Asset-Liability Management under time-varying Investment Opportunities

Ferstl, Robert and Weissensteiner, Alex

University of Regensburg, University of Innsbruck

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Robert Ferstl
Department of Finance
University of Regensburg, Germany
robert.ferstl@wiwi.uni-regensburg.de

Alex Weissensteiner
Department for Banking and Finance
University of Innsbruck, Austria
alex.weissensteiner@uibk.ac.at

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Abstract

In this paper, we propose multi-stage stochastic linear programming for asset-liability management under time-varying investment opportunities. We use a first-order unrestricted vector autoregressive process to model predictability in the asset returns and the state variables, where — additional to equity returns and dividend-price ratios — Nelson/Siegel parameters are included to account for the evolution of the yield curve. As objective function we minimize conditional value at risk of the shareholder value, i.e. the difference between the mark-to-market value of (financial) assets and the present value of future liabilities. Our results indicate strong hedging demands to mitigate interest rate risks.

Keywords: predictability, stochastic programming, scenario generation, VAR process

JEL Codes: C61, G11

1 Introduction

One of the classical problems in finance is to derive optimal dynamic investment strategies over a finite planning horizon, where the decision uncertainty is modeled with stochastic processes, which drive asset returns and state variables. Early work traces back to the pioneering papers of Samuelson (1969) and Merton (1969, 1971). The use of a geometric Brownian motion, i.e. a constant risk premia, ensures analytical tractability. If such a process is appropriate, investors should hold a constant asset allocation over time for a large class of utility functions. On the one hand this result is in contrast to the prevailing common practice, where for longer investment periods more risk-taking is recommended. On the other hand extensive empirical literature has found predictability in asset returns. A typical specification regresses an independent lagged predictor, e.g. dividend-price ratio, earnings-price ratio, interest rates and spreads, on the stock market return or on the equity premium. Beginning with contributions of Keim and Stambaugh (1986) and Campbell and Shiller (1988) the question was again actively discussed in the last decade (e.g. Cochrane and Piazzesi, 2005; Ang and Bekaert, 2007; Goyal and Welch, 2008; Campbell and Thompson, 2008).

Therefore, many different papers analyzed the impact of such time-varying investment opportunities on the optimal strategy of utility maximizing investors and found deviations from a pure myopic policy. The horizon effects in the asset allocation are called "hedging demands". However, analytical results as e.g. in Kim and Omberg (1996) and Wachter (2002) are the exception rather than the rule under such a setting. The overwhelming part of the works uses numerical methods. In addition to approximate analytical approaches (see e.g. Campbell and Viceira, 1999; Campbell et al., 2003), two main types of numerical solution techniques can be found in the literature. While the first approach discretizes the state space and solves the problem by backward induction (see e.g. Brennan et al., 1997; Barberis, 2000), the second method is simulation-based (e.g. Brandt et al., 2005; Detemple et al., 2003; Koijen et al., 2009).

In this paper we propose stochastic linear programming (SLP) for an asset-liability management (ALM) task where investment opportunities are time-varying. For successful applications of related problems see e.g. Cariño and Ziemba (1994); Gondzio and Kouwenberg (2001); Geyer and Ziemba (2008). Analogous to e.g. Campbell et al. (2003) and Brandt et al. (2005) we use a first-order unrestricted vector-autoregressive process, called VAR(1), to model asset returns and state variables. In the SLP approach the multivariate distribution of the process is approximated with a few mass points (nodes) (see e.g. Høyland and Wallace, 2001; Pflug, 2001; Heitsch and Römisch, 2003). Optimal decisions are then calculated for each node for the scenario tree. Given that the evolution of the whole term structure plays a decisive role in an ALM context a parametric approach seems appropriate to maintain computational tractability (see Boender et al., 2005). The most widely used one by researchers and practitioners is the Nelson and Siegel (1987) exponential component framework. In contrast to no-arbitrage and equilibrium models the entire yield curve is distilled into a parameter vector, which can be interpreted as level, slope and curvature of the term structure (for a discussion see e.g. Diebold and Li, 2006).[1] We use historic es-

[1] Similar to this approach Bertocchi et al. (2005) propose a multi-factor model to incorporate changes in credit risk in the SLP context and to develop immunization strategies for bonds with different credit ratings. Although

timates of Nelson/Siegel parameters with other asset returns and state variables, i.e. in our case log equity returns and log dividend-price ratios, to estimate a VAR(1) process for the scenario generation. To exploit predictability in the returns for the short run, we implement a multi-stage optimization setting. The impulse-response functions give evidence that the impact of shocks to the parameters of the VAR(1) process takes place within the first few periods.

The key parameter to keep under control in an ALM optimization task is the difference between the mark-to-market value of (financial) assets and the present value of future liabilities (or more general cash flows). While the market value of assets depends on the initial endowment of the fund, on past cash in- and outflows and on realized returns, the present value of future cash flows is a function of the current term structure of interest rates. E.g. a negative value indicates that the pension plan is underfunded. This difference can also be interpreted as the benefit owners' shareholder value (SV) of a pension plan. Therefore, we choose to include this parameter in our objective function. Following Pflug (2000) and Rockafellar and Uryasev (2000, 2002) we minimize the conditional value (CVaR) — a coherent risk measure (see Artzner et al., 1999) — of the SV under the constraint that some expected value for the SV is attained at the end of the planning horizon. Further, in addition to the classical budget, inventory and asset allocation constraints we enforce, that the possible drawdown in SV between two stages is above some prespecified level for all events in our scenario tree. This amount can be interpreted as the maximum loss in the SV a sponsor of the plan is willing to suffer.

The contribution to the existing literature is to propose a multi-stage ALM strategy in the context time-varying investment opportunities given by a VAR(1) process and to analyze horizon effects in this setting. The paper is organized as follows: In Section 2 we present the notation and the stochasting linear programming formulation. Section 3 explains the scenario generation procedure, which includes the estimation of Nelson/Siegel parameters, the calibration of the VAR(1) model and the generation of arbitrage-free asset returns. In Section 4 we present a numerical example, discuss the results and provide an economic interpretation of the proposed strategy. Section 5 concludes the paper.

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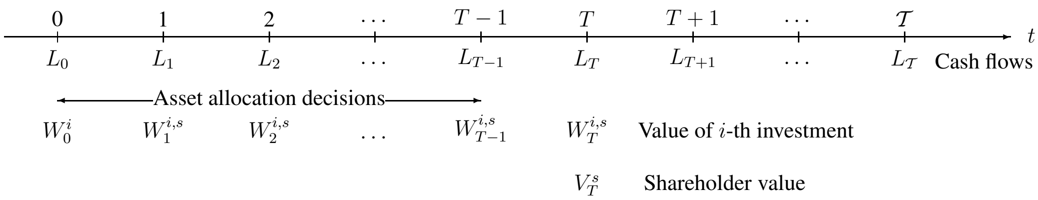
the first three factors explain more than 98% of the total variance, no financial or economic interpretation can be attached to the factors identified from their analysis.

2 Model

2.1 Asset liability management problem

We consider the following asset-liability-management model which is formulated as a multi-stage stochastic linear program with recourse.

Figure 1: Overview of decision model



A company plans to minimize the Conditional Value at Risk (CVaR) of its shareholder value V_T^s at at the end of a planning horizon T by making asset allocation decisions at discrete time stages $t = 0, \ldots, T-1$. It can choose between $i = 1, \ldots, N$ assets where $i = 1$ is an equity and $i = 2, \ldots, N$ are zero-coupon bonds. Further, a deterministic cash flow L_t is assumed at times m_t with $t = 0, \ldots, T, \ldots \mathcal{T}$ that can occur after the end of the planning horizon.[2] This kind of setting is typical for a defined benefit pension fund where the future payouts to its contributors are fixed. The simplest way to hedge the interest rate risk is by constructing a portfolio of zero-coupon bonds with appropriate maturities. However, this approach would not take into account the predictability of asset returns and state variables within the planning horizon. Therefore, we consider a set of scenarios $s = 1, \ldots, S$ and construct a scenario tree consisting of the stochastic returns $\tilde{R}_t^{i,s}$ for each asset. The scenarios are generated with the moment matching approach which uses a VAR(1) model incorporating stock returns, dividend-price ratios and level, slope and curvature of the term structure of interest rates. A detailed description follows in Section 3.

The initial value of the i–th asset before transactions is given by w_0^i and the total amount after investment by W_0^i and $W_t^{i,s}$. The non-negative variables P_0^i and $P_t^{i,s}$ denote the purchases and S_0^i and $S_t^{i,s}$ the sales in each stage. $\mathcal{L}_t^s$ is the scenario-dependent present value at time t of all cash flows occurring in the interval $t < t + \tau \leq \mathcal{T}$.

$$\mathcal{L}_t^s = \sum_{\tau=1}^{\mathcal{T}-t} L_{t+\tau}\delta(\boldsymbol{\beta}_t^s, m_\tau) \quad \forall s, \quad 0 \leq t < \mathcal{T}, \tag{1}$$

[2]Note that t is a time index and m_t is the corresponding time in years.

where the stochastic discount factors

$$\delta(\boldsymbol{\beta}_t^s, m_\tau) = e^{-y(\boldsymbol{\beta}_t^s, m_\tau) m\tau}$$

are calculated from a parametric spot rate function in (15) depending on the yield curve parameter vector and the time to maturity $m_\tau = m_{t+\tau} - m_t$.[3]

2.2 Stochastic linear programming formulation

The split variable formulation of the multi-stage stochastic program with recourse is given in (2)-(13). In the objective function (2) we minimize the Conditional Value at Risk CVaR_α of the shareholder value with confidence level $\alpha \in]0,1[$, which is a convex function of the assets in the portfolio. Further, it is a coherent risk measure as shown in Pflug (2000). For discrete distributions CVaR_α can be reduced to a linear programming formulation that follows Rockafellar and Uryasev (2000, 2002).

$$\mathrm{CVaR}_\alpha = \min \left\{ \phi + \frac{1}{1-\alpha} \sum_{s=1}^{S} p^s \psi_T^{+s} \right\} \tag{2}$$

subject to:

$$\sum_{i=1}^{N} W_t^{i,s} > 0 \quad \forall s \quad 0 \le t \le T-1 \tag{3}$$

$$l^i \le \frac{W_t^{i,s}}{\sum_{i=1}^{N} W_t^{i,s}} \le u^i \quad \forall i, s, \quad 0 \le t \le T-1 \tag{4}$$

$$W_0^i = w_0^i + P_0^i - S_0^i \quad \forall i \tag{5}$$

$$\sum_{i=1}^{N} P_0^i (1 + \tau_P^i) = \sum_{i=1}^{N} S_0^i (1 - \tau_S^i) + L_0 \tag{6}$$

$$W_t^{i,s} = R_t^{i,s} W_{t-1}^{i,s} + P_t^{i,s} - S_t^{i,s} \quad \forall i, s, \quad 1 \le t \le T-1 \tag{7}$$

$$\sum_{i=1}^{N} P_t^{i,s} (1 + \tau_P^i) = \sum_{i=1}^{N} S_t^{i,s} (1 - \tau_S^i) + L_t \quad \forall i, s, \quad 1 \le t \le T-1 \tag{8}$$

$$V_t^s = \sum_{i=1}^{N} W_t^{i,s} + \mathcal{L}_t^s \qquad \forall s, \quad 0 \le t \le T-1 \tag{9}$$

$$V_T^s = \sum_{i=1}^{N} W_T^{i,s} + L_T + \mathcal{L}_T^s \qquad \forall s \tag{10}$$

[3] Also here, τ is an index and m_τ measures the maturity of the future cash flow $L_{t+\tau}$ as the difference (in years) between $m_{t+\tau}$ and m_t.

$$V_t^s \delta(\boldsymbol{\beta}_{t-1}^s, m_\tau) - V_{t-1}^s + \gamma \geq 0 \quad \forall s, \quad 1 \leq t \leq T, \quad \tau = 1 \tag{11}$$

$$\psi_T^{+s} = -V_T^s - \phi + \psi_T^{-s} \qquad \forall s \tag{12}$$

$$\sum_{s=1}^{S} p^s V_T^s \geq \theta \tag{13}$$

In the optimal solution, ϕ represents the Value at Risk VaR_α and the second term accounts for the expected shortfall ψ_T^{+s} below the VaR_α for a prespecified confidence level α. The probabilities of the different scenarios are given by p^s. The variable $W_t^{i,s}$ denotes the mark-to-market value of the i-th asset in scenario s at time t. We restrict the total wealth in (3), i.e. the sum over all assets, to be positive in all time stages were asset allocation decisions are taken. However, the mark-to-market value of an individual asset can become negative and lower bounds l^i and upper bounds u^i are defined in (4).

The first-stage decision variables have to fulfill the inventory equations in (5), forcing the mark-to-market value of each asset to equal the initial holdings w_0^i plus purchases minus sales. The budget equations in (6) include proportional transaction costs τ_P^i and τ_S^i for purchases and sales of each asset and the deterministic cash flow. While the second-stage inventory equations in (7) also account for the gross returns $R_t^{i,s}$ on the holdings of the previous period $W_{t-1}^{i,s}$, the budget equation in (8) contains the scenario-dependent decision variables for purchases and sales. Further details on the return calculation can be found in (19) in Section 3.3.

We denote the shareholder value in (9) as sum of the total mark-to-market values of the assets plus the present value of all future cash flows. Note that the cash flow at the current stage is already included through the inventory equations, but these are only defined until $T-1$. Therefore, we add L_T to the shareholder value at the end of the planning horizon in (10). Constraint (11) ensures that the maximal drawdown in the shareholder value between two adjacent periods is above a given level γ. This leads to a stronger consideration of the interest rate risk within the planning horizon, as, in addition to the asset returns, also the present value of the cash flows is affected by the uncertain changes in the yield curve.

Given the shareholder value at stage T in (10), the portfolio shortfall in excess of Value at Risk[4] used in the objective function is $\psi_T^{+s} = \max[0, -W_T^s - \phi]$. To determine the value of the maximum operator in the linear programming formulation we introduce two non-negative auxiliary variables ψ_T^{+s} and ψ_T^{-s}. In (13) we ensure that the expected shareholder value exceeds some prespecified level θ, which we set to:

$$\theta = \left[\sum_{i=1}^{N} w_0^i + \sum_{t=0}^{T-1} L_t \delta(\boldsymbol{\beta}_0, m_t) \right] \delta(\boldsymbol{\beta}_0, m_T)^{-1} e^{\nu m_T} + L_T + \mathbb{E}[\mathcal{L}_T^s]. \tag{14}$$

[4]Note that with this formulation we minimize the negative value of VaR_α and CVaR_α.

The term in brackets takes the initial asset allocation and discount all cash flows that occur at future decision stages to $t = 0$. Then, we calculate the final value at $t = T$, where the required excess return is denoted by ν. The remaining two terms account for the L_T at the end of the planning horizon and the expected present value of all remaining cash flows. In Section 4 we derive an efficient frontier of risk return combinations by varying the level of θ.

Further, so-called "non-anticipativity constraints" are imposed to guarantee that a decision made at a specific node is identical for all scenarios leaving that node.

3 Modeling Uncertainty

3.1 Term structure of interest rates

In an ALM context, where the main objective is controlling the shareholder value, the term structure of interest rates plays a central role. While the evolution of the yield curve influences returns for the different bond holdings, it determines also the present value of future cash flows. We propose to use the Nelson/Siegel model here for at least two reasons. On the one hand, this parsimonious parametric model can represent the entire yield curve by only four parameters, restricting the size of our scenario tree and ensuring computational tractability. On the other hand, some extensions which include the Nelson/Siegel model as a special case may not be superior in out-of-sample forecasting due to their potential overfitting of in-sample data (see e.g. Diebold and Li, 2006). The three-factor model for the spot rates can be written as:

$$y(\boldsymbol{\beta}_t, m) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda_t m}}{\lambda_t m}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda_t m}}{\lambda_t m} - e^{-\lambda_t m}\right), \tag{15}$$

where $y(\boldsymbol{\beta}_t, m)$ indicates the (continuously compounded) spot rate for maturity m at stage t given the parameter vector $\boldsymbol{\beta}_t = [\beta_{1,t}, \beta_{2,t}, \beta_{3,t}]^\top$. Given this fixing of the loadings, the factors $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ can be interpreted as the level, slope and curvature of the term structure of interest rates. It can be seen that $\beta_{1,t}$ determines the long-term level of the spot rates as $y(\boldsymbol{\beta}_t, \infty) = \beta_{1,t}$, while the instantaneous yield depends on both the level and the slope factor by $y(\boldsymbol{\beta}_t, 0) = \beta_{1,t} + \beta_{2,t}$. This is due to the following facts: The factor loading of $\beta_{2,t}$ starts form a value of one and decreases asymptotically to zero for long maturities. The factor loading for the curvature is hump-shaped, approximating the value of zero for very short and very long maturities. The parameter λ_t determines the decay rate[5] and the time where the

[5] A small value of λ_t ensures a better fit for long maturities, while a large value enhances the fit for short maturities.

factor loading of the curvature archives its maximum. Following Diebold and Li (2006) we fix λ_t to 0.0609, which maximizes the factor loading at exactly 30 month. In this way, the estimation to the remaining parameters $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ simplifies to an ordinary least square regression (OLS) with the advantage of numerical stability.[6] These estimated Nelson-Siegel parameters are then included in the estimation of the VAR(1) process presented below.

3.2 Time-varying investment opportunities

We model time-varying investment opportunities with an unrestricted, stationary VAR(1)-process (for an application in asset allocation decision see e.g. Barberis, 2000; Campbell et al., 2003, 2004; Brandt et al., 2005), where stationarity is referred to time-invariant expected values, variances, and covariances. In this paper here we use the following $(K \times 1)$ parameter vector $\boldsymbol{\xi}_t$ (with $K = 5$):

$$\boldsymbol{\xi}_t = \begin{bmatrix} r_t^1 \\ d_t - p_t \\ \boldsymbol{\beta}_t \end{bmatrix},$$

where $r_t^1 \equiv \log(R_t^1)$ refers to the log equity return, $d_t - p_t$ to the log dividend-price ratio, and $\boldsymbol{\beta}_t$ to the (3×1) vector of the Nelson-Siegel parameters. The idea behind a vector-autoregressive process is that an economic variable is not only related to its predecessors in time but, in addition, it depends linearly on on past values of other variables. The functional form of the VAR(1) process can be written as:

$$\boldsymbol{\xi}_t = \mathbf{c} + \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{u}_t, \tag{16}$$

where $\mathbf{c}$ is the $(K \times 1)$ vector of intercepts, $\mathbf{A}$ is the $(K \times K)$ matrix of slope coefficients and $\mathbf{u}_t$ the $(K \times 1)$ vector of i.i.d innovations with $\mathbf{u} \sim N(0, \boldsymbol{\Sigma}_u)$. The covariance of the innovations $\boldsymbol{\Sigma}_u$ is given by $\mathbb{E}(\mathbf{u}\,\mathbf{u}^\top)$. Thus, we allow the shocks to be cross-sectionally correlated, but assume that they are homoskedastic and independently distributed over time. If all eigenvalues of $\mathbf{A}$ have modulus less than 1, as in our case below, the stochastic process in equation (16) is stable with unconditional expected mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Gamma}$ for the steady state at $t = \infty$ of (see e.g. Lütkepohl, 2005):

$$\boldsymbol{\mu} := (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c} \tag{17}$$

$$\mathrm{vec}(\boldsymbol{\Gamma}) := (\mathbf{I} - \mathbf{A} \otimes \mathbf{A})^{-1}\mathrm{vec}(\boldsymbol{\Sigma}_u), \tag{18}$$

[6] We don't have to optimize a non-linear objective function with multiple local optima.

where $\mathbf{I}$ refers to the identity matrix, $\otimes$ the Kronecker product and vec transforms a $(K \times K)$ matrix into an $(KK \times 1)$ vector by stacking the columns.

To estimate the intercepts and slope parameters of the VAR(1) process via OLS we use quarterly data from 1997.Q3 to 2007.Q4. Stock returns, which refer to the S&P 500 index, and the corresponding dividend-price ratios are taken from the Goyal and Welch (2008) data set, while the vector for the Nelson-Siegel parameters $\boldsymbol{\beta}_t$ are estimated from US spot rates provided by the Federal Reserve Bank. The autoregressive order of one in our process was selected by the Schwarz Criterion (also known as Bayesian Information Criterion). The corresponding parameters are reported in Table 1 (values for the t-statistics in parenthesis). The

Table 1: VAR(1) parameters and t-statistics for quarterly data 1997.Q3–2007.Q4

	c	r^1_{t-1}	$d_{t-1} - p_{t-1}$	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	R^2
r^1_t	0.3649 (1.6865)	-0.0641 (-0.6250)	0.0722 (1.8580)	-0.7643 (-0.6823)	-1.0413 (-1.5652)	-0.1791 (-0.4084)	0.0920
$d_t - p_t$	-0.1352 (-0.6200)	0.0970 (0.9379)	0.9658 (24.6577)	-0.2254 (-0.1997)	0.8352 (1.2461)	-0.2084 (-0.4716)	0.9631
$\beta_{1,t}$	0.0163 (0.3082)	0.0599 (2.3858)	0.0036 (0.3795)	0.8532 (3.1141)	0.3018 (1.8548)	-0.0714 (-0.6659)	0.7807
$\beta_{2,t}$	0.0034 (0.0719)	-0.0431 (-1.9419)	0.0002 (0.0278)	0.0401 (0.1658)	0.5919 (4.1165)	0.0655 (0.6912)	0.6181
$\beta_{3,t}$	-0.0087 (-0.0765)	-0.1190 (-2.2043)	-0.0039 (-0.1921)	0.1179 (0.2000)	-0.4921 (-1.4065)	1.0401 (4.5095)	0.8098

sample period of 80 quarters in our data set with five regression parameters and a confidence level of 95% gives a critical (absolute) t–value of 1.99. In line with current literature, it can be seen that the dividend-price ratio with a coefficient of 0.97 has very high persistent dynamics and — compared to other regressors — shows the greatest t-value for predicting equity returns (in Campbell et al., 2003 this value is equal to 2.32, and Brandt et al., 2005 report 0.87). A Granger-Causality test confirms this finding. The R^2 equals 9.2% (in Campbell et al., 2003 R^2 for equities is equal to 8.6%). Further, it can be seen that the first lags of the Nelson/Siegel parameters with t-values of 3.11, 4.12 and 4.51 are statistically significant in forecasting each of them.

In Table 2 we illustrate quarterly standard deviations (multiplied by 100) on and cross correlations of residuals above the main diagonal. As in Campbell et al. (2003) and the literature mentioned therein, unexpected log excess stock returns are highly negatively correlated with shocks to the log dividend–price ratio. All residuals pass the multivariate normality test (with Cholesky orthogonalization).

In Table 3 we indicate the unconditional expected mean $\boldsymbol{\mu}$ for the VAR parameters. The expected simple return per annum for equities equals 7.20%, while the term structure of interest rates (continuously compounded) given by the Nelson/Siegel parameters is illustrated

Table 2: Cross correlations and standard deviations of residuals for quarterly data 1997.Q3–2007.Q4

	r^1_{t-1}	$d_{t-1} - p_{t-1}$	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$
r^1_t	6.7203	-0.9829	0.0743	0.0202	-0.1473
$d_t - p_t$	–	6.7709	-0.0630	-0.0165	0.1219
$\beta_{1,t}$	–	–	1.6437	-0.9091	-0.9697
$\beta_{2,t}$	–	–	–	1.4526	0.8513
$\beta_{3,t}$	–	–	–	–	3.5343

in Figure 2. Furthermore, the impulse-response functions give evidence that the impact of

Table 3: Unconditional expected values $\boldsymbol{\mu}$ for the steady state

r^1	$d - p$	β_1	β_2	β_3
0.017374	-4.08700	0.011995	0.022203	0.105590

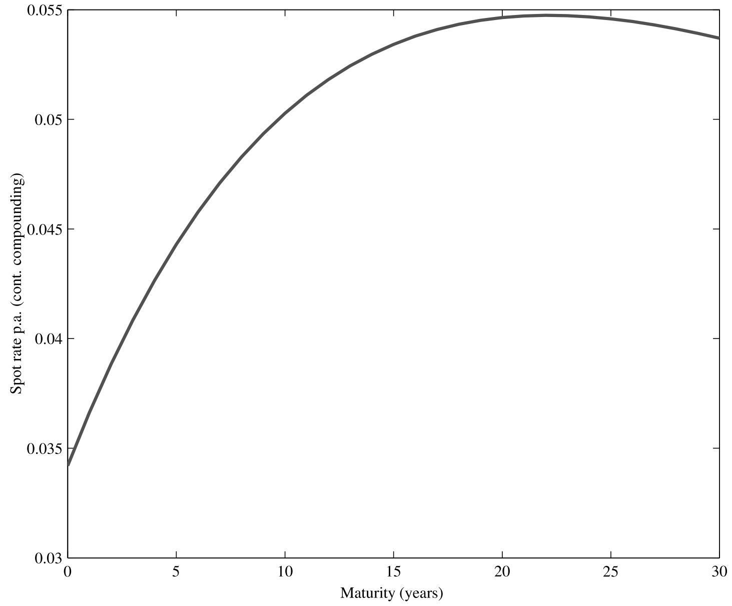
shocks to the parameters of the VAR(1) process takes place at the first few periods (quarters). Therefore, we try to exploit predictability in the near future by setting four decision stages with intervals of 3 months each (i.e. re-allocations at $m_t \in \{0, 0.25, 0.5, 0.75\}$).

3.3 Scenario Generation

For the scenario generation, additionally to the parameter estimation in Section 3.2, we have to set the starting values of our VAR(1)-process. For our exhibition here two choices are suitable: last realized parameters versus steady state values. On the one hand, for practical applications the SLP literature proposes a rolling-forward approach (see e.g. Dempster et al., 2003), where in every stage the process parameters are re-estimated and a new scenario tree is generated. In such a context, to predict and exploit returns, the starting values should coincide with the last realized parameters. On the other hand, numerical results based on this special setting do not allow to draw general conclusions at all. Therefore, it is not unusual to start the investigation with the unconditional expected values of the estimated process (e.g. Campbell et al., 2003). We follow this second approach by starting from the steady state in our numerical part.

The multivariate process in equation (16) evolves in discrete time, and the underlying probability distributions are approximated with a few mass-points in terms of a so-called scenario tree. Although different approaches have been discussed in the literature (see e.g.

Figure 2: Term structure of interest rates for the steady state



Pflug, 2001; Heitsch and Römisch, 2003), in our paper here we focus on the technique proposed by Høyland and Wallace (2001) and Hoyland et al. (2003) to match the first four conditional moments (including the correlations) of the process. This method uses an iterative procedure that combines simulation, Cholesky decomposition and various transformations to achieve the correct correlations without changing the marginal moments. More nodes emanating from on predecessor node (i.e. a higher branching factor) facilitate the matching of moments but increase the number of scenarios. In this application we use a constant branching factor of ten with four decision stages, resulting in a tree with a total number of scenarios S equal to $10,000$ $(= 10^4)$.

To make a long story short: Arbitrage opportunities are present in a market whenever investors — without investing own money and without taking risk — have a probability greater than zero to earn a positive portfolio return. Economists agree that such a condition is not a sound basis for financial models, as every optimization task without constraints on the asset allocation will become unbounded (infeasible) for the primal (dual) problem. With such constraints the asset allocation will be biased (see Geyer et al., 2009). Therefore, to avoid such opportunities in the generated scenarios we apply the arbitrage-check proposed by Klaassen (2002), which accounts for the return of traded assets in the different successor nodes. However, the simulated process in (16) does not only model asset returns, but includes also state variables in form of the dividend-price ratio as well as the Nelson/Siegel parameter vector. The last one is important for two reasons: First, the parameters determine the whole term structure of interest rates, and in this way the present value of the given liabilities at each stage and each scenario. Second, changes in the yield curve drive the realized gross returns $R_t^{i,s}$ in (7) of the different bond holdings. To check for arbitrage opportunities during the construction of our scenario tree we have to account also for these potential bond returns.

For zero-coupon bonds these returns can be easily calculated. We define $P^s(t,m)$ as the market price of the m-year maturity zero bond at stage t and scenario s. Then it follows, that the gross return, i.e. R, a short time period Δt later is given by:

$$R^s(t+\Delta t, m) = \frac{P^s(t+\Delta t, m-\Delta t)}{P^s(t,m)} = \frac{e^{my(\boldsymbol{\beta}_t^s, m)}}{e^{(m-\Delta t)y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t)}}, \tag{19}$$

where $y(\boldsymbol{\beta}_t^s, m)$ defines the term structure at stage t and scenario s, see (15). Moving to log holding-period returns results in:

$$\begin{aligned} r^s(t+\Delta t, m) &= my(\boldsymbol{\beta}_t^s, m) - (m-\Delta t)y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t) \\ &= \Delta t\, y(\boldsymbol{\beta}_t^s, m) - (m-\Delta t)\left(y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t) - y(\boldsymbol{\beta}_t^s, m)\right). \end{aligned} \tag{20}$$

Equation (20) shows that the continuously compounded return is a weighted sum given by the yield at the beginning of the period $y(\boldsymbol{\beta}_t^s, m)$ multiplied by the holding period Δt minus the yield change $\left(y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t) - y(\boldsymbol{\beta}_t^s, m)\right)$ multiplied by the remaining maturity of this zero bond $(m - \Delta t)$. A similar formula can be used to approximate the return of a coupon-bearing bond by substituting the maturity m with the duration of the bond, see Campbell et al. (1997).

We calculate these scenario-dependent returns of the different bonds[7] using (20) and include them, together with the returns modeled directly by the VAR(1) process (e.g. the equity returns), in the arbitrage-check proposed by Klaassen (2002). In this way we ensure an arbitrage-free scenario tree.

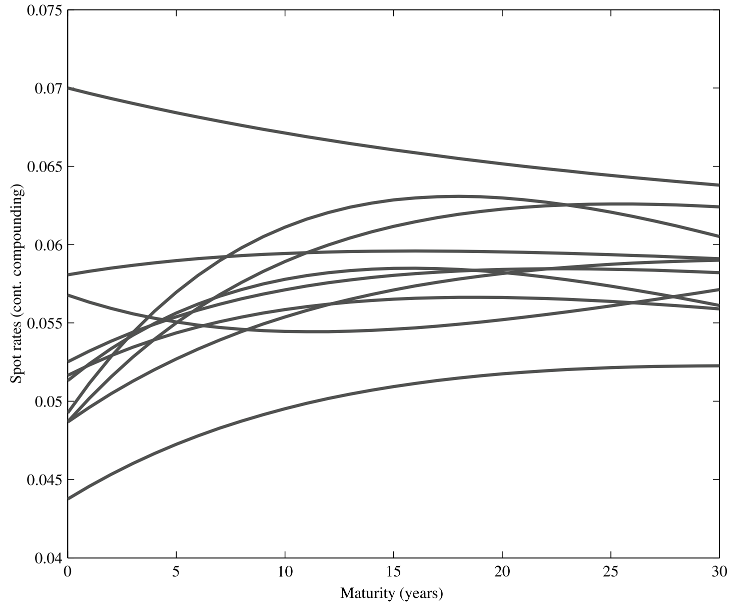
The evolution of the modeled yield curve in our scenario tree satisfies the most important stylized facts (see e.g. Diebold and Li, 2006):

1. The unconditional expected yield curve is increasing and concave, see Figure 2.

2. The yield curve assumes a variety of shapes through time, including upward sloping, downward sloping, humped, and inverted humped. In Figure 3 we illustrate different term structures of interest rates implied by e.g. the scenarios 51–60 at stage $m_t = 0.75$. Further, Table 4 reports quantile values of spot rates with different maturities for cumulative probabilities p_z, with $p_z \in \{0.025, 0.5, 0.975\}$, over all scenarios at $m_t = 0.75$. The upper and lower bounds give evidence that, although a variety of shapes are

[7] In the setting of Sections 4 we use bonds with three different maturities: 0.25, 5 and 10 years.

Figure 3: Term structure of interest rates for scenario 51–60 at $t = 0.75$



possible, the evolution of the term structure is well-behaved and economically sound.

3. Yield dynamics, reflected by the slope parameter for $\beta_{1,t-1}$, are much more persistent than spread dynamics given by $\beta_{2,t-1}$. This can be easily verified from Table 1, where the estimated parameters for $\beta_{1,t-1}$ and $\beta_{2,t-1}$ are 0.8532 versus 0.5919.

Table 4: Quantile values for the term structure of interest rates at $t = 0.75$

Maturity	1	5	10	15	20	25	30
$p_z = 0.025$	1.4803	2.7717	3.6415	4.0571	4.2575	4.3188	4.2879
$p_z = 0.5$	3.6638	4.4138	5.0169	5.3405	5.4623	5.4533	5.3649
$p_z = 0.975$	5.8811	6.1633	6.4625	6.6733	6.7106	6.6238	6.4721

4 Results

4.1 Initial setting

In this section, we give a numerical example with a cash flow structure typical for a defined benefit pension scheme. A company with a planning horizon of one year accumulates cash inflows (funding period) and takes asset allocations at the beginning of each quarter. After that a long period of cash outflows follows, see Figure 4.

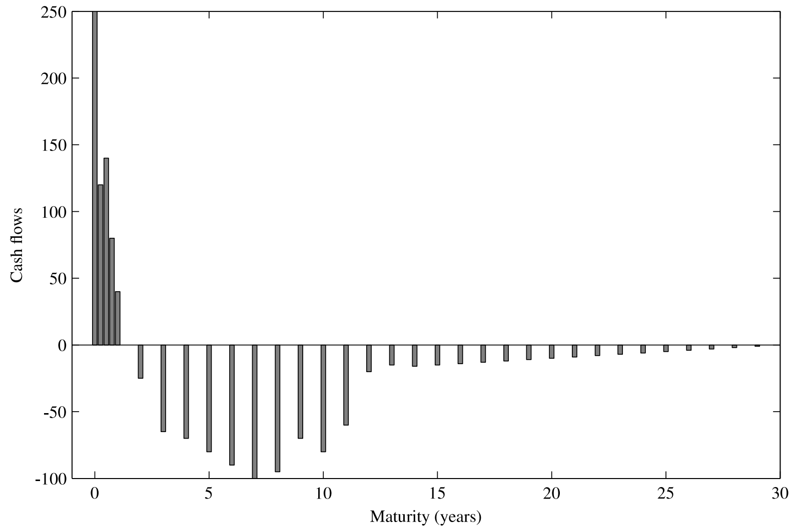


Figure 4: Cash flow structure

The company can choose from the following assets: an equity (which might be a broad index) and three zero-coupon bonds with maturities of 3 month, as well as 5 and 10 years. The short-term zero-coupon bond can be seen as an equivalent for a cash account. As its maturity matches perfectly the rebalancing intervals between two stages, the riskless return is known at the beginning of the period and equals the spot rate for that maturity. We use transaction costs for purchases and sales of $\tau_P^i = \tau_S^i = [1\%, 0\%, 0.5\%, 0.5\%]$. These are set to zero for the three-month bond. Further, the company has no initial holdings in any of the assets. For the first numerical experiment, we use the lower and upper bounds $l^i = [0\%, -30\%, 0\%, 0\%]$ and $u^i = [130\%, 100\%, 130\%, 130\%]$ in the asset allocation constraint (4). With such a setting, where modest leverage with a short-position in the riskless bond is allowed, we mimic a *prudent* version of the well-known "130/30" strategy. Compared to the traditional long-only approach, long-short strategies expand alpha opportunities for active portfolio management. Different contributions over the last decade report benefits from such an extensions (see e.g. Grinold and Kahn, 2000; Johnson et al., 2007).

By setting $\nu = 1.5\%$ we determine a feasible target θ equal to 16.97 using (14). The initial shareholder value is 18.82. The bound on the maximum drawdown of the shareholder value in (11) is set to $\gamma = 35$.

The scenario generation procedure was implemented in MATLAB and we formulated the optimization problem in AMPL. The solution time on a MacBook Pro 2.4 GHz Intel Core 2 Duo / 4 GB RAM with MOSEK was approximately 57 seconds with the interior-point solver.

Figure 5 shows the distribution of the shareholder value at the end of the planning horizon. In (13) we have constrained the mean to be greater or equal than the target θ. The first-stage solutions of the SLP are $W_0^i = [10.91\%, -30.00\%, 0.00\%, 119.09\%]$ with $\mathrm{CVaR}_{0.95} = 34.59$ and $\mathrm{VaR}_{0.95} = 24.13$. From Figure 5 we can also see the minimum shareholder value, i.e. the worst possible scenario for our company, which is at -109.79. As the risk measures VaR_α and CVaR_α can become positive or negative, it can be convenient to calculate so-called deviation measures. These were introduced in Rockafellar et al. (2006), and indicate the difference between the risk measure and the mean of the distribution. The CVaR Deviation $\mathrm{CVaR}^{\Delta}_{0.95} = 51.56$ and the VaR Deviation $\mathrm{VaR}^{\Delta}_{0.95} = 41.10$ can easily be calculated as shown in Figure 5. By definition these are always positive values and should be used for example in Sharpe-like ratios.

Figure 5: Histogram of shareholder value V_T^s at the end of planning horizon

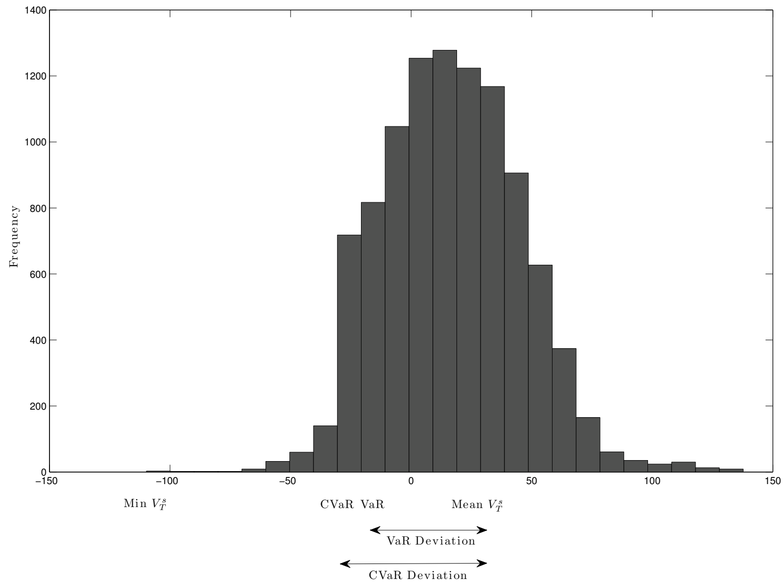


Figure 6 shows the efficient frontier of the expected final SV (i.e. target θ) versus the CVaR and the corresponding frontier portfolios. We can see that, even for low targets our company prefers to hold the maximum allowed short position in bond 1. We motivate this by hedging demands in our asset-liability management: Given the long-term payouts illustrated in Figure 4 one of the main sources of risk to the SV are term structures at a low level, which increases the present value of future liabilities $\mathcal{L}_t^s$ and decreases the SV in equation (9). To hedge against such scenarios the optimal policy proposes a strong exposure to the long-term bond, which will benefit from low interest rates.

Table 5 indicates that even for a high target of $\theta = 20$ the minimum shareholder value is bounded by maximum drawdown constraint, i.e. with an initial shareholder value of 18.82 the minimum possible value after four periods with $\gamma = 35$ is approximately 122 (including interest rate effects). Figure 6 shows that the company increases the equity position and lowers the holdings of the long-term bond when the target increases.

Table 5: First-stage asset allocation with shareholder value constraint

θ	$\mathrm{CVaR}_{0.95}$	$\mathrm{VaR}_{0.95}$	Min V_T^s	$W_0^1\%$	$W_0^2\%$	$W_0^3\%$	$W_0^4\%$
16.00	34.20	23.55	-96.38	3.59	-30.00	0.00	126.41
17.00	34.61	24.12	-110.42	11.19	-30.00	0.00	118.81
18.00	35.66	24.97	-117.49	17.31	-30.00	0.00	112.69
19.00	37.53	26.04	-119.53	24.84	-30.00	0.00	105.16
20.00	40.39	28.18	-121.93	28.11	-30.00	0.00	101.89

4.2 Sensitivity analysis

4.2.1 Maximum drawdown constraint

The maximal drawdown of SV is restricted by constraint (11). This amount is the highest potential loss a sponsor of the pension plan is willing or able to suffer during one period. In our base case of Section 4.1 we set $\gamma = 35$. Figure 7 illustrates a slack variables analysis for each rebalancing period and each scenario. If the slack variable equals zero then the boundary is active and will condition the solution. We can see that the constraint becomes binding from $t = 0.75$ on as the uncertainty in the asset returns increases. Further, to study the impact of this shareholder value constraint on the optimal investment policy we show in Table 6 results for the optimization task when (11) is disabled. As expected, without binding boundaries a better objective value (i.e. a lower CVaR_α) is found for all targets. However, compared to Table 5 the minimum possible SV at the end of the planning horizon (Min V_T^s) worsens due to unfavorable scenarios.

4.2.2 Constant-mix strategies

As one might ask for the practical relevance of stochastic dynamic programming in an ALM context, in this section here we analyze results for the setting of Section 4.1 when classical constant-mix strategies are applied. To motivate such a comparison we consider a pension fund manager, who erroneously neglects future cash flows (and the interest rate risk therein)

Table 6: First-stage asset allocation without shareholder value constraint

θ	$\mathrm{CVaR}_{0.95}$	$\mathrm{VaR}_{0.95}$	Min V_T^s	$W_0^1\%$	$W_0^2\%$	$W_0^3\%$	$W_0^4\%$
16.00	34.19	23.55	-99.18	3.39	-30.00	0.00	126.61
17.00	34.45	23.78	-125.67	9.92	-30.00	0.00	120.08
18.00	35.13	24.41	-136.02	15.05	-30.00	0.00	114.95
19.00	36.23	25.05	-147.13	22.18	-30.00	0.00	107.82
20.00	37.67	25.47	-165.62	24.85	-30.00	0.00	105.15

as well as predictability in asset returns, or is faced with too tight bounds on the allowed asset allocation. Table 7 shows the results for different constant-mix strategies. As one

Table 7: First-stage asset allocation of alternative strategies

Strategy	θ	$\mathrm{CVaR}_{0.95}$	$\mathrm{VaR}_{0.95}$	Min V_T^s	$W_0^1\%$	$W_0^2\%$	$W_0^3\%$	$W_0^4\%$
Equal weights	11.96	58.15	43.78	-108.36	25.00	25.00	25.00	25.00
Bond 1 only	6.85	64.23	49.57	-115.30	0.00	100.00	0.00	0.00
40/60	13.59	65.94	49.93	-116.41	40.00	20.00	20.00	20.00
30/70	12.50	60.46	45.60	-111.04	30.00	23.33	23.33	23.34

can immediately verify, these polities are suboptimal in a risk-return sense. Although the expected shareholder value θ is in all cases below our targets in Table 5, the $\mathrm{CVaR}_{0.95}$ and also the maximal loss in SV is always higher. In this way pension fund managers may be warned to copy well-established strategies from pure asset management.

4.2.3 Pure asset management

In the results of Table 5 we find for all levels of θ a short selling position of the riskless bond W_0^2 (in all cases the lower bound of -30% became active) and a heavy investment in the long-term bond W_0^4. We motivate these hedging demands as a consequence of active ALM: The huge risk of low term structures induced by the long time series of payouts can be mitigated by long-term bond investments. Further, equity investments W_0^1 are taken into account only when a higher expected SV is needed.

In the following, we compare these ALM-results with a pure asset management approach. To ensure comparability we take the same setting of the base case from Section 4.1 without future cash flows (i.e. $L_t = 0,\ \forall\, t > 0$) and without constraints of the scenario-dependent maximum loss in the SV, see (11). Our findings are reported in Table 8. Given that the interest-rate risk for payouts is no longer present in the optimization task, the short- and medium term bonds become attractive investments for low levels of θ. By increasing this target more wealth is allocated to the assets with higher expected returns, i.e. to long-term bonds and equities.

Table 8: First-stage asset allocation without external cash flows

θ	$\mathrm{CVaR}_{0.95}$	$\mathrm{VaR}_{0.95}$	Min V_T^s	$W_0^1\%$	$W_0^2\%$	$W_0^3\%$	$W_0^4\%$
260.00	-259.45	-259.65	256.03	0.00	21.53	78.47	0.00
262.00	-259.10	-259.33	254.23	0.00	-14.42	114.42	0.00
264.00	-257.80	-258.22	239.75	2.27	-30.00	99.41	28.32
266.00	-255.53	-256.54	206.52	2.81	-30.00	0.00	127.19
268.00	-251.26	-254.26	184.43	7.79	-30.00	0.00	122.21
270.00	-243.25	-249.12	177.39	36.59	-30.00	0.00	93.41

4.2.4 Confidence level

The confidence level α in (2) determines the left tail of the SV distribution, which is included in the objective function. Its choice clearly depends on the risk aversion of the decision taker. Here we check the impact of this parameter on the optimal policy for our base case in Section 4.1. From Table 9 we can see that — as expected — a higher confidence level with more extreme scenarios increases CVaR_α and VaR_α. Although the first-stage asset allocation is rather stable, an α greater than 0.9 reduces the allocation to the risky equities and increases the worst-case SV. Further, the smooth results with the increasing confidence level give evidence that our scenario generation of Section 3.3 is also well-suited to model the distribution in the heavy tails.

Table 9: First-stage asset allocation with shareholder value constraint

α	CVaR_α	VaR_α	Min V_T^s	$W_0^1\%$	$W_0^2\%$	$W_0^3\%$	$W_0^4\%$
0.80	16.74	1.29	-117.36	10.30	-30.00	0.00	119.70
0.85	21.04	6.68	-118.76	10.97	-30.00	0.00	119.03
0.90	26.57	13.89	-118.89	11.50	-30.00	0.00	118.50
0.95	34.59	24.13	-109.79	10.91	-30.00	0.00	119.09
0.99	48.29	41.27	-70.23	0.00	-30.00	0.00	130.00

Figure 6: Risk-return tradeoff and frontier portfolios

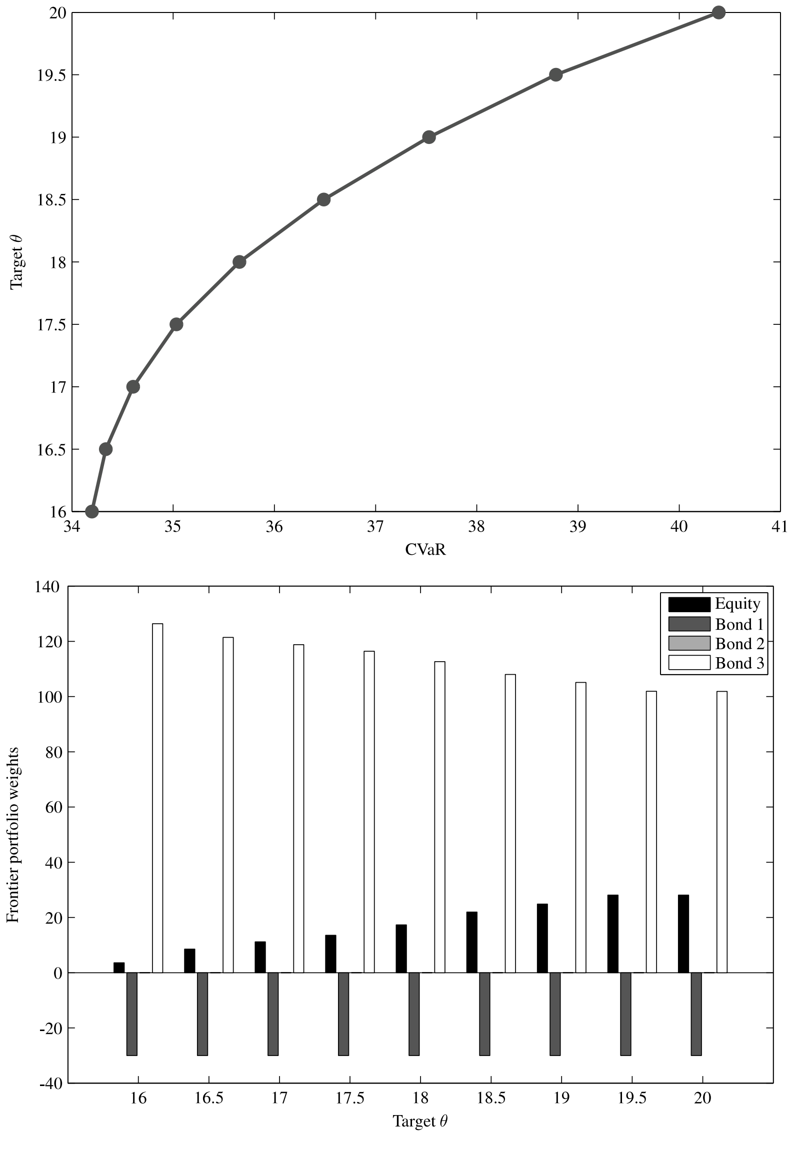
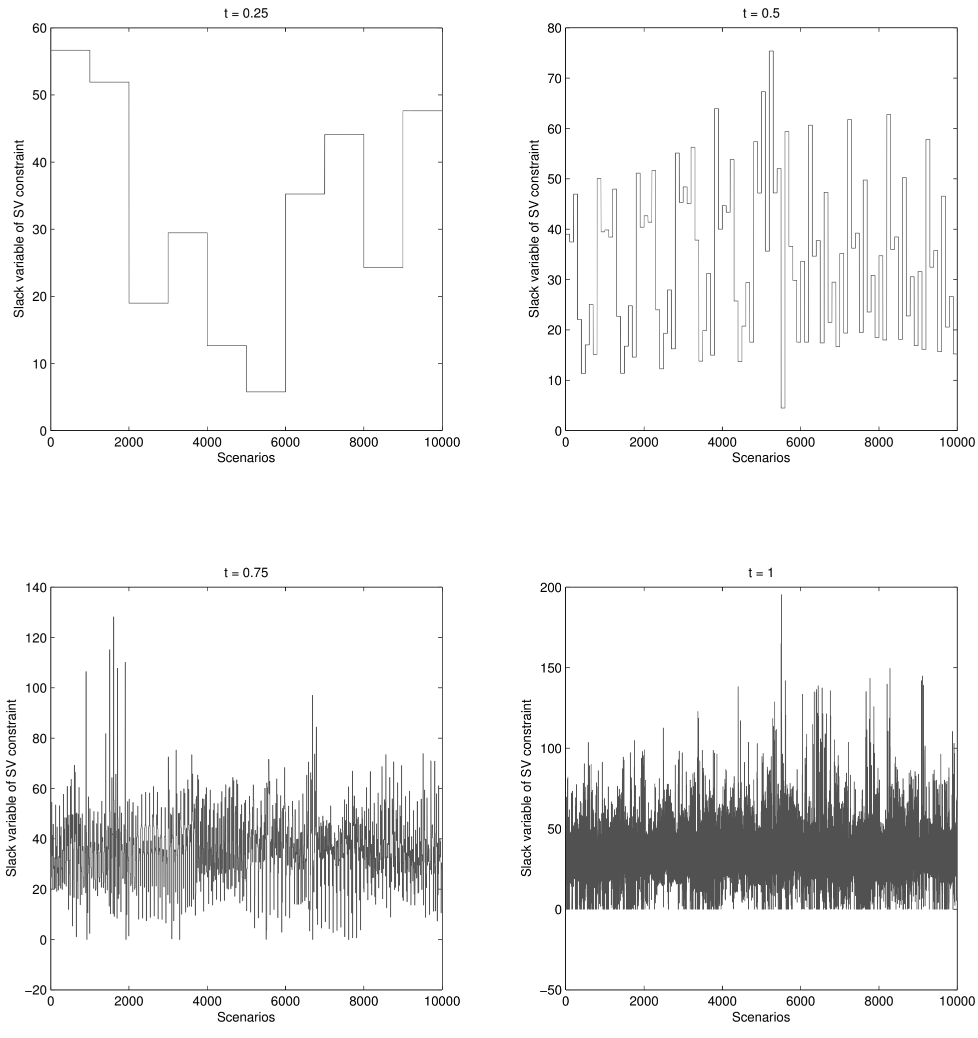


Figure 7: Slack variables for shareholder value constraint



5 Conclusion

In this paper we proposed a multi-stage dynamic strategy for ALM under time-varying investment opportunities. The uncertainty within the optimization task was driven by a first-order unrestricted vector autoregressive process, which determines the equity returns and state variables (i.e. the dividend-price ratio as predictor and the Nelson/Siegel coefficients for level, slope and curvature of the term structure). We estimated the VAR parameters from the Goyal and Welch (2008) and the Federal Reserve Bank dataset. Our analysis shows that the VAR-process is stable and is in line with previous results of the literature: As in Campbell et al. (2003) the dividend-price ratio indicates a high persistence and shows the greatest t-value in predicting equity returns. Further, in analogy to other studies, unexpected log excess returns are highly negatively correlated with shocks to the log dividend-price ratio. The unconditional expectation for the steady state gives economically sound values both for the equity returns and the term structure of interest rates.

In the SLP approach the multivariate distribution of the stochastic process is approximated with a few mass points. We applied the moment-matching algorithm proposed by Høyland and Wallace (2001) and Hoyland et al. (2003) and ruled out potential arbitrage opportunities in our scenario tree. The direct integration of the three Nelson/Siegel parameters was fascinating, as it reduces the size of tree and ensures computational tractability. The generated term structures satisfy the most important stylized facts (see e.g. Diebold and Li, 2006).

As objective function we minimized CVaR of final shareholder value under different asset allocation, budget and inventory constraints. The shareholder value, as the difference between mark-to-market value of financial asset and present value of future liabilities shows the funded status of a pension plan. It inherently includes investment risks as well as interest rate risks given by the future cash flows.

We gave a numerical example with an asset-liability management problem typical for a defined benefit pension scheme. A company is faced with a long stream of cash outflows after a short period of capital accumulation. Asset allocation bounds were set to allow a long/short “130/30” strategy. As an important result in our setting we found huge hedging demands against interest rate risk. The optimal policy shortened the riskless bond and leveraged the long-term bond to protect the SV against term structures at a low level. Further, our results indicated that an asset-liability mandate should not be misinterpreted as pure asset management with naive constant-mix strategies. Therefore, to test the sensitivity of the solution, we solved the optimization task also without cash flows. The results confirm economic intuition. Without the long series of future payouts the strong hedging demands for long-maturity bond are no longer present. For low targets also short- and medium-term bonds became attractive. Finally, we analyzed the impact of the confidence level on the outputs, which shows that the first-stage asset allocation remains rather stable. To sum up, our approach provides a computationally tractable method for asset-liability management and the results are intuitive and economically meaningful.

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