Information Theory and Knowledge-Gathering

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ABSTRACT

It is assumed that human knowledge-building depends on a discrete sequential decision-making process subjected to a stochastic information transmitting environment. This environment randomly transmits Shannon type information-packets to the decision-maker, who examines each of them for *relevancy* and then determines his optimal choices. Using this set of relevant information-packets, the decision-maker *adapts,* over time, to the stochastic nature of his environment, and optimizes the *subjective expected* rate-of-growth of knowledge. The decision-maker's optimal actions, lead to a decision function that involves his view of the subjective entropy of the environmental process and other important parameters at each stage of the process. Using this model of human behavior, one could create psychometric experiments using computer simulation and real decision-makers, to play programmed games to measure the resulting human performance.

KEYWORDS

decision-making, dynamic programming, entropy, epistemology, information theory, knowledge, sequential processes, subjective probability

Thinking About Knowledge

Chebyshev, as translated by Khinchin (1957), expressed the magical relationship between scientific theory and practice:

> *"The bringing together of theory and practice leads to the most favorable results; not only does practice benefit, but the sciences themselves develop under the influence of practice, which reveals new subjects for investigation and new aspects of familiar subjects."*

An objective for a scientist is to understand the phenomena of his or her environment. One of the means for understanding is to build hypothetical mathematical models that reflect our view of reality and hopefully reach the *laws of nature*. Without this understanding, how are we to conduct experiments to validate our hypothesis and then reach out for even more truths about nature? The risk for a scientific model builder is to be able to steer a narrow path between the naivety of oversimplification and the morass of overcomplication. The subject of *knowledge-gathering* in human decision-makers is the basis of this paper. Recently, the study of knowledge-gathering in *machines* has been a major part of the new science of Artificial Intelligence. Most of this AI research has been to the study knowledge as expressed along semantic or logic network lines to enable expert systems. Knowledge systems, as learning systems, have also had a long and successful history in the field of psychometrics. The influence of *information theory* on knowledge gathering systems that effect human decision-making has resulted in the application of several mathematical discrete sequential system analysis techniques that stem from the classical works of H. Poincaré, G.D.Birkhoff and on to the recent works of the late Richard Bellman (1960) of the RAND Corporation. These "dynamic programming" techniques have also been successfully in applied to many other diverse scientific endeavors. Applying these same techniques to the analysis of knowledge-gathering in *human* decision-makers may appear to be a "tail wagging the dog" to some, but sometimes new ideas come by strange paths. The subject of what human knowledge *is* has always occupied the old philosophers from Plato to Husserl. Recently, it has become possible to analyze knowledge-gathering as a discrete, sequential stochastic, adaptive decision-making process, subjected to the laws of probability, and the step-by-step actions of human decision-makers, who in Kenneth Arrow's words, are *learning by doing* (Arrow (1961).

In this paper, in its simplest form, I will develop the mathematical structure of a human knowledge-gathering mathematical process of this type. Despite the simplicity of this model, the mathematical details are still extensive and, for the sake of the balance between the simple and the complex, I have delegated those details to the Appendix, herein after.

Observation of a human, *decision-maker*, behaving in a knowledge-gathering process, indicates that, on the average, his intelligence increases over time and tends to simplify the decision-maker's gathering process by decreasing his relative entropy. This process can occur in the face of a stochastic *environment*, a "deus ex machina" that feeds the decision-maker a variety of random *information-packets*. Some of these information-packets are valuable to the decision-maker, but most are irrelevant. Somehow, the decision-maker is able to sort, these random information packets into a list of increasing *acceptance* from the irrelevant to the relevant, depending on the decision-maker's previous experience, stored in his *memory*. In this process, the decision-maker tends to *decrease*, both his relative entropy (his uncertainty or chaos), in the face of a world where the net entropy is believed to be always increasing. Non-intelligent dynamic systems, for example, Brownian motion of micro-particles in suspension, do not seem to behave in this manner. It appears that these non-intelligent systems lack a means of *organizing against*, or *adapting to* the hidden mechanisms in their environment. For the most part, except for some chemical processes that tend oscillate back and forth from chaos to order, with obvious consequences to the effect on their entropy, these non-intelligent processes, in the end, always *increase* their local entropy. Ilya Prigogine (1996) has given many examples of this kind of pseudo lifelike behavior.

In this paper, I postulate that lifelike systems have several *necessary* unique facilities to effect relative entropy decreases through their organized knowledge-gathering activity. The first of these is the basic ability to make *decisions.* This ability, in the face of complex world, enables living beings to *adapt* and *control* their decision-making through the *acceptance or rejection* of random information-packets transmitted by their environment. Within these systems, we find the human *knowledge-gathering* activity enables the optimal acceptance or rejection of random information-packets, received into their senses from their immediate stochastic environment and stored in their mind. As Shakespeare aptly put it for poets (and mathematicians),

> *"And as imagination bodies forth*
> *The forms of things unknown, the poet's pen*
> *Turns them to shapes, and gives to airy nothing*
> *A local habitation and a name."*
>
> "A Midsummer Night's Dream,"
> Act V, Scene I.

In general, human knowledge-gathering systems possess a unique and essential *memory* capability that enables them to optimize their personal and social decision-making behavior. The research of the late Hans Reichenbach (1999) in his book "The Direction of Time" has shown that memory (of some kind) is an essential element, forcing time to always seem to flow in an *increasing* direction. This human memory is the second *necessary* element that enables the human decision-maker, to *reduce* his relative entropy and *optimize* his decisions. The mathematical model, described herein, is a embodiment of how these key dependences, decision-making and memory, enable the human knowledge-gathering process and produce its decreasing effect on his relative entropy.

It takes nerve for an economist and computer scientist to leap into a world dominated by cognitive scientists and philosophers, and worse yet, propose yet another mathematical model for man's capability to accumulate knowledge. My previous research in mathematical statistics, information theory and economic decision processes forced me pass down this dangerous path. It all started when Claude Shannon (1948) developed an original idea. In his paper, he introduced a rational measure for information based on the needs of the communication engineers at Bell Telephone Labs and thus launched the formal ideas we now fondly call "Information Theory." In his original concept, Shannon considered *all* information-packets had *equal* value. Since his primary focus was on the efficiency of the transmission of electronic information over a bandwidth limited communications channel, he was not directly concerned with the *value* of that information. In Shannon's original view the *value* of a transmitted information-packet was always in the eyes of the *sender* or *receiver* - not the communications service provider. Leon Brillouin (1963) pointed out that the *value* of Shannon's measure of information is dependent on the potential *use* of that information by a decision-maker. Shannon and his associates soon rectified that omission in their later papers. J. Marshack (1960) advanced the classical theory of economics by his consideration the role of information as a new kind of "factor" in the production function and was soon followed by several other similar papers. Recently, Kenneth Arrow (1996) extended Shannon's cost of information theory to a remarkable level. Arrow has shown how information theory, applied to modern economic theory, causes the generation of strong increasing economic returns to scale in production that dominates today's new economic growth in this "information" age. Vittorio Somenzi (1965) has extended the aspects of entropy found in the information theory to the "Mind-Body" philosophic controversy. Finally, specially for the non-mathematical philosopher, Fred Dretske (1999) has recently drawn together, in great detail, all of these knowledge and information fragments . These new economic and philosophic applications of information theory provide the jumping off point for this paper.

In the early 1900's Henri Poincaré gave a lecture before the Societe de Psychologie in Paris on Discovery in Mathematics. This lecture forms chapter III of his book, "Science and Method." Poincaré (1953) , agreeing with Helmholtz, felt that the phenomenon of knowledge gathering was composed of three distinct phases. After attending Poincaré's famous lecture, Jacques Hadamand (1954) pulled all this together in his book, "The Psychology of Invention in the Mathematical Field." He named the three distinct phases as, "Preparation, Incubation and Illumination." I will paraphrase these three phase names in less formal terms:

1. Research it!
2. Sleep on it!
3. Aha!

Poincaré explained his view of these phases as follows:

> *"One is at once struck by these appearances of sudden illumination, obvious indications of a long course of previous unconscious work. The part* played *by this unconscious work in mathematical discovery seems to me indisputable, and we shall find traces of it in other cases where it is less evident. Often when a man is working at a difficult question, he accomplishes nothing the first time he sets to work. Then he takes more of less of a rest, and then sits down again at his table. During the first half hour he still finds nothing, and then all at once the decisive idea presents itself to his mind."*

Hadamand's concept of "Incubation," or as I call it, the "Sleep on it" phase, is a familiar experience for most scientists as they "invent" new concepts. The "Aha" phase always seems to appear later, almost any time *after,* the "Incubation" phase or as I call it the "Sleep on it" phase. I will postulate that each stage in my knowledge-gathering process passes through each of these three phases. Poincaré goes further and borrows

Epicurus's interesting "idea hooks" argument and gathers a call for wide diversity in research. He says:

> *"Among the combinations [of ideas] we choose, the most fruitful are often those which are formed of elements borrowed from widely separated domains."*

I might also add to Poincaré's concept of Epicurean "idea hooks," a reference to Leibnitz's concept of "monadology" where mysterious supernatural "beings" transmit ideas to humans! More realistically, a fascinating recent study on the "Sleep On It" phase, by Matthew Walker (2000), "Sleep to Remember," describes in detail the beneficial effect of multiple cycles of REM (Rapid Eye Movement) sleep on developing human memory. Many psychometric and neuropsychological studies are reported in the literature about the importance of REM sleep on the creation of ideas. It has been said that Einstein, when asked how much sleep he needed to create physical theory, impishly replied – at least ten hours a night!

Now, I must take a broader view of information, not only the *value* of information but also the roll of information as a *generator* of knowledge. Consider a TV viewer watching a news broadcast, a broad range of Shannon information-packets assail him, some of them are intensely important to him, others are possibly interesting and still others are virtuously useless. From experience we know that the TV watcher is able to *classify* the information-packets he receives, by some measure of *value*, and thus make decisions based on these value judgments.

I propose to define a measure, a "control" vector, that I will call the decision-maker's *information-relevance* vector. Each component of this vector is the relevance decision-maker attaches to each of the types of information-packets he receives from his environment. Using this information-relevance concept, consider the evaluation of the i^{th} information increment the decision-maker receives during the t^{th} stage in a discrete sequential decision process. We have for each component

$$k_{i,t} = r_{i,t}\, y_i \, , \qquad (1.1)$$

where

$r_{i,t}$ is the i^{th} component of the decision-maker's information-relevance vector assigned to the i^{th} kind of information-packet during stage *t* of his discrete sequential decision process,

y_i is the i^{th} component of an independent random valued vector that measures the *Shannon* information content of an information-packet for the i^{th} *kind* of information, and

$k_{i,t}$ is the i^{th} component, also an independent random valued vector, and is a measure of the decision-maker's *relevant* information from the i^{th} information-packet, transmitted to him by the environment, during the t^{th} stage of his sequential decision process.

So how can the decision-maker assign an *optimal* value to each $r_{i,t}$ for each information-packet from the i^{th} source during stage *t*? Since the information–packets are components of a random vector, determined solely by the decision-maker's environment, I propose that the decision-maker will base his optimal value of each $r_{i,t}$, partly, on the maximization of the statistical *expectation* of the compounded rate-of-growth of knowledge from each of the i^{th} information-packet sources during each stage of the process. Because of this "rate-of-growth" evaluation, the statistical expected value of the *compounded* growth of knowledge for the process takes on a natural logarithmic form.

Why "compounded" growth? Why a "logarithmic" function? The most familiar example of compounded rate-of-growth is in the conventional evaluation of an investment for a retirement fund. Suppose each year you reinvest the earned increment on your fund, so that the value of your fund will grow exponentially as time passes. So also can be viewed the decision-maker's investment in his knowledge, since his knowledge is a kind of investment for his life. One could say, "the more you know today, the more you are *capable* of knowing tomorrow;" or "what the decision-maker will know tomorrow is what you know today *plus* the contribution of any new *relevant* information you have gained today." In other words, a student must know about *numbers* to know about *algebra* and he must know about *algebra* to know about the *calculus,* and so on.

Let's take a look at a very simple discrete *deterministic* compounded sequential knowledge-gathering process for the compounded rate-of-growth of knowledge, K_T, given a constant increment of a decision-maker's relevant information during each stage of the process. We form the following deterministic discrete sequential process

$$K_1 = K_0\left(1+k\right),$$

$$K_2 = K_1\left(1+k\right),$$

$$\bullet\bullet\bullet$$

$$K_T = K_{T-1}\left(1+k\right).$$

By repeated substitution, this leads to a more concise result

$$K_T = K_0\left(1+k\right)^T,$$

taking the natural logarithm of this we get

$$g\left(K_0\right) \triangleq \frac{1}{T}\ln\frac{K_T}{K_0} = \ln\left(1+k\right), \tag{1.2}$$

where

$g\left(K_0\right)$ is defined as the *compounded rate-of-growth of knowledge* for a T stage process.

The Decision-Maker's "Research It" Phase Begins

Now we enter into the decision-maker's "Research it" phase and postulate a mathematical model for the decision-maker's compounded rate of growth of knowledge by considering a simple *single stage* process at stage t

$$\ln\frac{K_{t+1}}{K_t} = \sum_{i=1}^{M} n_{i,t}\ln\left(1+r_{i,t}y_i\right) \qquad i = 1, 2, \ldots M \tag{1.3}$$

where

M is the number of components in the random valued environmental vector, that is, it is the size of the set of information-packet types the environment can transmit,

i is the index for the i^{th} component or kind of information–packet,

t is the index signifying the t^{th} stage of the process, and

$n_{i,t}$ is the *frequency* (the number of) transmissions for the i^{th} kind of information-packet during stage t of the process.

Remembering that the y_i's are components of an independently chosen *random* vector and since that forces K_{t+1} to also be a random vector, it is customary to view this process *analytically* by considering the *statistical expectation* of the random process. So, we have for *one* typical stage of the decision-maker's knowledge process at time t

$$E\left\{\ln\frac{K_{t+1}}{K_t}\right\} = \sum_{i=1}^{M} E\left(n_{i,t}\right)\ln\left(1+r_{i,t}\,y_i\right),\ \text{for}\ i = 1,2,\ldots M\ . \tag{1.4}$$

Now it is realistic to assume that the probability distribution for the environment's random value vector generator to be chosen from a multinomial beta distribution. Therefore, in this case, the statistical expectation of $n_{i,t}$ is given by

$$E\left(n_{i,t}\right) = t\ p_i \tag{1.5}$$

where we also insist on the usual probability space conditions,

$$p_i \le 1 \text{ and } \sum_{i=1}^{M} p_i = 1\ ,\ i = 1,2,\ldots M\ , \tag{1.6}$$

and we also assume that this random process is quasi-ergodic – that is the p_i's will remain constant during the duration of the process.

So in this case, it can be shown that the rate-of-growth for our knowledge process at stage t is given by

$$\overline{g}_t \triangleq E\left\{\frac{1}{t}\ln\frac{K_t}{K_{t-1}}\right\} = \sum_{i=1}^{M} p_i \ln\left(1+r_{i,t}\,y_i\right),\ \text{for}\ i = 1,2,\ldots M\ , \tag{1.7}$$

where

the "$\frac{1}{t}$" signifies that this is for just *one* stage of a process after already experiencing t stages,

the information-relevance vector is subject to the additional special conditions

$r_{i,t} \geq 0$ and $\sum_{i=1}^{M} \mathrm{r}_{\mathrm{i,t}} = 1$, for stage t for $i = 1, 2, ...M$, (1.8)

and each of the environment's information-packages is assigned to a probability of p_i during stage *t*.

At this point equation (1.7) and the constraints of (1.6) and (1.8) completes the decision-maker's "Research it" phase for a single current t^{th} stage.

The Decision-maker's "Sleep-On-It" Phase Begins

The *optimal* expected compounded rate-of-growth of knowledge for a single stage *t* of a *T* stage process can now be evaluated. This operation begins our decision-maker's "Sleep on it" phase, where he mulls over during his "REM" sleep or in his subconscious, the information-packets he has received for *t* stages during his "Research it" phase and decides what to do about it. In the "Sleep on it" phase, I postulate that the decision-maker constructs within his mind an ordered list for each component of the information-relevance vector, the set of $r_{i,t}$'s, that tends to maximize the stage's rate of growth. Because of constraints, in (1.6) and (1.8), on the p_i's and the $r_{i,t}$'s , finding the optimal values for the decision-maker's, the $r_{i,t}$*'s, at stage *t* is a *non-linear* mathematical optimization procedure. The simple "marginal value" procedure for optimizing equation (1.7), one that is based only on the determination of the first and second derivatives of equation (1.7), will not assure that the special constraints on the p_i's and the $r_{i,t}$'s will hold.

Also negative components of the decision-maker's information-relevance vector are somewhat questionable. Since I have assumed that *zero* relevance is already an automatic *rejection* of an information packet, as far as an actual contribution to the decision-maker's optimal rate-of-growth of knowledge, for these rejected packets, nothing really happens! On the other hand, a negative $r_{i,t}$, could be considered some form of conscious "unknowing" or "forgetting" for that kind of information-packet. This expansion of the model seems overly complex and, at this point, and I will not consider it in this paper.

What is needed now is a prototype for a non-linear optimization technique that will enable a decision-maker to create this intuitive list of optimal *relevant* values for the $r_{i,t}$'s for each kind of information-packet. This should be a mathematical procedure that emulates what could possibly go on within the decision-maker's subconscious mind while he decides to *accept or reject* each information-packet delivered by his environment. There is such a non-linear optimization technique for determining a list of constrained optimal $r_{i,t}$'s that could be used to generate this list of relevant components under the non-linear constraints. This optimization technique is an expanded version of the classical method of Lagrange, coupled with special conditions that handle the non-linear constraints as set forth by the Kuhn-Tucker (1951) theorem. Other more "computer" oriented methods such as the "Branch and Bound" procedure, Wagner (1969), could also be used, but I will employ the typical Kuhn-Tucker procedure as outlined here in the Appendix. This method divides the indexes of the y_i's, into two subsets, $i = 1, 2, \ldots, M^+$, that is of size M^+, for the first set and $i = (+1), \ldots, M$ for the second set that is of size M^0. It is assumed that the first subset of the y_i's, will meet the special constraints for $r_{i,t} \in M^+$ and the second subset of y_i's will violates the special constraints for $r_{i,t} \in M^0$.

But wait, in reducing the components of the set of *i*'s we have altered the probability space upon which the p_i's were originally defined. In other words, it is possible that $\sum_{j=1}^{M^+} p_j < 1$. Therefore, we must *renormalize* the original probability space to that represented only by the set of $i = 1, 2, \ldots, M^+$ where $M^+ \leq M$. This can be done with the relations

$$q_i = \frac{p_i}{\sum_{j=1}^{M^+} p_j}, \qquad (1.9)$$

$$q_i \leq 1 \text{ , and } \sum_{i=1}^{M^+} q_i = 1.$$

The set $\{q_i\}$ now reflects the proper probability space over just the M^+ subset assigned to the information-packets that the decision-maker has determined that are *relevant* for the optimization of $\overline{g}_t$. Assuming that this procedure is now employed, we get the equation for determining the optimal relevant $r_{i,t}$'s

$$\overline{g}_t * \triangleq \max_{r_{i,t}} \left\{ E \ln \frac{K_t}{K_{t-1}} \right\} = \max_{r_{i,t}} E \sum_{i=1}^{M^+} \left[q_i \ln \left(1 + r_{i,t} y_i \right) \right]. \quad i = 1, 2, \ldots, M^+ . \tag{1.10}$$

By reference to the Appendix, and using the Kuhn-Tucker the non-linear optimization method, we can find the $r_{i,t}$*'s that optimize (1.10) by

$$r_{i,t} * = q_i \left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] - \frac{1}{y_i}, \qquad i = 1, 2, \ldots, M^+, \tag{1.11}$$

where

$M^+ \in M$ is the sub-set of the decision-maker's relevant information-packets.

The Decision-Maker Adapts to his Environment

We are still not done with the decision-maker's "Sleep on it" phase. There is a serious human problem with equations (1.9), (1.10) and (1.11). The decision-maker really *does not know* the values of the p_i's and thus also does not know the q_i's! What the decision-maker *does know,* is the memory resulting from his history of the past *frequency of occurrences* of the types of information-packets previously transmitted by the environment, in other words, the frequency of occurrence of the y_i's. I now assume that the decision-maker will *adapt* to his stochastic environment by using this mental history to *estimate* the value if the unknown $q_{i,t}$'s and thereby compute the optimal $r_{i,t}$'s in an equation similar to (1.11).

Note, that now the estimated or *subjective* values of the probabilities are subscripted with an addition of a "*t*" because these estimated values of the $q_{i,t}$'s will be developed by adapting *over time* to the history of the environment over a number of stages, say for example all *t* stages of a *T* stage process. An appropriate statistical method to enable the decision-maker to adapt to the environment is to assume that the unknown probabilities are defined by the multinomial beta distribution. Given this assumption, the decision-maker can determine or *estimate* the subjective $\hat{q}_{i,t}$'s (the hat "^" indicates that the $\hat{q}_{i,t}$ are the *subjective* probabilities) by the Bayes estimation procedure, Good(1965). The *actual* q_i's, of course, are known only to the "deus ex machina" (the random environment). There are many other statistical procedures that closely resemble this possible mental or subjective process, but I propose the method of Bayes, for several reasons. This is an *intuitive* procedure that would be natural for a human decision-maker to mentally or subjectively undertake. The Bayes estimator also has the useful property for being statistically *consistent* and *sufficient*.

This is one major reason to consider the Bayes estimator as rational procedure because with it the estimation can begin as a complete *guess* on the part of the decision-maker and still evolve over time into an approximation of the actual probabilities. Suppose initially the decision-maker has *no* history of any past occurrences of the y_i's, that is, he is at the initial stage of the process. Carnap (1950) has suggested an initial (a priori) equation, as a "hunch" or "guess," that can "start" the decision-makers learning experience to estimate the $\hat{q}_{i,t}$'s. Using his method we have then for the first stage of the process, where the decision-maker guesses the values for the subjective probabilities for the occurrence of each kind of information-packet

$$\hat{q}_{i,0} = \frac{\alpha_i}{\sum_{j=1}^{M^+} \alpha_j} \ , \ i, j = 1, 2, \ldots, M^+, \tag{1.12}$$

where the α_i's are Carnap's "logical width" parameters and represent the decision-makers *initial* guesses for the $\hat{q}_{i,t}$'s.

Now if the decision-maker has progressed as far as stage *t,* he should possesses a history of the actual environmental events over his *t* stages, he would have for the subjective estimate of the probabilities

$$\hat{q}_{i,t} = \frac{n_{i,t} + \alpha_i}{t + \sum_{i=1}^{M^+} \alpha_j}, \quad i, j = 1, 2, \ldots, M^+, \tag{1.13}$$

where the $n_{i,t}$'s are the decision-maker's observations of the frequency of occurrence for each of the random y_i's, up to and including the t^{th} stage.

One particular set of values for Carnap's parameters for an initial a priori estimate of the probabilities has important properties. If the decision-maker has *no* guess for the initial priorities it would be appropriate to assume that initially the probability for *every* information-packet is as likely as any other. In other words the set $\{\alpha_i\}$ has one identical value for α_i in (1.12) and (1.13) for *all* $i = 1, 2, ..., M^+$. In this case, such a decision-maker will be suffering from *total ignorance*. In total ignorance the a priori value of the probabilities would be simply be

$$\hat{q}_{i,0} = \frac{1}{M^+}, \text{ for } i = 1, 2, ..., M^+. \tag{1.14}$$

We will assume for simplicity that the decision-maker will start in total ignorance of these initial probabilities. Note that after a few stages of the process and $n_{i,t}$ observations for the information-packets have occurred, the $\hat{q}_{i,t}$'s are likely to deviate from this equal likely condition. On the first few stages of this *adaptive*, stochastic knowledge-gathering process, when the decision-maker still has few values for the $n_{i,t}$'s, the decision-maker may exhibit wild instability in his estimates for the $\hat{q}_{i,t}$'s. Fortunately, since the Bayes estimator is known to be both statistically consistent and sufficient, the estimator is one that will "settle down" and even have an "end," that is, if the estimation of the $\hat{q}_{i,t}$'s continues forever it will ultimately converge, in probability, to the true q_i's. Thus we can say

$$q_i = \underset{t \to \infty}{\text{plim}} \left(\hat{q}_{i,t} \right) \text{ for } i = 1, 2, \cdots, M^+.$$

We can now write the *adaptive* version of equations (1.10) and (1.11) that reflect the decision-maker's use of *subjective* probabilities to generate the optimal adaptive relevance factors, the $\hat{r}_{i,t}$'s, by using the decision-maker's *known* subjective probabilities. We have

$$\overline{g}_t * \triangleq \max_{r_{i,t}} \left\{ \hat{E}\left[\frac{1}{t} \ln \frac{K_t}{K_{t-1}} \right] \right\} = \max_{r_{i,t}} \hat{E} \sum_{i=1}^{M^+} \left[\hat{q}_{i,t} \ln\left(1 + r_{i,t} y_i\right) \right], \tag{1.15}$$

and

$$\hat{r}_{i,t} * = \hat{q}_{i,t} \left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] - \frac{1}{y_i}, \quad \text{for } i, j = 1, 2, \ldots, M^+, \tag{1.16}$$

also

$$\hat{q}_{i,t} = \frac{\hat{p}_{i,t}}{\sum_{j=1}^{M^+} \hat{p}_{i,j}}, \quad i, j \in M^+. \tag{1.17}$$

Remember that $\sum_{j=1}^{M^+} \hat{p}_{i,j}$ *may* not be equal to 1, since $M^+ \leq M$.

By substitution of (1.16) into (1.15) we can determine the optimal rate-of-growth of knowledge for this single t^{th} stage. This substitution will bring up several important relationships with respect to the entropy of the environmental process, but first we must consider the *whole sequential process* in detail. So, with the completion of the equations (1.15) (1.16) and (1.17) above we have come to the end of the "Sleep on it" phase for a *one stage*, the t^{th} , of a discrete, sequential, *adaptive,* stochastic knowledge-gathering process, we now are ready to turn to the "Aha" phase, where we finalize the whole sequence of *T* stages, where $t = 1, 2, \ldots, T$.

The Decision-Maker's "Aha!" Phase Begins

We have seen above that the estimates of the subjective probabilities requires some kind of history of the decision-maker's observations, so we must extend the concept developed above to an *entire* time sequence, where for each stage, *t*, the decision-maker's *relevant* sequential event history can be generated.

First, it is important that the basic mathematical structure for multistage sequential processes of this type be well understood before we continue and apply this technique to the knowledge-gathering process. A sequential discrete process is composed of a series of *linked*, causally ordered (irreversible) *stages*. Each stage begins and ends with one or more similar values, or vectors, called the *input state* and produces the *output state*. During the execution of a stage, one or more *events* occur that transform the input state into the output state. The input state is *transformed* into the output state by the events of the current stage. In our *stochastic* version of such a process, one or more of the state input vectors are independently randomly distributed vectors. In these cases, the stage's event transformation involves taking the *statistical expectation* of the random state variables. This use of the expectation function makes it possible to analyze, mathematically, stochastic processes.

The backbone of many of these discrete sequential processes is based on Markov's concept, see Doob (1952), for stage-by-stage transformations of expectation values of these random state variables. In general, for a Markovian discrete sequential process, after any number of stages, say t, we insist that the value of the remaining (T-t) stages will depend *only* on the values of the state variables at the end of the t^{th} stage. Thus, in a Markovian process, all the historical content that the decision-maker has gathered during the process up to the t^{th} stage is transformed and carried *forward* to the next $(t+1)^{th}$ stage as the current output state vector. This means that for a Markovian process we can be assured that the decision-maker will not have *lost* any relevant history by using the previous state vector at stage t, that might be required for the decision-maker in the following stages.

It is important to be clear that, because of the assumed Markov properties, the decision-maker does not need to *remember* the entire sequence of the history of his past events to determine his new, current subjective probabilities. He needs only to remember his last subjective probabilities to re-compute his newest subjective probabilities from the previous values by *transforming* the former Bayes estimates into the *new* Bayes estimates using the new frequency data he obtains due to the occurrence of the new environmental events. An objection voiced by many neurological scientists regarding the massive number of neurons required for the human brain to store complete historical records is avoided, at least in this mathematical model, thanks to the recursive nature of the Bayes estimation procedure and the Markovian property for sequential processes.

With the use of the details about sequential dynamic processes in the Appendix we see how the decision-maker can maximize an *entire* sequential knowledge-gathering process, that consists of T stages. This extension is not just a simple substitution of T and $T\text{-}1$ for the single stage t and $t\text{-}1$ equations. Fortunately, maximizing the *first* stage of the sequential process is relatively easy, since the decision-maker has, by definition, no previous history on which to base estimates of the subjective probabilities (and therefore he relies on his Carnap *guess* for the initial subjective probabilities) and only his initial conditions, his initial state of knowledge, K_0. As we saw in equations (1.15) and (1.16), the decision-maker has sufficient information to determination of the optimal values of his relevancy vector for the *first* stage of the T stage process, but just how *does* the decision-maker determine this optimal values for all the rest of the stages of the process?

Writing the equation for an entire T stage process in an explicit format we get,

$$\hat{g}^*(K_0) = \frac{1}{T}\ \max \hat{E}\left[\ln\frac{K_T}{K_{T-1}} + \ln\frac{K_{T-1}}{K_{T-2}} + \cdots + \ln\frac{K_2}{K_1} + \ln\frac{K_1}{K_0}\right]. \qquad (1.18)$$

Because of the additive property of statistical expectations we can rewrite (1.18) as,

$$\hat{g}^*(K_0) =$$
$$\frac{1}{T}\left[\max \hat{E}\ln\frac{K_T}{K_{T-1}} + \max \hat{E}\ \ln\frac{K_{T-1}}{K_{T-2}} + \cdots + \hat{E}\ \max\ \ln\frac{K_2}{K_1} + \hat{E}\ \max\ln\frac{K_1}{K_0}\right]\cdot \qquad (1.19)$$

Because *each* of the above terms is based on the immediately previous value of the rate-of-growth of knowledge and the previous values of the subjective probabilities for the information-packets, except the first term that is based on the initial conditions K_0 and the initial guess of the subjective probabilities (that the decision-maker already knows) we can maximize each of these terms with respect to the stages's information-relevance vector, one at a time, recursively. We have here a multistage Markov transformation equation where each element has an unknown input to produce an unknown output, except for the first stage, that the decision-maker already *knows*. We can formulate this function as a “dynamic program” as introduced by Richard Bellman. We define the Bellman functional relationship for the one, *first* stage process, given, K_0, and Carnap's, $\hat{q}_{i,0}$. We have

$$\hat{f}_1\left(K_0\right) \triangleq \max_{\left\{r_{i,0}\right\}} \hat{E}_0 \ln K_1 = \max_{\left\{r_{i,0}\right\}} \sum_{i=1}^{M^+} \hat{q}_{i,0} \ln\left(1 + r_{i,0} y_i\right) + \ln K_0 \tag{1.20}$$

and

$\hat{q}_{i,0} = \dfrac{\alpha_i}{\sum_{i=1}^{M^+} \alpha_i}$ where the α_i's are Carnap's "logical width", that is, the decision-maker's initial guesses of the estimators for the initial subjective probabilities. Continuing on the same Bellman functional scheme, we have for the second and following stages

$$\hat{f}_2\left(K_0\right) \triangleq \max_{\left\{r_{i,1}\right\}} \sum_{i=1}^{M^+} \hat{q}_{i,1} \ln\left(1 + r_{i,1} y_i\right) + \hat{f}_1\left(K_0\right) \tag{1.21}$$

$$\bullet \quad \bullet \quad \bullet$$

$$\hat{f}_T\left(K_0\right) \triangleq \max_{\left\{r_{i,T-1}\right\}} \sum_{i=1}^{M^+} \hat{q}_{i,T-1} \ln\left(1 + r_{i,T-1} y_i\right) + \hat{f}_{T-1}\left(K_0\right) \tag{1.22}$$

Following Bellman's approach to the solution of dynamic processes, equation (1.22) encloses the entire sequence of terms with in the brackets of equation (1.19) to one functional equation. Returning to the objective of the paper, the maximum compounded rate-of-return for knowledge-gathering, we form the final result

$$\hat{g}_T * \left(K_0\right) = \frac{1}{T}\left[\hat{f}_T\left(K_0\right)\right] \tag{1.23}$$

This terse functional is best analyzed using computer simulation.

Implications of the Results

Referring back to equations (1.15) and (1.16), it is appropriate to actually substitute the decision maker's optimal relevance vector from (1.16) into (1.15). For example, we can maximize a typical t^{th} stage of the compounded rate-of-growth for knowledge-gathering and obtain some interesting single stage results. After considerable algebra, we get the *optimal* functional for a *typical*, t^{th} stage as

$$f_{t+1}(K_0) = (H_t* - \hat{H}_t) + \hat{E}_t(\ln Y_t) + \hat{E}_t\left\{\ln\left[\frac{1}{M^+} + \frac{1}{\Phi}\right]\right\} + f_t(K_0) \tag{1.24}$$

where

$f_t(K_0)$ is the *optimal* value (already determined by the decision-maker) from the previous t-1 stages of the process,

$\hat{E}_t(\ln Y_t) = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln y_i$,

$\hat{H}_t \triangleq -\sum_{i=1}^{M^+} \hat{q}_{i,t} \ln \hat{q}_{i,t}$ (the decision-maker's subjective entropy for the environment at the current stage), and since $\sum_{i=1}^{M^+} \hat{q}_{i,t} = 1$,

$H_t* \triangleq \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln \frac{1}{M^+} = -\ln M^+$ (this is the maximum subjective entropy for the environmental process at the current stage).

Furthermore, Φ is the *geometric mean* for all the information-packets in set $\{Y_t\}$ for i $\in$ M$^+$, where Φ is defined by,

$$\Phi \triangleq \frac{M^+}{\sum_{j=1}^{M^+} \frac{1}{y_i}}.$$

The notation, $\hat{E}_t$ always means that we are taking the expectation with respect to the decision-maker's set of current subjective probabilities, $\{\hat{q}_{i,t}\}$, at stage *t*.

So then what is the entropy, $\hat{H}_t$, mean? According to Shannon,

"H is a measure of how much "choice" is involved in the selection of the event or how uncertain we are of the outcome."

In other words, $\hat{H}_t$ is the *decision-maker's* subjective measure of the *uncertainty* about the underlying stochastic environmental information-packet generation process at stage t . Now the maximum of H_t is H_t*, and it occurs when *all* the possible kinds information-packets at the current stage, appear to be *equal likely, as viewed by the decision-maker,* thus there is a *maximum* degree of uncertainty for the decision-maker's choice. So he is indifferent and may chooses to receive *all* or *none* of them. In other words, the decision-maker would be in the state of total ignorance! On the other hand, if any one information-packet is *for certain*, that is, its probability is unity, the rest must be zero (or not transmitted) so there is *no* uncertainty about the optimal relevance-vector for the certain information-packet, and therefore $\hat{H}_t$ is zero. (Note: 1 ln 1 is taken as 0). In this case the decision-maker just accepts only the single relevant information-packet and passes on to the next stage.

What then is the quantity ($H_t * - \hat{H}_t$)? It is often considered as a measure of marginal or *relative* uncertainty. We note that Shannon has defined

$$C \triangleq H * - H \tag{1.25}$$

as a system's "information channel capacity." The information capacity is zero if the probabilities defining H (that are generating the randomness) are equal likely. On the other hand the information capacity is unity if any one of these probabilities is one and therefore H is zero.

On the other hand, Heinz Von Foerster (1967) has suggested a similar function that he has called a measure of "order" of a stochastic process. Von Foerster's *order* function is given as

$$\Omega \triangleq \frac{H * - H}{H *} = 1 - \frac{H}{H *}. \tag{1.26}$$

A random process is in a completely *disorganized* state is when $H = H *$, the condition of total ignorance, where anything and everything is possible, that is, all probabilities are independent and equal; thus $\Omega = 0$. On the other hand, if the process is deterministic and thus in perfect order, H is zero and $\Omega = 1$.

Now consider a sequential decision process for knowledge-gathering process, where the decision-maker estimates the subjective probabilities based on his experiencing continuing reception of information-packets as he goes along during each stage of the transformational process. We note that there is an *actual* entropy for the knowledge-gathering process that, by our definition, is independently generated by the random process transmitting information-packets to the decision-maker. We have assumed that the decision-maker has no control over this environmental process and, therefore, has no control over the magnitude of the actual H of the process.

Now consider our sequential decision-making process for knowledge-gathering where the decision-maker estimates the subjective probabilities based on his reception of random information-packets during each stage. In the "flavor" of Shannon's definition of channel capacity, I propose a subjective measure of *relative entropy* or uncertainty for an adaptive sequential decision process given by

$$\hat{\theta}_t \triangleq |H - \hat{H}_t|, \tag{1.27}$$

where the actual entropy of the process, $H = -\sum_{i=1}^{M} p_i \ln p_i$, is fixed and based on the probabilities of the *entire* set of $\{y_i\}$, $i = 1, 2, ..., M$. If the actual entropy of the environmental process is H, no matter what the decision-maker's initial estimate of his subjective, $\hat{H}_t$, might be, then even though he might start totally ignorant, his relative entropy would fall over time. We have

$$\hat{H}_t \underset{t\to\infty}{\rightarrow} H \text{, and } \hat{\theta}_t \underset{t\to\infty}{\rightarrow} = 0,$$

and because the Bayes estimation procedure is *consistent*, the decision-maker's relative entropy for the knowledge-gathering process, in probability, would tend toward zero.

Some Concluding Remarks

In this hypothetical model, the key assumption made is the ability of a human *decision-maker* to draw out of his memory of the past events and adapt to the most rewarding information-packets available to him, while rejecting all the others, thereby, enabling him to optimize his compounded rate-of-growth of knowledge. At the risk of over simplifying an impossibly difficult human process, I hypothesize that:

Intelligent decision-makers attempt to maximize their subjective expectation of their compounded rate-of- growth of knowledge by partitioning, at each stage, a set of received information-packets into two sets; one of acceptable, and another of *unacceptable packets. The decision-maker does this selection by mentally determining an optimal* ***relevancy-vector****, and that provides a means for him to divide the information-packets into these two subsets. Acting on these relevant information-packets the decision-maker chooses an optimal set that, in probability,* ***decreases*** *his relative entropy during his sequence of experiential events as he maximizes his expected compounded-rate-of growth of knowledge over the duration of his activities.*

Many simplifying technical assumptions must be made to make this hypothesis amenable to mathematical analysis. Some of these assumptions are:

1. The process can be decomposed into a sequence of discrete causal *stages*, each of which receives the *a posteriori state* from the previous stage. During each stage a decision-maker takes an *action* that generates the *a priori state* for the following stage.
2. During this transformation the decision-maker optimizes his statistical expected compounded rate-of-growth of knowledge at each stage of the process. This assumption is based on the simple concept, "The more you know today; the more you are *capable* of knowing tomorrow."
3. The environment for this process is characterized by a hidden underlying stochastic mechanism that produces a discrete sequence of mutually independent *events,* the information-packets transmitted to the decision-maker and generated by a given quasi-stationary statistical process of the multinomial beta type.
4. The *events* produced by the environment at each stage of the discrete sequential process for each type of information-packet are the transmission of Shannon information-packets that are made available to the senses of the decision-maker. At each stage of this process, the environmental stochastic process exclusively controls the type of this information-packet transmitted and the actual probability of this transmission is unknown to the decision-maker.
5. The decision-maker can only *observe* the stochastic events produced by the environment, but not the underling probability generating parameters. This

observation of events is augmented by the decision-maker's memory and that is used to construct a Bayes estimate at each stage for the subjective probability for each relevant event, given the arbitrary a priori distribution function of the multinomial beta type. The Bayes estimator, is statistically sufficient and consistent, and is used by the decision-maker to compute a *relevance-vector* for the information-packet types, at each stage in the process.

6. The decision-maker, during a stage of the process, is capable of determining a set of optimal, relevant information-packets that maximizes the subjective value of his compounded rate-of-growth of knowledge.
7. In the process of knowledge-gathering, the statistical expectation of the stochastic vectors in the process must be "Markovian," that is, the *current* statistical expectation can be estimated *solely* by the single immediately proceeding probability estimates and the environmental events at the current stage. This means that the immediately proceeding statistical expectation must, in itself, contain a summation of all the estimation history of all the previous statistical expectations. This assumption makes the decision-making process an *adaptive* process that can be optimized stage-by-stage recursively.
8. Because Bayesian probability estimators are consistent, the subjective compounded rate-of- growth of knowledge approaches the theoretical rate-of-growth, in probability, as the number of stages in the process approaches infinity. Thus an intelligent decision-maker gets more intelligent as the number of observed events increase, and the decision-maker's relative entropy, as defined in (1.27) above decreases as he adapts to his environment over the course of the process.

None of these assumptions are, for the most part, restrictive for a real knowledge-gathering process; but, as a final word of advice from the Buddha,

Believe nothing, no matter where you read it,
or who said it,
no matter if I have said it,

unless it agrees with your own reason and
your own common sense.

Suggestions for Further Research

Testing the hypothesis presented in this paper is required to justify the assumptions made herein. The art of psychometric testing has entered a new technological phase. With modern personal computer technology and specialized computer game development software the psychometric researcher can create realistic decision-making situations with complex, but controlled, environments. Built-in, behind the scenes, random event generators, with probability driving functions known only to the experimenter, can easily be programmed. For example, presenting students with realistic decision-making computer games, coupled with behind the scenes data collection software can accumulation their reaction times and performance measures. Properly designed by the careful experimenters, these techniques can verify hypotheses for human behavior such as presented in this paper.

From the neuropsychological point of view, reverse memory searching, as the decision-maker relies on his memory to adapt and form estimates of subjective probabilities is in agreement with recent psychometric experiments with rats conducted by Davis J. Foster and Matthew A Wilson (2006) at MIT.[21] These researchers were able to actually observe the "replay" of the rat's experienced memories by *measuring* the activity of the neurons in the rat's hippocampus region of the brain. That is the location where *current* memory events are formed. Dr. Wilson seems to believes that the hippocampus region replays, in *reverse,* the rat's previous memorized events, then another part of the rat's brain, perhaps the prefrontal cortex, provides reward signals that enable a decision-function to determine which memory events are to be *retained* and which are to be *discarded* to generate an *advantageous* memory sequence for the rat.

APPENDIX

The Theory of Optimal Discrete Sequential Decision Processes

The study of dynamic processes as a sequence of stages linked with inputs (a priori) states and output (a posteriori) states where a sequence of *transformation* equations describing dynamic action for each stage was pioneered by H. Poincaré and extensively studied by G. D. Birkhoff (1927). This method is widely used with great success today, for example, in the mathematics of quantum physics.

Suppose at the n^{th} stage of a dynamic process, the a priori state is modified by a transformation function that is dependent on the action of a decision-maker during that stage and results in an a posteriori state for the stage. We write a typical transformation as

$$x_{n+1} = T_n\left(x_n, d_n\right),\ n = 1, 2, \cdots, \tag{2.1}$$

here the variable d_n is the *action* of the decision-maker at stage *n.* We write

The equations for *N* of these stages can be written as the sequence

$$\begin{aligned} x_2 &= T_1\left(x_1, d_1\right), \\ x_3 &= T_2\left(x_2, d_2\right), \\ &\bullet\bullet\bullet \\ x_n &= T_{n-1}\left(x_{n-1}, d_{n-1}\right), \\ &\bullet\bullet\bullet \\ x_N &= T_{N-1}\left(x_{N-1}, d_{N-1}\right), \end{aligned}$$

where x_1 is the initial condition for the process and is known.

These *N-1* equations can be collapsed into

$$x_N = T_{N-1}\left(T_{N-2}\left(\bullet\bullet\bullet T_2\left(T_1\left(x_1, d_1\right)\right)\right)\right). \tag{2.2}$$

Or, if the transformation functions are assumed to be identical in form for each stage, we have the special notation

$$x_N = T^N\left(x_1, d_1\right). \tag{2.3}$$

Now we choose a decision action set, $\left\{d_n\right\}, n = 1, 2, \cdots, N-1$, that achieves some particular result for the process, such as for the decision-maker to determine the set, $\left\{d\right\}$, that optimizes the process for each of the stages. Since the x_{n+1}'s are related to the x_n's through the transformation functions, (2.3), the decision-maker can maximize the entire system consisting of the variables, $x_1, x_2, \cdots, x_n, \cdots, x_{N-1}$ and $d_1, d_2, \cdots, d_n, \cdots, d_{N-1}$.

In many cases, and this is one of them, the Markovian property holds, so we can maximize this kind of function, *term by term,* individually, for each of the x's, with respect to the d's. So we have

$$\max_{\{d\}}\left[g\left(x_1,d_1\right)+g\left(x_2,d_2\right)+\cdots+g\left(x_n,d_n\right)+\cdots+g\left(x_{N-1},d_{N-1}\right)\right], \qquad (2.4)$$

where the transformation functions will be

$$x_{n+1}=T_n\left(x_n,d_n\right) \text{ for n = 2,3,}\cdots,N-1 \qquad (2.5)$$

For the decision-maker's maximization operation we define the functional, $f_N\left(x_1\right)$, and write

$$f_N\left(x_1\right)\triangleq\max_{\{d_n\}}\left[g\left(x_1,d_1\right)+g\left(x_2,d_2\right)+\cdots+g\left(x_{N-1},d_{N-1}\right)\right], \text{ for } N\ge 1. \qquad (2.6)$$

Note that the first term of (2.6), $g\left(x_1,d_1\right)$, can be easily maximized with respect to d_1 because this term is a simple function of the initial condition x_1 that is already known. But maximization of the remaining terms is not so easy, for example the term $g\left(x_2,d_2\right)$ requires the transformation of the $\max_{d_1}\left[g\left(x_1,d_1\right)\right]$ to generate the optimal value of x_2 *before* it can be further maximized with respect to d_2. Likewise, for all the remaining terms, each requires the transformation of the previous term in the sequence *before* they can be maximized. This sequential dependency is the outgrowth of the Markovian nature of the process, to wit:

> *After any number of decisions, say k, we must assure that the effect of the remaining N-k stages of the decision process, upon the total return, to depend* ***only*** *upon the state of the system at the end of the kth decision.*

The Markovian nature of the above equation applies, therefore, let us rewrite (2.6) and introduce the following form of the functional,

$$f_N\left(x_1\right)\triangleq\max_{d_1}\max_{d_2}\max_{d_3}\cdots\max_{d_{N-1}}\left[g\left(x_1,d_1\right)+g\left(x_2,d_2\right)+\cdots+g\left(x_{N-1},d_{N-1}\right)\right]. \quad (2.7)$$

This functional can be rewritten so as to identify and maximize the "easy" initial term separately

$$f_N(x_1) = \max_{d_1}\left[g(x_1,d_1) + \max_{d_2}\max_{d_3}\cdots\max_{d_{N-1}}\left\{g(x_2,d_2)+\cdots+g(x_{N-1},d_{N-1})\right\}\right] \quad (2.8)$$

Note that the maximization of the remaining terms

$\max_{d_2}\max_{d_3}\cdots\max_{d_{N-1}}\left\{g(x_2,d_2)+\cdots+g(x_{N-1},d_{N-1})\right\}$ for $N \geq 2$,

is not yet known and is identical to the maximizing an $(N-1)$ sequence starting with x_2.

So we define an (N-1) functional starting with x_2

$$f_{N-1}(x_2) \triangleq \max_{d_2}\max_{d_3}\cdots\max_{d_{N-1}}\left\{g(x_2,d_2)+\cdots+g(x_{N-1},d_{N-1})\right\} \text{ for } N \geq 2. \quad (2.9)$$

Using (2.9), equation (2.7) can be simplified and we get

$$f_N(x_1) = \max_{d_1}\left[g(x_1,d_1) + f_{N-1}(x_2)\right], \quad (2.10)$$

but from (2.3) we have

$$x_2 = T(x_1,d_1) \quad (2.11)$$

and so, continuing this procedure for each stage, we have

$$f_N(x_1) = \max_{d_1}\left[g(x_1,d_1) + f_{N-1}\left(T(x_1,d_1)\right)\right]. \quad (2.12)$$

Now we can write out for the whole sequence of *N* stage equations that can be evaluated one at a time from $f_1(x_1)$ to $f_2(x_2)$ to … $f_{N-1}(x_{N-1})$, and embed the transformation relations, (2.3), to get each equation in terms of the known initial condition, x_1

$$f_1(x_1) = \max_{d_1}\left(g(x_1,d_1)\right),$$

$$f_2(x_1) = \max_{d_1}\left(g(x_1,d_1) + f_1(x_1)\right),$$

$$\bullet \quad \bullet \quad \bullet$$

$$f_{N-1}(x_1) = \max_{d_1}\left[g(x_1,d_1) + f_{N-2}(x_1)\right],$$

$$f_N(x_1) = \max_{d_1}\left[g(x_1,d_1) + f_{N-1}(x_1)\right]. \quad (2.13)$$

This final functional, (2.13), is our goal; the optimization for the entire N set of functional equations for the dynamic decision process.

Procedure for Optimizing a Single Stage at *t*

Because of the Markovian properties of the sequential knowledge-gathering process, we have seen that each *individual* stage of the knowledge-gathering process can be maximized with respect to the decision-maker's information relevance-vector, and the resulting state from the *previous* stages. The recursive relationship between the stages will enable the accumulation of each of these maximized single stages into a total *process* maximization, since each maximized stage result will become the initial state for the following stage. Now I can turn to the *procedure* to maximize a *single* stage in the sequential process, at stage t, with respect to the decision-maker's determination of the relevance-vector for the environment's information-packets. The procedure, herein, follows a similar procedure found in Murphy (1965).

The subjective expectation, in general, of the compounded rate-of-growth at stage t, for one single stage is given by

$$\overline{g}_t = E \ln K_t - \ln K_{t-1} = \sum_{i=1}^{M} \hat{p}_{i,t} \ln\left(1 + r_{i,t} y_i\right), \tag{3.1}$$

where the decision-maker's relevance factors are subject to the constraints

$$\sum_{i=1}^{M_t} r_{i,t} \leq 1 \text{ and } r_{i,t} \geq 0 \text{ for } i = 1,2,\ldots, M, \tag{3.2}$$

and $\ln K_{t-1}$ is an initial condition determined from the previous stage.

The partial derivative of $\overline{g}_t$ (the rate of growth of knowledge during stage *t*) with respect to each of the i^{th} kinds of information-packets is given by

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}} = \frac{\hat{p}_{i,t} y_i}{\left(1 + r_{i,t} y_i\right)}, \text{ for } i = 1, 2, ..., M, \tag{3.3}$$

where the decision-maker's subjective probabilities are subject to the constraints

$$\hat{p}_{i,t} \geq 0 \text{ and } \sum_{i=1}^{M} \hat{p}_{i,t} = 1 \text{ for } i = 1, 2, ..., M. \tag{3.4}$$

We find that the maximization of (3.1) with respect to the relevance-vector under the constraints (3.2) is a *nonlinear* optimization problem. Consequently, we can use the method of Lagrange, but subjected to the Kuhn-Tucker special conditions. First, we split the set of indexes, M into two subsets, the subset M^+ (the elements soon to be determined) contain the indexes that apply to the decision-maker's relevance factors that are positive. The remainder of these indexes will be contained in the subset M^0 and are for the relevance-vector components that are zero or negative. Kuhn-Tucker recognizes two special kinds of constraints:

For the Kuhn-Tucker condition $\sum_{i=1}^{M} r_{i,t}* < 1$ we have

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}}\Big| r_{i,t}* = 0 \text{ for } i \in M^+ \tag{3.5}$$

and

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}}\Big| r_{i,t}* \le 0 \text{ for } i \in M^0 . \tag{3.6}$$

For the Kuhn-Tucker condition $\sum_{i=1}^{M} r_{i,t}* = 1$ (our case) we have

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}}\Big| r_{i,t}* - \lambda_{i,t} = 0 \text{ for } i \in M^+ \tag{3.7}$$

and

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}}\Big| r_{i,t}* - \lambda_{i,t} \le 0 \text{ for } i \in M^0 , \tag{3.8}$$

where the $\lambda_{i,t}$ is the Lagrangian for the i^{th} kind of information-packet during stage t^{th} and we note that it is to be a non-negative number. Also, in general

$$\frac{\partial \overline{g}_t}{\partial r_{i,t}}\Big| r_{i,t}* = \frac{\hat{p}_{i,t} y_i}{1 + r_{i,t} * y_i} \text{ for } i \in M . \tag{3.9}$$

Conditions (3.5) and (3.6) are not useful, since it is assumed that the decision-maker ignores the information-packets that are of zero or negative relevance and are not part of his optimality decisions. Therefore, we will limit the set of relevant information-packets to only those that have positive relevance, so

$$\sum_{i=1}^{M} r_{i,t}* = \sum_{i=1}^{M^+} r_{i,t}* . \tag{3.10}$$

Now, we can determine the index of the *boundary* between the subsets, M^+ and M^0. The complete Lagrangian can be determined as the sum of the individual Lagrangians, (3.7) using (3.9) and (3.10) for each relevant index in the complete set, $i \in M$. So

$$\lambda_t = \sum_{i=1}^{M} \lambda_{i,t} = \frac{\sum_{i=1}^{M^+} \hat{p}_{i,t}}{\sum_{i=1}^{M^+} r_{i,t} + \sum_{i=1}^{M^+} \frac{1}{y_{i,t}}} = \frac{\sum_{i=1}^{M^+} \hat{p}_{i,t}}{1 + \sum_{i=1}^{M^+} \frac{1}{y_{i,t}}} . \tag{3.11}$$

Substituting this value for λ_t into (3.7) into (3.8) we find that

$$r_{i,t}* = \frac{\hat{p}_{i,t}}{\sum_{j=1}^{M^+} \hat{p}_{j,t}} \left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] - \frac{1}{y_i} \text{, for } i \text{ and j} \in M^+ \tag{3.12}$$

and

$$0 \geq \frac{\hat{p}_{i,t}}{\sum_{j=1}^{M^+} \hat{p}_{j,t}} \left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] - \frac{1}{y_i} \quad \text{for } i \in M^0 \text{and } j \in M^+ . \tag{3.13}$$

Since $r_{i,t}* > 0$ for indexes in M^+ equations (3.12) and (3.13) can write

$$\hat{p}_{i,t} y_i > \frac{\sum_{i=1}^{M^+} \hat{p}_{j,t}}{1 + \sum_{j=1}^{M^+} \frac{1}{y_j}} \text{ for } i, j \in M^+ \tag{3.14}$$

and

$$\frac{\sum_{j=1}^{M^+} \hat{p}_{j,t}}{1 + \sum_{j=1}^{M^+} \frac{1}{y_j}} \geq \hat{p}_{i,t} y_i \text{ for } i \in M^0 \text{and } j \in M^+ . \tag{3.15}$$

Now we order the $r_{i,t}$'s of subset M^+ in descending magnitude starting with $r_{1,t}$ and ending at an arbitrary index, *m,* as $r_{m,t}$. Also in the subset M^0 the $r_{i,t}$'s are ordered descending from $r_{m+1,t}$ to $r_{M,t}$. We have from (3.14) and (3.15)

$$\hat{p}_{m,t} y_m > \frac{\sum_{i=1}^{M^+} \hat{p}_{j,t}}{1 + \sum_{j=1}^{M^+} \frac{1}{y_j}} \geq \hat{p}_{m+1,t} \mathrm{y}_{\mathrm{m+1}} \quad \text{for } i \in M^+ \tag{3.16}$$

Using this relation, we can find the index boundary, *m*, that separates the subset M^+ from the subset M^0 by substituting, one at a time, all the permissible values of *m* into (3.16) until the inequality holds. When the inequality holds, this *m* is index that divides the set *M* into the subsets M^+ and M^0.

As we assumed in the text, we have shown how the decision-maker can create a mental list, just for the relevant information-packets in subset M^+. Now that we know M^+, we can determine the optimal value of each relevance component in the subset M^+ only, but first we must *reduce* the probability space to match the size of the subset M^+ and this requires a redefinition of the probability space be established over only the space of M^+. So we *renormalize* the probability space to refer only to the set of *permissible* components and define,

$$\hat{q}_{i,t} = \frac{\hat{p}_{j,t}}{\sum_{j=1}^{M^+} \hat{p}_{j,t}} \quad \text{where } \sum_{i=1}^{M^+} \hat{q}_{i,t} = 1 \text{ and } \hat{\mathrm{q}}_{i,t} > 0 \text{ for } i, j \in M^+. \tag{3.17}$$

Also note that $\sum_{j=1}^{M^+} \hat{p}_{j,t}$ is *not* necessarily equal to 1, because the subset M^+ may not be equal to the complete set M upon which the original probabilities, $\{\hat{p}_{i,t}\}$ are based. Since it is now assured that $r_i^* > 0$ for $i \in M^+$ we can write (3.12) as

$$r_{i,t}{}^* = \hat{q}_{i,t}\left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j}\right] - \frac{1}{y_i} \quad \textit{for } i \in M^+. \tag{3.18}$$

Now we can do some more algebra to bring out certain important features of the optimal equation (3.18). First multiplying (3.18) by y_i we get

$$y_i r_{i,t}{*} = \hat{q}_{i,t} y_i \left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] - 1 \quad for\ i, j \in M_t^+ .$$

Transferring the "1" to the left side and taking the log, we get

$$\ln\left(1 + y_i r_{i,t}{*}\right) = \ln \hat{q}_{i,t} + \ln y_i + \ln\left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right] \quad for\ i, j \in M^+ ,$$

we then substitute this expression into the definition of the subjective, statistical expectation of the compounded rate-of-growth and get

$$\overline{g}_t{*} = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left(1 + y_i r_{i,t}{*}\right) = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln \hat{p}_{i,t} + \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln y_i + \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right]$$

$$for\ i, j \in M^+ . \tag{3.19}$$

Defining

$$-\hat{H}_t \triangleq \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln \hat{q}_{i,t} , \tag{3.20}$$

as the decision-maker's subjective entropy upon reaching stage *t,* and

$$\hat{E}\left\{y_t\right\} = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln y_i \tag{3.21}$$

as the subjective expectation of the set $\left\{y_t\right\}$ we get

$$\overline{g}_t{*} = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left(1 + r_{i,t}{*}\, y_i\right) = -\hat{H}_t + \hat{E}\left\{y_t\right\} + \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left[1 + \sum_{j=1}^{M^+} \frac{1}{y_j} \right]. \tag{3.22}$$

We note that $\Phi \triangleq \dfrac{M^+}{\sum_{j=1}^{M^+} \frac{1}{y_j}}$, is defined as the *harmonic mean* of $\left\{y_t\right\}$, so we get

$$1 + \sum_{j=1}^{M^+} \frac{1}{y_j} = \left[1 + \frac{M^+}{\Phi} \right] \tag{3.23}$$

and so then

$$\overline{g}_t{*} = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left(1 + r_{i,t} * y_i\right) = -\hat{H}_t + \hat{E}\{y_t\} + \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left[1 + \frac{M^+}{\Phi}\right]. \tag{3.24}$$

Dividing $\ln\left[1 + \frac{M^+}{\Phi}\right]$ by M^+ we get

$$\ln M^+ + \ln\left[\frac{1}{M^+} + \frac{1}{\Phi}\right].$$

Then

$$\overline{g}_t{*} = \sum_{i=1}^{M^+} \hat{q}_{i,t} \ln\left(1 + r_{i,t} * y_i\right) = -\hat{H}_t + \hat{E}\{\ln y_t\} + \ln M^+ + \hat{E}\left\{\ln\left[\frac{1}{M^+} + \frac{1}{\Phi}\right]\right\}, \tag{3.25}$$

but,

$$\ln M^+ = -\ln\frac{1}{M^+} \triangleq H_t{*} \ ,$$

(this is the *maximum* value of entropy for the process), so we finally get

$$\overline{g}_t{*} = \left(H_t{*} - \hat{H}_t\right) + \hat{E}\{\ln y_t\} + \hat{E}\left\{\ln\left[\frac{1}{M^+} + \frac{1}{\Phi}\right]\right\}. \tag{3.26}$$

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