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Abstract: In this paper we will use the Bayesian inference for the parameters that appear in the queueing systems. We will estimate these parameters and we will build confidence intervals and significance tests for them, considering the parameters of the exponential Poisson and geometric distribution.
We will also use the Neyman and the Neyman-Bayes inference for the exponential and Poisson distribution.
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1. Introduction

For the Bayesian inference the distributions depending on a parameter θ (which can be either a number or a vector) are considered as conditional distributions on this parameter, where θ is a random variable with the pdf φ. If we consider the sample $X_1,\dots,X_n$ for the random variable $X|\theta$, we denote by $f(x;\theta)$ its pdf and by $\hat{\varphi}(\theta;x_1,\dots,x_n)$ the pdf of $\theta|X_1,\dots,X_n$. For computing the last pdf we apply the Bayes formula and we obtain

$$\hat{\varphi}(\theta;x_1,\dots,x_n)=\frac{\varphi(\theta)\cdot\prod_{i=1}^{n}f(x_i;\theta)}{\int_{\Theta}\varphi(t)\cdot\prod_{i=1}^{n}f(x_i;t)dt},\tag{1}$$

where Θ is the domain of θ.

If X is a discrete random variable we replace $f(x;\theta)$ by $p(x;\theta)$ (the probability of having $X=x$ depending on the value of θ), and the formula (1) becomes

$$\hat{\varphi}(\theta;x_1,\dots,x_n)=\frac{\varphi(\theta)\cdot\prod_{i=1}^{n}p(x_i;\theta)}{\int_{\Theta}\varphi(t)\cdot\prod_{i=1}^{n}p(x_i;t)}.\tag{1'}$$

Definition 1. *The above pdf* φ *is called the prior pdf of* θ *, and the pdf* $\hat{\varphi}$ *is called the posterior pdf of* θ.

In the particular case of queueing systems parameters we will build sequences of random variables with the posterior probability density functions (1') and we will study the convergence in probability and in distribution in the second section. The method is analogous to the method used in the article of Lo (see [7]) for the rotationally symmetric spherical distributions.

We will also obtain Bayes estimators for these parameters: modulus estimator, expectation estimator and median estimator (see [8]). The first is the modulus of the posterior distribution, the second is the expectation and the third is the median.

The confidence intervals with the error ε for the Bayes inference are intervals so that a random variable having the posterior distribution of θ is in this interval with the probability $1-\varepsilon$. To distinguish these confidence intervals from the non-Bayes ones we call the firsts credible intervals (see [8]). In the third section we will find credible intervals for the parameters that appear in the queueing systems.

In [8] is presented for the expectation of the normal distribution if we know the variance a Bayes significance test. For building this test (see [8]) we use a prior cdf with a jump in θ_0 (the value of θ in the case of null hypothesis). In the fourth section we will build Bayes significance tests for the parameters that appear in the queueing systems.

In [4] are done for the expectation of the normal distribution if we know the variance a Neyman-Bayes inference and a Neyman inference.

First we compute the posterior distribution of θ conditioned by $\overline{X}$. Next we find for any value of θ chosen into an interval with the same distance between values the maximal value of its posterior distribution, $\overline{X}^*$, and an interval for $\overline{X}$ with the error ε (the probability of having $\overline{X}$ outside the interval).

After this we find two regression parables for the extremities (left and right) of these intervals and θ. We denote by Y the above extremities. If for a given $\theta = \theta_j$ we have an extremity equal to $-\infty$ or ∞ we replace the Y values by $Y \cdot (\theta - \theta_j)$ and we compute the regression parable for Y (new values) and θ. We consider the regression parables $Y = a_i \cdot \theta^2 + b_i \cdot \theta + c_i$, where i=1 for the left extremities and i=2 for the right extremities.

Finally, we compute the confidence interval with the error ε for θ using the section of the domain bordered by $\theta = \theta_{min}$, $\theta = \theta_{max}$ and $Y = \frac{a_i \cdot \theta^2 + b_i \cdot \theta + c_i}{\theta - \theta_{j_i}}$, where the extremity is $-\infty$ or ∞ for $\theta = \theta_{j_i}$, parallel to $O\theta$ in $Y = \overline{X}$. We denote this interval by $C_{n,1-\varepsilon}(\overline{X})$, where n is the sample volume. The estimator of θ with the error ε is

$$\hat{\theta}_{n,1-\varepsilon}(\overline{X}) = \frac{1}{1-\varepsilon} \cdot \int_{C_{n,1-\varepsilon}(\overline{X})} \theta \cdot g(\theta; \overline{X}) d\theta, \tag{2}$$

where g is the posterior pdf of θ.

The difference between the Neyman inference and the Neyman-Bayes inference is the following: in the first case we have no prior information, hence the prior information is considered uniform on $[L,U]$; in the second case we have another prior information given by some prior pdf with the value 0 outside the interval $[L,U]$. For both cases we compute the posterior pdf $g(\theta; \overline{X})$, where θ is the parameter for which we intend to do the Neyman inference or the Neyman-Bayes inference, and $\overline{X}$ is the sample expectation.

2. Parameter Estimation

Because the parameters that will appear are Gamma, restricted Gamma or Beta, we need the following proposition (see [3]).

Proposition 1. *a) If the posterior distribution of* θ *is* $\Gamma(\tilde{\alpha}, \tilde{\beta})$ *the Bayes modulus estimator is* $\hat{\theta}_{mode} = (\tilde{\alpha} - 1) \cdot \tilde{\beta}$*, and the Bayes expectation estimator is* $\hat{\theta}_{expect} = \tilde{\alpha} \cdot \tilde{\beta}$*. If we have* $2 \cdot \tilde{\alpha} \in N^*$*, the Bayes median estimator is* $\hat{\theta}_{median} = \frac{\tilde{\beta} \cdot \chi^2_{2\cdot\tilde{\alpha}}(0.5)}{2}$.

b) If the posterior distribution of θ *is* $\Gamma(\tilde{\alpha}, \tilde{\beta})$ *restricted to the interval* [0,u] *the Bayes modulus estimator is* $\hat{\theta}_{mode} = \min((\tilde{\alpha} - 1) \cdot \tilde{\beta}, u)$*. If we have* $\tilde{\alpha} \in N^*$*, the Bayes expectation estimator is*

$$\hat{\theta}_{expect} = \tilde{\alpha} \cdot \tilde{\beta} - \frac{\tilde{\beta} \cdot u^{\tilde{\alpha}} \cdot \exp\left(-\frac{u}{\tilde{\beta}}\right)}{E_{\tilde{\alpha}, \frac{1}{\tilde{\beta}}}(u)},$$

where $E_{\tilde{\alpha}, \frac{1}{\tilde{\beta}}}$ *is the Erlang cdf of the order* $\tilde{\alpha}$ *and parameter* $\frac{1}{\tilde{\beta}}$.

In the same conditions the Bayes median estimator is $\hat{\theta}_{median} = \frac{\tilde{\beta}}{2} \cdot \chi^2_{2\cdot\tilde{\alpha}}\left(\frac{E_{\tilde{\alpha},\frac{1}{\tilde{\beta}}}(u)}{2}\right)$.

c) If the posterior distribution of θ *is* $\beta(a,b)$ *the Bayes modulus estimator is* $\hat{\theta}_{mode} = \frac{a-1}{a+b-2}$*, and the Bayes expectation estimator is* $\hat{\theta}_{expect} = \frac{a}{a+b}$*. If we have* $2\cdot a \in \mathsf{N}^*$ *and* $2\cdot b \in \mathsf{N}^*$*, the Bayes median estimator is* $\hat{\theta}_{median} = \frac{a\cdot F_{2\cdot a,2\cdot b}(0.5)}{a\cdot F_{2\cdot a,2\cdot b}(0.5)+b}$*, where* $F_{2\cdot a,2\cdot b}(0.5)$ *is the* 0.5 *centil of the Snedecor-Fisher distribution with the orders* $2\cdot a$ *and* $2\cdot b$.

We consider now the distribution $\exp(\lambda)$ for *X*. We choose $\Gamma(\alpha,\beta)$ as the prior distribution of *λ*, using the maximum entropy principle (see [8]). We consider also the sample $X_1,...,X_n$ on the random variable *X*, and the posterior distribution of *λ* is

$$f(\lambda; X_1,...,X_n) = \frac{\lambda^{n+\alpha-1}\cdot\exp\left(-\frac{\lambda(n\cdot\beta\overline{X}+1)}{\beta}\right)}{\int_0^\infty t^{n+\alpha-1}\cdot\exp\left(-\frac{t(n\cdot\beta\overline{X}+1)}{\beta}\right)dt}, \quad (3)$$

hence the posterior distribution is $\Gamma\left(n+\alpha, \frac{\beta}{n\cdot\beta\overline{X}+1}\right)$.

We can prove that $\lambda|X_1,...,X_n$ tends in probability and in distribution to its true value λ_0. If we apply proposition 1 we obtain the following Bayes estimators: $\hat{\lambda}_{mode} = \frac{(n+\alpha-1)\cdot\beta}{n\cdot\beta\cdot\overline{X}+1}$, $\hat{\lambda}_{expect} = \frac{(n+\alpha)\cdot\beta}{n\cdot\beta\cdot\overline{X}+1}$, and if we have $2\cdot\alpha \in \mathsf{N}^*$ the median estimator is $\hat{\lambda}_{median} = \frac{\beta}{2\cdot n\cdot\beta\cdot\overline{X}+2}\cdot\chi^2_{2\cdot n+2\cdot\alpha}(0.5)$.

If the prior distribution of *λ* is uniform on $[0,\tilde{\lambda}]$ we can notice in the same manner that the posterior distribution of λ is Erlang of the order *n* and parameter $n\cdot\overline{X}$ restricted to $[0,\tilde{\lambda}]$. We can prove that $\lambda|X_1,...,X_n$ tends in probability and in distribution to $\min(\lambda_0,\tilde{\lambda})$. If we apply proposition 1 we obtain the following Bayes estimators: $\hat{\lambda}_{mode} = \min\left(\frac{n-1}{n\cdot\overline{X}},\tilde{\lambda}\right)$, $\hat{\lambda}_{expect} = \frac{1}{\overline{X}} - \frac{\tilde{\lambda}^n\cdot\exp(-n\cdot\tilde{\lambda}\cdot\overline{X})}{n\cdot\overline{X}\cdot E_{n,n\cdot\overline{X}}(\tilde{\lambda})}$ and $\hat{\lambda}_{median} = \frac{1}{2\cdot n\cdot\overline{X}}\cdot\chi^2_{2\cdot n}\left(\frac{E_{n,n\cdot\overline{X}}(\tilde{\lambda})}{2}\right)$.

We will study now the Poisson distribution *Po*(*λ*). If the prior distribution of *λ* is $\Gamma(\alpha,\beta)$ the posterior distribution is $\Gamma\left(n\cdot\overline{X}+\alpha, \frac{\beta}{n\cdot\beta+1}\right)$. We can prove that $\lambda|X_1,...,X_n$ tends in probability and in distribution to λ_0. If we apply proposition 1 we obtain the following Bayes estimators: $\hat{\lambda}_{mode} = \frac{(n\cdot\overline{X}+\alpha-1)\cdot\beta}{n\cdot\beta+1}$, $\hat{\lambda}_{expect} = \frac{(n\cdot\overline{X}+\alpha)\cdot\beta}{n\cdot\beta+1}$, and if $2\cdot\alpha \in \mathsf{N}^*$ we have $\hat{\lambda}_{median} = \frac{1}{2\cdot n}\cdot\chi^2_{2\cdot n\cdot\overline{X}+2\cdot\alpha}(0.5)$.

If the prior distribution of *λ* is uniform on $[0,\tilde{\lambda}]$ the posterior distribution of *λ* is Erlang of the order $n\cdot\overline{X}$ and the parameter *n* restricted to $[0,\tilde{\lambda}]$. We can prove that $\lambda|X_1,...,X_n$ tends in probability and in distribution to $\min(\lambda_0, \tilde{\lambda})$. If we apply proposition 1 we obtain the following Bayes estimators: $\hat{\lambda}_{mode} = \min\left(\frac{n \cdot \overline{X} - 1}{n}, \tilde{\lambda}\right)$, $\hat{\lambda}_{expect} = \overline{X} - \frac{\tilde{\lambda}^{n \cdot \overline{X}} \cdot \exp(-n \cdot \tilde{\lambda})}{n \cdot E_{n \cdot \overline{X}, n}(\tilde{\lambda})}$ and $\hat{\lambda}_{median} = \frac{1}{2 \cdot n} \cdot \chi^2_{2 \cdot n \cdot \overline{X}}\left(\frac{E_{n \cdot \overline{X}, n}(\tilde{\lambda})}{2}\right)$.

Finally, we will study the geometrical distribution of parameter ρ. We will consider the prior distribution uniform on [0,1]. The posterior distribution is in this case $\beta(n \cdot \overline{X} + 1, n + 1)$. We can prove that $\rho | X_1, \ldots, X_n$ tends in probability and in distribution to its true value ρ_0. If we apply proposition 1 we obtain the following Bayes estimators: $\hat{\rho}_{mode} = \frac{\overline{X}}{\overline{X} + 1}$, $\hat{\rho}_{expect} = \frac{n \cdot \overline{X} + 1}{n \cdot \overline{X} + n + 2}$ and $\hat{\rho}_{median} =$ $\frac{(n \cdot \overline{X} + 1) \cdot F_{2 \cdot n \cdot \overline{X} + 2, 2 \cdot n + 2}(0.5)}{(n \cdot \overline{X} + 1) \cdot F_{2 \cdot n \cdot \overline{X} + 2, 2 \cdot n + 2}(0.5) + n + 1}$.

3. Credible Intervals

We will build in this section credible intervals with the error ε for the parameters considered in the previous section. First we consider the distribution $\exp(\lambda)$ and the prior distribution for λ is $\Gamma(\alpha, \beta)$. Taking into account the results from the previous section, the posterior distribution of λ is $\Gamma\left(n + \alpha, \frac{\beta}{n \cdot \beta \cdot \overline{X} + 1}\right)$. It results that $\frac{2 \cdot \lambda (n \cdot \beta \cdot \overline{X} + 1)}{\beta}$ has the distribution $\Gamma\left(n + \alpha, \frac{1}{2}\right)$. If $\alpha = \frac{m}{2}$, where $m \in \mathsf{N}^*$ this distribution coincides with the $\chi^2_{2 \cdot n + m}$ distribution. It results that the credible interval is in this case

$$\lambda \in \left[\frac{\beta}{2(n \cdot \beta \cdot \overline{X} + 1)} \cdot \chi^2_{2 \cdot n + m}\left(\frac{\varepsilon}{2}\right), \frac{\beta}{2(n \cdot \beta \cdot \overline{X} + 1)} \cdot \chi^2_{2 \cdot n + m}\left(1 - \frac{\varepsilon}{2}\right)\right], \tag{4}$$

where $\chi^2_{2 \cdot n + m}\left(\frac{\varepsilon}{2}\right)$ and $\chi^2_{2 \cdot n + m}\left(1 - \frac{\varepsilon}{2}\right)$ are the centils of the orders $\frac{\varepsilon}{2}$ and $1 - \frac{\varepsilon}{2}$ for the $\chi^2_{2 \cdot n + m}$ distribution.

If the prior distribution of λ is uniform on $[0, \tilde{\lambda}]$, the posterior distribution of λ is Erlang of the order *n* and parameter $n \cdot \overline{X}$ restricted to $[0, \tilde{\lambda}]$. Therefore the posterior distribution of $2 \cdot n \cdot \lambda \cdot \overline{X}$ is χ^2_{2n} restricted to $[0, 2 \cdot n \cdot \tilde{\lambda} \cdot \overline{X}]$. We denote by H_{2n} the cdf of the χ^2_{2n} distribution, by $\varepsilon_1 = \frac{\varepsilon}{2} \cdot H_{2n}(2 \cdot n \cdot \tilde{\lambda} \cdot \overline{X})$ and by $\varepsilon_2 = \left(1 - \frac{\varepsilon}{2}\right) \cdot H_{2n}(2 \cdot n \cdot \tilde{\lambda} \cdot \overline{X})$. Using the above notations we obtain the credible interval

$$\lambda \in \left[\frac{\chi^2_{2n}(\varepsilon_1)}{2 \cdot n \cdot \overline{X}}, \frac{\chi^2_{2n}(\varepsilon_2)}{2 \cdot n \cdot \overline{X}}\right]. \tag{5}$$

For the Poisson distribution we obtain analogously the credible interval

$$\lambda \in \left[\frac{\beta}{2(n \cdot \beta + 1)} \cdot \chi^2_{2 \cdot n \cdot \overline{X} + m}\left(\frac{\varepsilon}{2}\right), \frac{\beta}{2(n \cdot \beta + 1)} \cdot \chi^2_{2 \cdot n \cdot \overline{X} + m}\left(1 - \frac{\varepsilon}{2}\right)\right], \tag{6}$$

if the prior distribution of λ is $\Gamma\left(\frac{m}{2}, \beta\right)$ with $m \in \mathsf{N}^*$.

If this prior distribution is uniform on $\left[0,\tilde{\lambda}\right]$ and we denote by $\varepsilon_1 = \frac{\varepsilon}{2} \cdot H_{2 \cdot n \cdot \overline{X}}\left(2 \cdot n \cdot \tilde{\lambda}\right)$ and by $\varepsilon_2 = \left(1 - \frac{\varepsilon}{2}\right) \cdot H_{2 \cdot n \cdot \overline{X}}\left(2 \cdot n \cdot \tilde{\lambda}\right)$, we obtain the credible interval

$$\lambda \in \left[\frac{\chi_{2 \cdot n \cdot \overline{X}}(\varepsilon_1)}{2 \cdot n}, \frac{\chi_{2 \cdot n \cdot \overline{X}}(\varepsilon_2)}{2 \cdot n} \right]. \tag{7}$$

If the parameter ρ of the geometrical distribution has the prior distribution uniform on [0,1], its posterior distribution is $\beta\left(n \cdot \overline{X} + 1, n + 1\right)$. The credible interval with the error ε is [*a*,*b*], where (*a*,*b*) is the solution of the problem

$$\begin{cases} \min(b - a) \\ \int_a^b \rho^{n \cdot \overline{X}} (1 - \rho)^n \, d\rho = (1 - \varepsilon) \beta\left(n \cdot \overline{X} + 1, n + 1\right) \end{cases}. \tag{8}$$

We apply the Lagrange multipliers method, and we obtain the unique solution (*a*,*b*) so that

$$\begin{cases} a^{\overline{X}}(1 - a) = b^{\overline{X}}(1 - b) \\ 0 < a < \frac{\overline{X}}{\overline{X}+1} < b < 1 \end{cases}. \tag{8'}$$

4. Signification Tests

We will verify first the null hypothesis $H_0 : \lambda = \lambda_0$ against the alternative hypothesis $H_1 : \lambda \neq \lambda_0$ with the first order error ε for the exponential distribution.

Definition 2 ([3]). *The above test that uses the Bayesian inference is called the Bayes two-sided test.*

We consider a continuous cdf $F_1 : [0, \infty) \to [0,1]$ and a real number $p_0 \in (0,1)$ (usualy we take $p_0 = 0.5$ using the maximum entropy principle). We denote also by

$$F(\lambda) = \begin{cases} (1 - p_0) F_1(\lambda) \text{ if } \lambda < \lambda_0 \\ p_0 + (1 - p_0) F_1(\lambda) \text{ if } \lambda \geq \lambda_0 \end{cases} \tag{9}$$

the prior cdf of λ and by φ the prior pdf of λ. Therefore we have $\varphi(\lambda) = (1 - p_0) \cdot \varphi_1(\lambda)$ for any $\lambda \neq \lambda_0$, where $\varphi_1 = F_1'$. p_0 is the prior probability for having $\lambda = \lambda_0$. The posterior probability of this is

$$\varphi(\lambda_0 | X_1, \ldots, X_n) = \frac{p_0 \cdot \lambda_0^n \cdot \exp\left(-\lambda_0 \cdot n \cdot \overline{X}\right)}{p_0 \cdot \lambda_0^n \cdot \exp\left(-\lambda_0 \cdot n \cdot \overline{X}\right) + (1 - p_0) \int_0^\infty \varphi_1(\lambda) \cdot \lambda^n \cdot \exp\left(-\lambda \cdot n \cdot \overline{X}\right) d\lambda}. \tag{10}$$

We denote by

$$\Psi_1(x) = \int_0^x \varphi(\lambda) \cdot \lambda^n \cdot \exp\left(-n \cdot \lambda \cdot \overline{X}\right) d\lambda. \tag{11}$$

We accept H_0 if the last probability is at least $1 - \varepsilon$, which is equivalent to

$$\varepsilon \cdot p_0 \cdot \lambda_0^n \cdot \exp\left(-\lambda_0 \cdot n \cdot \overline{X}\right) > (1 - \varepsilon)(1 - p_0) \Psi_1(\infty). \tag{12}$$

Definition 3 ([3]). *The test that verifies the null hypothesis* $H_0 : \lambda = \lambda_0$ *against the alternative hypo-*

thesis $H_1 : \lambda < \lambda_0$ *with the first order error* ε*, using the Bayesian inference is called the Bayes one-sided left test.*

Analogously, we accept H_0 if

$$\varepsilon \cdot p_0 \cdot \lambda_0^n \cdot \exp\left(-\lambda_0 \cdot n \cdot \overline{X}\right) > (1-\varepsilon)(1-p_0)\Psi_1(\lambda_0). \quad (13)$$

Definition 4 ([3]). *The test that verifies the null hypothesis* $H_0 : \lambda = \lambda_0$ *against the alternative hypothesis* $H_1 : \lambda > \lambda_0$ *with the first order error* ε*, using the Bayesian inference is called the Bayes one-sided right test.*

We accept H_0 if

$$\varepsilon \cdot p_0 \cdot \lambda_0^n \cdot \exp\left(-\lambda_0 \cdot n \cdot \overline{X}\right) > (1-\varepsilon)(1-p_0)\left(\Psi_1(\infty) - \Psi_1(\lambda_0)\right). \quad (14)$$

In the case of the *Po*(λ) distribution we do the same above tests, but we replace Ψ_1 by Ψ_2, where

$$\Psi_2(x) = \int_0^x \varphi(\lambda) \cdot \lambda^{n \cdot \overline{X}} \cdot \exp(-n \cdot \lambda) d\lambda . \quad (11')$$

For the geometrical distribution of the parameter ρ we denote by

$$\Psi_3(x) = \int_0^x \varphi(\rho) \cdot \rho^{n \cdot \overline{X}} (1-\rho)^n \, d\rho . \quad (11'')$$

For the Bayes two-sided test we accept H_0 if

$$\varepsilon \cdot p_0 \cdot \rho_0^{n \cdot \overline{X}} (1-\rho_0)^n > (1-\varepsilon)(1-p_0)\Psi_3(1). \quad (15)$$

For the Bayes one-sided left test we accept H_0 if

$$\varepsilon \cdot p_0 \cdot \rho_0^{n \cdot \overline{X}} (1-\rho_0)^n > (1-\varepsilon)(1-p_0)\Psi_3(\rho_0). \quad (15')$$

For the Bayes one-sided right test we accept H_0 if

$$\varepsilon \cdot p_0 \cdot \rho_0^{n \cdot \overline{X}} (1-\rho_0)^n > (1-\varepsilon)(1-p_0)\left(\Psi_3(1) - \Psi_3(\rho_0)\right). \quad (15'')$$

5. Neyman and Neyman-Bayes Inference

In the case of the normal distribution $N\left(m, \sigma^2\right)$ we have *θ*=*m* (see [4]). In the case of the Neyman inference the posterior distribution is $N\left(\overline{X}, \frac{\sigma^2}{n}\right)$ (where *n* is the sample size), and in the case of the Neyman-Bayes inference the posterior distribution is $N\left(\frac{\overline{X} \cdot \delta^2 + \gamma \cdot \frac{\sigma^2}{n}}{\delta^2 + \frac{\sigma^2}{n}}, \frac{\delta^2 \cdot \sigma^2}{n \cdot \delta^2 + \sigma^2}\right)$, where the prior distribution is $N\left(\gamma, \delta^2\right)$ (see [4]). All the above prior and posterior distributions are restricted to [*L*,*U*].

For the distribution $\exp(\lambda)$ we have *θ*=*λ*. In the case of the Neyman inference we obtain in the same manner the posterior distribution $\Gamma\left(n+1, \frac{1}{n \cdot \overline{X}}\right)$ restricted to [*L*,*U*]. If the prior distribution is $\Gamma(\alpha, \beta)$ restricted to [*L*,*U*] in the case of the Neyman-Bayes inference, the obtained posterior distribution is $\Gamma\left(n + \alpha, \frac{1}{n \cdot \overline{X}} + \frac{1}{\beta}\right)$ restricted to the same interval.

For the *Po*(λ) distribution the posterior distribution in the case of the Neyman inference is $\Gamma\left(n \cdot \overline{X} + 1, \frac{1}{n}\right)$ restricted to [*L*,*U*]. In the case of the Neyman-Bayes inference using the prior distribution $\Gamma(\alpha,\beta)$ restricted to [*L*,*U*], we obtain the posterior distribution $\Gamma\left(n\cdot\overline{X}+\alpha+1,\frac{1}{n}+\frac{1}{\beta}\right)$ restricted to the same interval.

If we have no prior information on θ we denote by $\overline{X}^*_\theta$ the value of $\overline{X}$ so that $g(\theta,\overline{X})$ is maximal (for a fixed $\theta\in[L,U]$, the distance between two fixed values of θ being the same). If we have this information we denote by $\overline{X}^{**}_\theta$ the same value.

If we have no prior information on θ we denote by $A^*_{n,1-\varepsilon}$ an interval (*a*,*b*) so that $P\left(\overline{X}\in A^*_{n,1-\varepsilon}\right)=1-\varepsilon$ and $g(\theta,a)=g(\theta,b)$. If we have this prior information we denote by $A^{**}_{n,1-\varepsilon}$ the above interval, and we have also $P\left(\overline{X}\in A^{**}_{n,1-\varepsilon}\right)=1-\varepsilon$ and $g(\theta,a)=g(\theta,b)$.

We will present now the modality to obtain the above values of $\overline{X}^*_\theta$ and $A^*_{n,1-\varepsilon}$, the confidence interval with the error ε for θ (denoted by $C^*_{n,1-\varepsilon}$) and the estimator with the error ε for θ (denoted by $\hat{\theta}^*_{n,1-\varepsilon}$) using the Monte Carlo method. If we have prior information on θ we obtain the values $\overline{X}^{**}_\theta$, $A^{**}_{n,1-\varepsilon}$, $C^{**}_{n,1-\varepsilon}$ and $\hat{\theta}^{**}_{n,1-\varepsilon}$ in an analogous manner.

We will generate 1000 groups of *n* random variables (normal, exponential or Poisson) and we will compute $\overline{X}$ and $g(\theta,\overline{X})$ for any fixed θ and any of the above groups.

From the 1000 values of $\overline{X}$ we choose $\overline{X}^*_\theta$ so that $g\left(\theta,\overline{X}^*_\theta\right)$ is the minimum of the computed values $g(\theta,\overline{X})$, and we sort ascending the values of $\overline{X}$. The interval $A^*_{n,1-\varepsilon}$ is so that it contains $1000(1-\varepsilon)$ sorted values of $\overline{X}$ and $g(\theta,\overline{X})$ has the same value for the first and for the last value of $\overline{X}$.

For computing $C^*_{n,1-\varepsilon}$ we find the regression parables in a classical manner and we solve a second degree equation. Using the formula (2) and the Monte Carlo method to compute the integral we obtain the estimator $\hat{\theta}^*_{n,1-\varepsilon}$ in the case of missing the prior information, and the estimator $\hat{\theta}^{**}_{n,1-\varepsilon}$ (using $C^{**}_{n,1-\varepsilon}$, computed in the same manner) in the contrary case. In both cases, the error of the estimator is ε.

A *C++* program called "BayesDlg.cpp" does these inferences using the Monte Carlo method.

In the case of the normal distribution $N\left(\theta,\sigma^2=10\right)$ we will divide the interval [-2,2] in *8* intervals with the same length and we take ε=0.1 (see [4]). For the Neyman-Bayes inference we consider also the prior distribution N(0,1). We obtain the following results:

m	−2	−1.5	−1	−0.5	0
$\overline{X}^*_m$	−2.15733	−0.03271	−2.64589	0.05353	−0.3318
$\overline{X}^{**}_m$	−5.24048	−2.25564	−1.10907	−0.47926	−0.00085
$A^*_{n,1-\varepsilon}$	$(-\infty,-0.7816)$	$(-3.28145,0.12837)$	$(-2.68002,0.66487)$	$(-2.07577,1.1636)$	$(-1.58886,1.66699)$
$A^{**}_{n,1-\varepsilon}$	$(-\infty,-0.69327)$	$(-4.77324,-0.1525)$	$(-3.22791,0.34815)$	$(-2.27858,1.00666)$	$(-1.63183,1.61071)$

m	0.5	1	1.5	2
$\overline{X}_m^*$	0.73337	0.53	0.11853	2.4221
$\overline{X}_m^{**}$	0.47155	1.13589	2.18006	6.0681
$A_{n,1-\varepsilon}^*$	$(-1.26872, 2.1955)$	$(-0.65323, 2.61196)$	$(-0.11817, 3.15921)$	$(0.77668, \infty)$
$A_{n,1-\varepsilon}^{**}$	$(-1.02357, 2.36536)$	$(-0.37977, 3.2941)$	$(0.2426, 4.71999)$	$(0.83486, \infty)$

$\overline{X}$	−2.5	−2	−1.5	−1
$C_{n,1-\varepsilon}^*$	$(-1.57218, -0.73086)$	$(-1.83376, -0.25426)$	$(-1.78607, 0.19138)$	$(-1.80736, 0.669)$
$C_{n,1-\varepsilon}^{**}$	$(-2, -0.63148)$	$(-2, -0.28322)$	$(-2, 0.09676)$	$(-2, 0.47369)$
$\hat{m}_{n,1-\varepsilon}^*$	−4.71086	−2.73414	−1.60098	−0.99119
$\hat{m}_{n,1-\varepsilon}^{**}$	−5.17705	−2.8116	−1.8942	−1.12156

$\overline{X}$	−0.5	0	0.5	1
$C_{n,1-\varepsilon}^*$	$(-1.84693, 1.05209)$	$(-1.48567, 1.47714)$	$(-1.08714, 1.86151)$	$(-0.65858, 1.83036)$
$C_{n,1-\varepsilon}^{**}$	$(-1.79207, 0.90438)$	$(-1.34598, 1.34651)$	$(-0.91735, 1.72103)$	$(-0.51336, 2)$
$\hat{m}_{n,1-\varepsilon}^*$	−0.52143	−0.00874	0.49933	1.01051
$\hat{m}_{n,1-\varepsilon}^{**}$	−0.57372	−0.01679	0.52058	1.14412

$\overline{X}$	1.5	2	2.5
$C_{n,1-\varepsilon}^*$	$(-0.20833, 1.81728)$	$(0.23875, 1.71761)$	$(0.73552, 1.6886)$
$C_{n,1-\varepsilon}^{**}$	$(-0.09206; 2)$	$(0.29407; 2)$	$(0.59477, 2)$
$\hat{m}_{n,1-\varepsilon}^*$	1.7029	2.46753	4.72744
$\hat{m}_{n,1-\varepsilon}^{**}$	1.88765	3.03897	5.50061

In fact the normal distribution has no direct connection to the queueing systems. Only if some interarrival times or services are log-normal, we can do these inferences after we compute the logarithm of the values. The above tables have the same entry data as in [4] for comparison. In that paper the software *R* is used and the results are as follows.

m	−2	−1.5	−1	−0.5	0	0.5
$\overline{X}_m^*$	?	−3.1311	−1.5168	−0.6695	0	0.6695
$\overline{X}_m^{**}$	?	−4.3829	−2.3136	−1.0659	0	1.0659
$A_{n,1-\varepsilon}^*$	$(-\infty, -0.7184)$	$(-9.09, 0.214)$	$(-4.09, 0.288)$	$(-2.53, 0.915)$	$(-1.65, 1.65)$	$(-0.915, 2.53)$
$A_{n,1-\varepsilon}^{**}$	$(-\infty, -0.7184)$	$(-12.927, -0.219)$	$(-5.759, 0.281)$	$(-3.286, 0.796)$	$(-1.65, 1.65)$	$(-0.796, 3.286)$

m	1	1.5	2
$\overline{X}_m^*$	1.5168	3.1311	?
$\overline{X}_m^{**}$	2.3136	4.3829	?
$A_{n,1-\varepsilon}^*$	$(-0.288, 4.09)$	$(0.214, 9.09)$	$(0.7184, \infty)$
$A_{n,1-\varepsilon}^{**}$	$(-0.281, 5.759)$	$(0.219, 12.927)$	$(0.7184, \infty)$

$\overline{X}$	−2.5	−2	−1.5	−1	−0.5
$C_{n,1-\varepsilon}^*$	$(-2, -0.42238)$	$(-2, -0.19051)$	$(-2, 0.08942)$	$(-2, 0.42342)$	$(-1.7655, 0.8148)$
$C_{n,1-\varepsilon}^{**}$	$(-2, -0.3447)$	$(-2, -0.15609)$	$(-2, 0.07409)$	$(-2, 0.3583)$	$(-1.715, 0.71269)$
$\hat{m}_{n,1-\varepsilon}^*$	−1.506	−1.3519	−1.1476	−0.88216	−0.46794
$\hat{m}_{n,1-\varepsilon}^{**}$	−1.1789	−1.0126	−0.82469	−0.61116	−0.33831

$\overline{X}$	0	0.5	1	1.5	2	2.5
$C_{n,1-\varepsilon}^*$	$(-1.2635, 1.2635)$	$(-0.8148, 1.7655)$	$(-0.42342, 2)$	$(-0.08942, 2)$	$(0.19051, 2)$	$(0.42238, 2)$
$C_{n,1-\varepsilon}^{**}$	$(-1.1573, 1.1573)$	$(-0.71269, 1.715)$	$(-0.3583, 2)$	$(-0.07409; 2)$	$(0.15609; 2)$	$(0.3447, 2)$
$\hat{m}_{n,1-\varepsilon}^*$	0	0.46794	0.88216	1.1476	1.3519	1.506
$\hat{m}_{n,1-\varepsilon}^{**}$	0	0.33831	0.61116	0.82469	1.0126	1.1789

In the above tables the question mark appears where there were no computation results in [4]. In our results using the Monte Carlo method we have no symmetry, but for $\overline{X} \in (L, U)$ we have the estimators closer to those obtained by the moments method (i.e. to $\overline{X}$). The only exception from this rule is $\overline{X} = 0$, where the estimator in [4] is exactly *0*. But this can be explained by symmetry. Of course, if $\overline{X} \notin (L, U)$ (in our case $\overline{X} \in \{\pm 2, \pm 2.5\}$) we do not obtain estimators as close as in the above paper.

In the cases of the distributions $\exp(\lambda)$ and $Po(\lambda)$ we will divide the interval [0.2,5] in 10 equal intervals. We take ε=0.05. For the Neyman-Bayes inference we consider the prior distribution $\Gamma(1,1)$. For the distribution $\exp(\lambda)$ we obtain the following results.

λ	0.2	0.68	1.16	1.64
$\overline{X}_\lambda^*$	10.345	1.64552	0.94839	0.65064
$\overline{X}_\lambda^{**}$	10.63952	1.47557	0.7538	0.55303
$A_{n,1-\varepsilon}^*$	$(2.71447, \infty)$	$(0.78777, 2.93968)$	$(0.46358, 1.70046)$	$(0.32521, 1.18417)$
$A_{n,1-\varepsilon}^{**}$	$(2.74375, \infty)$	$(0.77485, 2.58814)$	$(0.43314, 1.41979)$	$(0.2694, 0.98887)$

λ	2.12	2.6	3.08	3.56
$\overline{X}_{\lambda}^{*}$	0.51347	0.42714	0.33704	0.28078
$\overline{X}_{\lambda}^{**}$	0.42937	0.33516	0.23309	0.17586
$A_{n,1-\varepsilon}^{*}$	$(0.25432, 0.8893)$	$(0.19764, 0.68891)$	$(0.16199, 0.5374)$	$(0.09593, 0.45344)$
$A_{n,1-\varepsilon}^{**}$	$(0.18505, 0.74442)$	$(0.09963, 0.59272)$	$(0.09256, 0.50567)$	$(0.07827, 0.44373)$

λ	4.04	4.52	5
$\overline{X}_{\lambda}^{*}$	0.17632	0.09018	0.06515
$\overline{X}_{\lambda}^{**}$	0.10387	0.05558	0.04983
$A_{n,1-\varepsilon}^{*}$	$(0.08011, 0.39781)$	$(0.05709, 0.34869)$	$(0, 0.31787)$
$A_{n,1-\varepsilon}^{**}$	$(0.07222, 0.38294)$	$(0.05537, 0.34412)$	$(0, 0.3181)$

$\overline{X}$	0.2	0.5	1	1.5
$C_{n,1-\varepsilon}^{*}$	$(2.18724, 4.83244)$	$(1.69582, 3.44683)$	$(1.08231, 1.81712)$	$(0.59703, 1.2651)$
$C_{n,1-\varepsilon}^{**}$	$(2.08615, 4.79377)$	$(1.622, 3.14307)$	$(1.02809, 1.5887)$	$(0.58631, 1.10355)$
$\hat{\lambda}_{n,1-\varepsilon}^{*}$	10.003	2.47605	1.39555	0.8532
$\hat{\lambda}_{n,1-\varepsilon}^{**}$	9.26672	2.39259	1.32005	0.83324

$\overline{X}$	2	2.5	3	3.5
$C_{n,1-\varepsilon}^{*}$	$(0.2, 0.96504)$	$(0.2, 0.82446)$	$(0.2, 0.7182)$	$(0.2, 0.63361)$
$C_{n,1-\varepsilon}^{**}$	$(0.2, 0.86851)$	$(0.2, 0.72812)$	$(0.2, 0.63496)$	$(0.2, 0.56667)$
$\hat{\lambda}_{n,1-\varepsilon}^{*}$	0.5725	0.47069	0.40639	0.39011
$\hat{\lambda}_{n,1-\varepsilon}^{**}$	0.56914	0.45854	0.4092	0.39071

$\overline{X}$	4	4.5	5
$C_{n,1-\varepsilon}^{*}$	$(0.2, 0.57554)$	$(0.2, 0.52998)$	$(0.2, 0.49852)$
$C_{n,1-\varepsilon}^{**}$	$(0.2, 0.5269)$	$(0.2, 0.48774)$	$(0.2, 0.46401)$
$\hat{\lambda}_{n,1-\varepsilon}^{*}$	0.38085	0.41511	0.48451
$\hat{\lambda}_{n,1-\varepsilon}^{**}$	0.38223	0.41311	0.46674

For the *Po*(λ) distribution we obtain the following results.

λ	0.2	0.68	1.16	1.64	2.12	2.6
$\overline{X}_{\lambda}^{*}$	0	0.6	1.1	1.6	2.1	2.6
$\overline{X}_{\lambda}^{**}$	0	0.6	1.5	1.7	2.1	2.8
$A_{n,1-\varepsilon}^{*}$	$(0, 0.4)$	$(0.2, 1.11484)$	$(0.5, 1.84922)$	$(0.8, 2.53828)$	$(1.2, 3.07891)$	$(1.62422, 3.6)$
$A_{n,1-\varepsilon}^{**}$	$(0, 0.5)$	$(0.12734, 1.12734)$	$(0.5, 1.9)$	$(0.9, 2.53672)$	$(1.33984, 3.13984)$	$(1.7, 3.85703)$

λ	3.08	3.56	4.04	4.52	5
$\overline{X}^*_\lambda$	3.1	3.6	4.2	5.4	7.2
$\overline{X}^{**}_\lambda$	3.2	3.8	4.4	5.2	7.3
$A^*_{n,1-\varepsilon}$	$(2, 4.24141)$	$(2.5, 4.8)$	$(3.02266, 5.62266)$	$(3.4, 7.49922)$	$(3.9, \infty)$
$A^{**}_{n,1-\varepsilon}$	$(2.17578, 4.47578)$	$(2.6, 5.30078)$	$(3, 6.29922)$	$(3.5, 6.89922)$	$(3.9, \infty)$

$\overline{X}$	0.2	0.5	1	1.5	2	2.5
$C^*_{n,1-\varepsilon}$	$(0.2, 5.01549)$	$(0.2, 1.23146)$	$(0.5723, 1.55582)$	$(0.94242, 1.83114)$	$(1.3332, 1.99052)$	$(1.7398, 2.18731)$
$C^{**}_{n,1-\varepsilon}$	$(0.2, 0.90671)$	$(0.21651, 1.1706)$	$(0.55998, 1.5006)$	$(0.95059, 1.73134)$	$(1.28237, 1.94195)$	$(1.63439, 2.12617)$
$\hat{\lambda}^*_{n,1-\varepsilon}$	0.59407	0.62852	1.10202	1.53244	1.82271	2.10584
$\hat{\lambda}^{**}_{n,1-\varepsilon}$	0.56576	0.63525	1.08371	1.47164	1.76591	2.04629

$\overline{X}$	3	3.5	4	4.5	5
$C^*_{n,1-\varepsilon}$	$(2.11454, 2.33864)$	$(2.47443, 2.50082)$	$(2.62295, 2.88553)$	$(2.73229, 3.23747)$	$(2.87175, 3.59522)$
$C^{**}_{n,1-\varepsilon}$	$(1.99849, 2.25366)$	$(2.32951, 2.43421)$	$(2.56261, 2.71161)$	$(2.69825, 3.00338)$	$(2.80995, 3.34727)$
$\hat{\lambda}^*_{n,1-\varepsilon}$	2.36567	2.64267	3.21074	4.32562	7.51598
$\hat{\lambda}^{**}_{n,1-\varepsilon}$	2.28096	2.5751	3.05617	4.27609	7.6295

6. Conclusions

In the case of parameter estimation we can notice that if we consider the prior distribution Gamma for the parameter λ of the exponential or the Poisson distribution and in the case of parameter ρ of the geometrical distribution the prior distribution uniform on [0,1], the parameter conditioned by the sample expectation tends in probability and in distribution to its true value.

If we consider in the first two cases the prior distribution uniform on $\left[0, \tilde{\lambda}\right]$ the parameter tends in probability and in distribution to $\min\left(\lambda_0, \tilde{\lambda}\right)$, where λ_0 is the true value of λ.

We notice also that the three Bayes estimators are asymptotically equivalent: each of them has the same limit in probability and in distribution. If these limits are the true values of the parameters, these estimators are asymptotically equivalent to those obtained by the maximum likelihood method and by the moments method.

The credible intervals are built in a more simple manner than the classical confidence intervals. For the first ones we do not need statistics and intervals of these statistics: we use only the posterior distribution.

For the Bayes tests we use a cdf with a jump in the value of that parameter for which the null hypothesis is true. This distribution is restricted to the left of λ_0 for the one-sided left tests and to the right of λ_0 for the one-sided right tests. The condition for accepting H_0 is that the posterior probability to have $\theta = \theta_0$ is greater than $1 - \varepsilon$.

The Neyman and Neyman-Bayes inferences are more natural in the case of the exponential and Poisson distribution, because $\lambda \to 0$ means a very slow service (or interarrival time), and $\lambda \to \infty$ means a very fast one.

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