CAPM and capital budgeting: present versus future, equilibrium versus disequilibrium, decision versus valuation

Magni, Carlo Alberto

Dipartimento di Economia politica, University di Modena e Reggio Emilia, Italy

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Carlo Alberto Magni

Department of Economics, University of Modena and Reggio Emilia,

Modena viale Berengario 51, 41100 Modena, Italy

magni@unimo.it

tel. +39-0059-2056777

Abstract. This paper deals with the use of the CAPM for investment decisions and evaluations. Four different measures are deductively drawn from this model: the disequilibrium Net Present Value, the equilibrium Net Present Value, the disequilibrium Net Future Value, the equilibrium Net Future Value. It is shown that all of them may be used for accept-reject *decisions*, but only the equilibrium Net Present Value and the disequilibrium Net Future Value may be used for *valuation*, given that they enjoy the additivity property. The two nonadditive indexes cannot be deducted from the CAPM assumptions if the decision problem "invest/no invest" is reframed as "invest in *Z*/invest in *Y*". Despite their additivity, the equilibrium Net Present Value and the disequilibrium Net Future Value are unreliable for both valuation and decision, because they do not signal arbitrage opportunities whenever there is some state of nature for which they are decreasing functions with respect to the end-of-period cash flow. In this case, the equilibrium value of a project is not the price it would have if it were traded in the security market. This result is the capital-budgeting counterpart of Dybvig and Ingersoll's (1982) result.

Keywords. Modelling, investment, decision**,** valuation, CAPM, equilibrium, disequilibrium, additivity, arbitrage.

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Introduction

The use of the CAPM for capital budgeting purposes traces back to the 60s and 70s, when various authors developed a theoretical link between this asset pricing model and corporate capital budgeting decisions. Among the several contributions we find classical papers of foremost authorities such as Tuttle and Litzenberger (1968), Hamada (1969), Mossin (1969), Litzenberger and Budd (1970), Stapleton (1971, 1974), Rubinstein (1973), Bierman and Hass (1973, 1974), Bogue and Roll (1974). The decision criteria these authors present are seemingly different, but, logically, they are equivalent (see Senbet and Thompson, 1978) and may be framed in terms of risk-adjusted cost of capital (see Magni, 2007a): the resulting capital budgeting criterion suggests that, as long as the CAPM assumptions are met, a firm aiming at maximizing share price should undertake a project if and only if the project's risk-adjusted cost of capital exceeds the project's expected internal rate of return. These classical papers are aimed at formally deducting a *decision* rule from the CAPM, but do not particularly focus on project *valuation*; although the net-present-value rule is often reminded, no explicit claim appears that the risk-adjusted cost of capital may or may not be used for *valuing* projects. The risk-adjusted cost of capital is presented as depending on a disequilibrium (cost-based) systematic risk (see Rubinstein, 1973), but project value is often framed in a certainty-equivalent form (Bogue and Roll, 1974), which implies that an equilibrium systematic risk is used. As a result, ambiguities arise on the use of the project NPV as a *decision* rule or as a *valuation* tool, and uncertainties arise regarding the correct calculation of the NPV, using either the equilibrium or the disequilibrium systematic risk. Furthermore, while most of the contributions deal with net present values, no thorough analysis is found in the literature concerning the relation between present value and excess return (but see Weston and Chen, 1980). Few contributions have drawn attention on these topics. Among these, we find Rendleman's (1978) paper, which deals with the use of cost-based (disequilibrium) covariance terms as opposed to market-determined (equilibrium) covariance terms. The author suggests that if a firm were to rank projects on the basis of excess of internal return over equilibrium (market-determined) return, an incorrect decision would be reached. Haley and Schall (1979, pp. 182-183) show that the disequilibrium NPV is unreliable in ranking projects. Weston and Chen (1980) state that either the disequilibrium or equilibrium return may be used for ranking projects, if appropriate use is made of both. And while the equilibrium form of NPV is widespread for valuation purposes (in the classical certainty-equivalent form), the disequilibrium form of NPV has its own upholders as well among scholars. For example, Lewellen

(1977) uses the disequilibrium NPV to value projects; Copeland and Weston use cost-based betas, and therefore disequilibrium NPVs, for valuing projects in various occasions (Copeland and Weston, 1983, 1988, Weston and Copeland, 1988); Bossaerts and Ødegaard (2001) endorse the use of the disequilibrium NPV for valuing projects. Some other authors are aware that the disequilibrium NPV is often used in finance, and warn against it claiming that this kind of NPV is a common misuse of the NPV rule: Ang and Lewellen (1982, p. 9) explicitly claim that the disequilibrium NPV is the "standard discounting approach" in finance for valuing projects, and show that such a method is incorrect for it leads to nonadditive valuations. Grinblatt and Titman (1998), being aware that the use of disequilibrium NPVs is extensive, present an example where cost-based betas are used (see their example 10.5) and claim that their example deliberately shows an incorrect procedure. Ekern (2006) distinguishes between NPV as a decision rule and NPV as a valuation tool; he states that the disequilibrium NPV is correct for decision but not for valuation, and suggests the use of the equilibrium NPV as well as other several equivalent methods. Magni (2007b) focuses on the relation between disequilibrium NPV and absence of arbitrage, showing that while deductively valid as a decision tool, the former is incompatible with the latter.

This paper, limiting its scope to one-period projects and accept-reject situations, aims at giving some clarification on these topics. In particular it shows that three conceptual categories are involved when the CAPM is used for capital budgeting: equilibrium/disequilibrium, present/future, decision/valuation. The results obtained inform that if the CAPM assumptions are met in the security market and a firm's objective is to maximize share price, the investor may reliably employ either present of future values, either in equilibrium or disequilibrium format, as long as the resulting values are used for *decision-making* purposes. If, instead, the purpose is *valuation*, only the disequilibrium NFV and the equilibrium NPV may be used, because the disequilibrium NPV and the equilibrium NFV are not additive. This also makes their use unsafe for decision-making as well: whenever decision makers face a portfolio of projects (or a project composed of several sub-projects) they may separately compute each project's NPV (NFV) and then sum the values obtained or sum the cash flows and then compute the portfolio NPV. Changing the order in which summation and discounting are made, different results are obtained. This result is a conundrum, because two nonadditive indexes are validly deducted from the CAPM assumptions. However, the same two indexes may not be deducted if the decision problem is reframed: instead of coping with the problem "invest in project Z/do not invest in project Y" one may consider the problem "invest in project Z/invest in alternative Y". The latter case is more general and it boils down to the former case whenever "project Y" is the null alternative, that is, the project with zero cash flows. This makes the equilibrium NPV and the disequilibrium NFV the only capital budgeting criteria validly deducted from the CAPM. Nevertheless, despite their additivity, they have serious pitfalls as well: if there is a state of nature for which they are decreasing functions with respect to the end-of-period cash flow, then valuation (and decision) is unreliable. This result is just the capital-budgeting version of a result found in Dybvig and Ingersoll (1982) concerning asset pricing in complete markets, and explains why the equilibrium value of a project is not always the price it would have if it were traded in the security market.

The paper is structured as follows. In section 1 definitions of net present values and net future values, in either equilibrium or disequilibrium format, are given. In section 2 four decision criteria are formally deducted assuming that the CAPM assumptions are met. In section 3 the equilibrium NPV and the disequilibrium NFV are shown to be additive, whereas the disequilibrium NPV and the equilibrium NFV are shown to be nonadditive. Section 4 shows that by reframing the decision problem the nonadditive measures are dismissed. Section 5 shows that additivity does not guarantee absence of arbitrage and that the two additive measures previously found may be in some cases misleading. Section 6 shows that the equilibrium value of a project is not necessarily the value a project would have if it were traded in the security market. Some remarks conclude the paper.

Equilibrium in the security market is assumed throughout the paper, unless otherwise specified. To avoid pedantry, main notational conventions are placed in Table 0.

1. Equilibrium and disequilibrium, present and future

This section introduces the notions of Net Present Value (NPV) and Net Future Value (NFV) and shows that, under uncertainty, they are not univocal.

Under certainty, Net Present Value and Net Future Value are equivalent notions. In particular, let $V_Z = F_Z /(1+i)$ be the project's value, where i is the (opportunity) cost of capital. The NPV of a project Z with cost I_Z and end-of-period cash flow F_Z is given by

$$\text{NPV}_Z = -I_Z + V_Z = -I_Z + \frac{F_Z}{1+i} . \tag{1.1}$$

The NFV of project Z is just the NPV compounded at the cost of capital:

$$\text{NFV}_Z = \text{NPV}_Z (1+i) = -I_Z(1+i) + F_Z . \tag{1.2}$$

As $r_Z = F_Z / I_Z - 1$ is the project rate of return, the NFV may be rewritten in excess-return form:

$$\text{excess return} = I_Z (r_Z - i) = \text{NFV}_Z . \tag{1.3}$$

Therefore, the NPV is just the present value of the project excess return, calculated at the cost of capital:

$$\text{NPV}_Z = \frac{I_Z (r_Z - i)}{1+i} . \tag{1.4}$$

Under certainty, the NPV is the current project (net) value, the NFV (excess return) is the end-of-period project (net) value. In terms of decisions, the NPV and the NFV have the same sign (as long as $(1+i) > 0$) so that a project is worth undertaking if and only if the NPV and the NFV are positive. The NPV and NFV are twin notions: both may interchangeably be used as decision rules and valuation tools.

Under uncertainty, if the CAPM is used for measuring risk, the notions of NPV and NFV (and the very notion of value) are not univocal. Depending on whether disequilibrium covariance terms or equilibrium covariance terms are used, we find disequilibrium or equilibrium NPVs and NFVs. We then give the following definitions:

Definition 1.1. *The disequilibrium NPV (dNPV) is the net discounted expected cash flow, where the discount rate is the disequilibrium (cost-based) rate of return of the project* $r_Z^d = r_f + \lambda \operatorname{cov}(F_Z, r_m)/I_Z$:

$$\mathrm{dNPV}_Z := \frac{\overline{F}_Z}{R_f + \dfrac{\lambda}{I_Z}\operatorname{cov}(F_Z, r_m)} - I_Z. \tag{1.5}$$

The first addend is the disequilibrium value of the project, so that $\mathrm{dNPV}_Z := V_Z^d - I_Z$.

Definition 1.2. *The equilibrium NPV (eNPV) is the net discounted expected cash flow, where the discount rate is the equilibrium rate of return* $r_Z^e = r_f + \lambda \operatorname{cov}(F_Z, r_m)/V_Z^e$ *(with V_Z^e being the equilibrium value of the project):*

$$\mathrm{eNPV}_Z := \frac{\overline{F}_Z}{R_f + \dfrac{\lambda}{V_Z^e}\operatorname{cov}(F_Z, r_m)} - I_Z. \tag{1.6}$$

As widely known, we have $V_Z^e := \big(\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)\big)/R_f$ so that we may alternatively reframe the eNPV in a certainty-equivalent form

$$\mathrm{eNPV}_Z := \frac{\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)}{R_f} - I_Z \tag{1.7}$$

Using eq. (2.1) we give the following

Definition 1.3. *The disequilibrium NFV (dNFV) is given by the compounded disequilibrium Net Present Value:* $\mathrm{dNFV}_Z = \mathrm{dNPV}_Z(1 + r_Z^d) = \mathrm{dNPV}_Z(R_f + \lambda \operatorname{cov}(F_Z, r_m)/I_Z)$. *Therefore, we may write, in an excess-return format,*

$$\mathrm{dNFV}_Z := I_Z\big(\overline{r}_Z - r_Z^d\big) = I_Z\left(\overline{r}_Z - r_f - \frac{\lambda \operatorname{cov}(F_Z, r_m)}{I_Z}\right) \tag{1.6}$$

Definition 1.4. *The equilibrium NFV (eNFV) is given by the compounded eNPV$_Z$:* $\text{eNFV}_Z = \text{eNPV}_Z (1 + r_Z^e)$. *Therefore, we may write, in an excess-return format,*

$$\text{eNFV}_Z = I_Z\left(\bar{r}_Z - r_Z^e\right) = I_Z\left(\bar{r}_Z - r_f - \frac{\lambda \text{cov}(F_Z, r_m)}{V_Z^e}\right) \tag{1.9}$$

or, using the relation $\overline{F}_Z - I_Z = \bar{r}_Z I_Z$,

$$\text{eNFV}_Z = (\overline{F}_Z - I_Z) - \left(r_f + \frac{\lambda \text{cov}(F_Z, r_m)}{V_Z^e}\right) I_Z. \tag{1.10}$$

Remark 1.1 It is worth reminding that the project's expected rate of return differs from both the disequilibrium rate of return and the equilibrium rate of return. For the sake of clarity, the three rates of return may be written as

$$\bar{r}_Z = \frac{\overline{F}_Z}{I_Z} - 1 \qquad\qquad \text{expected rate of return} \tag{1.11}$$

$$r_Z^d = \frac{\overline{F}_Z}{V_Z^d} - 1 = r_f + \frac{\lambda \text{cov}(F_Z, r_m)}{I_Z} \qquad\qquad \text{disequilibrium rate of return} \tag{1.12}$$

$$r_Z^e = \frac{\overline{F}_Z}{V_Z^e} - 1 = r_f + \frac{\lambda \text{cov}(F_Z, r_m)}{V_Z^e} \qquad\qquad \text{equilibrium rate of return} \tag{1.13}$$

(see also Weston and Chen, 1980, p. 12). The disequilibrium rate of return in (1.12) is the risk-adjusted cost of capital introduced in the classical contributions cited above (see Rubinstein, 1973, and Magni, 2007a). Using (1.11)-(1.13), Table 1 collects various ways of representing NPVs and NFVs, in either equilibrium or disequilibrium format, which are equivalent to those presented in Definitions (1.1)-(1.4) above.[1]

The following section shows that the proliferation of measures under uncertainty, while surprising, is harmless in accept-reject decisions, for all of them are validly deducted by the CAPM and the assumption of share price maximization.

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[1] It is worth reminding that if the project lies on the Security Market Line (SML), then $I_Z = V_Z^d = V_Z^e$ and $\bar{r}_Z = r_Z^d = r_Z^e$, i.e. the three notions of rate of return collapse into one.

2. The four decision criteria

This section shows that the four indexes above introduced are logically equivalent as decision rules in accept-reject situations. To begin with, we have the following

Lemma 2.1 *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. Then, after acceptance of the project,*

$$\overline{F}_Z - R_f I_Z - \lambda \operatorname{cov}(F_Z, r_m) = R_f N_l (P_l^\circ - P_l). \tag{2.1}$$

Proof: Consider firm *l*. Before acceptance of the project, we have, due to the Security Market Line,

$$\overline{r}_l = r_f + \lambda \operatorname{cov}(r_l, r_m).$$

Reminding that $1 + \overline{r}_l = \overline{F}_l / V_l$, we have

$$\frac{\overline{F}_l}{V_l} = R_f + \lambda \operatorname{cov}(r_l, r_m)$$

and, multiplying by the firm value V_l, we obtain

$$\overline{F}_l = R_f V_l + \lambda \operatorname{cov}(F_l, r_m) = R_f N_l P_l + \lambda \operatorname{cov}(F_l, r_m). \tag{2.2}$$

After acceptance of the project, the new equilibrium value is set as

$$V_l^\circ = \frac{\overline{F}_l + \overline{F}_Z - \lambda \operatorname{cov}(F_l + F_Z, r_m)}{R_f}.$$

The existing shares are N_l, so the new resulting price P_l° is such that $V_l^\circ - I_Z = N_l P_l^\circ$, which

determines $P_l^\circ = \dfrac{V_l^\circ - I_Z}{N_l}$. To actually make the investment the firm shall issue $N_l^\circ = \dfrac{I_Z}{P_l^\circ}$ shares at the

price P_l°. The Security Market Line is now such that

$$\frac{\overline{F}_l + \overline{F}_Z}{V_l^\circ} = R_f + \lambda \operatorname{cov}\left(\frac{F_l + F_Z}{V_l^\circ}, r_m\right)$$

whence

$$\overline{F}_l + \overline{F}_Z = R_f V_l^\circ + \lambda \operatorname{cov}(F_l + F_Z, r_m).$$

Having determined the new price P_l° and the number N_l° of stocks issued, the latter boils down

$$\overline{F}_l + \overline{F}_Z = R_f (N_l + N_l^\circ) P_l^\circ + \lambda \operatorname{cov}(F_l + F_Z, r_m). \tag{2.3}$$

Subtracting (2.3) from (2.2) we get to

$$-\overline{F}_Z = R_f N_l P_l + \lambda \operatorname{cov}(F_l, r_m) - R_f (N_l + N_l^\circ) P_l^\circ - \lambda \operatorname{cov}(F_l + F_Z, r_m)$$

and, using $N_l^\circ P_l^\circ = I_Z$,

$$\overline{F}_Z - R_f I_Z - \lambda \operatorname{cov}(F_Z, r_m) = R_f N_l (P_l^\circ - P_l).$$

<div align="right">Q.E.D.</div>

From Lemma 2.1, four decision rules are deducted. In particular, we have the following

Proposition 2.1 *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs* I_Z *and generates the end-of-period payoff* F_Z*. The firm's share price increases if and only if the project disequilibrium Net Present Value is positive:* [2]

$$\mathrm{dNPV}_Z := \frac{\overline{F}_Z}{R_f + \dfrac{\lambda}{I_Z} \operatorname{cov}(F_Z, r_m)} - I_Z > 0. \tag{2.4}$$

Proof: From eq. (2.1) we find

$$\overline{F}_Z - I_Z \left[R_f + \lambda \operatorname{cov}\left(\frac{F_Z}{I_Z}, r_m \right) \right] = R_f N_l (P_l^\circ - P_l).$$

whence

$$\frac{\overline{F}_Z}{R_f + \lambda \operatorname{cov}\left(\dfrac{F_Z}{I_Z}, r_m \right)} - I_Z = \frac{R_f N_l (P_l^\circ - P_l)}{R_f + \lambda \operatorname{cov}\left(\dfrac{F_Z}{I_Z}, r_m \right)}.$$

Therefore,

$$P_l^\circ > P_l \text{ if and only if } V_Z^d - I_Z = \mathrm{dNPV}_Z > 0.$$

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[2] It is assumed that R_f and $R_f + (\lambda / I_Z) \operatorname{cov}(F_Z, r_m)$ have equal sign. If this condition is not met, the thesis holds with the sign of (2.4) reversed.

Proposition 2.2 *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs* I_Z *and generates the end-of-period payoff* F_Z. *The firm's share price increases if and only if the project equilibrium Net Present Value is positive*:

$$\text{eNPV}_Z := \frac{\overline{F}_Z}{R_f + \dfrac{\lambda}{V_Z^e}\text{cov}(F_Z, r_m)} - I_Z > 0. \tag{2.5}$$

Proof: Using eq. (2.1) and the fact that $\overline{F}_Z - \lambda\,\text{cov}(F_Z, r_m) = R_f V_Z^e$, we have

$$R_f V_Z^e - R_f I_Z = R_f N_l(P_l^\circ - P_l)$$

whence, dividing by R_f,

$$\text{eNPV}_Z = N_l(P_l^\circ - P_l). \tag{2.6}$$

Finally, we have

$$P_l^\circ > P_l \text{ if and only if } \text{eNPV}_Z > 0.$$

Proposition 2.3 *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs* I_Z *and generates the end-of-period payoff* F_Z. *The firm's share price increases if and only if the project disequilibrium Net Future Value is positive*:

$$\text{dNFV}_Z = I_Z\left(\overline{r}_Z - r_Z^d\right) > 0 \tag{2.7}$$

Proof: From eq. (2.1) we have

$$\overline{F}_Z - I_Z(R_f + \lambda\,\text{cov}(r_Z, r_m)) = R_f N_l(P_l^\circ - P_l). \tag{2.8}$$

Given that

$$\text{dNFV}_Z = I_Z(\overline{r}_Z - r_Z^d) = \overline{F}_Z - I_Z(R_f + \lambda\,\text{cov}(r_Z, r_m)) \tag{2.9}$$

we have

$$P_l^\circ > P_l \text{ if and only if } \text{dNFV}_Z > 0.$$

Proposition 2.4. *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if the project equilibrium Net Future Value is positive:*[3]

$$\text{eNFV}_Z = I_Z\left(\bar{r}_Z - r_Z^e\right) > 0 . \tag{2.10}$$

Proof: Using eq. (2.1) and the equalities $\bar{F}_Z - \lambda\operatorname{cov}(F_Z, r_m) = R_f V_Z^e = R_f \bar{F}_Z /(1 + r_Z^e)$, we have

$$R_f \frac{\bar{F}_Z}{(1 + r_Z^e)} - R_f I_Z = R_f N_l(P_l^\circ - P_l)$$

and therefore

$$\bar{F}_Z R_f - R_f (1 + r_Z^e)I_Z = (1 + r_Z^e)R_f N_l(P_l^\circ - P_l) .$$

whence, dividing by R_f,

$$\bar{F}_Z - I_Z (1 + r_Z^e) = (1 + r_Z^e)N_l(P_l^\circ - P_l)$$

which leads to

$$P_l^\circ > P_l \text{ if and only if } \text{eNFV}_Z > 0 .$$

<div align="right">Q.E.D.</div>

Remark 2.1 Propositions 2.1-2.4 show four ways of using the CAPM for capital budgeting purposes. All of them are CAPM-consistent. In particular, it is worth stressing that: (a) the disequilibrium NPV is indeed a correct decision rule, despite some claims against its use (e.g. De Reyck, 2005); (b) the Net Present Value rule may be safely replaced by a Net Future Value (excess return) rule, either in equilibrium or disequilibrium format.

Remark 2.2 The results obtained have some practical consequences. In real life, investors face several different situations in capital budgeting. In particular, information about the project may be extensive or partial so that project analysis may or may not rely on a scenario basis, and there may or may not be assets in the security market having economic characteristics similar to those of the project under consideration (representative assets). If appropriate information on the project is available (so that scenario analysis is possible) and/or there are not representative assets in the market, the investor must rely on an ex ante probability distribution to compute the covariance between the end-of-period cash flow and the market

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[3] It is here assumed $(1 + r_Z^e) > 0$. If this condition is not met, then the thesis holds with the sign of (2.10) reversed.

return, $\text{cov}(F_Z, r_m)/I_Z$; this means that he will equivalently employ the disequilibrium NPV or the disequilibrium NFV to decide whether investing or not in the project. If appropriate information is somehow lacking and there are representative assets in the security market, the decision maker may measure the covariance from historical return data of representative assets. The covariance so obtained is a proxy for the equilibrium covariance $\text{cov}(r_Z^e, r_m) = \text{cov}(F_Z, r_m)/V_Z^e$ (assuming the market is in equilibrium)[4] and the investor will therefore employ the equilibrium NPV or the equilibrium NFV. In both cases the decision maker is reliably supported by a pair of metrics that lead to correct decisions.

3. Nonadditivity

This section shows that the disequilibrium NFV and the equilibrium NPV are additive, whereas the disequilibrium NPV and the equilibrium NFV are nonadditive. NPV additivity means

$$\text{NPV}_{Z_1} + \text{NPV}_{Z_2} = \text{NPV}_{Z_1 + Z_2} \quad \text{for any pair of projects } Z_1, Z_2 \tag{3.1}$$

(analogously for the NFV). Therefore, to show nonadditivity it suffices to provide a counterexample, i.e. a pair of projects (or a class of pairs of projects) for which eq. (3.1) does not hold.

Proposition 3.1 *The disequilibrium NPV is nonadditive.*

Proof: Consider a pair of projects Z_1 and Z_2 such that $Z_1 = (-h, k)$ and $Z_2 = (-I_Z + h, F_Z - k)$ with h, k being any nonzero real numbers (note that $Z = Z_1 + Z_2$). Consider the function

$$f(h,k) := \overbrace{\left(-(I_Z - h) + \frac{\overline{F}_Z - k}{R_f + \dfrac{\lambda \text{cov}(F_Z, r_m)}{(I_Z - h)}} \right)}^{\text{dNPV}_{Z_1}} + \overbrace{\left(-h + \frac{k}{R_f} \right)}^{\text{dNPV}_{Z_2}}.$$

If the disequilibrium NPV were additive, then eq. (3.1) would hold and $f(h,k)$ would be constant under changes in h and k (in particular, we would have $f(h,k) = f(0,0) = \text{dNPV}_Z$ for all h, k). But

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$$\frac{\partial f(h,k)}{\partial h} = -\frac{\lambda \operatorname{cov}(F_Z, r_m)(\overline{F}_Z - k)}{[R_f(I_Z - h) + \lambda \operatorname{cov}(F_Z, r_m)]^2}$$

$$\frac{\partial f(h,k)}{\partial k} = \frac{1}{R_f} - \frac{1}{R_f + \dfrac{\lambda \operatorname{cov}(F_Z, r_m)}{(I_Z - h)}}$$

which, in general, are not identically zero; therefore $f(h,k)$ is not invariant with respect to h and k.

<div align="right">Q.E.D.</div>

Proposition 3.2 *The equilibrium NPV is additive.*

Proof: Consider any pair of projects Z_1 and Z_2, with I_{Z_1} and I_{Z_2} being the respective outlays, while F_{Z_1} and F_{Z_2} are the respective end-of-period outcomes. Let $I_Z := I_{Z_1} + I_{Z_2}$ and $F_Z := F_{Z_1} + F_{Z_1}$. Using the certainty-equivalent form of the equilibrium NPV (see eq. (1.7)) we have

$$\operatorname{eNPV}_{Z_1} + \operatorname{eNPV}_{Z_2} = \frac{\overline{F}_{Z_1} - \lambda \operatorname{cov}(F_{Z_1}, r_m)}{R_f} - I_{Z_1} + \frac{\overline{F}_{Z_2} - \lambda \operatorname{cov}(F_{Z_2}, r_m)}{R_f} - I_{Z_2}$$

$$= \frac{\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)}{R_f} - I_Z = \operatorname{eNPV}_Z$$

<div align="right">Q.E.D.</div>

Proposition 3.3. *The disequilibrium NFV is additive.*

Proof: Reminding that $\operatorname{dNFV}_Z = \overline{F}_Z - I_Z(1 + r_Z^d)$ (see Table 1) we have

$$\operatorname{dNFV}_{Z_1} + \operatorname{dNFV}_{Z_2} = \left(\overline{F}_{Z_1} - R_f I_{Z_1} - \frac{\lambda}{I_{Z_1}} \operatorname{cov}(F_{Z_1}, r_m) I_{Z_1}\right) + \left(\overline{F}_{Z_2} - R_f I_{Z_2} - \frac{\lambda}{I_{Z_2}} \operatorname{cov}(F_{Z_2}, r_m) I_{Z_2}\right)$$

$$= (\overline{F}_Z - I_Z) - r_f I_Z - \lambda \operatorname{cov}(F_Z, r_m)$$

$$= \operatorname{dNFV}_Z$$

<div align="right">Q.E.D.</div>

Proposition 3.4. *The equilibrium NFV is nonadditive.*

Proof: Consider a pair of projects Z_1 and Z_2 such that $Z_1 = (-h, k)$ and $Z_2 = (-I_Z + h, F_Z - k)$ with h, k being any nonzero real numbers (note that $Z = Z_1 + Z_2$). Taking into consideration eq. (1.10) and reminding that $\text{cov}(k, r_m) = 0$ for all $k \in R$, consider the function

$$g(h,k) = \overbrace{\left[(\overline{F}_Z - k) - (I_Z - h) - \left(r_f + \frac{\lambda \, \text{cov}(F_Z - k, r_m)}{V_{Z_1}^e} \right)(I_Z - h) \right]}^{\text{eNFV}_{Z_1}} + \overbrace{\left(k - h - r_f h \right)}^{\text{eNFV}_{Z_2}}.$$

Manipulating algebraically, we find

$$g(h,k) = \overline{F}_Z - R_f I_Z - \frac{\lambda \, \text{cov}(F_Z, r_m)(I_Z - h)}{V_{Z_1}^e}$$

with

$$V_{Z_1}^e = V_{Z_1}^e(k) = \frac{\overline{F}_Z - k - \lambda \, \text{cov}(F_Z - k, r_m)}{R_f}$$

so that

$$\frac{\partial g(h,k)}{\partial h} = \frac{\lambda \, \text{cov}(F_Z, r_m)}{V_{Z_1}^e(k)}$$

$$\frac{\partial g(h,k)}{\partial k} = -\frac{\lambda \, \text{cov}(F_Z, r_m)(I - h)}{R_f \left[V_{Z_1}^e(k) \right]^2}$$

which, in general, are not identically zero.

Q.E.D.

Table 2 summarizes the results obtained, showing that additivity is, so to say, two-dimensional, depending on the two pairs equilibrium/disequilibrium and present/future.

Table 3 illustrates a numerical example where a decision maker is supposed to be evaluating two risky projects. The security market is composed, for the sake of simplicity, of a single risky security (so that its rate of return coincides with the market rate of return r_m); one of three states of nature may occur with probabilities equal to 0.4, 0.3, 0.4 respectively. The risk-free security has a face value of 120 and a price of 90. The risk-free rate is therefore 33.33% (=120/90-1). To compute the four net values, we use eqs. (1.5) (dNPV) and (1.7) (eNPV), while the dNFV (eq. (1.8)) and the eNFV (eq. (1.9)) are found by multiplying the former by $(1 + r_Z^d)$ and the latter by $(1 + r_Z^e)$ (eqs. (1.12) and (1.13)). Consistently with the Propositions above, the sum of the dNPVs (eNFVs) of the two projects is not equal to the dNPV (eNFV) of the project obtained by summing the two projects' cash flows. Conversely, the eNPV and the dNFV are

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additive, which confirms the economic interpretation of these indexes as valuation tools: eq. (2.6) just represents the eNPV as the price increase times the number of shares outstanding, which exactly measures the increase in shareholders' wealth if project is undertaken.

Remark 3.1 It is worth noting that the the dNFV and the eNPV are risk-free-related, so to say, in the sense that the equilibrium Net Present Value is just the discounted value of the disequilibrium Net Future Value, where the discount rate is the risk-free rate of the security market:

$$\frac{\text{dNFV}}{R_f} = \frac{\overline{F}_Z - I_Z\left(R_f + \frac{\lambda}{I_Z}\text{cov}(F_Z, r_m)\right)}{R_f} = \frac{\overline{F}_Z - \lambda\,\text{cov}(F_Z, r_m)}{R_f} - I_Z = \text{eNPV}. \qquad (3.2)$$

Referring to the example of Table 3 and, in particular, to projects Z_1 and Z_2, we have 6.41=8.55/1.3333 and 9.66=12.88/1.3333.) This fact may be interpreted in an arbitrage perspective. Suppose a shareholder owns n shares of the firm; before acceptance of the project the value of his portfolio is nP_l, after acceptance it becomes nP_l°. Suppose he sells m shares, with $m = n(P_l^\circ - P_l)/P_l^\circ$; then the value of his investment in the firm gets back to $nP_l^\circ - mP_l^\circ = nP_l$ as before acceptance of the project. If he invests the proceeds at the risk-free rate, he will have, at the end of the period, a certain amount equal to

$$R_f m P_l^\circ = R_f n(P_l^\circ - P_l) = \left(\frac{n}{N_l}\right)R_f N_l(P_l^\circ - P_l) = \left(\frac{n}{N_l}\right)\text{dNFV}$$

where we have used eq. (2.6) and eq. (3.2). By undoing the increase in the firm value, the investor will assure himself an arbitrage profit equal to that part of the dNFV corresponding to his investment in the firm. To put it differently, the dFNV is the (total) arbitrage profit shareholders get at the end of the period if the project is undertaken.

Remark 3.2 The dNPV and the eNFV may only be used as decision rules.[5] However, nonadditivity has something to do with decision as well. *Given* an investment, eq. (2.4) does hold, but dealing with two investments to be both accepted or rejected (or an investment composed of two sub-investments), one may not deduce that the portfolio of the two projects is profitable if the sum of the two NPVs is positive. In other words, before applying eq. (2.4), one must first consider the overall cash flows deriving from the two investments, and only afterwards compute the NPV. To calculate the NPV of each investment and then sum the NPVs is not compatible with Proposition 2.1. This boils down to saying that the disequilibrium NPV is dangerous if used for decision purposes, because decision makers coping with two or more projects

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[5] Given that the disequilibrium Net Present Value and the equilibrium Net Future Value are not valuation tools, to use the term "value" for labelling them is admittedly improper.

(or a single project that is composed of several sub-projects) may be tempted to first compute the NPV of each project and then sum the NPVs. This procedure may lead to a different sign than the one obtained with the correct procedure. It is easy to show that there may be instances where the sign of $\text{NPV}_{Z_1} + \text{NPV}_{Z_2}$ does not coincide with the sign of $\text{NPV}_{Z_1 + Z_2}$: consider again the example in Table 3 and suppose the cost of project Z_2 is equal to 48 (other things unvaried). A simple calculation shows that $\text{dNPV}_{Z_1} + \text{dNPV}_{Z_2} = 8.15 + (-5.86) = 2.29 > 0$ while $\text{dNPV}_{Z_1 + Z_2} = -1.84 < 0$ (i.e. this portfolio of projects is profitable or not depending on how the investor computes the overall NPV).

The same remarks obviously hold for the equilibrium NFV. For example, if one sets the cost of project Z_2 at 45 euros (other things unvaried) we have

$$\text{eNFV}_{Z_1} + \text{eNFV}_{Z_2} = 6.88 + (-10.77) = -3.89 < 0 \text{ and } \text{eNFV}_{Z_1 + Z_2} = 1.5 > 0.$$

4. Re-modelling the decision problem

Though the dNPV and the eNFV are nonadditive, they are impeccably deducted from the CAPM assumptions. One may well dismiss them by invoking additivity. Additivity is a cardinal assumption in finance and nonadditive measures are unacceptable. However, in modelling a decision criterion, one should preferably *obtain* rather than *assume* additivity; that is, one should not resort to additivity as an ad hoc assumption to get rid of unpleasant (though logical) results: additivity should be a logical consequence of the criterion at hand. This section is devoted to showing that the dNPV and the eNFV cannot be logically derived the CAPM assumptions if the decision process is reframed in a more general way.

First, note that Lemma 2.1 is based on a well-determined problem:

> *An economic agent faces the opportunity of investing in project Z.*
> *Should the decision maker invest in Z or not?* (DP-1)

The dichotomy is: undertake *Z*/do not undertake *Z*. Formally, the two alternatives are described in the equilibrium relations (2.2) and (2.3), which we rewrite below for the benefit of the reader:

$$Z \text{ is undertaken} \rightarrow \overline{F}_l + \overline{F}_Z = R_f (N_l + N_l^\circ) P_l^\circ + \lambda \text{cov}(F_l + F_Z, r_m). \tag{2.3}$$

$$Z \text{ is not undertaken} \rightarrow \overline{F}_l = R_f V_l + \lambda \text{cov}(F_l, r_m) = R_f N_l P_l + \lambda \text{cov}(F_l, r_m). \tag{2.2}$$

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The difference between the two equations leads to eq. (2.1), which logically implies Propositions 2.1 and 2.4 (which in turn legitimate the use of the dNPV and the eNFV for decision-making). Let us now change the framing of the problem into the following one:

> *An economic agent faces the opportunity of investing in project Z or in project Y.*
> *Should the decision maker invest in Z or in Y?*   (DP-2)

The decision problem (DP-2) states that the decision maker faces two alternatives, named "project Z" and "project Y". The problem (DP-2) is a generalization of (DP-1): the latter may be obtained by the former by stating that "project Y" is the null alternative, that is, a project with zero cash flows. It is just this general framing that prevents the dNPV and the eNFV to be deducted from the CAPM assumptions.

Proposition 4.1. *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking either project Z, which costs I_Z and generates the end-of-period payoff F_Z, or project. Y, which costs I_Y and generates the end-of-period payoff F_Y. The firm's share price increases if and only if project Z's eNPV (respectively, dNFV) is greater than project Y's eNPV (respectively, dNFV).*

Proof. If Z is undertaken, the equilibrium relation will be

$$Z \text{ is undertaken} \rightarrow \overline{F}_l + \overline{F}_Z = R_f(N_l + N_l^\circ)P_l^\circ + \lambda \operatorname{cov}(F_l + F_Z, r_m).$$ (2.3)

If Y is undertaken, an analogous equilibrium relation will hold, where Y replaces Z:

$$Y \text{ is undertaken} \rightarrow \overline{F}_l + \overline{F}_Y = R_f(N_l + N_l^\bullet)P_l^\bullet + \lambda \operatorname{cov}(F_l + F_Y, r_m).$$ (2.3-bis)

Subtracting (2.3-bis) from (2.3), one gets

$$\left[\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m) - R_f I_Z\right] - \left[\overline{F}_Y - \lambda \operatorname{cov}(F_Y, r_m) - R_f I_Y\right] = R_f N_l(P_l^\circ - P_l^\bullet)$$

where we have used the equality $N_l^\bullet P_l^\bullet = I_Y$. Therefore,

$$R_f\left[(V_Z^e - I_Z) - (V_Y^e - I_Y)\right] = R_f N_l(P_l^\circ - P_l^\bullet)$$ (4.1)

so that

$$P_l^\circ > P_l^\bullet \quad \text{if and only if} \quad \text{eNPV}_Z > \text{eNPV}_Y. \tag{4.2a}$$

so proving the first part of the proposition. Owing to eq. (3.2). we also have

$$P_l^\circ > P_l^\bullet \quad \text{if and only if} \quad \text{dNFV}_Z > \text{dNFV}_Y \tag{4.2b}$$

(as long as $R_f > 0$), so proving the second part.

Q.E.D.

Poposition 4.1 tells us that the decision rule deducted from the CAPM assumptions and (DP-2) is

invest in Z if and only if its eNPV (dNFV) is greater than Y's eNFV (dNFV).

Corollary 4.1. *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking project Z, which costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if project Z's eNPV (resp. dNFV) is positive.*

Proof. The assumptions are the same as in Proposition 4.1, with *Y* being the null alternative (with cash flows equal to zero). Then, the net present value in (4.2a) (the net finale value in (4.2b)) is zero, and the criterion becomes

invest in Z if and only if the eNPV (dNFV) is positive.

Q.E.D.

We now prove that eqs. (2.4) and (2.10) cannot be deduced from (DP-2)

Proposition 4.2. *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking either project Z, which costs I_Z and generates the end-of-period payoff F_Z or project. Y, which costs I_Y and generates the end-of-period payoff F_Y. The dNPV rule and the eNFV rule cannot be derived.*

Proof. As seen, problem (DP-2) implies eqs. (2.3) and (2.3-bis). If the dNPV rule is deductable from these equations, then it must be

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$$P_l^\circ > P_l^\bullet \text{ if and only if } \quad \frac{\overline{F}_Z}{R_f + \lambda \text{cov}(\frac{F_Z}{I_Z}, r_m)} - I_Z > \frac{\overline{F}_Y}{R_f + \lambda \text{cov}(\frac{F_Y}{I_Y}, r_m)} - I_Y \tag{4.3}$$

However, this excludes the case where Y is the null alternative, because I_Y cannot be zero. Furthermore, subtracting (2.3-bis) from (2.3) and manipulating, one gets to

$$\overline{F}_Z - \overline{F}_Y - (I_Z - I_Y)\left(R_f + \lambda \text{cov}(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m)\right) = N_l R_f (P_l^\circ - P_l^\bullet)$$

whence

$$P_l^\circ > P_l^\bullet \text{ se e solo se } \quad \frac{\overline{F}_Z}{R_f + \lambda \text{cov}(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m)} - I_Z > \frac{\overline{F}_Y}{R_f + \lambda \text{cov}(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m)} - I_Y \tag{4.4}$$

which is not equivalent to (4.3) . As for the eNFV rule to be valid, it must be

$$P_l^\circ > P_l^\bullet \text{ if and only if } \quad (\overline{F}_Z - I_Z) - r_Z^e I_Z > (\overline{F}_Y - I_Y) - r_Y^e I_Y. \tag{4.5}$$

But subtracting (2.3-bis) from (2.3) and using the equalities $\overline{F}_j - \lambda \text{cov}(F_j, r_m) = V_j^e R_f$ and $1 + r_j^e = \dfrac{\overline{F}_j}{V_j^e}$, $j=Z, Y,$ algebraic manipulations lead to

$$P_l^\circ > P_l^\bullet \text{ if and only if } \quad \frac{(\overline{F}_Z - I_Z) - r_Z^e I_Z}{1 + r_Z^e} > \frac{(\overline{F}_Y - I_Y) - r_Y^e I_Y}{1 + r_Y^e} \tag{4.6}$$

which is not equivalent to (4.5).

Q.E.D.

The eNFV and the dNPV rule are thus removed with no need of invoking additivity: simply, they are not deductable from the CAPM assumptions if the decision problem is (DP-2), which transforms (and generalizes) the dychotomy "undertake Z/do not undertake Z" into "undertake Z/undertake Y".

5. Decreasing net values and project valuation

The previous sections have shown that *only* the eNPV and the dNFV are legitimately deducted from the CAPM and an appropriate decision problem: they are deducted not only as decision rules but also as

valuation tools. In other words, they provide the project value (current and future respectively). This section shows that, despite their additivity, the eNPV or dNFV may be misleading in some cases.

Consider a project whose random end-of-period payoff is $F_Z^k \in R$ if state k occurs, $k=1, 2, \ldots, n$. The project disequilibrium NFV and the project equilibrium NPV may be represented as functions of n variables:

$$\mathrm{dNFV}(F_Z^1, F_Z^2, \ldots, F_Z^n) = \overline{F}_Z - I_Z \left(R_f + \frac{\lambda}{I_Z} \mathrm{cov}(F_Z, r_m) \right)$$

$$= \sum_{k=1}^{n} p_k F_Z^k - I_Z R_f - \lambda \left(\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \overline{r}_m \sum_{k=1}^{n} p_k F_Z^k \right)$$

(5.1)

$$\mathrm{eNFV}(F_Z^1, F_Z^2, \ldots, F_Z^n) = \frac{\overline{F}_Z - \lambda \, \mathrm{cov}(F, r_m)}{R_f} - I_Z$$

$$= \frac{\sum_{k=1}^{n} p_k F_Z^k - \lambda \left(\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \overline{r}_m \sum_{k=1}^{n} p_k F_Z^k \right)}{R_f} - I_Z$$

(5.2)

where p_k is the probability of state k. For functions (5.1) and (5.2) to provide correct (net) values, they must abide by the no-arbitrage principle. In other words, increasing end-of-period cash flows should lead to increasing values, ceteris paribus. Consider two assets Z and W that may be purchased at the same price. Suppose $F_Z^k = F_W^k$ for all k but s, with $F_Z^s < F_W^s$. Asset W may then be seen as asset Z plus an arbitrage profit paying off nonnegative amounts in all states and a strictly positive amount $(F_W^s - F_Z^s)$ in state s. Asset W's value must therefore be higher than asset Z's, otherwise arbitrage opportunities arise.[6] From a capital budgeting perspective, given a determined eNPV and dNFV for project Z, project W must have higher eNPV and dNFV (assuming their costs are equal), which boils down to $\dfrac{\partial}{\partial F_Z^k} \mathrm{dNFV} > 0$ and $\dfrac{\partial}{\partial F_Z^k} \mathrm{eNPV} > 0$ for every $k=1, 2, \ldots, n$. If, instead, the project under consideration is such that

$$\frac{\partial}{\partial F_Z^s} \mathrm{dNFV} < 0 \quad \text{and} \quad \frac{\partial}{\partial F_Z^s} \mathrm{eNPV} < 0 \text{ for some } s$$

(5.3)

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[6]From a stochastic dominance perspective, note that asset W dominates Z according to both first-order and second-order stochastic dominance).

the dNFV and the eNPV do not provide a reliable valuation, because they are inconsistent with the no-arbitrage principle. From eq. (5.1) we have that

$$\frac{\partial}{\partial F_Z^s} \text{dNFV} = \frac{\partial}{\partial F_Z^s}\left[\sum_{k=1}^n p_k F_Z^k - I_Z R_f - \lambda\left(\sum_{k=1}^n p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^n p_k F_Z^k\right)\right] \tag{5.4}$$

and, owing to eq. (3.2),

$$\frac{\partial}{\partial F_Z^s} \text{eNPV} = \frac{1}{R_f}\left(\frac{\partial}{\partial F_Z^s}\text{dNFV}\right). \tag{5.5}$$

Also, it is evident that

$$\frac{\partial}{\partial F_Z^s}\sum_{k=1}^n p_k F_Z^k = p_s \tag{5.6}$$

$$\frac{\partial}{\partial F_Z^s}\left[\sum_{k=1}^n p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^n p_k F_Z^k\right] = p_s r_{ms} - \bar{r}_m p_s \tag{5.7}$$

with r_{mk} being the market rate of return if state k occurs. Therefore, we may write

$$\frac{\partial}{\partial F_Z^s}\text{dNFV} = p_s - \lambda p_s r_{ms} + \bar{r}_m \lambda p_s \quad \text{and} \quad \frac{\partial}{\partial F_Z^s}\text{eNPV} = (p_s - \lambda p_s r_{ms} + \bar{r}_m \lambda p_s)/R_f . \tag{5.8}$$

Let us now consider project Z in Table 4. Considering its dNFV and eNPV as functions of F_Z^3 (end-of-period cash flow if state 3 occurs) and using eq. (5.8), we find that condition (5.3) is satisfied for $s=3$:

$$\frac{\partial}{\partial F_Z^3}\text{dNFV} = 0.3 - 4.52(0.3)(0.8518 - 0.6259) < 0$$

$$\frac{\partial}{\partial F_Z^3}\text{eNPV} = \frac{1}{1.3333}\left[0.3 - 4.52(0.3)(0.8518 - 0.6259)\right] < 0.$$

This means "the more the payoff, the less the value", which is incompatible with an arbitrage-free evaluation. Note that project Z may be seen as the risky security plus an arbitrage profit that pays off nonnegative cash flows in all states and a strictly positive amount of 250 if state 3 occurs.[7] Therefore, project Z must have a higher (net) value than the risky security. Given that the net values of the risky security are zero (for the risky security lies on the SML), project Z's net values must be positive. Both first-order and second-order stochastic dominance confirm the natural intuition according to which Z dominates

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[7] It is possible to set the project's cost lower than the risky security's price, so that the arbitrage becomes a *strong* arbitrage, with a positive net cash flow at time 0 and nonnegative amount (possibly positive) at time 1.

the risky security. Yet, both the equilibrium NPV and the disequilibrium NFV are negative. They signal nonprofitability for project Z (the equilibrium value is 52.808, smaller than the cost) or, equivalently, they do not signal that the project gives the investor an arbitrage opportunity. This means that, if the dNPV and the eNFV are not additive, the eNPV and the dNFV have pitfalls as well, even though they are additive.

This enables us to state the following

Proposition 5.1. *Suppose that*

    (a) the security market is in equilibrium

    (b) condition (5.3) holds, i.e. $\dfrac{\partial}{\partial F_Z^s}\,\mathrm{dNFV} < 0$ *and* $\dfrac{\partial}{\partial F_Z^s}\,\mathrm{eNPV} < 0$ *for some s*

Then, the eNPV and the dNFV may not be used for valuation (nor decision) purposes.

Proposition 5.1 bears relation to a previous result found by Dybvig and Ingersoll (1982, p. 237). The authors, dealing with pricing of marketed assets in a complete market, prove the following:

Dybivg and Ingersoll's Proposition (DIP).

Suppose that

    (i) mean-variance pricing holds for all assets, that is, $\bar{r}_l - r_f = \lambda\,\mathrm{cov}(r_l, r_m)$ with $r_f, \lambda > 0$

    (ii) markets are complete so that any payoff across states can be purchased as some   portfolio of marketed securities; and

    (iii) the market portfolio generates sufficiently large returns in some state(s), that is,

    $\mathrm{prob}\,(r_m > \bar{r}_m + 1/\lambda) > 0$.

Then there exists an arbitrage opportunity.

Remark 5.1 It is worth noting that condition (b) of Proposition 4.1 is equivalent to Dybvig and Ingersoll's condition (iii), because $\mathrm{prob}\,(r_m > \bar{r}_m + 1/\lambda) > 0$ if and only if $r_{ms} > \bar{r}_m + 1/\lambda$ for some s, which means $\lambda(r_{ms} - \bar{r}_m) > 1$ for some s, and, owing to eq. (4.8) and the fact that $p_s > 0$ and $R_f > 0$, the latter holds if and only if $\dfrac{\partial}{\partial F_Z^s}\,\mathrm{dNFV} < 0$ and $\dfrac{\partial}{\partial F_Z^s}\,\mathrm{dNFV} < 0$ for some s.[8] As a result, the two assumption (a) and (b) in Proposition 4.1 imply that the market is not complete. To understand why, consider that if the market were complete and (b) held, then condition (ii) and (iii) of DIP would hold. But then the market would not be in equilibrium, otherwise arbitrage opportunities would arise (see Dybvig and Ingersoll, 1982, p. 238). Therefore assumptions (a) and (b) are only compatible with an incomplete market.

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[8] The implicit assumption is that $\lambda > 0$. If not, the two conditions are not equivalent. In our particular case described in Table 4, we have $\lambda(r_{m3} - \bar{r}_m) = 4.52(0.8518 - 0.6259) = 1.02 > 1$.

The result presented in Proposition 5.1 is, so to say, the capital-budgeting counterpart of DIP. In particular, while the latter deals with pricing of *marketed* assets when the security market is *complete*, the former deals with valuation of *nonmarketed* assets (projects) when the security market is *incomplete*. The two Propositions are the two sides of the same coin and the two perspectives are perfectly reconciled (see Table 5).

6. Equilibrium value and counterfactual equilibrium price

This section shows that the equilibrium value of a project is not necessarily the value the project would have if it were traded.

Let us consider eq. (1.7) in section 1 above. It says that the eNPV is just the difference between the equilibrium value and the cost of the project: $\text{eNPV} = V_Z^e - I_Z$ where

$$V_Z^e = \frac{\overline{F}_Z - \lambda \text{cov}(F_Z, r_m)}{R_f} . \tag{6.1}$$

In finance, V_Z^e is known as the "equilibrium value" of the project. It is commonly believed that it is the price the project would have in equilibrium if it were traded in the security market (e.g. Mason and Merton, 1985, pp. 38-39, Smith and Nau, 1995, p. 800). But this equivalence does not always hold, as Smith and Nau (1995) clearly point out:

> We also have some semantic problems defining exactly what is meant by the value of a non-traded project. Earlier the … value of a project was defined as the price the project would have if it were traded in an arbitrage-free market …. This definition does not work well in general because the introduction of the project into the market may create new investment opportunities and change the prices of the traded securities. (Smith and Nau, 1995, p. 804, footnote 7)

Let us call "counterfactual equilibrium price" the price the project would have if it were traded: we now illustrate a counterexample where the equilibrium value V_Z^e differs from the counterfactual equilibrium price. Let us consider project Z introduced in Table 4. What if one counterfactually assumes that Z is traded in the security market?[9] First of all, note that the introduction of the project in the security market renders the latter a *complete* market. It is thus evident that project Z's counterfactual equilibrium price cannot coincide with the equilibrium value $V_Z^e = 52.808$ previously found, otherwise conditions (i)-(iii) of DIP would be satisfied, and arbitrage opportunities would arise (which implies that the market would not be in equilibrium). This means that when the project is introduced in the security market, market prices

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[9] This assumption is equivalent to the assumption that a security with the same payoff as project Z is traded in the market.

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shift so that the market moves toward a new equilibrium. How does the resulting new equilibrium turns out to be? Intuition would tell us that the risky security's price should decrease to avoid arbitrage (given that the project dominates it), but this is not the case. It is easy to verify that, to avoid condition (iii) of DIP and achieve an equilibrium, the risky security's price must increase and project Z's equilibrium price must increase to a larger extent so as to be greater than the risky security's price.[10] Suppose the new equilibrium is as represented in Table 6. The (counterfactual) equilibrium price of project Z is 121.57 and the price of the risky security is now 65.76. The market is now complete and arbitrage is not possible. The counterfactual equilibrium price of the project differs from the equilibrium value of the project ($121.57 \neq 52.808$). The conclusion is that the equilibrium value in eq. (6.1) is not the price the project would have if it were traded in the market. Contrary to the equilibrium value, the counterfactual equilibrium price is *rational* by definition, in the sense that arbitrage is not possible in the resulting equilibrium. This means that the counterfactual equilibrium price is obviously the correct value of the project.

One might think that, for valuation to be correct, one should replace the equilibrium value with the counterfactual equilibrium price. Unfortunately the counterfactual equilibrium price cannot be univocally determined. Table 7 shows another possible equilibrium for the market where project Z is traded. The equilibrium counterfactual price in this second equilibrium is equal to 76.197, which not simply conflicts with the equilibrium value of the project, but differs form the counterfactual equilibrium price previously found. Which one of the two counterfactual equilibrium prices is the one to be used for valuation? The answer is not possible, because there is no way of anticipating how equilibrium is reached from a disequilibrium situation. That is, one cannot compute ex ante "the" equilibrium price the project would have if it were traded in the security market. However, from a practical point of view, one may collect statistical data and make an ex ante estimation of the most probable equilibrium the market would reach. In this case, the *estimated* counterfactual equilibrium price could be taken as the correct project value.[11]

Remark 5.1 Proposition 5.1 just gives us the reason why the equilibrium value may sometimes turn out to be incorrect. The correct value measuring increase in shareholders' wealth is indeed given by the equilibrium value if the market is complete and in equilibrium. Problems in project valuation arise only when the market is not complete and condition (5.3) holds.[12] In this case, equilibrium value and counterfactual equilibrium price are not equal. A project's equilibrium value is therefore reliable only if

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[10] This result holds regardless of the number of shares of project Z (or of the security having the same payoff as Z) that are traded in the market.

[11] From a theoretical point of view, upper and lower bound can be computed for the counterfactual equilibrium price (Smith and Nau, 1995), but whenever the cost is greater than the lower limit and smaller than the upper limit, the "optimal strategy is unclear"(Smith and Nau, 1995, p. 805), and decision is not possible (a further analysis must be conducted to reach a single estimated value).

[12] It is worth reminding that if the market is complete and in equilibrium, condition (5.3) may not hold (given that the equivalent condition (iii) of DIP may not hold). Conversely, if the market is not complete and in equilibrium, condition (5.3) may hold, as we have seen.

the market is complete; in this case it does represent the (counterfactual) equilibrium price that the project would actually have if it were traded.

Conclusions

The CAPM is a theoretical model aimed at valuing financial assets in a security market under the assumption that the market is in equilibrium. As widely known, the CAPM may also be used as a decision criterion: an investment is worth undertaking if and only if the investment expected rate of return is greater than the (cost-based) risk-adjusted cost of capital (Rubinstein, 1973). However, the role of this simple criterion has not been thoroughly investigated, so that errors and misunderstanding often arise in financial textbooks and papers, where the CAPM is incorporated in the net present value criterion in an unclear way, with no explicit indication of

- o   the way it should be computed (use of disequilibrium data versus equilibrium data),
- o   the purpose it serves (decision or valuation)
- o   the relation excess return (net future value) bears to present value.

This paper, focusing on accept-reject situations and one-period projects, aims at providing a clarification of these issues. In particular, it shows that:

- from the CAPM four *decision* rules are validly deducted: the disequilibrium Net Present Value, the equilibrium Net Future Value, the equilibrium Net Present Value, the disequilibrium Net Future Value. All of them may be interchangeably used for *decision-making*

- while logically impeccable as decision tools, the disequilibrium NPV (equilibrium NFV) may lead to incorrect decisions if decision makers facing a portfolio of several projects (or a project composed of several sub-projects) separately compute each project's NPV (NFV) and then sum the values obtained. The correct procedure is: to sum the cash flows of the projects and then compute the disequilibrium NPV (equilibrium NFV)

- only the equilibrium NPV and the disequilibrium NFV are additive, which means that they may be used for *valuation* purposes. The other two are not valuation tools, because they are nonadditive

- the deduction of the disequilibrium NPV (equilibrium NFV) from the CAPM assumptions is possible because the decision problem is shaped as "undertake Z/do not undertake Z". If the problem is reframed in a more general way as "undertake Z/undertake *Y*", the two nonadditive decision rules may not be deduced from the CAPM assumptions

- even if the market is in equilibrium, the project's equilibrium NPV and disequilibrium NFV lead to an incorrect valuation whenever they are decreasing functions with respect to end-of-period cash flow in some state of nature (which implies that the security market is incomplete). This result is

the capital-budgeting equivalent of Dybvig and Ingersoll's (1982) result, which they find under the assumption of a complete market

- if the above stated condition holds, the correct value would be given by the (counterfactual) equilibrium price the project would have if it were traded in the security market. Unfortunately, this price is not univocally determined ex ante and one can only rely on an *estimated* equilibrium price based on exogenous data about the market.

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Table 0. Main Notational Conventions

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F_j $(\overline{F}_j)$	Asset j's end-of-period random (expected) cash flow
I_j	Cost of project j
r_j $(\overline{r}_j)$	Asset j's random (expected) rate of return
V_j^e (V_j^d)	Equilibrium (disequilibrium) value of asset j
r_Z^d	Disequilibrium (cost-based) rate of return of project Z (aka risk-adjusted cost of capital)
r_Z^e (r_Y^e)	Equilibrium rate of return of project Z (Y)
r_f (R_f)	Risk-free rate (1+risk-free rate)
σ_m^2	Variance of the market rate of return
cov	Covariance
$\lambda := \dfrac{\overline{r}_m - r_f}{\sigma_m^2}$	Market price of risk
P_l	Price of firm l's shares before acceptance of the project
P_l° $(P_l^{\bullet})$	Price of firm l's shares before acceptance of project Z (Y)
N_l	Number of firm l's outstanding shares
N_l° $(N_l^{\bullet})$	Additional shares issued at price P_l° $(P_l^{\bullet})$ to finance project Z (Y)
V_l	Firm value before acceptance of the project
V_l°	Firm value after acceptance of project Z
dNPV$_j$, eNPV$_j$	Disequilibrium (equilibrium) net present value of project j
dNFV$_j$, eNFV$_j$	Disequilibrium (equilibrium) net future value of project j

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j=Z, Y, Z₁, Z₂, l, m

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Table 1. Equilibrium and disequilibrium net values		
	Equilibrium	**Disequilibrium**
Net Present Value	$\dfrac{\overline{F}_Z}{(1+r_Z^e)} - I_Z$ $\dfrac{I_Z(\bar{r}_Z - r_Z^e)}{1+r_Z^e}$	$\dfrac{\overline{F}_Z}{(1+r_Z^d)} - I_Z$ $\dfrac{I_Z(\bar{r}_Z - r_Z^d)}{1+r_Z^d}$
Net Future Value **(excess return)**	$\overline{F}_Z - I_Z(1+r_Z^e)$ $I_Z(\bar{r}_Z - r_Z^e)$	$\overline{F}_Z - I_Z(1+r_Z^d)$ $I_Z(\bar{r}_Z - r_Z^d)$

Table 2. Additive and nonadditive net values		
	Equilibrium	**Disequilibrium**
Net Present Value	Additive	Nonadditive
Net Future Value	Nonadditive	Additive

	Proj. Z_1	Proj. Z_2	Proj. Z_1+ Z_2	Proj. Z_1 +Proj. Z_2	Risky security	Risk-free security	Market (3000 sh.)
	Table 3. Project valuation with the CAPM						
F	$\begin{cases} 130 \\ 100 \\ 10 \end{cases}$	$\begin{cases} 120 \\ 50 \\ 80 \end{cases}$	$\begin{cases} 250 \\ 150 \\ 90 \end{cases}$		$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294000 \\ 213000 \\ 300000 \end{cases}$
Cost/price	70	30	100		54	90	162000
Rate of return (%)	$\begin{cases} 85.7 \\ 42.8 \\ -85.7 \end{cases}$	$\begin{cases} 300 \\ 66.6 \\ 166.6 \end{cases}$	$\begin{cases} 150 \\ 50 \\ -10 \end{cases}$		$\begin{cases} 81.48 \\ 31.48 \\ 85.18 \end{cases}$	$\begin{cases} 33.3 \\ 33.3 \\ 33.3 \end{cases}$	$\begin{cases} 81.48 \\ 31.48 \\ 85.18 \end{cases}$
$\overline{F}$	82	80	162		87.8	120	263400
$\overline{r}$ (%)	17.14	166.66	62		62.59	33.33	62.59
λ	4.52						
$\lambda \operatorname{cov}(F, r_m)$	−19.88	27.12	7.23		15.8	0	47400
$\overline{F} - \lambda \operatorname{cov}(F, r_m)$	82	80	162		87.8	120	263400
Disequilibrium value	78.15	35.75	115.25	113.9	54	90	162000
Equilibrium value	76.41	39.66	116.07	116.07	54	90	162000
Disequilibrium NPV	8.15	5.75	15.24	13.9	0	0	0
Equilibrium NPV	6.41	9.66	16.07	16.07	0	0	0
Disequilibrium NFV	8.55	12.88	21.43	21.43	0	0	0
Equilibrium NFV	6.88	19.49	22.44	26.37	0	0	0

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	Project Z	Risky security	Risk-free security	Market (000,000) (3 mil. shares)	Probability
				Table 4. Decreasing net values	

	Project Z	Risky security	Risk-free security	Market (000,000) (3 mil. shares)	Probability
F	$\begin{cases} 98 \\ 71 \\ 350 \end{cases}$	$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294 \\ 213 \\ 300 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
Cost/Price	54	54	90	162	
r (%)	$\begin{cases} 81.48 \\ 31.48 \\ 548.1 \end{cases}$	$\begin{cases} 81.48 \\ 31.48 \\ 85.18 \end{cases}$	$\begin{cases} 33.3 \\ 33.3 \\ 33.3 \end{cases}$	$\begin{cases} 81.48 \\ 31.48 \\ 85.18 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
$\bar{r}$ (%)	201.48	62.59	33.33	62.59	
$\overline{F}$	162.8	87.8	120	263.4	
$\operatorname{cov}(F, r_m)$	20.44	3.49	0	10.486	
λ	4.52				
$\lambda \operatorname{cov}(F, r_m)$	92.39	15.8	0	47.4	
$\overline{F} - \lambda \operatorname{cov}(F, r_m)$	70.41	72	120	216	
V^e	52.808	54	90	162	
Equilibrium NPV	−1.19	0	0	0	
Disequilibrium NFV	−1.59	0	0	0	

	Security market	Type of assets
Table 5. Range of applicability of DIP and Proposition 4.1		
Dybvig and Ingersoll's Proposition	Complete	Securities (marketed assets)
Proposition 4.1	Incomplete	Projects (nonmarketed assets)

	Project is traded in the Market (1 share)	Risky security (3 mil. Shares)	Risk-free security	Market (000,000)	Probability
Table 6. Project *Z* is traded in the market (first equilibrium)					
F	$\begin{cases} 98 \\ 71 \\ 350 \end{cases}$	$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294.000098 \\ 213.000071 \\ 300.000350 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
Price	121.57	65.76	90	197.28	
$\bar{r}$ (%)	33.91	33.51	33.33	33.51	
NPV	0	0	0	0	

	Table 7. Project Z is traded in the market (second equilibrium)				
	Project is traded in the Market (1 share)	**Risky security (3 mil. shares)**	**Risk-free security**	**Market (000,000)**	**Probability**
F	$\begin{cases} 98 \\ 71 \\ 350 \end{cases}$	$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294.000098 \\ 213.000071 \\ 300.000350 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
Price	76.197	58	90	174	
$\bar{r}$ (%)	113.65	51.37	33.33	51.37	
NPV	0	0	0	0	