A Characterization of the Shapley Value for Cooperative Games

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30 March 2009

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Abstract. Motivated by a case of discrimination against some particular players happened in the real world, we define the partially consistent property of the solutions for cooperative games and use the property to characterize the Shapley value. This is different from the characterization of the Shapley value by applying the consistency property proposed by Hart and Mas-Colell.

Primary and Motivation. In 1989, Hart and Mas-Colell [1] were the first to introduce the potential approach to traditional TU games. In consequence, they proved that the traditional Shapley value [5] can result as the vector of marginal contributions of a potential. The potential approach is also shown to yield a characterization for the Shapley value, particularly in terms of an internal consistency property.

Let U be the universe set of players, we have the following definitions and notations from [1] and chapter 9 in [6].

Let $N \subset U$ be a finite set of players and $|N|$ denote the number of players in N.

A cooperative game with side payments - in short, *a game* - consists of a pair (N, v), where N is a finite set of players and $v : 2^N \to R$ is the *characteristic function* satisfying

$$v(\emptyset) = 0.$$

[1]We are grateful to the participants of the 18th International Conference on Game Theory at Stony-Brook for many valuable discussions.

A subset $S \subset N$ is called a *coalition*.

Let $\mathbf{G}$ denote the set of all games. Formally, a solution function ϕ is a function defined on $\mathbf{G}$ that associated to every $(N, v) \in \mathbf{G}$ a payoff vector $\phi(N, v)= (\phi^i(N, v))_{i\in N} \in R^n$. We denote the well-known Shapley value by $\psi(N, v)= (\psi^i(N, v))_{i\in N} \in R^n$.

Definition 0. (Hart and Mas-Colell) Given a solution function ϕ, a game (N, v) and a coalition $T \subset N$, the reduced game is defined by

$$v_T^{\phi}(S) = v(S \cup T^c) - \sum_{i\in T^c} \phi^i(S \cup T^c, v)$$

for all $S \subset T$, where $T^c = N \backslash T$. The solution function ϕ is *consistent* if

$$\phi^j(T, v_T^{\phi}) = \phi^j(N, v)$$

for every game (N, v), every coalition $T \subset N$ and all $j \in T$.

Remark 1. Readers may easily find the typo in **Definition 0** as follows,

$$v_T^{\phi}(\emptyset) = v(T^c) - \sum_{i\in T^c} \phi^i(T^c, v) \text{ not necessarily be zero for each } T \subseteq N.$$

That is the "reduced games" defined in **Definition 0** are not necessarily belong to $\mathbf{G}$. However, the solution function ϕ is a function defined on $\mathbf{G}$, therefore the typo is noteworthy. Some authors fixed the typo by assuming $2^{|N|}$ additional equations hold as the following,

$$v_T^{\phi}(\emptyset) = 0, \text{ for each one of the } 2^{|N|} \text{ subsets, } T\text{'s, of } N.$$

Even if we exclude the trivial cases which $T = \emptyset$ or N, indexed by T where $T \subset N$, there are still $2^{|N|-1}$ additional equations $v_T^{\phi}(\emptyset) = 0$ given for fixing the typo. Furthermore, it will be very controversial if $v(\emptyset \cup T^c) - \sum_{i\in T^c} \phi^i(\emptyset \cup T^c, v) \neq 0$ and we assume $v_T^{\phi}(\emptyset) = 0$.

Another way to fix the typo is to assume that ϕ is efficient which makes the characterization of the Shapley value less desirable.

In this article, we will leave the typo alone, and characterize the Shapley value by a natural way. We use **only one** natural equation called *simple prato optimal* as a substitute for the $2^{|N|}$ equations.

The game $(\{1\}, v)$ which $v(\emptyset) = 0$ and $v(\{1\} = 1$ is obviously belong to $\mathbf{G}$.

Definition 1. (Simple Prato Optimal) A solution function ϕ defined on $\mathbf{G}$ is said to be *simple Prato optimal* if and only if $\phi^1(\{1\}, v) = 1$ for the game $(\{1\}, v)$ with $v(\emptyset) = 0$ and $v(\{1\} = 1$.

We will characterize the Shapley value by the motivation of the following example happened in the real world. We now separate the universe set of players U into two parts as: $U = U_c \cup U_t$ where U_c and U_t are mutually exclusive.

Example 1. Given a game (N, v) where $N \subset U = U_t \cup U_c$ and $U_c \cap U_t = \emptyset$. A government gives a solution concept (as a law) $\kappa(N, v)$= $(\kappa^i(N, v))_{i \in N} \in R^n$ as the following:

(i) If $N \subset U_c$, then $\kappa^i(N, v) = \psi^i(N, v)$ for all $i \in N \subset U_c$, i.e. the Shapley value.

(ii) If $N \cap U_t \neq \emptyset$ and $N \cap U_c \neq \emptyset$, then

$$\kappa^i(N, v) = \frac{35}{36} \cdot \frac{v(N) - 1}{|N \cap U_c|}, \text{ for all the players } i \in N \cap U_c$$

and

$$\kappa^i(N, v) = \frac{1}{36} \cdot \frac{v(N) - 1}{|N \cap U_t|}, \text{ for all the players } i \in N \cap U_t.$$

(iii) If $N \subset U_t$, then

$$\kappa^i(N, v) = \frac{1}{36} \cdot \frac{v(N)}{|N \cap U_t|}, \text{ for all the players } i \in N \cap U_t$$

Apparently, κ is consistent for the games purely defined on $N \subset U_c$ and is not consistent for the games where some players come from U_t. This example gives us the motivation to define the *partially consistent property* for the solutions of the cooperative games.

Main Results. Motivated by Example 1, we suggest the following definitions.

Definition 2. Given a solution function ϕ, a game (N, v) and a coalition $T \subset N$ and $T \neq \emptyset$ the reduced function with respect to T and ϕ is defined by

$$v_T^\phi(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi^i(S \cup T^c, v)$$

for all $S \subseteq T$, where $T^c = N \backslash T$. Furthermore, if v_T^ϕ satisfies

$$v_T^\phi(\emptyset) = v(\emptyset \cup T^c) - \sum_{i \in T^c} \phi^i(\emptyset \cup T^c, v) = 0,$$

then we call v_T^ϕ a reduced game.

Definition 3. Let ϕ be a solution function defined on $\mathbf{G}$ such that for some $(N, v) \in \mathbf{G}$ and some $T \subset N$

$$\phi^j(T, v_T^\phi) = \phi^j(N, v), \tag{1}$$

holds for all $j \in T$ whenever the reduced function v_T^ϕ is a reduced game, then ϕ is said to be *partially consistent*.

If every reduced function v_T^ϕ is a reduced game for every game (N, v) and every coalition $T \subset N$ and (1) holds for all $j \in T$, then ϕ is said to be *consistent*.

Note 1. Obviously, the solution concept κ in Example 1 is partially consistent rather than consistent.

Definition 4. Given a game (N, v) a player i is said to be a *non-essential player* if $v(\{i\}) = k$ for some constant k and $v(S \cup \{i\}) = v(S) + v(\{i\}) = v(S) + k$ for all $S \subset N$ with $i \notin S$. If $k = 0$, we call player i a *dummy player*. *Dummy player* is a special case of *non-essential player*.

Given $(N, v) \in \mathbf{G}$ where $N = \{1, 2, ..., n\}$, allow a new player, say $(n + 1)$, to join the game, then we have a new set of players $N^* = N \cup \{n + 1\}$.

Let $\bar{v}(S) = v(S)$, for all $S \subseteq N$. Assign $\bar{v}(\{n + 1\})$ a value k not necessarily zero. Then we can define a new game $(N^*, \bar{v})$, such that $n+1$ is a non-essential player in $(N^*, \bar{v})$. We call $(N^*, \bar{v})$ a non-essential extension of (N, v). A solution ϕ of (N, v) is said to be *independent of non-essential players* if $\phi^i(N, v) = \phi^i(N^*, \bar{v})$, for all $i \in N$. Otherwise, ϕ is said to be *dependent of non-essential players*.

In case the player $n+1$ is dummy in $(N^*, \bar{v})$, then we say $(N^*, \bar{v})$ is a dummy extension of (N, v). Accordingly, ϕ is said to be *dummy free* if $\phi^i(N, v) = \phi^i(N^*, \bar{v})$, for all $i \in N$. Otherwise, ϕ is said to be *dependent of dummy players*.

To make this article self-contained, we copy the following definition form [1].

Definition 5. A solution is *standard for two-person games* if

$$\phi^i(\{i,j\}, v) = v(\{i\}) + \frac{1}{2}[v(\{i,j\}) - v(\{i\}) - v(\{j\})] \tag{2}$$

for all $i \neq j$ and all v. Thus, the "surplus " $[v(\{i,j\}) - v(\{i\}) - v(\{j\})]$ is equally divided among the two players.

If readers fix the typo by assuming $v^{\phi}_T(\emptyset) = 0$ for each $T \subset N$, then the following proof can be omitted. Since, in this article we leave the typo alone, the proof of the following Theorem is essential.

Theorem 1. Let ϕ be a solution function. If ϕ is (i) simple prato optimal (ii) standard for two-person games and (iii) partially consistent, then ϕ is efficient, accordingly ϕ is efficient for all one-person games.

Proof. Omitted.

Corollary 1. Let ϕ be a solution function. If ϕ is (i)simple prato optimal(ii) standard for two-person games and (iii) partially consistent, then ϕ consistent.

Proof. Following the proof of Theorem 1, by mathematical induction on $|T^c|$, we can show that every reduced function (T, v^{ϕ}_T) is a reduced game for every game (N, v) and every coalition $T \subset N$. Then by (iii) ϕ is consistent. $\diamondsuit$

The following characterization of the Shapley value is different from that in [1].

Theorem 2. Let ϕ be a solution function. Then ϕ is (i) simple prato optimal (ii) standard for two-person games (iii) partially consistent, if only if ϕ is the Shapley value.

Proof. Omitted.

Now, by Theorem 1 and Theorem 2, we can easily see the following Corollary.

Corollary 2. The Shapley value is independent of non-essential players, in particular, dummy free.

Proof. Omitted.

Conclusion and Suggestion. In this article, we leave the typo in [1] alone, define the partially consistent property and use the simple prato optimal property as a substitute for the $2^{|N|}$ equations $v_T^{\phi}(\emptyset) = 0$ to characterize the Shapley value. As a matter of fact, a game theorist may confuse readers not only by typos, but also by mathematical notations. For example, some game theorists never use bold face letters to denote vectors as the traditional mathematical notations always do. We suggest that authors who are interested in multi-choice cooperative games revise their papers with traditional mathematical notations to check if there is any typo.

References

[1] Hart, S and Mas-Colell (1989), Potential, value, and Consistency. Econometrica, vol. 57, No. 3, pp. 589 614.

[2] Chang, C and Kan CY (1992),The kernel, the bargaining set and the reduced game. **21** International Journal of Game Theory, pp. 75-83.

[3] Hsiao CR (1995), A Note on Non-essential Players in Multi-Choice Cooperative Games.*Games and Economic Behavior*, **8**(1995), 424-432.

[4] Hsiao CR and Liao YS (2008), The potential and consistency property for multi-choice Shapley value. Taiwanese Journal of Mathematics, 12 (2): 545-559.

[5] Shapley LS, A Value for n-person Game.In: Kuhn HW, Tucker AW(eds.) Contributions to the Theory of Games II, Annals of Mathematics Studies 28, Princeton University Press, Princeton, 307-317.

[6] Roth, A.: *The Shapley Value.* Essays in honor of L.S.Shapley, Edited by A. Roth, Cambridge University Press, 1988.