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Abstract

When in “exact” present value (PV) relations the decision variables do not Granger cause the explanatory variables and a VAR process is used to derive restrictions, the system embodies explosive roots. Hence any test of the PV restrictions would reject the null if the system incorporates Granger non-causality constraints. This paper investigates the issue.

1 Problem and motivation

An “exact” present value (PV) relation between the $m \times 1$ vector of decision variables, y_t, and the $q \times 1$ vector of explanatory (forcing) variables, z_t, can be formulated as

$$y_t = \omega_0 y_{t-1} + \sum_{j=0}^{\infty} \omega_1^j \gamma \theta E_t z_{t+j} + \varsigma \tag{1}$$

where ω_h, $h = 0, 1$ and γ are $m \times m$ matrices, θ is $m \times q$, ς is an $m \times 1$ constant, and $E_t \cdot := E(\cdot \mid \Omega_t)$ denotes expectations conditional on the sigma algebra Ω_t, $\Omega_t \subseteq \Omega_{t+1}$, summarizing agent's information at time t.[1] It is assumed that the coefficients of the system of equations (1) satisfy:

A1 for each $i, h = 1, ..., m$, the elements γ_{ih} of γ depend on $\varpi_0 = vec(\omega_0)$ and $\varpi_1 = vec(\omega_1)$ through differentiable functions $\gamma_{ih} := \gamma_{i,h}(\varpi_0' \colon \varpi_1')$, $\gamma_{i,h}(\cdot) : S \to R$, with S open subset of R^{m^2};

A2 ω_1 is non-singular and has stable eigenvalues, i.e. lying within the unit circle in the complex plane;

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[1] Further deterministic terms might be included in (1) without any significant change to the concepts and results that follow.

A3 ω_0 has stable eigenvalues.

The elements of the θ matrix in the relation (1) reflect the link between y_t and z_t and usually depend on agents' preferences and technology. The matrices ω_h, $h = 0, 1$ and γ typically embody adjustment-type parameters. Usually the elements of ω_1 depend on a time-invariant discount factor δ, $(0 < \delta < 1)$, and may depend on the elements of ω_0. Given assumption A1, the number of free parameters of model (1) is (at most) $2m^2 + mq$, and correspond to those in the matrices $(\omega_0, \omega_1, \theta)$.

Two important aspects of the PV model specified in (1) are worth stressing. First, we have deliberately left for the moment the stochastic process generating z_t unspecified. In general, however, the process generating z_t plays a key role in determining solution properties and the identification of the PV model. Second, the PV relation reads as an "exact" rational expectations model in the sense of Hansen and Sargent (1991). This means that the model does not include stochastic processes which are unobservable to the econometrician. Examples of "exact" PV models include, *inter alia*, Sargent (1979), Campbell and Shiller (1987), Baillie (1989), Hansen and Sargent (1991), Engsted and Haldrup (1994), Johansen and Swensen (1999) and Fanelli (2002, 2006*a*, 2006*b*).

It might be argued that assumption A3 rules out some interesting situations, e.g. $\omega_0 := I_m$. Actually, the PV model (1) with A1-A3 is general enough to cover several special cases of interest, provided that the variables are opportunely transformed. For instance, when $\omega_0 := I_m$ one may specify a new PV relation (1) with $y_t^* := \Delta y_t$ as vector of decision variables, and $\omega_0 := 0_{m\times m}$. More generally, when one faces a model of the form (1) where the ω_0 matrix has eigenvalues equal to one, it is possible to resort to transformations of the system which preserve the form (1) and assumptions A1-A3.

A convenient formulation of (1), for given z_t, reads as

$$(I_m + \omega_1\omega_0)y_t = \omega_1 E_t y_{t+1} + \omega_0 y_{t-1} + \gamma\theta z_t + \varsigma^* \tag{2}$$

where $\varsigma^* := (I - \omega_1)\varsigma$. The system of Euler equations (2) can be obtained from (1) through the following steps: (i) write (1) at time $t + 1$ and multiply both sides by ω_1; (ii) condition with respect to the information set Ω_t and apply the law of iterated expectations; (iii) subtract the system of equations derived in (ii) from (1) and rearrange terms. Observe that while (1) implies (2), the reverse is not generally true, unless a suitable transversality condition is imposed on, or satisfied by the stochastic process generating y_t, see e.g. Sargent (1987).

Testable implications of (1) or of its counterpart (2) can be derived by assuming that the form of the rational expectations solution of (1) belongs to the class of VAR processes for the $p \times 1$ $(p = m + q)$ vector $X_t := (y_t' \vdots\, z_t')'$, and then applying the method of undetermined coefficients, see e.g. Bekaert and Hodrick (2001). This paper shows that when y_t does not Granger causes z_t, the restrictions that the PV relation entails on the VAR are inconsistent with a stable process for X_t. In other words, the VAR embodies explosive roots under both PV and Granger non-causality restrictions. Aside from the so-called "rational bubbles" (e.g. Diba and Grossman,

1988) and other special circumstances and episodes, explosive (unstable) roots usually imply dynamic patterns for the variables which can be hardly reconciled with the typical features observed in most macroeconomic and financial time-series. Thus, one can reasonably expect that any VAR-based test of the restrictions implied by (1) tend to reject the null if the system incorporates Granger non-causality constraints.

In general, there are strong economic grounds to expects feedbacks from y_t to z_t in PV relations, see e.g. Timmermann (1994).[2] Nonetheless, PV models where z_t behaves (or is treated by the econometrician) as a strongly exogenous vector of variables with respect to the parameters of interest, $(\omega_0, \omega_1, \theta)$, are frequently used in both theoretical and applied research. For instance, in the PV model for stock prices, the finance literature typically treats dividends as given exogenously, and generally specifies univariate ARMA processes to describe their law of motion (Campbell and Shiller, 1988). In dynamic factor demand models, it is often assumed - and found empirically, see e.g. Meese (1980) - that firms face stochastic processes where real wages and the rental price of capital, $z_t = (w_t \colon c_t)'$, are not Granger caused by labour and capital, $y_t = (n_t \colon k_t)'$.

We conclude this section by showing that the PV formulation (1) under A1-A3 covers many of the models typically used in financial and macroeconomic time-series. In the expectations theory of interest rates, $y_t = R_t$ $(m = 1)$ is a long-term yield, $z_t = r_t$ $(q = 1)$ the one-period rate, $\omega_0 := 0$, $\omega_1 := \delta$ is the discount factor, $\gamma := (1 - \omega_1) = (1 - \delta)$, θ a (scalar) proportionality parameter, and ς a constant risk premium, see Campbell and Shiller (1987). In modern macroeconomic sticky pricing theories (1) follows from the assumption of forward-looking price-setting firms (Calvo model), see e.g. Galì and Gertler (1999), with $y_t = \pi_t$ $(m = 1)$ being the inflation rate, $z_t = mc_t$ $(q = 1)$ a measure of firm's marginal costs, θ a structural parameter related to the degree of firm's price rigidity, and $\omega_0 := \lambda_1$, $\omega_1 := \lambda_2^{-1}$, $\gamma := (\lambda_1 + \lambda_2)(\lambda_1 \lambda_2^2)^{-1}$, with λ_1 and λ_2 the stable and unstable root of a second-order equation, respectively. Other examples may be found in e.g. Baillie (1989), Fanelli (2006*a*) and references therein.

2 Main result

Let $X_t := (y_t' \colon z_t')'$ be generated by a VAR(k), $A(L)X_t = \mu + \varepsilon_t$, with $A(L) := I_p - A_1 L - \cdots - A_k L^k$, L being the lag operator, A_i $p \times p$ matrices of parameters, μ a $p \times 1$ constant, and ε_t an $iidN(0, \Sigma)$ $p \times 1$ disturbance term. It is assumed that:

A4 $k \geq 2$.

A5 every root of the characteristic equation $\det(A(s)) = 0$ is such that $\mid s \mid > 1$ or $s = 1$, where the symbol $\det(\cdot)$ denotes the determinant of a matrix.

[2] Feedback mechanisms can be interpreted as proxies for equilibrium forces which should be modelled directly. Timmermann (1994) provides a thorough analysis of the impact of feedbacks on the solutions to PV models. Granger causality from y_t to z_t is often regarded as a "weak implication" of the PV model, see e.g. Campbell and Shiller (1987) and Engsted and Haldrup (1994).

A6 When there are roots at $s = 1$, their number is equal to $q = p - r$, $r := m$, and y_t and z_t are cointegrated with cointegration matrix $\beta' := [I_m : \ -\theta]$, i.e. $v_t := \beta' X_t = y_t - \theta z_t$ is an I(0) process.

See e.g. Fanelli (2002) for an explanation of the role of A4. Condition A5 states that both stationary and possibly I(1) processes are allowed; condition A6, which is subordinate to the existence of unit roots in A5, confines the analysis of I(1) processes to the case where y_t and z_t are cointegrated with θ restricted to the cointegration space, covering therefore a well known empirical implication of PV models (Campbell and Shiller, 1987).

The state space representation of the VAR is given by

$$\widetilde{X}_t = A\widetilde{X}_{t-1} + \widetilde{\mu} + \widetilde{\varepsilon}_t \tag{3}$$

where $\widetilde{X}_t := (X_t' \colon X_{t-1}' \colon ... \colon X_{t-k+1}')'$ is the $g \times 1$ state vector, $g = pk$, and

$$A := \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ I_p & 0_{p\times p} & \cdots & 0_{p\times p} \\ & \ddots & & \vdots \\ 0_{p\times p} & & I_p & 0_{p\times p} \end{bmatrix} = \begin{bmatrix} \dfrac{G}{\Xi} \end{bmatrix} \tag{4}$$

is the $g \times g$ companion matrix, the sub-matrix $G = [A_1 : A_2 : \cdots : A_k]$ has dimension $p \times g$, the sub-matrix Ξ has dimension $(g - p) \times g$ and contains "1" and "0" only; $\widetilde{\mu}$ and $\widetilde{\varepsilon}_t$ are defined accordingly. It is assumed that when A6 holds, the sub-matrix G in (3)-(4) is restricted as $G := G^c = \Psi W - F$, where $\Psi := [\alpha\beta' : \Gamma_1 : \cdots : \Gamma_{k-1}]$, $F := [I_p : 0_{p\times p} : \cdots : 0_{p\times p}]$,

$$W := \begin{bmatrix} I_p & 0_{p\times p} & \cdots & 0_{p\times p} \\ I_p & -I_p & \cdots & 0_{p\times p} \\ 0_{p\times p} & I_p & \cdots & 0_{p\times p} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{p\times p} & 0_{p\times p} & & -I_p \end{bmatrix},$$

α is a $p\times m$ matrix that satisfies $\alpha\beta' := \alpha[I_m \colon -\theta] := -A(1) = \sum_{j=1}^{k} A_j - I_p$, and the Γ_is are $p \times p$ matrices such that $\Gamma_i := -\sum_{j=i+1}^{k} A_j$, $i = 1, ..., k-1$ (Johansen, 1996). The $p \times g$ matrix Ψ contains the coefficients associated with the Vector Equilibrium Correction (VEqC) representation of the cointegrated VAR, and $G := G^c$ maps VEqC to VAR coefficients. Therefore results derived with respect to the general state space representation (3)-(4) of the VAR, automatically hold when X_t is cointegrated as in A6, provided that

$$A := A^c = \begin{bmatrix} \dfrac{G^c}{\Xi} \end{bmatrix}.$$

Expectations conditional on the sigma algebra $H_t := \sigma(X_t, X_{t-1}, ..., X_1)$, $H_t \subseteq \Omega_t$, can be easily computed from the VAR (3)-(4). More precisely, $E(X_{t+1} \mid H_t) :=$

$J_x' E(\widetilde{X}_{t+1} \mid H_t) := J_x' A\widetilde{X}_t + J_x'\widetilde{\mu} = \sum_{h=0}^{k-1} A_h X_{t-h} + \mu$, where $J_x' := [I_p \colon 0_{p\times(g-p)}]$ is a selection matrix; when $A := A^c$ because of A6, $E(X_{t+1} \mid H_t) := J_x' A^c \widetilde{X}_t + J_x'\widetilde{\mu} := X_t + \alpha\beta' X_t + \sum_{h=0}^{k-2} \Gamma_h \Delta X_{t-h} + \mu$. Hence, by using $J_y' := [I_m \colon 0_{m\times(g-m)}]$, one can replace expectations in (2) by the forecast $J_y' E(\widetilde{X}_{t+1} \mid H_t) := J_y' A\widetilde{X}_t + J_y'\widetilde{\mu}$ and retrieve a set of restrictions.

We now introduce some further notation. Partition matrices A_i, $i = 1, 2, ..., k$ and the constant μ in the VAR (3)-(4) conformably with $X_t = (y_t' \colon z_t')'$:

$$A_i = \left[\begin{array}{c|c} A_i^{yy} & A_i^{yz} \\ \hline A_i^{zy} & A_i^{zz} \end{array}\right] \begin{array}{cc} {\scriptstyle m\times m} & {\scriptstyle m\times q} \\ {\scriptstyle q\times m} & {\scriptstyle q\times q} \end{array} \quad , \quad \mu = \left[\begin{array}{c} \mu^y \\ \hline \mu^z \end{array}\right] \begin{array}{c} {\scriptstyle m\times 1} \\ {\scriptstyle q\times 1} \end{array} . \tag{5}$$

Hereafter the absence of Granger causality from y_t to z_t will be denoted by "y_t GNC z_t" and will correspond to the VAR restrictions

$$A_i^{zy} := 0_{q\times m} \quad , \quad i = 1, 2, ..., k. \tag{6}$$

Likewise, "y_t GC z_t" means that y_t Granger causes z_t, i.e. that it exists at least a matrix $A_{\widetilde{i}}^{zy}$, $1 \leq \widetilde{i} \leq k$, such that $A_{\widetilde{i}}^{zy} \neq 0_{q\times m}$.

The following proposition and corollary establish the relation between "exact" PV models, Granger causality and VAR stability.

PROPOSITION 2.1

Assume that X_t is generated by (3)-(4) with the assumption A4, and that y_t GNC z_t, i.e. that (6) holds. If X_t satisfies the restrictions implied by the PV model (2) with A1-A3, then X_t does not match A5.

Proof. Using A2, write (2) as

$$E_t y_{t+1} = \omega_1^{-1}(I_m + \omega_1\omega_0)y_t - \omega_1^{-1}\omega_0 y_{t-1} - \omega_1^{-1}\gamma\theta z_t + \varsigma^{**}$$

where $\varsigma^{**} := \omega_1^{-1}\varsigma^*$. Condition both sides with respect to H_t, apply the law of iterated expectations and replace the quantity on the left hand-side by the VAR forecast $E(y_{t+1} \mid H_t) = J_y' A\widetilde{X}_t + J_y'\widetilde{\mu}$, obtaining, in light of (5), the relation

$$\begin{aligned} & A_1^{yy} y_t + A_1^{yz} z_t + A_2^{yy} y_{t-1} + A_2^{yz} z_{t-1} + \sum_{i=3}^{k} [A_i^{yy} \colon A_i^{yz}] X_{t-i+1} + \mu^y \\ = \ & \omega_1^{-1}(I_m + \omega_1\omega_0)y_t - \omega_1^{-1}\omega_0 y_{t-1} - \omega_1^{-1}\gamma\theta z_t + \varsigma^*. \end{aligned}$$

As $X_t \neq 0_{p\times 1}$ (a.s.) $\forall\ t$, for the equality above to be satisfied the following set of cross-restrictions must hold:

$$\begin{aligned} A_1^{yy} &:= \omega_1^{-1}(I_m + \omega_1\omega_0) \quad , \quad A_1^{yz} := -\omega_1^{-1}\gamma\theta && (7) \\ A_2^{yy} &:= -\omega_1^{-1}\omega_0 \quad , \qquad A_2^{yz} := 0_{m\times q} \\ A_i^{yy} &:= 0_{m\times m} \quad , \qquad A_i^{yz} := 0_{m\times q} \ , \quad i = 3, ..., k \\ \mu^y &:= \varsigma^*. && (8) \end{aligned}$$

Given (6) and (7)-(8), the characteristic polynomial associated with the VAR (3)-(4), $\det(A(s))$, reduces to

$$\det(\widetilde{A}(s)) = \det\left(\begin{bmatrix} \widetilde{A}_{yy}(s) & \widetilde{A}_{yz}(s) \\ 0_{q\times m} & A_{zz}(s) \end{bmatrix}\right) = \det(\widetilde{A}_{yy}(s))\det(A_{zz}(s)) \tag{9}$$

where " $\sim$ " indicates that the VAR coefficients are constrained, $\widetilde{A}_{yy}(s) := I_m - (\omega_1^{-1} + \omega_0)s + \omega_1^{-1}\omega_0 s^2$, $\widetilde{A}_{yz}(s) := -\omega_1^{-1}\gamma\theta s$, and $A_{zz}(s) := I_q - \sum_{i=1}^{k} A_i^{zz} s^i$. It can be recognized that $\widetilde{A}_{yy}(s) = (I_m - \omega_1^{-1}s)(I_m - \omega_0 s)$, and that $\det(\widetilde{A}_{yy}(0)) \neq 0$. Moreover, $\det(\widetilde{A}_{yy}(1)) \neq 0$, as $\det(I_m - \omega_1^{-1}) \neq 0$ given A2, and $\det(I_m - \omega_0) \neq 0$ given A3 and a Jordan decomposition of ω_0. Since if λ is a non-zero eigenvalue of the square matrix P, then $s := \lambda^{-1}$ is a root of $\det(I - Ps) = 0$, it turns out that $\det(I_m - \omega_1^{-1}s) = 0$ has exactly m (non-zero) roots inside the unit circle because of A2, whereas $\det(I_m - \omega_0 s) = 0$ has roots outside the unit circle (when $\omega_0 \neq 0$) because of A3. Thus the characteristic equation (9) of the constrained VAR process has m roots inside the unit circle and X_t does not match A5.■

COROLLARY

Consider the PV relation (2) with A1-A3, and the VAR (3)-(4) with A4. Necessary condition for X_t to match the stability condition A5 under the PV restrictions is that y_t GC z_t.

Remark. Proposition 2.1 can be easily extended to the situation where the specified VAR is of the form $X_t := (y_t' \vdots z_t' \vdots w_t')'$, where w_t is a vector of "additional" variables that help to forecast z_t. Indeed, define the new vector $z_t^* := (z_t' \vdots w_t')$ of dimension $q^* \times 1$ $(q^* > q)$; for a suitable definition of the $m \times q^*$ matrix θ, the PV model (1) and its counterpart (2) can be re-written, other things remaining fixed, by replacing z_t by z_t^*.

A natural fix to the shortcoming sketched in Propositions 2.1 is to appeal to "inexact" formulations of (1), other than looking outside VAR processes. For instance, one may add an exogenous $m \times 1$ MDS, u_t, on the right hand side of (1). In principle, it is possible to interpret such a component as a process capturing temporary unexplained deviations from the theory, however, there are circumstances where a precise motivation for u_t stems from theory itself.[3] In general, Proposition 2.1 does not apply to the class of "inexact" PV models, see e.g. Hansen and Sargent (1981).

References

[1] Baillie, R. T. (1989). Econometric tests of rationality and market efficiency, *Econometric Reviews* 8, 151-186.

[3]For instance, in the term structure of interest rates u_t might capture a time-varying risk premium; in the permanent income theory of consumption u_t may proxy a transitory consumption components; in New Keynesian pricing theory u_t may represent a shock in firms' markup. u_t might also account for measurement errors affecting the PV relation.

[2] Bekaert, G., Hodrick, R. (2001), Expectations hypotheses tests, *Journal of Finance* 56, 1357-1394.

[3] Campbell, J.Y., Shiller, R.J. (1987). Cointegration and tests of present-value models, *Journal of Political Economy* 95, 1052-1088.

[4] Campbell, J.Y., Shiller, R.J. (1988). The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195-228.

[5] Diba, B.T., Grossman, H.I. (1988), Explosive rational bubbles in stock prices?, *American Economic Review* 78, 520-530.

[6] Engsted, T., Haldrup, N. (1994). The linear quadratic adjustment cost model and the demand for labour, *Journal of Applied Econometrics* 9, 145-159.

[7] Fanelli, L. (2002). A new approach for estimating and testing the linear quadratic adjustment cost model under rational expectations and I(1) variables, *Journal of Economic Dynamics and Control* 26, 117-139.

[8] Fanelli, L. (2006*a*), Dynamic adjustment cost models with forward-looking behavior, *Econometrics Journal* 9, 23-47.

[9] Fanelli, L. (2006*b*), Multi-equational linear quadratic adjustment cost models with rational expectations and cointegration, *Journal of Economic Dynamics and Control*, 30, 445-456.

[10] Galí, J., Gertler, M. (1999), Inflation dynamics: a structural econometric analysis, *Journal of Monetary Economics* 44, 195-222.

[11] Hansen, L. P., T. J. Sargent (1981), *Linear rational expectations models for dynamically interrelated variables*, in: Lucas Jr. R.E., Sargent, T.J. (eds.), *Rational Expectations and Econometric Practise*, University of Minnesota Press, Minneapolis, pp. 127-156.

[12] Hansen, L.P., Sargent, T.J. (1991), *Exact linear rational expectations models: specification and estimation*, in: Hansen, L.P., Sargent, T.J. (eds.), *Rational Expectations Econometrics*. Westview Press, Boulder, pp. 45-75.

[13] Johansen, S. (1996). *Likelihood based inference in cointegrated vector autoregressive models*, Oxford: Oxford University Press.

[14] Johansen, S., Swensen, A. R. (1999), Testing exact rational expectations in cointegrated vector autoregressive models, *Journal of Econometrics* 93, 73-91.

[15] Meese, R. (1980), Dynamic factor demand schedules for labour and capital under rational expectations, *Journal of Econometrics* 14, 141-158.

[16] Sargent, T. J. (1979), A note on the maximum likelihood estimation of the rational expectations model of the term structure, *Journal of Monetary Economics* 5, 133-143.

[17] Sargent, T. J. (1989), *Macroeconomic Theory*, San Diego: Academic Press.

[18] Timmermann, A. (1994), Present value models with feedback. Solutions, stability, bubbles and some empirical evidence, *Journal of Economic Dynamics and Control* 18, 1093-1119.