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Abstract

This paper proves the existence, for any preorder on a real vector space satisfying translation invariance, of a complete preorder extending the preorder and satisfying translation invariance. As application, the existence of a translation-invariant complete preorder on infinite utility streams satisfying *strong Pareto* and *fixed-step anonymity*, is established.

JEL classification numbers: C60, D71.

1 Introduction

[Szpilrajn 1930] proved a theorem, known as Szpilrajn theorem, that may be stated as follows [Hansson 1968]: For any preorder (reflexive and transitive binary relation) on a given set, there exists a complete preorder which is an extension of the preorder (i.e. the preorder is a subrelation to the extension, see definition 1 for subrelation). Szpilrajn theorem proved of a great utility in mathematical social choice theory as in some other branches of pure and applied mathematics. There exists today stronger versions of Szpilrajn theorem, requiring weaker conditions on the initial binary relation, for example consistency instead of transitivity [Suzumura 1976]. There also exists a literature interested in the question of knowing up to what point Szpilrajn theorem remains valid if one imposes on the extension additional conditions, like continuity or representability or the existence and invariance of optima. I refer to [Andrikopoulos 2009] for a survey of that literature. The present paper establishes the existence, for any preorder on a real vector space satisfying translation invariance, of a complete preorder extending the preorder and satisfying translation invariance.

*I thank an anonymous referee who, when reviewing another paper, guided me towards the issue of extending a preorder under translation invariance.

In social choice theory, the property of completeness of a preorder is regarded as desirable, for example in [Arrow 1950], a paper that founded social choice theory. However, since [Zame 2007, Lauwers 2009] proved that, in certain concrete applications, completeness is necessarily accompanied by nonconstructiveness, the property of completeness lost a little of its attraction. In spite of that, it is always useful to have an extension theorem asserting the existence of a complete preorder satisfying certain conditions. Indeed, some theorems on the properties of preorders either apply only to complete preorders, or are less convenient when applied to noncomplete preorders, for example, in social choice theory, the weak weighted utilitarianism theorem of [d'Aspremont-Gevers 2002], or, on the question of the existence of a utility function, the representability theorem of [Monteiro 1987]. By extending the preorder to a complete preorder, an extension theorem would make it possible to exploit these theorems although the extension might not be completely constructed.

Other existence theorems, although nonconstructive, proved very useful. These theorems often paved the way for constructive results. For example, Brouwer's fixed point theorem made it possible to build the general equilibrium theory. Another example, the existence theorem of [Svensson 1980] (theorem 2) made it possible for social choice theory to progress in the search for an equitable and efficient intergenerational criterion. This last theorem is a direct application of Szpilrajn theorem.

In social choice theory, translation invariance is often met. It is related to one extensively researched class of SWRs (social welfare relations): utilitarian SWRs.

The proof of the suggested extension theorem (section 3) follows the same diagram as the proof of Szpilrajn theorem. Starting from a preorder satisfying translation invariance, one adds comparisons on some pairs of alternatives in such a way that translation invariance remains satisfied. Then, an argument based on Zorn's lemma makes it possible to extend the procedure to the whole space.

As example of application, I show in section 4 the existence of a translation-invariant complete preorder on infinite utility streams satisfying the axioms *strong Pareto* and *fixed-step anonymity.*

2 Preliminary

$\mathbb{N}^*$ is the set of positive integers. $\mathbb{R}$ is the real line. A preorder is a reflexive and transitive binary relation. An order is a complete preorder. The set of alternatives is a nonempty real vector space denoted X , equipped with a preorder R.

B being a binary relation on X and x,y two alternatives in X, [xBy and non(yBx)] is denoted $x \succ_B y$, xBy is denoted $x \succsim_B y$ and [xBy and

$yBx]$ is denoted $x \sim_B y$.

Definition 1 *A binary relation B_1 is said to be a subrelation to a binary relation B_2 if, for all x, y in X,*

$$x \succsim_{B_1} y \Longrightarrow x \succsim_{B_2} y$$

and

$$x \succ_{B_1} y \Longrightarrow x \succ_{B_2} y$$

Condition 2 *TI: A preorder R satisfies translation invariance (TI) if:*

$$\forall (x, y) \in \; X \times X, \forall u \in X, \;\; [x \succsim_R y \Longrightarrow x + u \succsim_R y + u]$$

Condition 3 *DI: A preorder R satisfies division invariance (DI) if:*

$$\textit{For every } x \textit{ in } X \textit{ and every positive integer } n, \;\; x \succsim_R y \Longrightarrow \frac{1}{n} x \succsim_R \frac{1}{n} y$$

Lemma 4 *If a preorder R on X satisfies TI (condition 2), then there exists a preorder $\widehat{R}$ on X of which R is a subrelation and such that $\widehat{R}$ satisfies TI and DI (condition 3).*

Proof. First, notice that under R, it is possible to sum inequalities. Indeed, by TI, if a, b, u, v are such that $a \succsim_R b$ and $u \succsim_R v$, then $a + u \succsim_R b + u$ and $b + u \succsim_R b + v$. By transitivity, $a + u \succsim_R b + v$. For each positive integer n, consider the binary relation R_n defined by

$$x \succsim_{R_n} y \text{ iff } nx \succsim_R ny$$

If x, y are such that $x \succsim_R y$, we can sum n times this inequality. Thus, $x \succsim_{R_n} y$. Likewise, it is easily seen that $x \sim_R y$ implies $x \sim_{R_n} y$. As a result, R is a subrelation to R_n. Moreover, R_n is reflexive and transitive. It is a preorder. It is easily checked that R_n satisfies TI. Consider the binary relation

$$\widehat{R} = \cup_{n \in \mathbb{N}^*} R_n$$

defined on X by: $x \succsim_{\widehat{R}} y$ iff there is n such that $x \succsim_{R_n} y$.

R is a subrelation to $\widehat{R}$. Moreover, $\widehat{R}$ is reflexive and transitive. It is a preorder. Since for each positive integer n, R_n satisfies TI, we deduce that $\widehat{R}$ satisfies TI. The lemma is proved if we show that $\widehat{R}$ satisfies DI. Let n be a positive integer, and x, y such that $x \succsim_{\widehat{R}} y$. There exists a positive integer m such that $x \succsim_{R_m} y$. Thus, $mx \succsim_R my$. We can write that as $mn(\frac{1}{n}x) \succsim_R mn(\frac{1}{n}y)$. Thus $\frac{1}{n}x \succsim_{R_{mn}} \frac{1}{n}y$, what implies $\frac{1}{n}x \succsim_{\widehat{R}} \frac{1}{n}y$. $\widehat{R}$ satisfies DI. ■

Remark 5 *(1) It is easily seen that $\widehat{R}$ is the minimal preorder satisfying TI and DI, of which R is a subrelation. (2) If R is complete, since R is a subrelation to $\widehat{R}$, we have necessarily $R = \widehat{R}$. This shows that if the preorder is complete, TI implies DI.*

3 The translation-invariant extension theorem

Theorem 6 *Let R be a preorder on X satisfying TI (condition 2). Then there exists an order on X satifying TI , of which R is a subrelation.*

Proof. If R is an order, there is nothing to prove. Suppose that R is not complete. Consider the preorder $\widehat{R}$ built in the proof of lemma 4, and the set $\Re$ of all preorders on X satisfying TI (condition 2) and DI (condition 3), and of which R is a subrelation. $\Re$ is not empty since $\widehat{R} \in \Re$. Let (R_α) be a chain in $\Re$, i.e. for any $\alpha, \alpha', R_\alpha$ is a subrelation to $R_{\alpha'}$ or $R_{\alpha'}$ is a subrelation to R_α. Notice that (1) the relation $\cup_\alpha (R_\alpha)$ defined on X by: $x [\cup_\alpha (R_\alpha)] y$ iff there is α such that $xR_\alpha y$, is a preorder, (2) it satisfies TI and DI, (3) R is a subrelation to $\cup (R_\alpha)$, (4) for all α, R_α is a subrelation to $\cup (R_\alpha)$. Hence, in the set $\Re$, every chain admits a maximal element in $\Re$. According to Zorn's lemma, $\Re$ admits at least a maximal element. Denote M such a maximal element in $\Re$. Suppose we can prove the following claim:

Claim 7 *For any non complete $\widetilde{R}$ in $\Re$ and any pair of $\widetilde{R}-$incomparable alternatives (x_0, y_0), there exists a preorder $\widetilde{R}_1$ in $\Re$ to which $\widetilde{R}$ is a subrelation and such that x_0 and y_0 are $\widetilde{R}_1-$comparable.*

Then, if M were not complete, there would exist a preorder in $\Re$ to which M is a strict subrelation. This would contradict that M is maximal in $\Re$. Therefore, if the claim holds, M would be necessarily complete. M would be the order we are looking for.

What remains of the proof is devoted to establish claim 7. This is done through the 6 following steps.

If there is no non complete preorder in $\Re$, the theorem is proved since $\Re$ is not empty. Let $\widetilde{R}$ be a non complete preorder in $\Re$ and x_0,y_0 be two elements of X, $\widetilde{R}-$incomparable.

Consider the binary relation B on X: $x \succsim_B y$ iff either $x \succsim_{\widetilde{R}} y$ or there is a positive rational q such that $x - y = q(x_0 - y_0)$.

I prove successively that the two clauses of the definition of B are exclusive (step1), that the indifference relations are equal (step 2), that $\widetilde{R}$ is a subrelation to B (step 3), that B is weakly acyclic (this prepares for transitivity) (step 4), that $\widetilde{R}$ is a subrelation to the transitive closure of B (step 5), that the transitive closure of B satisfies TI and DI (step 6). The transitive closure of B is then the required preorder.

Step 1: If there is a positive rational q such that $x-y=q(x_0-y_0)$, then x,y are $\widetilde{R}-$incomparable. Suppose not. For instance suppose $x \succsim_{\widetilde{R}} y$. By TI, $x-y \succsim_{\widetilde{R}} 0$. By DI, for all positive integer n, $\frac{1}{n}(x-y) \succsim_{\widetilde{R}} 0$. Recall that it is possible to sum inequalities (see the proof of lemma 4). We can sum m times the inequality $\frac{1}{n}(x-y) \succsim_{\widetilde{R}} 0$, m being a positive integer. We obtain $\frac{m}{n}(x-y) \succsim_{\widetilde{R}} 0$. Take $\frac{m}{n}=\frac{1}{q}$. It gives $x_0-y_0 \succsim_{\widetilde{R}} 0$, what contradicts x_0, y_0 being $\widetilde{R}-$incomparable. The case $y \succsim_{\widetilde{R}} x$ is similar.

Step 2: Equivalence of indifferences: Clearly, $x \sim_{\widetilde{R}} y \Longrightarrow x \sim_B y$. I show now that $x \sim_B y$ entails $x \sim_{\widetilde{R}} y$. According to the definition of B, it is enough to prove that x and y are necessarily $\widetilde{R}-$comparable. Suppose that x and y are $\widetilde{R}-$incomparable. According to the definition of B, $x \succsim_B y$ implies that there is some positive rational q such that $x-y=q(x_0-y_0)$. We have also $y \succsim_B x$. Thus for some positive rational q', $y-x=q'(x_0-y_0)$. We see that this gives $q(y_0-x_0)=-q'(y_0-x_0)$, what implies $y_0=x_0$ (q and q' are positive).But that contradicts x_0, y_0 being $\widetilde{R}-$incomparable.

Step 3: $\widetilde{R}$ is a subrelation to B: This is a direct consequence of

$$x \succsim_{\widetilde{R}} y \Longrightarrow x \succsim_B y \text{ (definition of } B\text{)}$$

and

$$x \sim_{\widetilde{R}} y \Leftrightarrow x \sim_B y \text{ (step 2)}$$

Step 4: Weak acyclicity of B: I show that for all x,y,z in X :

$$x \succsim_B y \text{ and } y \succsim_B z \implies x \succsim_B z \text{ or non } (z \succsim_B x)$$

One of the four following cases is implied by $x \succsim_B y$ and $y \succsim_B z$: (1) $x \succsim_{\widetilde{R}} y$ and $y \succsim_{\widetilde{R}} z$,(2) there are q,q' such that $x-y=q(x_0-y_0)$ and $y-z=q'(x_0-y_0)$, (3) $x \succsim_{\widetilde{R}} y$ and there is q' such that $y-z=q'(x_0-y_0)$, (4) there is q such that $x-y=q(x_0-y_0)$ and $y \succsim_{\widetilde{R}} z$. Consider successively the four cases:

(1) $x \succsim_{\widetilde{R}} y$ and $y \succsim_{\widetilde{R}} z$. By transitivity of $\widetilde{R}$: $x \succsim_{\widetilde{R}} z$. Thus, according to the definition of B, $x \succsim_B z$.

(2) $x-y=q(x_0-y_0)$ and $y-z=q'(x_0-y_0)$. This gives $x-z=(q+q')(x_0-y_0)$. Thus $x \succsim_B z$.

(3) $x \succsim_{\widetilde{R}} y$ and $y-z=q'(x_0-y_0)$. Suppose we have $z \succsim_B x$. We would have either $z \succsim_{\widetilde{R}} x$ or $z-x=q"(x_0-y_0)$. Both possibilities contradict $x \succsim_{\widetilde{R}} y$ and $y-z=q'(x_0-y_0)$. Indeed, with $x \succsim_{\widetilde{R}} y$, $z \succsim_{\widetilde{R}} x$ gives $z \succsim_{\widetilde{R}} y$ what contradicts $y-z=q'(x_0-y_0)$ (step 1); whereas

$y-z = q'(x_0-y_0)$ with $z-x = q"(x_0-y_0)$ implies $y-x = (q"+q')(x_0-y_0)$, what contradicts $x \succsim_{\widetilde{R}} y$. As a result, we have non$(z \succsim_B x)$.

(4) $x - y = q(x_0 - y_0)$ and $y \succsim_{\widetilde{R}} z$. This case is similar to case (3). For the sake of clarity, I give the argument. Suppose we had $z \succsim_B x$. We would have either $z \succsim_{\widetilde{R}} x$ or $z - x = q"(x_0 - y_0)$. Both possibilities contradict $x - y = q(x_0 - y_0)$ and $y \succsim_{\widetilde{R}} z$. Indeed, with $y \succsim_{\widetilde{R}} z$, $z \succsim_{\widetilde{R}} x$ gives $y \succsim_{\widetilde{R}} x$, what contradicts $x - y = q(x_0 - y_0)$ (step 1); whereas $z-x = q"(x_0-y_0)$ with $x-y = q(x_0-y_0)$ implies $z-y = (q+q")(x_0-y_0)$, what contradicts $y \succsim_{\widetilde{R}} z$. As a result, we have non$(z \succsim_B x)$ and weak acyclicity is established.

Remark 8 *Let x, y, z be such that $x \succsim_B y$ and $y \succsim_B z$. Weak acyclicity entails that if one of the judgments $x \succsim_B y$ and $y \succsim_B z$ is a strict preference, then either the judgment on (x, z) is $x \succ_B z$ or x and z are $B-$incomparable.*

Step 5: $\widetilde{R}$ is a subrelation to the transitive closure of B : Consider $\overline{B}$ the transitive closure of B, defined by:

$$x \succsim_{\overline{B}} y \text{ if there is a sequence } (z_i)_{i=1}^n \text{ such that } x \succsim_B z_1,\ z_1 \succsim_B z_2 \text{...and } z_n \succsim_B y$$

It is clear that $x \succsim_{\widetilde{R}} y$ implies $x \succsim_{\overline{B}} y$ (step 3: $\widetilde{R}$ is a subrelation to B). It is enough to prove that $x \succsim_{\overline{B}} y$ implies non$(y \succ_{\widetilde{R}} x)$.

For a positive integer n, consider the statement Q_n: " If there is a sequence $(z_i)_{i=1}^n$ such that $x \succsim_B z_1 \succsim_B z_2 ... \succsim_B z_n \succsim_B y$, then non$(y \succ_{\widetilde{R}} x)$." Let's prove by induction that Q_n is true for all positive integer. Notice that when the sequence (z_i) has n terms, there is $n+1$ successive judgments.

$n = 1$: The sequence $(z_i)_{i=1}^n$given by the definition of $\overline{B}$ have only one term. We have $x \succsim_B z_1 \succsim_B y$. By step 4, we have $x \succsim_B y$ or non$(y \succsim_B x)$. Both possibilities contradict $y \succ_{\widetilde{R}} x$. So, we have non$(y \succ_{\widetilde{R}} x)$.

Suppose Q_n true for n and let's show that Q_{n+1} is true: Consider the sequence of $n + 2$ judgments:

$$x \succsim_B z_1 \succsim_B z_2 ... \succsim_B z_{n+1} \succsim_B y$$

Each one of these judgments comes either from the clause $x \succsim_{\widetilde{R}} y$ or the clause $x - y = q(x_0 - y_0)$ of the definition of B.

If there is two successive judgments coming from the clause $x \succsim_{\widetilde{R}} y$, say $z_p \succsim_{\widetilde{R}} z_{p+1} \succsim_{\widetilde{R}} z_{p+2}$ (with $p = 0, .., n+2$ and the convention: $z_0 = x$ and $z_{n+2} = y$), by transitivity of $\widetilde{R}$, we have:

$$x \succsim_B ... z_P \succsim_B z_{p+2} ... \succsim_B y$$

which constitutes a sequence of $n+1$ judgments. By Q_n, we have non$(y \succ_{\widetilde{R}} x)$.

If there is two successive judgments coming from the clause $x - y = q(x_0 - y_0)$, say $z_p \succsim_B z_{p+1} \succsim_B z_{p+2}$, then $z_p - z_{p+1} = q(x_0 - y_0)$ and $z_{p+1} - z_{p+2} = q'(x_0 - y_0)$.Thus, $z_p - z_{p+2} = (q + q')(x_0 - y_0)$ so that $z_p \succsim_B z_{p+2}$. We have again reduced the number of judgments to $n+1$. Thus, we have also non$(y \succ_{\widetilde{R}} x)$.

It remains to consider the cases where the judgments are alternate. Two cases must be considered: $n+2$ even and $n+2$ odd.

$n+2$ even: The sequence of judgments either begin or ends with a judgment from $\widetilde{R}$. Suppose it begins with a judgment from $\widetilde{R}$:

$$x \succsim_{\widetilde{R}} z_1 \succsim_B z_2 ... \succsim_{\widetilde{R}} z_{n+1} \succsim_B y$$

Apply Q_n to

$$z_1 \succsim_B z_2 ... \succsim_{\widetilde{R}} z_{n+1} \succsim_B y$$

It gives non$(y \succ_{\widetilde{R}} z_1)$. Since $x \succsim_{\widetilde{R}} z_1$, we cannot have $y \succsim_{\widetilde{R}} x$.

If the sequence of judgments ends with a judgment from $\widetilde{R}$, the proof is similar. So it is omitted.

$n+2$ odd: If the sequence of judgments begins by a judgment from $\widetilde{R}$, the proof is also similar. So it is omitted.

If the sequence of judgments begins from a judgment from the clause $x - y = q(x_0 - y_0)$, we have

$$x \succsim_B z_1 \succsim_{\widetilde{R}} z_2 ... \succsim_{\widetilde{R}} z_{n+1} \succsim_B y \tag{1}$$

Denote (x, z_1) by (α_1, β_1), (z_2, z_3) by (α_2, β_2)... $\left(z_{2(p-1)}, z_{2p-1}\right)$ by (α_p, β_p) with $p = 1, .., \frac{n+3}{2}$ and the convention: $z_0 = x$ and $z_{n+2} = y$. Since judgments $x \succsim_B z_1, z_2 \succsim_B z_3, ..., z_{n-1} \succsim_B z_n, z_{n+1} \succsim_B y$ come from the clause $x - y = q(x_0 - y_0)$, we have $\alpha_p - \beta_p = q_p(x_0 - y_0)$ for $p = 1, .., \frac{n+3}{2}$. Moreover, according to (1) , $\beta_p \succsim_{\widetilde{R}} \alpha_{p+1}$ for $p = 1, .., \frac{n+1}{2}$. Thus

$$\begin{aligned}
\alpha_1 - q_1(x_0 - y_0) &\succsim_{\widetilde{R}} \alpha_2 \\
\alpha_2 - q_2(x_0 - y_0) &\succsim_{\widetilde{R}} \alpha_3 \\
&....\\
\alpha_{(n+1)/2} - q_{(n+1)/2}(x_0 - y_0) &\succsim_{\widetilde{R}} \alpha_{(n+3)/2}
\end{aligned}$$

We can sum these inequalities (this is established in the proof of lemma 4). We obtain

$$\alpha_1 + \left(\sum_{2}^{(n+1)/2} \alpha_p\right) - \left(\sum_{1}^{(n+1)/2} q_p\right)(x_0 - y_0) \succsim_{\widetilde{R}} \left(\sum_{2}^{(n+1)/2} \alpha_p\right) + \alpha_{(n+3)/2}$$

By TI, we obtain

$$\alpha_1 - \left(\sum_{1}^{(n+1)/2} q_p \right) (x_0 - y_0) \succsim_{\widetilde{R}} \alpha_{(n+3)/2}$$

But $\alpha_1 = x$ and $\alpha_{(n+3)/2} = y$. Denote $q = \left(\sum_1^{(n+1)/2} q_p \right)$. Thus

$$x - q(x_0 - y_0) \succsim_{\widetilde{R}} y$$

By TI, $x - y \succsim_{\widetilde{R}} q(x_0 - y_0)$. If we had $y \succ_{\widetilde{R}} x$, it would give $0 \succ_{\widetilde{R}} x - y \succsim_{\widetilde{R}} q(x_0 - y_0)$. By transitivity of $\widetilde{R}$ and by TI, x_0 and y_0 would be $\widetilde{R}-$comparable, which is not the case. As a result, we have non$(y \succ_{\widetilde{R}} x)$. Thus we have non$(y \succ_{\widetilde{R}} x)$. Step 5 is proved.

Remark 9 *$\widetilde{R}$ is a subrelation to $\overline{B}$, but B is not.*

Step 6: $\overline{B}$ satisfies TI: As $\widetilde{R}$ is translation-invariant, B is clearly translation-invariant. It is easily deduced that $\overline{B}$ is also translation-invariant. Likewise, it is easily seen that $\overline{B}$ satisfies DI. Thus, $\overline{B}$ is the required preorder. ■

4 Application

As application of theorem 6, I propose the following corollary asserting the existence of a translation-invariant order on infinite utility streams satisfying the axioms *strong Pareto* and *fixed-step anonymity*. The set of alternatives is $\mathbb{R}^{\mathbb{N}^*}$, equipped with the usual addition and scalar multiplication. This space, or a subset of it, often constitutes the space of alternatives in intergenerational choice theory. Let R be a preorder on $\mathbb{R}^{\mathbb{N}^*}$. It is necessary to introduce some definitions and properties.

Axiom 10 *strong Pareto: R is strong Pareto if, $\forall x, y \in \mathbb{R}^{\mathbb{N}^*}$ such that $\forall i \in \mathbb{N}^*$ $x_i \geq y_i$ and there exists $j \in \mathbb{N}^*$ such that $x_j > y_j$, we have $x \succ_R y$ (x_i, y_i denote the i^{th} components of resp. x, y).*

Axiom 11 *[Mitra-Basu 2007] $Q-$anonymity : Let Q be a set of permutations on $\mathbb{N}^*$. Denote $\sigma(x)$ the sequence obtained by permuting the components of x according to the permutation σ. R is $Q-$anonymous if $\forall x \in \mathbb{R}^{\mathbb{N}^*}$ and $\sigma \in Q$, we have $x \sim_R \sigma(x)$.*

[Mitra-Basu 2007] showed that there exists a preorder satisfying at the same time *strong Pareto* and $Q-$*anonymity* iff Q contains only cyclic permutations. Thus we have to impose on Q the assumption that it contains only cyclic permutations. A permutation is cyclic iff $\forall x \in \mathbb{R}^{\mathbb{N}^*}$, there exists $n \in \mathbb{N}^*$ such that $\sigma^n(x) = x$.

Definition 12 *[Mitra-Basu 2007] The $Q-$grading principle : $\forall x, y \in \mathbb{R}^{\mathbb{N}^*}$, $x \succsim_Q y$ iff there is $\sigma \in Q$ such that $\forall i \in \mathbb{N}^*$, $x_{\sigma(i)} \geq y_i$.*

Lemma 13 *([Banerjee 2006], proposition 2) A preorder R on $\mathbb{R}^{\mathbb{N}^*}$ satisfies strong Pareto and $Q-$anonymity iff it admits the $Q-$grading principle as subrelation.*

Notice that if $[Q]$ denotes the group generated by Q, the $Q-$grading principle and the $[Q]-$grading principle are the same preorder (a consequence of proposition 2 of [Mitra-Basu 2007]).

Several authors gave a special attention to fixed-step permutations, for example [Fleurbaey-Michel 2003] and [Mitra-Basu 2007]. A permutation σ on $\mathbb{N}^*$ is said to be *fixed step* iff there exists a partition of $\mathbb{N}^*$: N_1, N_2...such that $\forall i, j, |N_i| = |N_j|$ and σ can be written as the composition of permutations $\sigma_1 \circ \sigma_2 \circ$ ...where for all i and j such that $j \neq i$, σ_i leaves invariant all the elements of N_j. Denote S the set of fixed-step permutations. S is a group [Mitra-Basu 2007, Lauwers 2009]. The axiom *$S-$anonymity* corresponds to *fixed-step anonymity.*

Definition 14 *[Fleurbaey-Michel 2003] The fixed-step catching-up $\succsim_{SC}$: $\forall x, y$ in $\mathbb{R}^{\mathbb{N}^*}$, $x \succsim_{SC} y$ iff there exist two positive integers k, K such that for all integer $n > K$*

$$\sum_{i=1}^{kn} x_i \geq \sum_{i=1}^{kn} y_i$$

Lemma 15 *([Fleurbaey-Michel 2003], the table on page 788) $\succsim_{SC}$ is fixed-step anonymous.*

Corollary 16 *There exists a translation-invariant, strong-Pareto and fixed-step anonymous order on $\mathbb{R}^{\mathbb{N}^*}$.*

Proof. $\succsim_{SC}$ is obviously translation-invariant. Apply theorem 6 to $\succsim_{SC}$. There exists a translation-invariant order on $\mathbb{R}^{\mathbb{N}^*}$ of which $\succsim_{SC}$ is a subrelation. Let A be such an order. Obviously, $\succsim_{SC}$ is *strong Pareto.* Lemma 13 with $Q = S$ and lemma 15 entail that the $S-$grading principle is a subrelation to $\succsim_{SC}$. As a result, the $S-$grading principle is also a subrelation to A. By lemma 13, A satisfies *strong Pareto* and *fixed-step anonymity.* ■

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