events arriving before the time horizon. For a homogeneous collection, some improvement in efficiency can be achieved by using just one each of Λ_i and c_i variables and working with a sorted list of u_is to know the order of defaults. Significant improvement in efficiency is achieved by using quasi random sequences such as Sobol sequences to generate each of the independent random numbers as is done in this study.

4 Simulation Results

Calibration is carried out on a homogeneous collection of credit names assuming Pareto distributed Lévy densities. All parameters, except $\overline{\lambda}(t)$ and $h(t)$, are assumed to be time-independent. $\overline{\lambda}(t)$ and $h(t)$ are assumed to be piecewise constant, taken to be flat in between maturities. The choice $h(t) = p(t)$ is implemented by setting $h(t)$ to be the piecewise constant hazard rate curve obtained by matching the index default swap spreads outside of the Monte Carlo algorithm (an approximation since the model implied hazard rate curve $p(t)$ is not expected to be piecewise constant). Best fits are obtained by minimizing the sum of squares of percentage errors in the model spreads with more weight given to the index default swap spreads to obtain a better fit to them. Recovery rate is assumed to be flat[4] at 35%.

Table 1 shows the results of calibrating the model to CDX.NA.IG and iTraxx Europe CDOs for the market quotes on October 2, 2006 (source: Brigo, Pallavicini and Torresetti [2006b]). It is a simultaneous fit to all three maturities, 5, 7 and 10 year. It is also calibrated to the index default swap spread for 3-year along with the other three maturities. The results look reasonable. They are obtained with 100,000 runs or scenarios of the Monte Carlo algorithm. Reasonable results can also be obtained by calibrating with as few as 25,000 scenarios that takes just about a second for one simulation.

Calibration results suggest a finite value for μ. A simultaneous fit to maturities in the $\mu \to \infty$ model of simultaneous defaults is found to be less satisfactory (though the 5-year maturity alone can be fit equally well). It is possible to improve the fit by allowing for time-dependent α or other model parameters.

In Balakrishna[2006,2007] and some of the literature in the field, it is found that the modeled loss distribution displays one or more bumps along its tail. Even if such a distribution is able to reproduce the market prices for CDOs providing an explanation of the correlation smile, it is not immediately obvious whether the bumps are a realistic feature of the distribution or an artifact of the model. Their presence can give rise to significant differences in the predicted prices for bespoke CDOs. Models capable of reproducing the market prices without any such bumps can potentially give rise to better behaved prices and sensitivities for nonstandard products. As can be seen from figures 1 and 2, the present model exhibits no discernible bumps along the tail of the default probability distribution.

5 Conclusion

The article discusses a modeling framework to handle default dependency among a collection of credit names based on pure jump processes for the default intensities involving a Lévy density describing the distribution of jump sizes. A Monte Carlo algorithm to generate default scenarios and price default correlation products makes this framework viable. An important feature of the model is that the jump in the default intensity of a credit name is

[4]If desired, one may use a recovery rate $R(\lambda)$ modeled to be inversely related to the default intensity λ in line with some of the empirical findings. Altman, Brady, Resti and Sironi [2005] present the data and the best fits with goodness of fit measure $\mathcal{R}^2 = 0.65$ supporting this relationship. Given their data, one can look for a best fit that is well-behaved for both small and large default rates such as $R(\lambda) = 0.27 + 0.38e^{-39\lambda}$ or $R(\lambda) = 0.26/(1 - 0.64e^{-22\lambda})$ with $\mathcal{R}^2 = 0.64$. Some modifications are expected for these to be consistent with the recovery rate conventions or to be applicable for the λ's appearing in our jump processes.

taken to be proportional to its hazard rate from the credit curve. This leads to an increasing dependence of default correlation on the hazard rate in agreement with our expectations, also accounting for a correlation term structure over the three maturities: 5, 7 and 10 year. It is found that the model can be calibrated reasonably well to market prices for CDX.NA.IG and iTraxx Europe CDOs, to all three maturities simultaneously. The behavior $y^{-1-\alpha}$ of the jump distribution of y, a characteristic feature of stable Lévy processes, appears to be largely responsible for the fits. The underlying jump process appears to be closely related to a maximally skewed stable Lévy process with index of stability $\alpha \sim 1.5$.

A Explicit Solutions

As is known under various contexts, processes of the kind (1) admit some explicit solutions. The following is an adaptation of the results presented in Balakrishna[2007]. This involves looking for an explicit solution to the following expectation taken over the common Poisson processes with ordered times $t = t_0 \leq t_1 \leq t_2 \leq ... \leq t_n$,

$$f(t, \lambda_1(t), ...) = \mathrm{E}_t \left\{ \prod_{i=1}^{n} \exp\left[-\int_t^{t_i} ds \ \lambda_i(s) \right] \right\}. \tag{27}$$

Its differential can be written down using Ito's calculus leading to

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left[(\phi_i - \mu_i \lambda_i) \frac{\partial f}{\partial \lambda_i} - \lambda_i f \right] + \int_x dx \ \zeta(x,t) \left[f(\lambda_1 + h_1, ...) - f(\lambda_1, ...) \right] = 0. \tag{28}$$

This can be solved with the ansatz of the form

$$f\left(t, \lambda_1, ...\right) = \exp\left[-\alpha(t) - \sum_{i=1}^{n} \beta_i(t) \lambda_i \right]. \tag{29}$$

Equating coefficients of f independent of λ_i's and those linear in λ_i's separately gives

$$\begin{aligned} \frac{d\beta_i(t)}{dt} - \mu_i(t)\beta_i(t) + 1 &= 0, \\ \frac{d\alpha(t)}{dt} + \sum_{i=1}^{n} \phi_i(t)\beta_i(t) + \int_x dx \ \zeta(x,t) \left\{ 1 - \exp\left[-\sum_{i=1}^{n} h_i(x,t)\beta_i(t) \right] \right\} &= 0. \end{aligned} \tag{30}$$

These can be solved requiring $\beta_i(t) = 0$ for $t \geq t_i$ that ensures continuity as t is allowed to vary crossing various t_i's. The solutions are

$$\begin{aligned} \beta_i(t) &= \int_t^{t_i} d\tau \ \exp\left(-\int_t^{\tau} ds \ \mu_i(s) \right), \\ \alpha(t) &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ds \left\{ \sum_{k=i}^{n} \phi_k(s)\beta_k(s) + \int_x dx \ \zeta(x,s) \left[1 - \exp\left[-\sum_{k=i}^{n} h_k(x,s)\beta_k(s) \right] \right] \right\}. \end{aligned} \tag{31}$$

Expressing $\phi_k(s)$ in terms of $\overline{\lambda}_k(s)$, $\phi_k(s)\beta_k(s)$ can be written as

$$\phi_k(s)\beta_k(s) = \frac{d}{ds} \left[\overline{\lambda}_k(s)\beta_k(s) \right] + \overline{\lambda}_k(s). \tag{32}$$

In the following, $\beta_k(s)$ is denoted as $\beta_k(s,t_k)$ making its t_k dependence explicit. In the article, μ's are assumed to be time-independent and the resulting $\beta_k(s,t_k)$ is denoted as $\beta_k(t_k-s)$.

After these steps, one obtains an expression for $f(t,\lambda_1,...)$. For the joint survival probability $Q(t_1,...,t_n)\equiv f(0,\lambda_1(0),..)$ this gives, in terms of $\overline{\lambda}_k(s)$,

$$\begin{aligned}
Q(t_1,...,t_n) &= \exp\left[-\sum_{i=1}^{n}\left(\lambda_i(0)-\overline{\lambda}_i(0)\right)\beta_i(0,t_i)-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} ds\ \pi_{i...n}(s,t_i,...)\right], \quad \text{where} \\
\pi_{i...n}(s,t_i,...) &= \sum_{k=i}^{n}\overline{\lambda}_k(s)+\int_x dx\ \zeta(x,s)\left\{1-\exp\left[-\sum_{k=i}^{n}h_k(x,s)\beta_k(s,t_k)\right]\right\}.
\end{aligned} \tag{33}$$

Also interesting is the joint survival probability up to time t for credit names in the list $\Omega=\{i,j,...\}$ given by

$$Q_\Omega(t)\equiv \mathrm{E}_0\left\{\prod_{k\in\Omega}\exp\left[-\int_0^t ds\ \lambda_k(s)\right]\right\}. \tag{34}$$

This can be obtained from $Q(t_1,...,t_n)$ with appropriate time ordering by setting t_k's to t for all k in the list $\Omega=\{i,j,...\}$ and to zero for the rest,

$$\begin{aligned}
Q_\Omega(t) &= \exp\left[-\sum_{k\in\Omega}\left(\lambda_k(0)-\overline{\lambda}_k(0)\right)\beta_k(0,t)-\int_0^t ds\ \pi_\Omega(s,t)\right], \\
\pi_\Omega(s,t) &= \sum_{k\in\Omega}\overline{\lambda}_k(s)+\int_x dx\ \zeta(x,s)\left\{1-\exp\left[-\sum_{k\in\Omega}h_k(x,s)\beta_k(s,t)\right]\right\}.
\end{aligned} \tag{35}$$

The above results can also be obtained directly by substituting the following solution of the process into the concerned expectations,

$$\int_0^t ds\ \lambda_i(s)=\left(\lambda_i(0)-\overline{\lambda}_i(0)\right)\beta_i(0,t)+\int_0^t ds\ \overline{\lambda}_i(s)+\int_0^t\int_x h_i(x,s)\beta_i(s,t)dN(dx,s). \tag{36}$$

Increments $dN(dx,s)$ are all independent of each other and

$$\mathrm{E}\left\{\exp\left[-udN(dx,s)\right]\right\}\simeq\exp\left[-dxds\zeta(x,s)\left(1-e^{-u}\right)\right]. \tag{37}$$

For the concerned expectations, u is a sum of one or more terms of the kind $h_k(x,s)\beta_k(s,t)$.

The limiting model of simultaneous defaults is obtained by taking μ_i and h_i to ∞ such that $h_i/\mu_i\to h_i'$. Terms involving $\lambda_k(0)-\overline{\lambda}_k(0)$ then get suppressed and $h_k(x,s)\beta_k(s,t_k)$ (or $h_k(x,s)\beta_k(s,t)$) gets replaced by $h_k'(x,s)$ so that $\pi_{i...n}$ (or π_Ω) becomes independent of $t_i,...,t_n$ (or t). The π's, say $\pi_\Omega dt$, can then be given a nice interpretation as the conditional probability that at least one of the names listed in Ω default during an infinitesimal interval $(t,t+dt)$ (the rest are not looked at). As detailed in Balakrishna [2006], the probability that all the names listed in Ω default during an infinitesimal interval $(t,t+dt)$ (the rest are not looked at) is then given by

$$p_\Omega(t)=\int_x dx\ \zeta(x,t)\prod_{k\in\Omega}\left\{1-\exp\left[-h_k'(x,t)\right]\right\}. \tag{38}$$

An additional term $\overline{\lambda}_k(t)$ is to be included when Ω has just one element, say name k.

B The u-Expansion

Explicit solutions of the model contain an integral involving a parameter, say u, representable as Laplace transform of the density $\nu_Y(y) = a^\alpha g(a/y) y^{-1-\alpha}$. Here, let us derive the leading terms in the u-expansion of this integral,

$$I(u) = \int_0^\infty dy\ \nu_Y(y) e^{-uy} = \int_0^\infty dz\ g(z) z^{\alpha-1} e^{-au/z}. \tag{39}$$

The a factor multiplying u is dropped below for simplicity of presentation. Index α is as usual assumed to lie in between 1 and 2. Function $g(z)$ is assumed to go to zero as $z \to \infty$ faster than any power of z. For $\alpha > 1$, the integral to $\mathcal{O}(u)$ is

$$I(u) = 1 - u \int_0^\infty dz\ g(z) z^{\alpha-2} + I_1(u), \tag{40}$$

where

$$I_1(u) = \int_0^\infty dz\ g(z) z^{\alpha-1} \left[e^{-u/z} + u/z - 1 \right]. \tag{41}$$

To expand $I_1(u)$, it is convenient to consider its second derivative,

$$\begin{aligned} \frac{d^2 I_1(u)}{du^2} &= g(0) \int_0^\infty dz\ z^{\alpha-3} e^{-u/z} + \int_0^\infty dz\ \left[g(z) - g(0) \right] z^{\alpha-3} e^{-u/z} \\ &= g(0)\Gamma(2-\alpha) u^{\alpha-2} + \int_0^\infty dz\ \left[g(z) - g(0) \right] z^{\alpha-3} + I_2(u), \end{aligned} \tag{42}$$

where

$$I_2(u) = \int_0^\infty dz\ \left[g(z) - g(0) \right] z^{\alpha-3} \left[e^{-u/z} - 1 \right]. \tag{43}$$

If $g(z) - g(0) \sim z^\omega$ as $z \to 0$ for some $\omega > 2 - \alpha$, which is the case for instance for $\omega = 1$, the integral term in (42) is finite and can be rewritten after a partial integration as

$$\int_0^\infty dz\ \left[g(z) - g(0) \right] z^{\alpha-3} = \frac{1}{2-\alpha} \int_0^\infty dz\ g'(z) z^{\alpha-2}, \tag{44}$$

where a prime denotes differentiation. This integral makes an $\mathcal{O}(u)^2$ contribution to $I(u)$ (if $\omega < 2 - \alpha$, a case ignored here, a contribution of $\mathcal{O}(u)^{\alpha+\omega}$ will occur first). To see the behavior of $I_2(u)$ as $u \to 0$, consider

$$\frac{dI_2(u)}{du} = -\int_0^\infty dz\ \left[g(z) - g(0) \right] z^{\alpha-4} e^{-u/z}. \tag{45}$$

This is finite as $u \to 0$ if $\omega > 3 - \alpha$. $I_2(u)$ then contributes a $\mathcal{O}(u^3)$ term to $I(u)$. If instead $\omega < 3 - \alpha$, $I_2(u)$ contributes an $\mathcal{O}(u^{\alpha+\omega})$ term to $I(u)$ obtainable from the above by replacing $g(z) - g(0) \sim z^\omega$. The remainder, say $I_3(u)$, can be examined further to obtain the next order term. We thus find

$$I(u) = 1 + \frac{u}{\alpha-1} \int_0^\infty dz\ g'(z) z^{\alpha-1} + g(0)\Gamma(-\alpha) u^\alpha + \frac{u^2}{2(2-\alpha)} \int_0^\infty dz\ g'(z) z^{\alpha-2} + \mathcal{O}(u^\theta), \tag{46}$$

where θ is $\alpha+\omega$ or 3 whichever is smaller and the first integral has been rewritten after a partial integration. The procedure can be continued to determine the other higher order terms in this expansion if necessary.

For $g(z)$ regular in z at $z=0$, the procedure can be continued by examining the u-derivative of the remainders to obtain

$$I(u) = 1 + \sum_{k=0}^{\infty} g^k(0)\Gamma(-\alpha-k)u^{\alpha+k} + \sum_{k=1}^{\infty} \frac{(-u)^k}{(1-\alpha)...(k-\alpha)k!} \int_0^{\infty} dz\ g^k(z) z^{\alpha-1}, \tag{47}$$

where $g^k(z)$ is k^{th}-derivative of $g(z)$.

For $g(z)$ that is flat near $z=0$, perhaps because it is forced to be $g(z_0)$ for $z \leq z_0$, the expansion takes a simpler form that can obtained either from (47) or by expanding $e^{-u/z}$ inside the integral in $I_2(u)$ in powers of $-u/z$ so that

$$I(u) = 1 + g(0)\Gamma(-\alpha)u^{\alpha} + \sum_{k=1}^{\infty} \frac{(-u)^k}{(k-\alpha)k!} \int_0^{\infty} dz\ g'(z) z^{\alpha-k}. \tag{48}$$

For a piecewise constant $g(z)$, $g'(z)$ makes Dirac-delta contributions. For the Pareto density arising from $g(z)=\alpha$, $z \leq 1$ and zero otherwise, this gives an expansion for the incomplete gamma function as expected,

$$I(u) = \alpha\Gamma(-\alpha, u)u^{\alpha} = \alpha\Gamma(-\alpha)u^{\alpha} - \alpha \sum_{k=0}^{\infty} \frac{(-u)^k}{(k-\alpha)k!}. \tag{49}$$

Analogous expansions can be derived for $\alpha < 1$. They can also be obtained directly from the above results by replacing α with $\alpha+1$ in the expansion for $-dI(u)/du$. Expansion (46) then holds for $\alpha < 1$ up to $\mathcal{O}(u)$ and the remainder is of $\mathcal{O}(u^{\theta})$ where θ is $\alpha+\omega$ or 2 whichever is smaller. Expansions (47) and (48) hold as such.

If the y-integral in (39) has a finite upper limit b'/u that is still kept large with u in the denominator, the expansion needs to be corrected by adding

$$\begin{aligned} -\int_0^{u/b'} dz\ g(z) z^{\alpha-1} e^{-u/z} &= -u^{\alpha} \int_0^{1/b'} dz\ g(uz) z^{\alpha-1} e^{-1/z} \\ &= -u^{\alpha} g(0)\Gamma(-\alpha, b') + \mathcal{O}(u^{\alpha+\omega}). \end{aligned} \tag{50}$$

If $g(z)$ can expanded in appropriate powers of z, the integral can be expressed as a sum of incomplete gamma functions.

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Table 1: Simultaneous fit to the five tranches and the three maturities of CDX.NA.IG and iTraxx Europe CDOs for the market quotes on October 2, 2006 (source: Brigo, Pallavicini and Torresetti [2006b]). Recovery rate is assumed to be 35%, and interest rate at a constant 5% for CDX.NA.IG and 3.5% for iTraxx Europe CDOs. Three year Index default swap spreads of 24bp and 18bp respectively are also calibrated to exactly. A flat $\overline{\lambda}(t)$ is assumed in between maturities. Equity tranche is quoted as an upfront fee in percent (plus 500bp per year running) and the other tranches are quoted as spreads per year in bp.

CDX.NA.IG							
Tranches :	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	0-100%
Quotes 5y :	30.50	102.0	22.50	10.25	5.00		40.00
7y :	45.63	240.0	53.00	23.00	7.20		49.00
10y :	55.00	535.0	123.00	59.00	15.50		61.00
Results 5y :	31.16	95.36	22.81	10.16	4.95	2.31	40.00
7y :	46.98	229.45	55.09	22.73	7.34	2.81	49.02
10y :	56.75	502.23	131.08	55.89	15.42	4.26	61.04
$\mu = 0.45$, $\alpha = 1.58$, $a = 6.42$, b=0.54, $\zeta = 257$bp, $\overline{\lambda}$(t) = (29.72, 78.01, 84.02, 103.2)bp							

iTraxx Europe							
Tranches :	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%	0-100%
Quotes 5y :	19.75	75.0	22.25	10.50	4.00	1.50	30.00
7y :	37.12	189.0	54.25	26.75	9.00	2.85	40.00
10y :	49.75	474.0	125.50	56.50	19.50	3.95	51.00
Results 5y :	19.95	75.1	22.38	10.28	4.10	1.29	29.99
7y :	38.44	186.9	54.47	24.64	9.18	2.00	39.95
10y :	53.64	445.8	129.74	58.17	19.69	3.29	51.03
$\mu = 1.23$, $\alpha = 1.53$, $a = 4.42$, b=0.52, $\zeta = 305$bp, $\overline{\lambda}$(t) = (20.40, 56.67, 74.71, 87.62)bp							

Figure 1: Logarithmic plot of the 5-year joint default probability distributions computed with 100,000 and one million Monte Carlo scenarios using model parameters from Table 1 calibrated to iTraxx Europe CDOs. Also shown is the tail at the far end of the distribution retained by setting $b = \infty$.

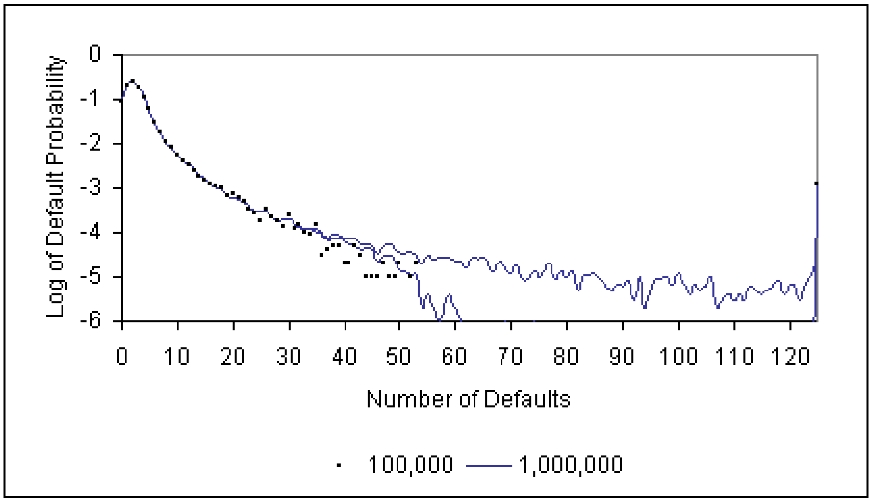

Figure 2: Logarithmic plot of the joint default probability distributions over 5, 7 and 10 years computed with one million Monte Carlo scenarios using model parameters from Table 1 calibrated to iTraxx Europe CDOs. Also shown are the tails at the far end of the distributions retained by setting $b = \infty$.