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The Growth of Business Firms:
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Abstract

We introduce a model of proportional growth to explain the distribution $P_g(g)$ of business firm growth rates. The model predicts that $P_g(g)$ is exponential in the central part and depicts an asymptotic power-law behavior in the tails with an exponent $\zeta = 3$. Because of data limitations, previous studies in this field have been focusing exclusively on the Laplace shape of the body of the distribution. In this article, we test the model at different levels of aggregation in the economy, from products to firms to countries, and we find that the model's predictions agree with empirical growth distributions and size-variance relationships.

I. INTRODUCTION

Gibrat [1, 2], building upon the work of the astronomers Kapteyn [3], assumed the expected value of the growth rate of a business firm's size to be proportional to the current size of the firm, which is called "Law of Proportionate Effect" [4, 5]. Several models of proportional growth have been subsequently introduced in economics in order to explain the growth of business firms [6, 7, 8]. Simon and co-authors [9, 10, 11, 12] extended Gibrat's model by introducing an entry process according to which the number of firms rise over time. In Simon's framework, the market consists of a sequence of many independent "opportunities" which arise over time, each of size unity. Models in this tradition have been challenged by many researchers [13, 14, 15, 16, 17] who found that the firm growth distribution is not Gaussian but displays a tent shape.

Here we introduce a general framework that provides an unifying explanation for the growth of business firms based on the number and size distribution of their elementary constituent components [18, 19, 20, 21, 22, 23, 24, 25]. Specifically we present a model of proportional growth in both the number of units and their size and we draw some general implications on the mechanisms which sustain business firm growth [7, 11, 21, 26, 27, 28]. According to the model, the probability density function (PDF) of growth rates is Laplace in the center [13] with power law tails [29, 30] decaying as $P_g(g) \sim g^{-\zeta}$ where $\zeta = 3$.

Because of data limitations, previous studies in this field focus on the Laplace shape of the body of the distribution [31]. Using a database on the size and growth of firms and products, we characterize the shape of the whole growth rate distribution.

We test our model by analyzing different levels of aggregation of economic systems, from the "micro" level of products to the "macro" level of industrial sectors and national economies. We find that the model accurately predicts the shape of the PDF of growth rate at all levels of aggregation studied.

II. THE THEORETICAL FRAMEWORK

We model business firms as classes consisting of a random number of units. According to this view, a firm is represented as the aggregation of its constituent units such as divisions [22], businesses [20], or products [21]. Accordingly, on a different level of

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coarse-graining, a class can represent a national economy composed by economic units such as firms. In this article we study the logarithm of the one-year growth rate of classes $g \equiv \log(S(t+1)/S(t))$ where $S(t)$ and $S(t+1)$ are the sizes of classes in the year t and $t+1$ measured in monetary values (GDP for countries, sales for firms and products). Our model is illustrated in Fig. 1. Two key sets of assumptions in the model are (A) the number of units in a class grows in proportion to the existing number of units and (B) the size of each unit fluctuates in proportion to its size.

The first set of assumptions is:

(A1) Each class α consists of $K_\alpha(t)$ number of units. At time $t = 0$, time step measured by year generally, there are $N(0)$ classes consisting of $n(0)$ total number of units. The initial average number of units in a class is thus $n(0)/N(0)$.

(A2) At each time step a new unit is created. Thus the number of units at time t is $n(t) = n(0) + t$.

(A3) With birth probability b, this new unit is assigned to a new class, so that the average number of classes at time t is $N(t) = N(0) + bt$.

(A4) With probability $1 - b$, a new unit is assigned to an existing class α with probability $P_\alpha = (1 - b)K_\alpha(t)/n(t)$, so $K_\alpha(t + 1) = K_\alpha(t) + 1$.

For simplicity, we do not consider the decrease of the number of units in a class. In reality, elementary units enter and exit. Because we are considering the case of a growing economy, it is legitimate to assume the entry rate being higher than the exit rate. On the average, the net entry rate of units can be simplified as a positive constant. In the model, the net entry rate of units is fixed at 1. Thus, at large t, it gives results equivalent to the ones that would have been obtained considering a value for the exit rate of units.

Our goal is to find $P(K)$, the probability distribution of the number of units in the classes at large t. This model in two limiting cases (i) $b = 0$, $K_\alpha = 1$ ($\alpha = 1, 2 \ldots N(0)$) and (ii) $b \neq 0$, $N(0) = 1$, $n(0) = 1$ has exact analytical solutions $P(K) = N(0)/t(t/(t + N(0)))^K(1 + O(1/t))$ [32, 33] and $\lim_{t \to \infty} P(K) = (1 + b)\Gamma(K)\Gamma(2 + b)/\Gamma(K + 2 + b)$ [34] respectively.

In the general case, the exact analytical solution is not known and we obtain a numerical solution by computer simulations and compare it with the approximate mean field solution. (see, e.g., Chapter 6 of [35] and Appendix A)

Our results are consistent with the exactly solvable limiting cases as well as with the empirical data on the number of products in the pharmaceutical firms and can summarized as follows. In the limit of large t, the distribution of K in the old classes that existed at $t = 0$ converges to an exponential distribution [36]

$$P_{\text{old}}(K) = \lambda^K \frac{1}{K(t) - 1} \approx \frac{1}{K(t)} \exp(-K/K(t)), \tag{1}$$

where $\lambda = 1 - 1/K(t)$ and $K(t)$ is the average number of units in the old classes at time t, $K(t) = [(n(0)+t)/n(0)]^{1-b} \cdot n(0)^b/N(0)$. The distribution of units in the new classes created at $t > 0$ converges to a power law with an exponential cutoff

$$P_{\text{new}}(K) \sim K^{-\varphi} f(K), \tag{2}$$

where $\varphi \approx 2 + b$ for small b, and $f(K)$ decays for $K \to \infty$ faster than $P_{\text{old}}(K)$. The distribution of units in all classes is given by

$$P(K) = \frac{N(0)}{N(0) + bt} P_{old}(K) + \frac{bt}{N(0) + bt} P_{new}(K). \tag{3}$$

The mean field approximation for $P_{new}(K)$ is given by

$$P_{new}(K) \approx \frac{n(0)/t + 1}{1 - b} K^{(-1/(1-b)-1)} \int_{K'}^{K} e^{-y} \, y^{\frac{1}{1-b}} \, dy. \tag{4}$$

where $K' = K[n(0)/(n(0) + t)]^{1-b}$.

The second set of assumptions is:

(B1) At time t, each class α has $K_\alpha(t)$ units of size $\xi_i(t)$, $i = 1, 2, ... K_\alpha(t)$ where K_α and $\xi_i > 0$ are independent random variables taken from the distributions $P(K_\alpha)$ and $P_\xi(\xi_i)$ respectively. $P(K_\alpha)$ is defined by Eq. (3) and $P_\xi(\xi_i)$ is a given distribution with finite mean and standard deviation and $\ln \xi_i$ has finite mean $\mu_\xi = \langle \ln \xi_i \rangle$ and variance $V_\xi = \langle (\ln \xi_i)^2 \rangle - \mu_\xi^2$. The size of a class is defined as $S_\alpha(t) \equiv \sum_{i=1}^{K_\alpha} \xi_i(t)$.

(B2) At time $t + 1$, the size of each unit is decreased or increased by a random factor $\eta_i(t) > 0$ so that

$$\xi_i(t + 1) = \xi_i(t) \, \eta_i(t), \tag{5}$$

where $\eta_i(t)$, the growth rate of unit i, is independent random variable taken from a distribution $P_\eta(\eta_i)$, which has a finite mean. We also assume that $\ln \eta_i$ has finite mean $\mu_\eta \equiv \langle \ln \eta_i \rangle$ and variance $V_\eta \equiv \langle (\ln \eta_i)^2 \rangle - \mu_\eta^2$.

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The growth rate of each class is defined as

$$g_\alpha \equiv \log\left(\frac{S_\alpha(t+1)}{S_\alpha(t)}\right) = \log\sum_{i=1}^{K_\alpha} \xi_i(t+1) - \log\sum_{i=1}^{K_\alpha} \xi_i(t). \tag{6}$$

Here we neglect the influx of the new units, so $K_\alpha = K_\alpha(t+1) = K_\alpha(t)$. The resulting distribution of the growth rates of all classes is determined by

$$P_g(g) \equiv \sum_{K=1}^{\infty} P(K)P_g(g|K), \tag{7}$$

where $P(K)$ is the distribution of the number of units in the classes, computed in the previous stage of the model and $P_g(g|K)$ is the conditional distribution of growth rates of classes with given number of units determined by the distribution $P_\xi(\xi)$ and $P_\eta(\eta)$.

The analytical solution of this model can be obtained only for certain limiting cases but a numerical solution can be easily computed for any set of assumptions. We investigate the model numerically and analytically (see Appendix B) and find:

(1) The conditional distribution of the logarithmic growth rates $P_g(g|K)$ for the firms consisting of a fixed number of units converges to a Gaussian distribution for $K \to \infty$:

$$P_g(g|K) \approx \frac{\sqrt{K}}{\sqrt{2\pi V_g}} \exp\left((g-\bar{g})^2 K/2V_g\right), \tag{8}$$

where V_g is a function of parameters of the distribution $P_\xi(\xi)$ and $P_\eta(\eta)$, and $\bar{g}$ is mean logarithmic growth rate of a unit, $\bar{g} = \langle \ln \eta_i \rangle$.

Thus the width of this distribution decreases as $1/\sqrt{K}$. This result is consistent with the observation that large firms with many production units fluctuate less than small firms [7, 18, 22, 37].

(2) For $g \gg V_\eta$, the distribution $P_g(g)$ coincides with the distribution of the logarithms of the growth rates of the units:

$$P_g(g) \approx P_\eta(\ln \eta). \tag{9}$$

In the case of power law distribution $P(K) \sim K^{-\varphi}$ which dramatically increases for $K \to 1$, the distribution $P_g(g)$ is dominated by the growth rates of classes consisting of a single unit $K = 1$, thus the distribution $P_g(g)$ practically coincides with $P_\eta(\ln \eta_i)$ for all g. Indeed, our empirical observations confirm this result.

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(3) If the distribution $P(K) \sim K^{-\varphi}$, $\varphi > 2$ for $K \to \infty$, as happens in the presence of the influx of new units $b \neq 0$, $P_g(g) = C_1 - C_2|g|^{2\varphi-3}$, for $g \to 0$ which in the limiting case $b \to 0$, $\varphi \to 2$ gives the cusp $P_g(g) \sim C_1 - C_2|g|$ (C_1 and C_2 are positive constants), similar to the behavior of the Laplace distribution $P_{\mathrm{L}}(g) \sim \exp(-|g|C_2)$ for $g \to 0$.

(4) If the distribution $P(K)$ weakly depends on K for $K \to 1$, the distribution of $P_g(g)$ can be approximated by a power law of g: $P_g(g) \sim |g|^{-3}$ in wide range $\sqrt{V_g/K(t)} \ll g \ll \sqrt{V}$, where $K(t)$ is the average number of units in a class. This case is realized for $b = 0$, $t \to \infty$ when the distribution of $P(K)$ is dominated by the exponential distribution and $K(t) \to \infty$ as defined by Eq. (1). In this particular case, $P_g(g)$ for $g \ll \sqrt{V_g}$ can be approximated by

$$P_g(g) \approx \frac{\sqrt{K(t)}}{2\sqrt{2V_g}} \left(1 + \frac{K(t)}{2V_g}g^2\right)^{-3/2}. \tag{10}$$

(5) In the case in which the distribution $P(K)$ is not dominated by one-unit classes but for $K \to \infty$ behaves as a power law, which is the result of the mean field solution for our model when $t \to \infty$, the resulting distribution $P_g(g)$ has three regimes, $P_g(g) \sim C_1 - C_2|g|^{2\varphi-3}$ for small g, $P_g(g) \sim |g|^{-3}$ for intermediate g, and $P_g(g) \sim P(\ln \eta)$ for $g \to \infty$. The approximate solution of $P_g(g)$ is obtained by using Eq. (8) for $P_g(g|K)$ for finite K, mean field solution Eq. (4) in the limit $t \to \infty$ for $P(K)$ and replacing summation by integration in Eq. (7):

$$P_g(g) = \frac{1}{1-b} \frac{1}{\sqrt{2\pi V}} \int_0^\infty \exp(-y) \, y^{\frac{1}{1-b}} \, dy \int_y^\infty \exp(-g^2 K/2V) \, K^{(-\frac{1}{2} - \frac{1}{1-b})} \, dK. \tag{11}$$

For $b \neq 0$ the integral above can be expressed in elementary functions. In the $b \to 0$ case, Eq. (11) yields the main result

$$P_g(g) \approx \frac{2V_g}{\sqrt{g^2 + 2V_g} \, (|g| + \sqrt{g^2 + 2V_g})^2}, \qquad (b \to 0) \tag{12}$$

which combines the Laplace cusp for $g \to 0$ and the power law decay $|g|^{-3}$ for $g \to \infty$. Note that due to replacement of summation by integration in Eq. (7), the approximation Eq. (12) holds only for $g < \sqrt{V_\eta}$.

In Fig. 2a we compare the distributions given by Eq. (10), the mean field approximation Eq. (11) for $b = 0.1$ and Eq. (12) for $b \to 0$. We find that all three distributions have very

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similar tent shape behavior in the central part. In Fig. 2b we also compare the distribution Eq. (12) with its asymptotic behaviors for $g \to 0$ (Laplace cusp) and $g \to \infty$ (power law), and find the crossover region between these two regimes.

III.   THE EMPIRICAL EVIDENCE

To test our model, we analyze different levels of aggregation of economic systems, from the micro level of products to the macro level of industrial sectors and national economies.

First, we analyze a new and unique database, the pharmaceutical industry database (PHID), which records sales figures of the 189,303 products commercialized by 7,184 pharmaceutical firms in 21 countries from 1994 to 2004, covering the whole size distribution for products and firms and monitoring the flows of entry and exit at both levels kindly provided by the EPRIS program. Then, we study the growth rates of all U.S. publicly-traded firms from 1973 to 2004 in all industries, based on Security Exchange Commission filings (Compustat). Finally, at the macro level, we study the growth rates of the gross domestic product (GDP) of 195 countries from 1960 to 2004 (World Bank).

Fig. 3 shows that the growth distributions of countries, firms, and products are well fitted by the distribution in Eq. (12) with different values of V_g. Indeed, growth distributions at any level of aggregation depict marked departures from a Gaussian shape. Moreover, even if the $P_g(g)$ of GDP can be approximated by a Laplace distribution, the $P_g(g)$ of firms and products are clearly more leptokurtic than Laplace. Based on our model, the growth distribution is Laplace in the body, with power-law tails. In fact, Fig. 4 show that the central body part of the growth rate distributions at any level of aggregation is well approximated by a double exponential fit. Fig. 5 reveals that the asymptotic behaviors of g at any level of aggregation can be well fitted by power-law with an exponent $\zeta = 3$.

Our analysis in Sec. II predicts that the power law regime of $P_g(g)$ may vary depending on the behavior of $P(K)$ for $K \to 1$, and the distribution of the growth rates of units. In case of PHID, for which $P(1) \gg P(2) \gg P(3) \ldots$ the growth rate distribution of firms must be almost the same as the growth rate distribution of products, as we stated in Sec. II. Hence the power law wings of $P_g(g)$ for firms originate on the level of products. Because PHID does not contain information on the subunits of products we can not test our prediction directly, but we can hypothesize that the distribution of the product subunits (number

of customers or shipping ways) is less dominated by small K, but has a sufficiently wide power law regime due to the influx of new products. These rather plausible assumptions are sufficient to explain the shape of the distribution of the product growth rates, which is well described by Eq. (12).

The PHID database allows us to test the empirical conditional distribution $P_g(g|K)$ and the dependence of its variance σ^2 on K, where K is the number of products. We find that $\sigma \sim K^{-0.28}$, which is significantly smaller than $1/\sqrt{K}$ behavior. This result does not imply correlations among product growth rates on the firm level [21], but can be explained by the fact that for skewed distributions of product sizes $P_\xi(\xi)$ characterized by large V_ξ, the convergence of $P_g(g|K)$ to its Gaussian limit Eq. (8) is slow and the growth rates of the firms are determined by the growth of the few large products. Using the empirical values for the PHID $\mu_\xi = 3.44$, $V_\xi = 5.13$, $\mu_\eta = 0.016$, $V_\eta = 0.36$ and assuming lognormality of the distributions $P_\xi(\xi)$ and $P_\eta(\eta)$ we find that the behavior of σ can be well approximated by a power law $\sigma \sim K^{-0.20}$ for $K < 10^3$. For this set of parameters, the convergence of $P_g(g|K)$ to a Gaussian distribution takes place only for $K > 10^5$. This result is consistent with the observations of the power law relationship between firm size and growth rate variance reported earlier [13, 18, 19, 38].

IV.   DISCUSSION

Business firms grow by increasing their scale and scope. The scope of a firm is given by the number of its products. The scale of a firm is given by the size of its products. A firm like Microsoft gets few big products while Amazon sells a huge variety of goods, each of small size in terms of sales. In this article we argue that both mechanisms of growth are proportional. The number of products a firm can successfully launch is proportional to the number of products it has already commercialized. Once a product has been launched its success depends on the number of customers who buy it and the price they are willing to pay. To a large extent, if products are different enough, the success of a product is independent from other products commercialized by the same company. Hence, the sales of products can be modeled as independent stochastic processes. Moreover, sometimes, new products are commercialized by new companies. As a result, small companies with few products can experience sudden jerks of growth due to the successful launch of a new product.

In this article, we find that the empirical distribution of firm growth rates exhibits a central part which is distributed according to a Laplace distribution and power-law wings $P_g(g) \sim g^{-\zeta}$ where $\zeta = 3$. If the distribution of number of units K is dominated by single unit classes, the tails of firm growth rate are primarily due to smaller firms composed of one or few products. The Laplace center of the distribution is shaped by big multiproduct firms. We find that the shape of the distribution of firm growth is almost the same in presence of a small entry rate and with zero entry. We also find that the model's predictions are accurate also in the case of product growth rates, which implies that products can be considered as composed by elementary sale units, which evolve according to a random multiplicative process [6]. Although there are several plausible explanations for the Laplace body of the distribution [18, 31], the power law decay of the tails has not previously been observed. We introduce a simple and general model that accounts for both the central part and the tails of the distribution. The shape of the business growth rate distribution is due to the proportional growth of both the number and the size of the constituent units in the class. This result holds in the case of an *open* economy (with entry of new firms) as well as in the case of a *closed* economy (with no entry of new firms).

Acknowledgment

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Appendix A: The distribution of units in old and new classes

Assume that at the beginning there are $N(0)$ classes with $n(0)$ units. Because at every time step one unit is added to the system and a new class is added with probability b, at moment t there are $n(t) = n(0) + t$ units and $N(t) = N(0) + bt$ classes, among which there are bt new classes with n_{new} units and $N(0)$ old classes with n_{old} units, such that $n_{old} + n_{new} = n(0) + t$.

Because of the preferential attachment assumption, we have

$$\frac{dn_{new}}{dt} = b + (1 - b)\frac{n_{new}}{n(0) + t}, \tag{A1}$$

$$\frac{dn_{old}}{dt} = (1 - b)\frac{n_{old}}{n(0) + t}. \tag{A2}$$

Solving the second differential equation and taking into account initial condition $n_{old}(0) = n(0)$, we obtain

$$n_{old}(t) = (n(0) + t)^{1-b}\, n(0)^b. \tag{A3}$$

Analogously, the number of units at time t in the classes existing at time t_0 is

$$n_e(t_0, t) = (n(0) + t)^{1-b}(n(0) + t_0)^b,$$

where the subscript 'e' means "existing". The average number of units in old classes is

$$K(t) = \frac{n_{old}(t)}{N(0)} = \frac{(n(0) + t)^{1-b}}{N(0)}\, n(0)^b. \tag{A4}$$

It is known that [36] for $t \to \infty$ the preferential attachment model converges to the exponential distribution:

$$P_{app}(K) \approx \exp(-K/K(t))/K(t). \tag{A5}$$

Thus, we obtain

$$P_{old}(K) \approx \frac{N(0)}{(n(0) + t)^{1-b}n(0)^b}\ \exp\left(-\frac{K\,N(0)}{(n(0) + t)^{1-b}n(0)^b}\right), \tag{A6}$$

and the part of $P(K)$ of old classes is

$$\tilde{P}_{old}(K) \approx P_{old}(K)\frac{N(0)}{N(0) + t}. \tag{A7}$$

The number of units in the classes that appear at t_0 is $b\,dt$ and the number of these classes is $b\,dt$. Because the probability that a class captures a new unit is proportional to

the number of units it has already gotten at time t, the number of units in the classes that appear at time t_0 is

$$n_{new}(t_0, t) = n_e(t_0, t) \cdot \frac{bdt}{n(0) + t_0}.$$

The average number of units in these classes is $K(t_0, t) = n_{new}(t_0, t)/b\,dt = (n(0) + t)^{1-b}/(n(0) + t_0)^{1-b}$. Assuming that the distribution of units in these classes is given by a continuous approximation in Eq. (A5):

$$P_{new}(K) \approx \frac{1}{K(t_0, t)} \exp\left(-K/K(t_0, t)\right) \tag{A8}$$

Thus, their contribution to the total distribution is

$$\frac{b\,dt_0}{N(0) + b\,t} \frac{1}{K(t_0, t)} \exp\left(-K/K(t_0, t)\right)$$

The contribution of all new classes to the distribution $P(K)$ is

$$\tilde{P}_{new}(K) \approx \frac{b}{N(0) + b\,t} \int_0^t \frac{1}{K(t_0, t)} \exp\left(-K/K(t_0, t)\right)\, dt_0. \tag{A9}$$

If we let $y = K/K(t_0, t)$, then

$$\tilde{P}_{new}(K) \approx \frac{b}{1 - b} K^{\left(-\frac{1}{1-b} - 1\right)} \frac{n(0) + t}{N(0) + bt} \int_{K\left(\frac{n(0)}{n(0)+t}\right)^{1-b}}^K e^{-y}\, y^{\frac{1}{1-b}}\, dy. \tag{A10}$$

Note that Eq. (A7) and Eq. (A10) are not exact solutions but continuous approximations which assume K is a real number. Now we investigate the distribution in Eq. (A10).

1. At fixed K when $t \to \infty$, the low limit of integration in Eq. (A10) goes to zero and we have

$$P_{new}(K) = \frac{K^{-1-\frac{1}{1-b}}}{1 - b} \int_0^K e^{-y} y^{\frac{1}{1-b}} dy. \tag{A11}$$

As $K \to \infty$,

$$P_{new}(K) = K^{-1-\frac{1}{b}} \left(\frac{1}{1 - b}\right) \Gamma\left(1 + \frac{1}{1 - b}\right). \tag{A12}$$

As $K \to 0$,

$$P_{new}(K) = \frac{1}{1 - b} K^{\left(-\frac{1}{1-b} - 1\right)} \frac{K^{\left(1 + \frac{1}{1-b}\right)}}{1 + \frac{1}{1-b}} = \frac{1}{2 - b}. \tag{A13}$$

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2. At fixed t when $K \to \infty$, we use the partial integration to evaluate the incomplete Γ function:

$$\int_x^\infty e^{-y} \, y^\alpha \, dy = -e^{-y} \, y^\alpha|_x^\infty + \alpha \int_x^\infty e^{-y} \, y^{\alpha-1} \, dy \approx e^{-x} \, x^\alpha.$$

Therefore, from Eq. (A10) we obtain

$$\begin{aligned}
P_{new}(K) &\approx \frac{n(0)+t}{N(0)+bt} \frac{b}{1-b} \, K^{-\frac{1}{1-b}-1} \int_{K\left(\frac{n(0)}{n(0)+t}\right)^{1-b}}^\infty e^{-y} \, y^{\frac{1}{1-b}} \, dy, \\
&= \frac{n(0)}{N(0)+bt} \frac{b}{1-b} \frac{1}{K} \, \exp\left(-K\left(\frac{n(0)}{n(0)+t}\right)^{1-b}\right),
\end{aligned} \qquad (A14)$$

which always decays faster than Eq. (A6).

Appendix B: Calculation of the growth distribution of classes $P(g)$

Let us assume both the size and growth of units (ξ_i and η_i respectively) are distributed lognormally

$$p(\xi_i) = \frac{1}{\sqrt{2\pi V_\xi}} \frac{1}{\xi_i} \, \exp\left(-(\ln \xi_i - m_\xi)^2/2V_\xi\right), \qquad (A15)$$

$$p(\eta_i) = \frac{1}{\sqrt{2\pi V_\eta}} \frac{1}{\eta_i} \, \exp\left(-(\ln \eta_i - m_\eta)^2/2V_\eta\right). \qquad (A16)$$

If units grow according to a multiplicative process, the size of units $\xi_i' = \xi_i \eta_i$ is distributed lognormally with $V_{\xi'} = V_\xi + V_\eta$ and $m_{\xi'} = m_\xi + m_\eta$.

The nth moment of the variable x distributed lognormally is given by

$$\mu_x(n) = \int_0^\infty \frac{1}{\sqrt{2\pi V}} \frac{x^n}{x} \, dx \, \exp\left(-(\ln x - m)^2/2V\right) = \exp\left(nm_x + n^2 V_x/2\right). \qquad (A17)$$

Thus, its mean is $\mu_x \equiv \mu_x(n=1) = \exp(m_x + V_x/2)$ and its variance is $\sigma_x^2 \equiv \mu_2 - \mu_1^2 = \mu_1^2\left(\exp(V_x) - 1\right)$.

Let us now find the distribution of g growth rate of classes. It is defined as

$$g \equiv \ln \frac{S(t+1)}{S(t)} = \ln \sum_{i=1}^K \xi_i' - \ln \sum_{i=1}^K \xi_i. \qquad (A18)$$

Here we neglect the influx of new units. According to the central limit theorem, the sum of K independent random variables with mean $\mu_\xi \equiv \mu_\xi(1)$ and finite variance σ_ξ^2 is

$$\sum_{i=1}^K \xi_i = K\mu_\xi + \sqrt{K}\nu_K, \qquad (A19)$$

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where ν_K is the random variable with the distribution converging to Gaussian

$$\lim_{K \to \infty} P(\nu_K) \to \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left(-\nu_K^2/2\sigma_\xi^2\right). \tag{A20}$$

Because $\ln \mu_\eta = m_\eta + V_\eta/2$ and $\ln \mu_{\xi'} = \ln \mu_\xi + \ln \mu_\eta$ we have

$$
\begin{aligned}
g \equiv \ln S(t+1) - \ln S(t) &= \ln(K\mu_{\xi'}) + \frac{\nu_K'}{\sqrt{K}\mu_{\xi'}} - \ln(K\mu_\xi) - \frac{\nu_K}{\sqrt{K}\mu_\xi}, \\
&= m_\eta + \frac{V_\eta}{2} + \frac{\nu_K'\mu_\xi - \nu_K\mu_{\xi'}}{\sqrt{K}\mu_\xi\mu_{\xi'}}.
\end{aligned} \tag{A21}
$$

For large K the last term in Eq. (A21) is the difference of two Gaussian variables and that is a Gaussian variable itself.

$$\ln \mu_{\xi'} = m_{\xi'} + V_{\xi'}/2 = \ln \mu_\xi + \ln \mu_\eta, \tag{A22}$$

where $\ln \mu_\eta = m_\eta + V_\eta/2$ is the average growth rate. To find the distribution of g we must find its mean and variance. In order to do this, we rewrite

$$\frac{\nu_K'}{\sqrt{K}\,\mu_{\xi'}} = \frac{\sum_{i=1}^{K}(\xi_i' - \mu_{\xi'})}{K\,\mu_{\xi'}},$$

and

$$\frac{\nu_K}{\sqrt{K}\,\mu_\xi} = \frac{\sum_{i=1}^{K}(\xi_i - \mu_\xi)}{K\,\mu_\xi}.$$

Thus

$$
\begin{aligned}
g &= m_\eta + \frac{V_\eta}{2} + \frac{\sum_{i=1}^{K}\xi_i(\eta_i\mu_\xi - \mu_{\xi'})}{K\mu_\xi\mu_{\xi'}}, \\
&= m_\eta + \frac{V_\eta}{2} + \frac{\sum_{i=1}^{K}\xi_i(\eta_i - \mu_\eta)}{K\mu_{\xi'}}.
\end{aligned} \tag{A23}
$$

Because $\mu_{\xi'} = \mu_\xi\mu_\eta$, the average of each term in the sum is $\mu_{\xi'} - \mu_\xi\,\mu_\eta = 0$. The variance of each term in the sum is $\langle(\xi_i\,\eta_i)^2\rangle - \langle 2\xi_i^2\,\eta_i\,\mu_\eta\rangle + \langle\xi_i^2\,\mu_\eta^2\rangle$ where $\xi_i\eta_i$, $\xi_i^2\eta_i$ and ξ_i^2 are all lognormal independent random variables. Particularly, $(\xi_i\eta_i)^2$ is lognormal with $V = 4V_\eta + 4V_\xi$ and $m = 2m_\eta + 2m_\xi$; $\xi_i^2\eta_i$ is lognormal with $V = 4V_\xi + V_\eta$ and $m = 2m_\xi + m_\eta$; ξ_i^2 is lognormal with $V = 4V_\xi$ and $m = 2m_\xi$. Using Eq. (A17) and Eq. (A23)

$$\langle(\xi_i\eta_i)^2\rangle = \exp(2m_\eta + 2m_\xi + 2V_\eta + 2V_\xi), \tag{A24a}$$

$$\langle\xi_i^2\eta_i\rangle = \exp(m_\eta + 2m_\xi + 2V_\xi + V_\eta/2), \tag{A24b}$$

$$\langle\xi_i^2\rangle = \exp(2m_\xi + 2V_\xi). \tag{A24c}$$

Collecting all terms in Eqs. (A24a-A24c) together and using Eq. (A23) we can find the variance of g:

$$
\begin{aligned}
\sigma_g^2 &= \frac{K \, \exp(2m_\xi + 2V_\xi + 2m_\eta + V_\eta)(\exp(V_\eta) - 1)}{K^2 \exp(2m_\xi + V_\xi + 2m_\eta + V_\eta)}, \\
&= \frac{1}{K} \exp(V_\xi) \, (\exp(V_\eta) - 1).
\end{aligned}
\tag{A25}
$$

Therefore, for large K, g has a Gaussian distribution

$$
P(g|K) = \frac{\sqrt{K}}{\sqrt{2\pi V}} \, \exp\left(-\frac{(g - m)^2 K}{2V}\right),
\tag{A26}
$$

where $m = m_\eta + V_\eta/2$, $V = \exp(V_\xi)(\exp(V_\eta) - 1)$ and $\mu_\eta = \exp(m_\eta + V_\eta/2)$.

The distribution of the growth rate of the old classes can be found by Eq. (7) in the text. In order to find a close form approximation, we replace the summation in Eq. (7) by integration and replace the distributions $P(K)$ by Eq. (A6) and $P(g)$ by the Eq. (A26) assuming $m = 0$:

$$
\begin{aligned}
P_{old}(g) &\approx \frac{1}{\sqrt{2\pi V}} \int_0^\infty \frac{1}{K(t)} \exp(\frac{-K}{K(t)}) \exp(-\frac{g^2 K}{2V}) \sqrt{K} \; dK, \\
&= \frac{\sqrt{K(t)}}{2\sqrt{2V}} \left(1 + \frac{K(t)}{2V} g^2\right)^{-\frac{3}{2}},
\end{aligned}
\tag{A27}
$$

where $K(t)$ is the average number of units in the old classes (see Eq. (A4)). This distribution decays as $1/g^3$ and thus does not have finite variance. In fact, we approximate the distribution of number of units in the old classes by a continuous function $\exp(-K/K(t))/K(t)$, while in reality it is a discrete distribution

$$
P_{\mathrm{old}}(K) = \lambda^K \left(\frac{1}{\lambda} - 1\right),
\tag{A28}
$$

where $\lambda = \exp(-1/K(t))$. The corrected distribution of growth rates is then given by the sum

$$
P_{\mathrm{old}}(g) \approx \frac{1}{\sqrt{2\pi V}} \frac{1 - \lambda}{\lambda} \sum_{K=1}^\infty \lambda^K \sqrt{K} \, \exp(-g^2 K/2V).
\tag{A29}
$$

The slowest decaying term is

$$
\frac{1}{\sqrt{2\pi V}} \, (1 - \lambda) \, \exp(-g^2/2V),
\tag{A30}
$$

which describes the behavior of the distribution when $g \to \infty$. Thus there is a crossover from Eq. (A27) to Eq. (A30) when $g \approx \sqrt{2V}$.

For the new classes, when $t \to \infty$ the distribution of number of units is approximated by

$$P_{\text{new}}(K) \approx \frac{1}{1-b} K^{-1-\frac{1}{1-b}} \int_0^K y^{\frac{1}{1-b}} e^{-y} \, dy. \tag{A31}$$

Again replacing summation in Eq. (7) in the text by integration and $P(g|K)$ by Eq. (A26) and after the switching the order of integration we have:

$$P_{new}(g) \approx \frac{1}{1-b} \frac{1}{\sqrt{2\pi V}} \int_0^\infty \exp(-y) \, y^{\frac{1}{1-b}} \, dy \int_y^\infty \exp(-g^2 K/2V) \, K^{(-\frac{1}{2}-\frac{1}{1-b})} \, dK. \tag{A32}$$

As $g \to \infty$, we can evaluate the second integral in Eq. (A32) by partial integration:

$$\begin{aligned} P_{new}(g) &\approx \frac{1}{1-b} \int_0^\infty \frac{1}{\sqrt{2\pi V}} \frac{2V}{g^2} \, y^{-\frac{1}{1-b}-\frac{1}{2}} \, y^{\frac{1}{1-b}} \, \exp(-y) \, \exp(-y \, g^2/2V) \, dy, \\ &= \frac{1}{1-b} \frac{1}{\sqrt{2\pi V}} \frac{2V}{g^2} \frac{1}{\sqrt{g^2/2V+1}} \sqrt{\pi} \sim \frac{1}{g^3}. \end{aligned} \tag{A33}$$

We compute the first derivative of the distribution (A32) by differentiating the integrand in the second integral with respect to g. The second integral converges as $y \to 0$, and we find the behavior of the derivative for $g \to 0$ by the substitution $K' = Kg^2/(2V)$. As $g \to 0$, the derivative behaves as $g \cdot g^{2[-(3/2)+1/(1-b)]} \sim g^{1/(1-b)-2}$, which means that the function itself behaves as $C_2 - C_1 |g|^{2b/(1-b)+1}$, where C_2 and C_1 are positive constants. For small b this behavior is similar to the behavior of a Laplace distribution with variance V: $\exp(-\sqrt{2}|g|/\sqrt{V})/\sqrt{2V} = 1/\sqrt{2V} - |g|/V$.

When $b \to 0$, Eq. (A32) can be simplified:

$$\begin{aligned} P_{new}(g)|_{b\to 0} &\approx \frac{1}{\sqrt{2\pi V}} \int_0^\infty K^{-3/2} \exp(-K \, g^2/2\,V) \, dK \int_0^K \exp(-y) \, y \, dy, \\ &\approx \frac{1}{\sqrt{2\,V}} \left(-\frac{1}{\sqrt{1+g^2/2\,V}} + \frac{2}{|g|/\sqrt{2\,V} + \sqrt{g^2/2\,V+1}} \right). \end{aligned}$$

Finally we find

$$P_{\text{new}}(g)|_{b\to 0} \approx \frac{2V}{\sqrt{g^2+2V} \, (|g| + \sqrt{g^2+2V})^2}. \tag{A34}$$

which behaves for $g \to 0$ as $1/\sqrt{2V} - |g|/V$ and for $g \to \infty$ as $V/(2g^3)$. Thus the distribution is well approximated by a Laplace distribution in the body with power-law tails.

Because of the discrete nature of the distribution of the number of units, when $g \gg \sqrt{2V}$ the behavior for $g \to \infty$ is dominated by $const \cdot \exp(-g^2/2V)$.
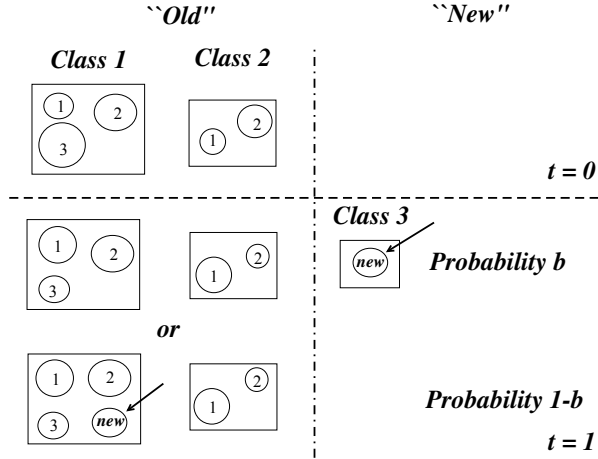
FIG. 1: Schematic representation of the model of proportional growth. At time $t = 0$, there are $N(0) = 2$ classes ($\square$) and $n(0) = 5$ units ($\bigcirc$) (Assumption A1). The area of each circle is proportional to the size ξ of the unit, and the size of each class is the sum of the areas of its constituent units (see Assumption B1). At the next time step, $t = 1$, a new unit is created (Assumption A2). With probability b the new unit is assigned to a new class (class 3 in this example) (Assumption A3). With probability $1 - b$ the new unit is assigned to an existing class with probability proportional to the number of units in the class (Assumption A4). In this example, a new unit is assigned to class 1 with probability 3/5 or to class 2 with probability 2/5. Finally, at each time step, each circle i grows or shrinks by a random factor η_i (Assumption B2).
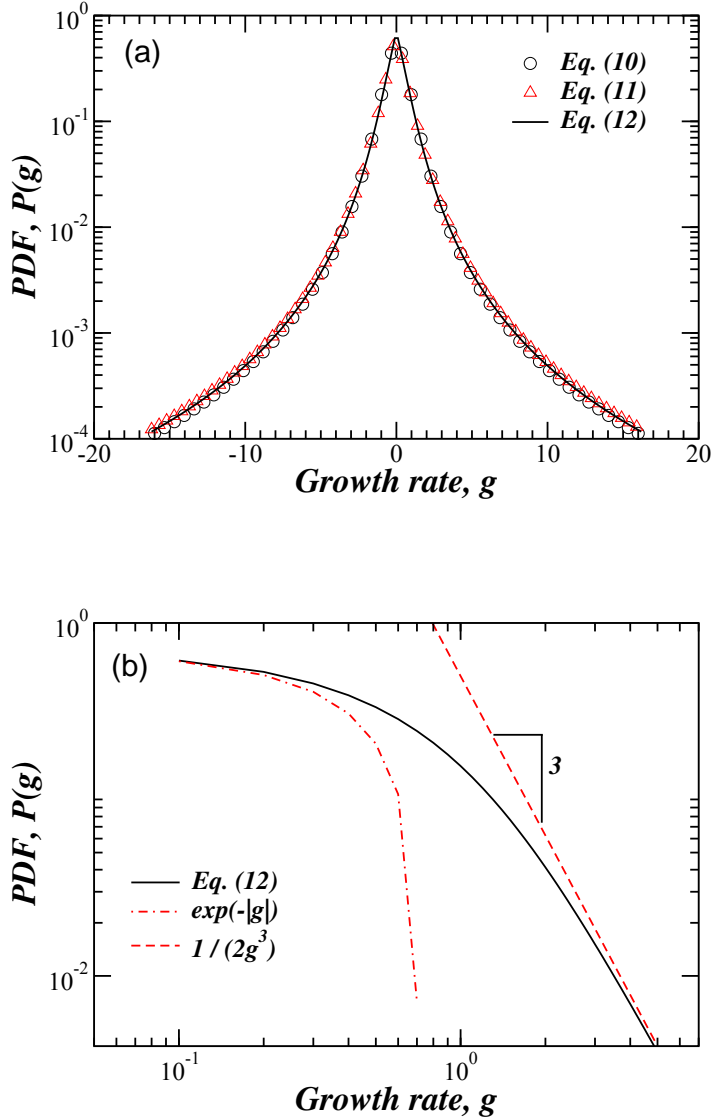
FIG. 2: (a) Comparison of three different approximations for the growth rate PDF, $P_g(g)$, given by Eq. (10), mean field approximation Eq. (11) for $b = 0.1$ and Eq. (12). Each $P_g(g)$ shows similar tent shape behavior in the central part. We see there is little difference between the three cases, $b = 0$ (no entry), $b = 0.1$ (with entry) and the mean field approximation. This means that entry of new classes ($b > 0$) does not perceptibly change the shape of $P_g(g)$. Note that we use $K(t)/V_g = 2.16$ for Eq. (10) and $V_g = 1$ for Eq. (12). (b) The crossover of $P_g(g)$ given by Eq. (12) between the Laplace distribution in the center and power law in the tails. For small g, $P_g(g)$ follows a Laplace distribution $P_g(g) \sim \exp(-|g|)$, and for large g, $P_g(g)$ asymptotically follows an inverse cubic power law $P_g(g) \sim g^{-3}$.
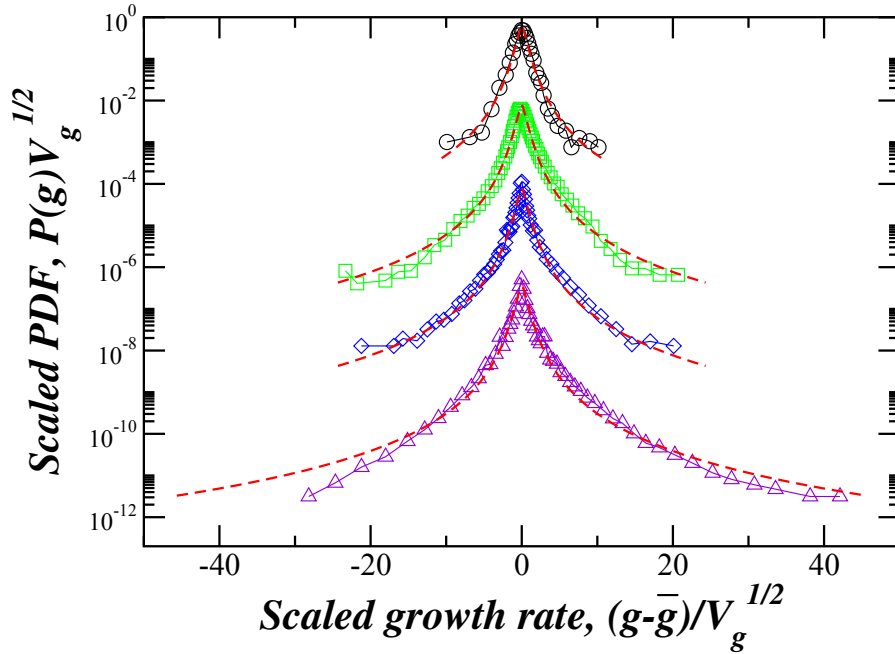
FIG. 3: Empirical tests of Eq. (12) for the probability density function (PDF) $P_g(g)$ of growth rates rescaled by $\sqrt{V_g}$. Shown are country GDP ($\bigcirc$), pharmaceutical firms ($\square$), manufacturing firms ($\diamond$), and pharmaceutical products ($\triangle$). The shapes of $P_g(g)$ for all four levels of aggregation are well approximated by the PDF predicted by the model (dashed lines). Dashed lines are obtained based on Eq. (12) with $V_g \approx 4 \times 10^{-4}$ for GDP, $V_g \approx 0.014$ for pharmaceutical firms, $V_g \approx 0.019$ for manufacturing firms, and $V_g \approx 0.01$ for products. After rescaling, the four PDFs can be fit by the same function. For clarity, the pharmaceutical firms are offset by a factor of 10^2, manufacturing firms by a factor of 10^4 and the pharmaceutical products by a factor of 10^6. Note that the data for pharmaceutical products extend from $P_g(g) = 1$ to $P_g(g) \approx 10^{-4}$ and the mismatch in the tail parts is because $P_g(g)$ for large g is mainly determined by the logarithmic growth rates of units $\ln \eta$.
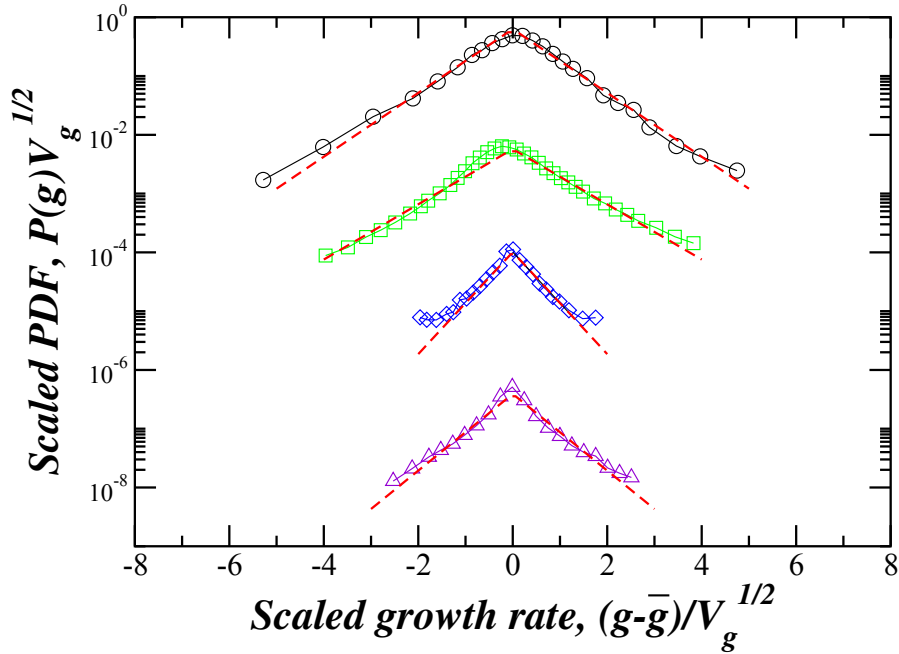
FIG. 4: Empirical tests of Eq. (12) for the *central* part in the PDF $P(g)$ of growth rates rescaled by $\sqrt{V_g}$. Shown are 4 symbols: country GDP ($\bigcirc$), pharmaceutical firms ($\square$), manufacturing firms ($\diamondsuit$), and pharmaceutical products ($\triangle$). The shape of central parts for all four levels of aggregation can be well fit by a Laplace distribution (dashed lines). Note that Laplace distribution can fit $P_g(g)$ only over a restricted range, from $P_g(g) = 1$ to $P_g(g) \approx 10^{-1}$.
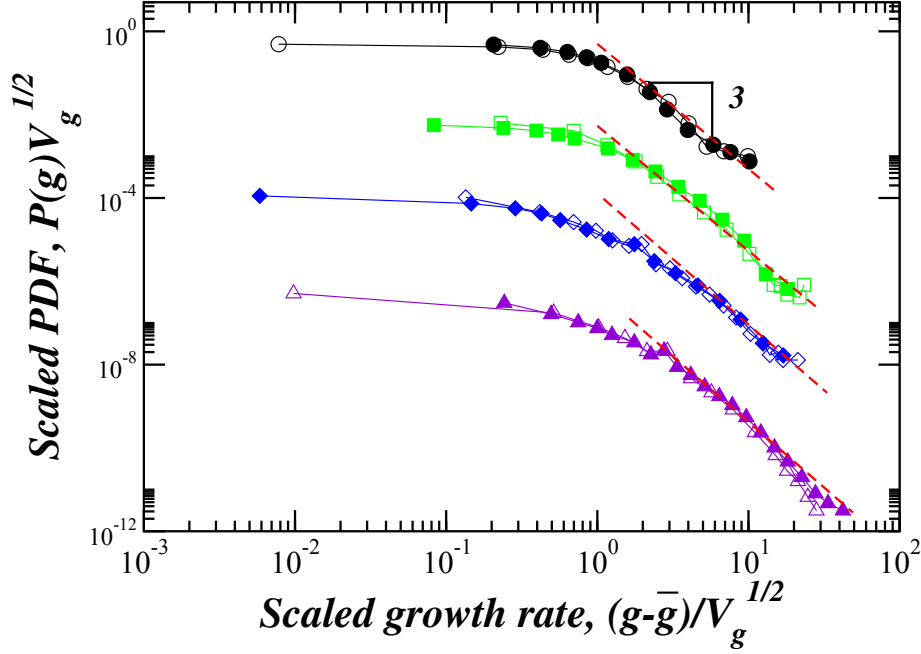
FIG. 5: Empirical tests of Eq. (12) for the *tail* parts of the PDF of growth rates rescaled by $\sqrt{V_g}$. The asymptotic behavior of g at any level of aggregation can be well approximated by power laws with exponents $\zeta \approx 3$ (dashed lines). The symbols are as follows: Country GDP (left tail: ◯, right tail: ●), pharmaceutical firms (left tail: □, right tail: ■), manufacturing firms (left tail: ◇, right tail: ♦), pharmaceutical products (left tail: △, right tail: ▲).