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Abstract. We defined the potential for multi-choice cooperative games, and found the relationship between the potential and the multi-choice Shapley value. Moreover, we show that the multi-choice Shapley is consistent.

Introduction. In [1](1991), we extended the traditional cooperative game to the multi-choice cooperative game and extended the traditional Shapley value to the Shapley value for multi-choice cooperative games. In short, we call the shapley value for multi-choice cooperative games the *multi-choice Shapley value*. In 1990, Shapley asked "what is the potential for multi-choice games ?". In this article, we would answer Shapley's question.

In [1], [2], [3], we assumed that the players in a multi-choice cooperative game have the same number of options, or say actions. As a matter of fact, from the point of view of the *multi-choice Shapley value*, it makes no difference whether the players have the same number of options or not. Therefore, by just rewriting the definitions and the proofs in [1], [2], [3], we may define the multi-choice Shapley value for a game where the players have different numbers of actions.

In this article, We would first rewrite the definition of the multi-choice Shapley value and define the potential of multi-choice cooperative games, then, show the relationship between the multi-choice Shapley value and the potential, and prove that the multi-choice Shapley value is consistent.

Finally, we would we define " w-proportional for two-working-player games ", and show that a solution for multi-choice cooperative game is the multi-choice Shapley value if and only if it satisfies consistency and "w-proportional for two-working-player games".

Definitions and Notations.

Let I_+ denote the set of all finite non-negative integers. Let $N = \{1, 2, ..., n\}$ be the set of players. We allow player j to have $(m_j + 1)$ actions, say σ_0, σ_1, σ_2, ..., σ_{m_j}, where σ_0 is the action to do nothing, while σ_k is the option to work at level k, which is better than σ_{k-1}.In this article, we assume that all the players have finitely many choices.

For convenience, we will use non-negative integers to denote the players' actions. Given $\mathbf{m} = (m_1, m_2, ..., m_n) \in I_+^n$, the action space of N is defined by $\Gamma(\mathbf{m}) = \{(x_1, ..., x_n) \mid x_i \leq m_i$ and $x_i \in I_+$, for all $i \in N\}$. Thus $(x_1, ...x_n)$ is called an action vector of N, and $x_i = k$ if and only if player i takes action σ_k.

Given $\mathbf{z} = (z_1, z_2, ..., z_n)$, $\mathbf{m} = (m_1, m_2, ..., m_n) \in I_+^n$, we define $\mathbf{z} \leq \mathbf{m}$ if and only if $z_i \leq m_i$ for all $i \in N$. It is clear that $\Gamma(\mathbf{z}) \subseteq \Gamma(\mathbf{m})$ whenever $\mathbf{z} \leq \mathbf{m}$.

Definition 1. A multi-choice cooperative game in characteristic function from is the pair $(\mathbf{m}, v)$ defined by, $v : \Gamma(\mathbf{m}) \to R$, such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, 0..., 0)$.

Player j is called a *useless player* if and only if $m_j = 0$. Moreover, player j is called an *working player* if and only if $m_j > 0$. We may consider $v(\mathbf{x})$ as the payoff or the cost for the players whenever the players take action vector $\mathbf{x}$. Sometimes, we will denote $v(\mathbf{x})$ by $(\mathbf{m}, v)(\mathbf{x})$ in order to emphasis that the domain of v is $\Gamma(\mathbf{m})$.

Given $\mathbf{z} \in I_+^n$ with $\mathbf{z} \leq \mathbf{m}$, a sub-game of $(\mathbf{m}, v)$ is obtained by restricting the domain of v to $\Gamma(\mathbf{z})$. We denote the sub-game by $(\mathbf{z}, v)$. In other words, Let $\mathbf{z} \in I_+^n$ with $\mathbf{z} \leq \mathbf{m}$, we call $(\mathbf{z}, v)$ a sub-game of $(\mathbf{m}, v)$, if and only if $(\mathbf{z}, v)(\mathbf{x}) = (\mathbf{m}, v)(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{z})$.

We can identify the set of all multi-choice cooperative games defined on $\Gamma(\mathbf{m})$ by $G \simeq R^{\prod_{j=1}^n (m_j+1)-1}$.

Since we do not assume that action σ_2 is say, twice as powerful as action σ_1, and since we do not assume that the difference between σ_{k-1} and σ_k is the same as the difference between σ_k and σ_{k+1}, etc., giving weights (discrimination) to actions is necessary.

Let $m = \max_{j \in N}\{m_j\}$, and let $w : \{0, 1, ..., m\} \to R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \leq w(2) \leq \ldots \leq w(m)$, then w is called a **weight function** and $w(i)$ is said to be a **weight** of σ_i.

Given a weight function w for the actions, we define a value, or say a solution of a multi-choice cooperative game $(\mathbf{m}, v)$ by a $\sum_{j=1}^{n} m_j$ dimensional vector $\phi^w : G \to R^{\sum_{j=1}^{n} m_j}$ be such that

$$\phi^w(v) = (\phi^w_{11}(v), .., \phi^w_{m_1 1}(v), \phi^w_{12}(v), .., \phi^w_{m_2 2}(v), .., \phi^w_{1n}(v), .., \phi^w_{m_n n}(v))$$

Here $\phi^w_{ij}(v)$ is the power index or the value of player j when he takes action σ_i in game v.

Rewrite [3], we can show that when w is given, given, there exists a unique ϕ^w satisfying the following four axioms.

Axiom 1. Suppose $w(0), w(1), \ldots, w(m)$ are given. If v is of the form

$$v(\mathbf{y}) = \begin{cases} c > 0 & \text{if } \mathbf{y} \geq \mathbf{x} \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^w_{x_i,i}(v)$ is proportional to $w(x_i)$.

Axiom 1 states that for binary valued (0 or c) games that stipulate a minimal exertion from players, the reward, for players using the minimal exertion level is proportional to the weight of his minimal level action.

We denote $(\mathbf{x} \mid x_i = k)$ as an action vector with $x_i = k$.

. Given $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbf{m})$, we define $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$ where $x_i \vee y_i = \max\{x_i, y_i\}$ for each i. Similarly, we define $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$ where $x_i \wedge y_i = \min\{x_i, y_i\}$ for each i.

Definition 2. A vector $\mathbf{x}^* \in \Gamma(\mathbf{m})$ is called a **carrier** of v, if $v(\mathbf{x}^* \wedge \mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$. We call $\mathbf{x}^0$ a *minimal carrier* of v if $\sum x_i^0 = \min\{\sum x_i \mid \mathbf{x} \text{ is a carrier of } v\}$.

Definition 3. Player i is said to be a **dummy player** if $v((\mathbf{x} \mid x_i = k)) = v((\mathbf{x} \mid x_i = 0))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and for all $k = 0, 1, 2, ..., m_i$.

A useless player is of course a dummy player. The following is a version of the usual efficiency axiom that combines the carrier and the notions of dummy player.

Axiom 2. If $\mathbf{x}^*$ is a carrier of v then, for $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ we have

$$\sum_{\substack{x_i^* \neq 0 \\ x_i^* \in \mathbf{x}^*}} \phi^w_{x_i^*,i}(v) = v(\mathbf{m}).$$

By $x_i^* \in \mathbf{x}^*$ we mean x_i^* is the i-th component of $\mathbf{x}^*$.

Axiom 3. $\phi^w(v^1+v^2)=\phi^w(v^1)+\phi^w(v^2)$, where $(v^1+v^2)(\mathbf{x})=v^1(\mathbf{x})+v^2(\mathbf{x})$.

Axiom 4. Given $\mathbf{x}^0 \in \Gamma(\mathbf{m})$ if $v(\mathbf{x})=0$, whenever $\mathbf{x} \ngeq \mathbf{x}^0$, then for each $i \in N$ $\phi^w_{k,i}(v)=0$, for all $k < x_i^0$.

Axiom 4 states that in games that stipulate a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded.

Definition 4. Given $\mathbf{x} \in \Gamma(\mathbf{m})$, let $S(\mathbf{x}) = \{i \mid x_i \neq 0, x_i \text{ is a component of } \mathbf{x}\}$. Given $S \subseteq N$, let $\mathbf{e}(S)$ be the binary vector with components $e_i(S)$ satisfying

$$e_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we let the standard unit vectors $\mathbf{e}(\{i\}) = \mathbf{e}_i$, for all $i \in N$, and let $|S|$ be the number of elements of S.

Definition 5. Given $\Gamma(\mathbf{m})$ and $w(0)=0,\ w(1),\dots,w(m)$, for any $\mathbf{x} \in \Gamma(\mathbf{m})$, we define $\|\mathbf{x}\|_w = \sum\limits_{r=1}^{n} w(x_r)$.

Definition 6. Given $\mathbf{x} \in \Gamma(\mathbf{m})$ and $j \in N = \{1,2,\dots,n\}$, we define $M_j(\mathbf{x};\mathbf{m}) = \{i \mid x_i \neq m_i,\ i \neq j\}$.

From Theorem 2 in [3], we have

$$\phi^w_{ij}(v) = \sum_{k=1}^{i} \sum_{\substack{x_j=k \\ \mathbf{x}\neq 0 \\ \mathbf{x}\in\Gamma(\mathbf{m})}} \left[\sum_{T \subseteq M_j(\mathbf{x};\mathbf{m})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum\limits_{r\in T}[w(x_r+1)-w(x_r)]} \right] \times [v(\mathbf{x}) - v(\mathbf{x}-\mathbf{e}_j)]. \qquad (\star)$$

The Potential.

Given $\mathbf{m} = (m_1,\ m_2,\ \dots,m_n\ ) \in I_+^n$ and an n-person multi-choice cooperative game $(\mathbf{m}, v)$. We denote the set of all the sub-games of $(\mathbf{m}, v)$ by

$$G^* = \{(\mathbf{z}, v) \mid \mathbf{z} \in I_+^n \ \text{ and } \ \mathbf{z} \leq \mathbf{m}\}$$

.

Given a weight function w for $\{0,1,..,m\}$, we define a function $P_w : G^* \to R$ which associates a real number $P_w((\mathbf{x},v))$.

Given $P_w((\mathbf{x},v))$, we define the following operators.

$$D_{i,j}P_w((\mathbf{x}),v)) = w(i)\cdot\Big[P_w(\ ((\mathbf{x}|x_j=i),v)) - P_w(\ ((\mathbf{x}|x_j=i-1),v))\Big],$$

and

$$H_{x_j,j} = \sum_{\ell=1}^{\ell=x_j} D_{\ell,j}.$$

Definition 8. A function $P_w : G \to R$ with $P_w((\mathbf{0},v)) = 0$ is called a w-potential function if it satisfies the following condition: for each fixed $\mathbf{x}\in\Gamma(\mathbf{m})$

$$\sum_{j\in S(\mathbf{x})} H_{x_j,j}P_w((\mathbf{x},v)) = (\mathbf{x},v)(\mathbf{x}) \qquad (\star\star)$$

Given $j\in N$ and $v(\mathbf{x})$, we define

$$d_jv(\mathbf{x}) = v(\mathbf{x}) - v(\mathbf{x}-\mathbf{e}_j),$$

then d_j is associative, i.e. $d_k(d_jv(\mathbf{x})) = d_j(d_kv(\mathbf{x}))$. For convenience, we denote $d_id_j = d_{ij}$, $d_{ijk} = d_id_jd_k$, ..., etc. We also denote $d_{i_1,i_2,\ldots,i_\ell} = d_T$ whenever $\{i_1,i_2,...,i_\ell\} = T$. Furthermore, for brevity, we denote $d_{S(\mathbf{x})}$by $d_{\mathbf{x}}$.

Theorem 1. The Potential of multi-choice cooperative games is unique, and

$$P_w((\mathbf{x},v)) = \sum_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}(\mathbf{x},v)(\mathbf{y}) \qquad (1)$$

Proof. Consider $(\mathbf{m},v)$ and all its sub-games $(\mathbf{x},v)$. It is easy to see that $P_w((\mathbf{0},v)) = 0$. Let $|\mathbf{x}| = \sum_{i\in N} x_i$, by mathematical induction on $|\mathbf{x}|$, by equation $(\star\star)$, we can easily see that the potential is unique. Now, we show the following claim in order to prove our theorem.

Claim: Given any multi-choice cooperative game $(\mathbf{x},v)$, let

$$\psi_w((\mathbf{x},v)) = \sum_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}(\mathbf{x},v)(\mathbf{y}),$$

then

$$H_{ij}\psi_w((\mathbf{x},v)) = \phi_{ij}^w((\mathbf{x},v)).$$

$$\begin{aligned}
H_{i,j}\psi_w((\mathbf{x},v)) &= \sum_{k=1}^{i} D_{k,j}\psi_w((\mathbf{x},v)) \\
&= \sum_{k=1}^{i} w(k)\cdot[\psi_w(((\mathbf{x}|x_j = k)),v)) - \psi_w(((\mathbf{x}|x_j = k-1),v))] \\
&= \sum_{k=1}^{i} w(k)\cdot[\sum\nolimits_{\substack{\mathbf{y}\le\ (\mathbf{x}|x_j=k)\\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}v(\mathbf{y}) - \sum\nolimits_{\substack{\mathbf{y}\le\ (\mathbf{x}|x_j=k-1)\\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}v(\mathbf{y})] \\
&= \sum_{k=1}^{i} w(k)\cdot[\sum\nolimits_{\substack{y_j=k\\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}v(\mathbf{y})] \\
&= \sum_{k=1}^{i} w(k)\cdot[\sum\nolimits_{\substack{y_j=k\\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} \sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r\in T}\mathbf{e}_r] \qquad (1.1)
\end{aligned}$$

Consider $\sum\limits_{S(\mathbf{y})} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T))$, where $\mathbf{e}(T) = \sum\limits_{r\in T}\mathbf{e}_r$,we have

$$\begin{aligned}
&\sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T)) \\
&= \sum_{\substack{j\in T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T)) + \sum_{\substack{j\notin T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T)) \\
&= \sum_{\substack{j\notin T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|+1}v(\mathbf{y}-\mathbf{e}(T\cup\{j\})) + \sum_{\substack{j\notin T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T)) \\
&= \sum_{\substack{j\notin T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|}\left[v(\mathbf{y}-\mathbf{e}(T)) - v(\mathbf{y}-\mathbf{e}(T\cup\{j\}))\right]
\end{aligned}$$

Let $\mathbf{z} = \mathbf{y} - \mathbf{e}(T)$, then we have

$$\sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|}v(\mathbf{y}-\mathbf{e}(T)) = \sum_{\substack{j\notin T\\ T\subseteq S(\mathbf{y})}} (-1)^{|T|}\left[v(\mathbf{z}) - v(\mathbf{z}-\mathbf{e}_j)\right] \qquad (1.2)$$

Since

$$\mathbf{y} = \mathbf{z} + \mathbf{e}(T), \mathbf{y} \neq \mathbf{0} \text{ and } j \notin T,$$

we have $\{T \subseteq S(\mathbf{y})\} = \{T \subseteq M_j(\mathbf{z};\mathbf{x})\}$.

Hence (1.2) can be written as

$$\sum_{T \subset M_j(\mathbf{z};\mathbf{x})} (-1)^{|T|} \left[v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}_j)\right] \tag{1.3}$$

From (1.3), we know that (1.1) can be written as

$$\sum_{k=1} \sum_{\substack{z_j=k \\ \mathbf{z}\neq\mathbf{0} \\ \mathbf{z}\in\Gamma(\mathbf{x})}} \left[\sum_{T \subseteq M_j(\mathbf{z};\mathbf{x})} (-1)^{|T|} \frac{w(k)}{||\mathbf{z} + \mathbf{e}(T)||_w} \right] \cdot [v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}_j)]$$
$$= \sum_{i} \sum_{k=1} \sum_{\substack{z_j=k \\ \mathbf{z}\neq\mathbf{0} \\ \mathbf{z}\in\Gamma(\mathbf{x})}} \left[\sum_{T \subseteq M_j(\mathbf{z};\mathbf{x})} (-1)^{|T|} \frac{w(k)}{||\mathbf{z}||_w + \sum_{r\in T} [w(z_r+1) - w(z_r)]} \right] \cdot [v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}_j)]$$
$$= \phi^w_{i,j}((\mathbf{x}, v)).$$

Since $(\mathbf{x}, v)$ is arbitrarily given, then, by the above claim and (*), we have,

$$\sum_{j\in S(\mathbf{x})} H_{x_j,j}\psi_w((\mathbf{x}, v)) = \sum_{j\in S(\mathbf{x})} \phi^w_{x_j,j}((\mathbf{x}, v)) = (\mathbf{x}, v)(\mathbf{x}),$$

for each fixed $\mathbf{x} \in \Gamma(\mathbf{m})$

Since the potential of $(\mathbf{x}, v)$ is unique and $\psi_w((\mathbf{x}, v))$ satisfies $(\star\star)$, then $\psi_w((\mathbf{x}, v))$ is the potential.

The proof is complete $\diamondsuit$

From the above proof, we can easily see the following theorem.

Theorem 2. Given a multi-choice cooperative game $(\mathbf{m}, v)$ then the Shapley value and the Potential of $(\mathbf{m}, v)$ have the following relationship.

$$\phi^w_{ij}((\mathbf{m}, v)) = H_{ij}P_w((\mathbf{m}, v)). \tag{2}$$

Consistency Property of the Multi-choice Shapley value.

Given a multi-choice cooperative game $(\mathbf{m}, v)$ and its solution,

$$(\psi_{11}^w(v), .., \psi_{m_1 1}^w(v), \psi_{12}^w(v), .., \psi_{m_2 2}^w(v), .., \psi_{1n}^w(v), .., \psi_{m_n n}^w(v)),$$

for each $\mathbf{z} \in \Gamma(\mathbf{m})$, we define an action vector $\mathbf{z}^* = (z_1^*, z_2^*, ..., z_n^*)$ where

$$\begin{cases} z_j^* = m_j \text{ if } z_j < m_j \\ z_j^* = 0 \ \text{ if } z_i = m_j. \end{cases}$$

Furthermore, we define a new game $v_{\mathbf{z}}^{\psi} : \Gamma(\mathbf{z}) \to R$ such that

$$v_{\mathbf{z}}^{\psi}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z_j^* \neq 0} \psi_{m_j, j}((by \vee \mathbf{z}^*, v).$$

We call $v_{\mathbf{z}}^{\psi}$ a reduced game of v with respect to $\mathbf{z}$ and the solution ψ. Furthermore, we say that the solution ψ is *consistent* if $\psi_{i,j}(v) = \psi_{i,j}(v_{\mathbf{z}}^{\psi})$ for all $i \leq z_j$ and all $j \in N - S(\mathbf{z}^*)$.

Theorem 3. The multi-choice Shapley value ϕ^w is consistent.

Proof. Given a multi-choice cooperative game $(\mathbf{m}, v)$ and its Shapley value ϕ^w. Given $\mathbf{z} \in \Gamma(\mathbf{m})$, the reduced game of v with respect to $\mathbf{z}$ and the Shapley is :
$v_{\mathbf{z}}^{\phi^w} : \Gamma(\mathbf{m}) \to R$ such that

$$v_{\mathbf{z}}^{\phi^w}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z_j^* \neq 0} \phi_{m_j, j}^w((\mathbf{y} \vee \mathbf{z}^*, v)).$$

Let $\mathbf{b} = (b_1, ..., b_n) = \mathbf{y} \vee \mathbf{z}^*$, since the Shapley value satisfies Axiom 2, we have

$$\begin{aligned} v_{\mathbf{z}}^{\phi^w}(\mathbf{y}) &= v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z_j^* \neq 0} \phi_{m_j, j}^w((\mathbf{y} \vee \mathbf{z}^*, v) \\ &= \sum_{b_j \neq 0} \phi_{b_j, j}^w((\mathbf{y} \vee \mathbf{z}^*, v) - \sum_{z_j^* \neq 0} \phi_{m_j, j}^w((\mathbf{y} \vee \mathbf{z}^*, v)). \end{aligned}$$

Now, since z_j^* is either 0 or m_j, for any $\mathbf{y} \leq \mathbf{z}$, by Theorem 2, we have

$$\begin{aligned} v_{\mathbf{z}}^{\phi^w}(\mathbf{y}) &= \sum_{y_j \neq 0} \phi_{y_j, j}^w((\mathbf{y} \vee \mathbf{z}^*, v)) \\ &= \sum_{y_j \neq 0} H_{y_j, j} P_w((\mathbf{y} \vee \mathbf{z}^*), v)). \end{aligned} \tag{3.1}$$

By Theorem 1, the potential of the game $(\mathbf{z}, v_{\mathbf{z}}^{\phi^w})$ is uniquely determined by formula $(\star\star)$ applied to the game and all its sub games. Comparing this with (3.1), we know that

$$P_w((\mathbf{y}, v_{\mathbf{z}}^{\phi^w})) = P_w((\mathbf{y} \vee \mathbf{z}^*, v)) + c$$

for all $\mathbf{y} \leq \mathbf{z}$, where c is a suitable constant so as to make $P_w((\mathbf{0}, v_{\mathbf{z}}^{\phi^w})) = 0$. It is clear that $\mathbf{z} \vee \mathbf{z}^* = \mathbf{m}$, hence,

$$\phi_{i,j}^w((\mathbf{z}, v_{\mathbf{z}}^{\phi^w})) = H_{i,j} P_w((\mathbf{z}, v_{\mathbf{z}}^{\phi})) = H_{i,j} P_w((\mathbf{m}, v))$$

for all $i \leq z_j$ and all $j \in N - S(\mathbf{z}^*)$.

The proof is complete $\diamondsuit$

w-proportional for two-working-player games.

In the beginning of this article, we define σ_0 as the action to do nothing.

Since the solution concept of the *multi-choice Shapley value*, is *dummy free* the useless player does not affect the value no matter if he is regard as a player or not, see [2] for detail. However, not all solution concepts are dummy free.

W.L.O.G. suppose $i < j$, given $N_1 = \{i, j\}$ and a two person cooperative game $(\mathbf{m}_1, v_1)$ with $\mathbf{m}_1 = (m_i, m_j)$, furthermore, given $N_2 = \{1, ..., i, ..., j, ...n\}$ and an n-person cooperative game $(\mathbf{m}_2, v_2)$ with $\mathbf{m}_2 = (0, 0, ..., 0, m_i, 0, ..., 0, m_j, 0, ..., 0)$. Suppose $v_1((x_i, x_j)) = v_2((0, 0, ..., 0, x_i, 0, ..., 0, x_j, 0, ..., 0))$ for all $x_i = 0, 1, ..., m_i$ and $x_j = 0, 1, ..., m_j$, we can easily make up some solution concept where the solution for v_1 is different from the solution for v_2 i.e. we can easily make up a solution concept where the value for player i in v_1 is different from the value for player i in v_2. Therefore, we can not regard v_1 and v_2 as the same game. To avoid ambiguity, we call v_1 a two person game and call v_2 a two-working-player game. When there is only one $m_j > 0$, we called $(\mathbf{m}, v)$ an one-working-player game.

Given an n-person multi-choice cooperative game $(\mathbf{m}, v)$, and its sub-game $(\mathbf{x}, v)$ where $\mathbf{x} \leq \mathbf{m}$, as usual, we let $m = \max\{m_1, ..., m_n\}$ Let $(\mathbf{0}|x_i = k, x_j = \ell)$ be an action vector where player i takes action σ_k, player takes action σ_ℓ and all the other players take action σ_0, we have the following definition.

Definition 9. Given $w(0) = 0$, $w(1)$, ..., $w(m)$, a solution function ψ is said to satisfies "w-proportional for two-working-player games" if for any two-working-player game $(\mathbf{m}, v)$ with$\mathbf{m}$=(0,..,m_i,0,..,m_j,0,..0), ψ satisfies the following.

$$
\begin{aligned}
\psi_{k,i}((\mathbf{m},v)) = & \sum_{t=1}^{k} v((\mathbf{0}|x_i = t, x_j = 0) + \\
& [\frac{w(t)}{w(t)+w(1)}] \cdot [v((\mathbf{0}|x_i = t, x_j = 1)) - v((\mathbf{0}|x_i = t-1, x_j = 1)) - v((\mathbf{0}|x_i = t, x_j = 0))] + \\
& [\frac{w(t)}{w(t)+w(2)}] \cdot [v((\mathbf{0}|x_i = t, x_j = 2)) - v((\mathbf{0}|x_i = t-1, x_j = 2)) - v((\mathbf{0}|x_i = t, x_j = 1))] + \\
& \vdots \\
& + \\
& [\frac{w(t)}{w(t)+w(m_j)}] \cdot [v((\mathbf{0}|x_i = t, x_j = m_j)) - v((\mathbf{0}|x_i = t-1, x_j = m_j)) \\
& \qquad - v((\mathbf{0}|x_i = t, x_j = m_j - 1))]
\end{aligned} \tag{4.1}
$$

For player j, we have a formula of $\psi_{\ell,j}((\mathbf{m},v))$ similar to (4.1) which is omitted.

It is easy to see that (4.1) is an extension of the definition of standard for two-person games in [1]. For convenience, we reformulate (4.1) as follows.

$$
\begin{aligned}
\psi_{k,i}((\mathbf{m},v)) = & \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ \mathbf{z}\in\Gamma(\mathbf{m}) \\ z_j = m_j}} \left[\frac{w(z_i)}{w(z_i)+w(z_j)}\right] \cdot [v(\mathbf{z}) - v(\mathbf{z}-\mathbf{e}_i))] + \\
& \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ \mathbf{z}\in\Gamma(\mathbf{m}) \\ z_j \neq m_j}} \left[\frac{w(z_i)}{w(z_i)+w(z_j)}\right] \cdot [v(\mathbf{z}) - v(\mathbf{z}-\mathbf{e}_i))] + \\
& \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ \mathbf{z}\in\Gamma(\mathbf{m}) \\ z_j \neq m_j}} \left[\frac{-w(z_i)}{w(z_i)+w(z_j+1)}\right] \cdot [v(\mathbf{z}) - v(\mathbf{z}-\mathbf{e}_i)]
\end{aligned} \tag{4.2}
$$

We have the following conjecture:

Conjecture. Given a n-person multi-choice cooperative game $(\mathbf{m},v)$ with all its sub-games and a weight function for $\{0,1,...,m\}$, let ϕ^w be a solution function. Then:
(i) ϕ^w is consistence; and
(ii)ϕ^w is"w-proportional for two-working-player cooperative games";
if and only if ϕ^w is the multi-choice Shapley value.

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