R/S analysis and DFA: finite sample properties and confidence intervals

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20 July 2009

R/S Analysis and DFA:
Finite Sample Properties and Confidence Intervals

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Abstract: *We focus on finite sample properties of two mostly used methods of Hurst exponent H estimation – R/S analysis and DFA. Even though both methods have been widely applied on different types of financial assets, only several papers have dealt with finite sample properties which are crucial as the properties differ significantly from the asymptotic ones. Recently, R/S analysis has been shown to overestimate H when compared with DFA. However, we show on the random time series with lengths from 2^9 to 2^{17} that even though the estimates of R/S are truly significantly higher than an asymptotic limit of 0.5, they remain very close to the estimates proposed by Anis & Lloyd and the estimated standard deviations are lower than the ones of DFA. On the other hand, DFA estimates are very close to 0.5. The results propose that R/S still remains useful and robust method even when compared to newer method of DFA which is usually preferred in recent literature.*

Keywords: *rescaled range analysis, detrended fluctuation analysis, Hurst exponent, long-range dependence, confidence intervals*

JEL Classification: *C01, C12, C59*

1 Introduction

Long-range dependence and its presence in the financial time series has been discussed in several recent papers (e.g. Czarnecki, Grech & Pamula, 2008; Grech & Mazur, 2004; Carbone, Castelli & Stanley, 2004; Matos *et al.*, 2008; Vandewalle, Ausloos & Boveroux, 1997; and Alvarez-Ramirez *et al.*, 2008, Peters, 1994; Di Matteo, Aste & Dacorogna, 2005; Di Matteo, 2007). However, most authors interpret the results on the basis of comparison of estimated Hurst exponent H with the theoretical value for independent process of 0.5. In more detail, Hurst exponent of 0.5 indicates two possible processes – either independent (Beran, 1994) or short-range dependent process (Lillo & Farmer, 2004). If $H > 0.5$, the process has significantly positive correlations at all lags and is said to be persistent (Mandelbrot & van Ness, 1968). On the other hand, if $H < 0.5$, it has significantly

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negative correlations at all lags and the process is said to be anti-persistent (Barkoulas, Baum & Travlos, 2000).

However, the estimates for pure Gaussian process can strongly deviate from the limit of 0.5 (Weron, 2002; and Couillard & Davison, 2005). Moreover, the estimates are influenced by choice of minimum and maximum scale (Weron, 2002). There have been several papers dealing with finite sample properties of estimators of Hurst exponent (Peters, 1994; Couillard & Davison, 2005; Grech & Mazur, 2005; and Weron, 2002). However, none of the papers use the proposition for optimal scales presented elsewhere (Grech & Mazur, 2004; Matos *et al.*, 2008; Alvarez-Ramirez, Rodriguez & Echeverria, 2005; and Einstein, Wu & Gil, 2001). This paper attempts to fill this gap and presents results of Monte Carlo simulations for two mostly used techniques – rescaled range analysis and detrended fluctuation analysis.

In Section 2, we present and describe both techniques in detail. In Section 3, we show results of Monte Carlo simulations for time series lengths from 512 to 131072 observations and support that R/S overestimates Hurst exponent for all examined time series lengths. The overestimation decreases significantly with growing length. In Section 4, we present results for simulations for time series of length from 256 to 131072 observations but this time, on the same series, both procedures are applied and we comment on differences. We find out that even if R/S shows higher values of Hurst exponent than DFA, the standard deviations are lower for R/S so that the confidence intervals are narrower. Nevertheless, both methods show very similar estimates, when the bias is taken into consideration, whereas they are more correlated with growing time series length.

2 Hurst exponent estimation methods

2.1. Rescaled range analysis

Rescaled range analysis (R/S) was developed by Harold E. Hurst while working as a water engineer in Egypt (Hurst, 1951) and was later applied to financial time series by Mandelbrot (1970). In the procedure, one takes returns of the time series of length T and divides them into N adjacent sub-periods of length υ while $N * \upsilon = T$. Each sub-period is labeled as I_n with $n = 1,2,...,N$. Moreover, each element in I_n is labeled $r_{k,n}$ with $k = 1,2,...,\upsilon$. For each sub-period, one calculates the average value and constructs new series of accumulated deviations from the arithmetic mean values (profile).

The procedure follows in calculation of the range, which is defined as a difference between maximum and minimum value of profile $X_{k,n}$, and standard deviation of the profile for each sub-period. Each range R_{I_n} is standardized by corresponding standard deviation S_{I_n} and forms the rescaled range as

$$(R/S)_{I_n} = \frac{R_{I_n}}{S_{I_n}}.$$

The process is repeated for each sub-period of length υ. The length υ is increased and the whole process is repeated. We use the procedure used in recent papers (e.g. Weron, 2002) so that we use the length υ equal to the power of a set integer value. Thus, we set a basis b, a minimum power *pmin* and a maximum power *pmax* so that we get $\upsilon = b^{p\min}, b^{p\min+1}, \ldots, b^{p\max}$ where $b^{p\max} \leq T$.

We get average rescaled ranges $(R/S)_\upsilon$ for each sub-interval of length υ. Rescaled range then scales as

$$(R/S)_\upsilon \approx c * \upsilon^H \quad (1)$$

where c is a finite constant independent of υ (Taqqu, Teverovsky & Willinger, 1995; Di Matteo, 2007). The linear relationship in double-logarithmic scale indicates the power scaling (Weron, 2002). To uncover the scaling law, we use a simple ordinary least squares regression on logarithms of each side of (1). We suggest using logarithm with basis equal to b. Thus, we get

$$\log_b (R/S)_\upsilon \approx \log_b c + H \log_b \upsilon,$$

where H is Hurst exponent.

2.2. Detrended fluctuation analysis

Detrended fluctuation analysis (DFA) was firstly proposed by Peng *et al.* (1994) while examining series of DNA nucleotides. Compared to the R/S analysis examined above, the DFA focuses on fluctuations around trend rather than a range of signal. Therefore, DFA can be used for non-stationary time series contrary to R/S.

Starting steps of the procedure are the same as the ones of R/S analysis as the whole series is divided into non-overlapping periods of length υ which is again set on the same basis as in the mentioned procedure and the series profile is constructed. The following steps are based on Grech & Mazur (2005). Polynomial fit $X_{\upsilon,l}$ of the profile is estimated for each sub-period I_n. The choice of order l of the polynomial is rather a rule of thumb but is mostly set as the first or the second order polynomial trend as higher orders do not any significant information (Vandewalle, Ausloos & Boveroux, 1997). The procedure is then labeled as DFA-0, DFA-1 and DFA-2 according to the order of the filtering trend (Hu *et al.*, 2001). We stick to the linear trend filtering and thus use DFA-1 in the paper. A detrended signal $Y_{\upsilon,l}$ is then constructed as

$$Y_{\upsilon,l}(t) = X(t) - X_{\upsilon,.}(t).$$

Fluctuation $F_{DFA}(\upsilon,l)$, which is defined as

$$F_{DFA}(\upsilon,l) = \sqrt{\frac{1}{T}\sum_{t=1}^{T} Y_{\upsilon,l}^{2}(t)},$$

scales as

$$F_{DFA}(\upsilon,l) \approx c * \upsilon^{H(l)}, \tag{2}$$

where again c is a constant independent of υ (Weron, 2002).

We again run an ordinary least squares regression on logarithms of (2) and estimate Hurst exponent $H(l)$ for set l-degree of polynomial trend in same way as for R/S as

$$\log_b F_{DFA}(\upsilon,l) \approx \log_b c + H(l)\log_b \upsilon. \tag{2.7}$$

3 Finite sample properties of R/S and DFA

3.1. R/S analysis

R/S analysis has one significant advantage compared to the other methods – as it is known and tested for over 50 years, the methods for testing have been well developed and applied (Peters, 1991).

The condition for a time series to reject long-term dependence is that $H = 0.5$. However, it holds only for infinite samples and therefore is an asymptotic limit. The correction for finite samples is thoroughly tested in Couillard & Davison (2005). Anis & Lloyd (1976), which we note AL76, states the expected value of rescaled range as

$$E(R/S)_{\upsilon} = \frac{\Gamma\left(\frac{\upsilon-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\upsilon}{2}\right)}\sum_{i=1}^{\upsilon-1}\sqrt{\frac{\upsilon-i}{i}}. \tag{3.1}$$

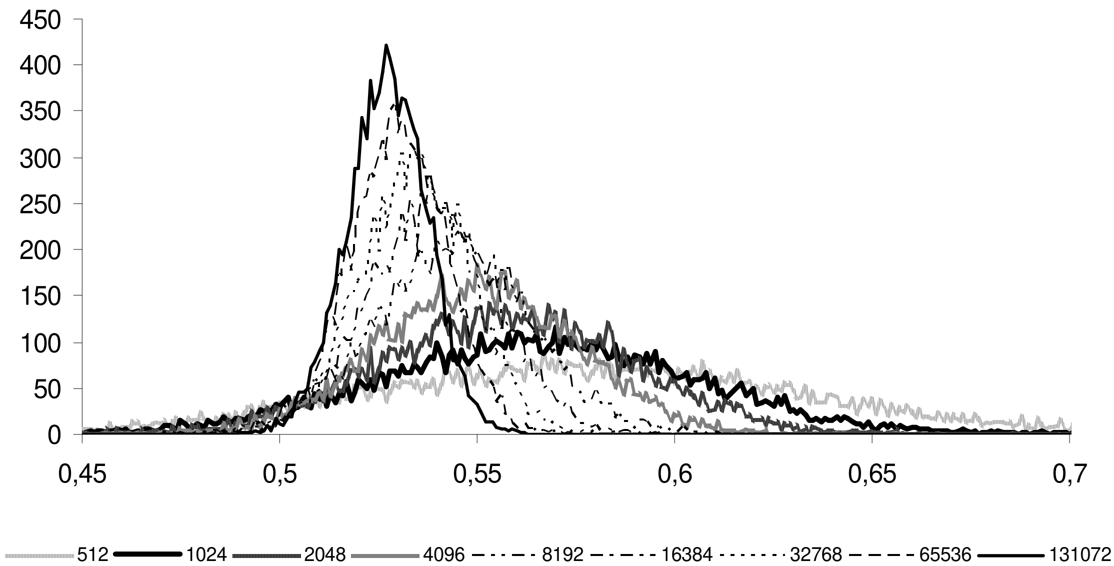
Gamma functions $\Gamma(\bullet)$ can be used up to $\upsilon = 2^8 = 256$ and approximation for higher ones since gamma functions cannot be estimated for high values even by modern analytical software. The approximation is based on relationship between gamma and beta functions together with Stirling's approximation so that we get $\Gamma((\upsilon-1)/2)/\Gamma(\upsilon/2) \approx \sqrt{2/(\upsilon-1)}$ (Boisvert *et al.*, 2008) to eventually obtain

$$E(R/S)_\upsilon = \sqrt{\frac{2}{(\upsilon-1)\pi}} \sum_{i=1}^{\upsilon-1} \sqrt{\frac{\upsilon-i}{i}}\,. \tag{3.2}$$

We performed original tests for time series lengths from $T = 512 = 2^9$ up to $T = 131072 = 2^{17}$. All steps of R/S analysis on 10000 time series drawn from standardized normal distribution $N(0,1)$ were performed. Hurst exponent was estimated by log-log regression according to the presented procedure. Averaged rescaled ranges applied in the regression were the ones for $2^4 \leq \upsilon \leq 2^{T-2}$. The logic behind this step is rather intuitive – very small scales can bias the estimate as standard deviations are based on very few observations; on the other hand, large scales can bias the estimate as outliers or simply extreme values are not averaged out (Peters, 1994; Grech & Mazur, 2004; Matos *et al.*, 2008; Alvarez-Ramirez, Rodriguez & Echeverria, 2005; a Einstein, Wu & Gil, 2001). The same procedure is applied for DFA-1 later.

The expected values of Hurst exponent and corresponding descriptive statistics together with Jarque-Bera test (Jarque & Bera, 1981) for normality are summed in Table 1 and histograms are showed in Chart 1.

Chart 1 Histogram of Monte Carlo simulations (R/S)



The estimates of Hurst exponent are not equal to 0.5 as predicted by asymptotic theory. Therefore, one must be careful when accepting or rejecting hypotheses about long-term dependence present in time series solely on its divergence from 0.5. This statement is most valid for short time series. However, the Jarque-Bera test rejected normality of Hurst exponent estimates for time series lengths of 512, 65536 and 131072 and therefore, we should use percentiles rather than standard deviations for the estimation of confidence intervals (Weron, 2002). Nevertheless, the differences for mentioned estimates not normally distributed are only of the order of the tenths of the thousandth and therefore, we present confidence intervals based on standard deviations for R/S. Standard deviation can be estimated as

$$\hat{\sigma}(H) \approx 1/\pi T^{0.3}$$

with R^2 of 98,55% so that the estimates are very reliable (Chart 2). Therefore, we propose (3.3) for other time series length but for the same minimum and maximum scales only as the estimates can vary for different scales choice (Peters, 1994; Weron, 2002; and Couillard & Davison, 2005).

Chart 2 Standard deviations based on Monte Carlo simulations (R/S)

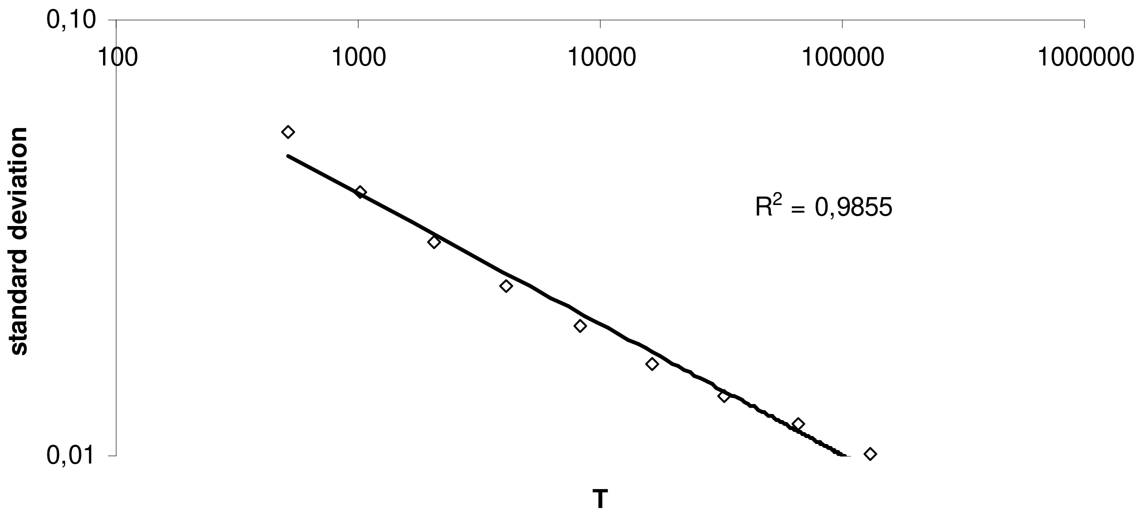


Chart 2: Standard deviations in double-logarithmic scale show decreasing trend with growing time series length. The evolution is well fitted with linear approximation with high squared R. The use of the approximation is suggested for time series lengths different from the ones we present.

Table 1 Monte Carlo simulations descriptive statistics (R/S)

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,5763	0,5647	0,557	0,5494	0,543	0,538	0,5338	0,5296	0,5267
AL76	0,5686	0,5611	0,5513	0,5455	0,5411	0,5361	0,5318	0,5282	0,5254
SD	0,0551	0,0404	0,031	0,0246	0,0199	0,0162	0,0138	0,0118	0,0102
skewness	0,0104	0,0003	-0,0231	-0,0316	-0,0223	-0,0331	-0,0329	0,0068	-0,0762
excess kurtosis	-0,1316	0,073	-0,0595	-0,0567	0,022	-0,0271	0,0136	-0,1108	0,0237
JB statistic	7,4569	2,18	2,3895	3,0314	1,0196	2,144	1,8737	5,2405	9,908
p-value	0,0240	0,3362	0,3028	0,2197	0,6006	0,3423	0,3919	0,0728	0,0071

In Chart 3, we present the estimated confidence intervals for 90%, 95% and 99% two-tailed significance level. From the chart, we can see that all shown confidence intervals are quite wide for short time series. Even if time series of 512 observations yields *H* equal to 0.65, we can't reject the hypothesis of an independent process even at 90% significance level.

Chart 3 Confidence intervals for R/S

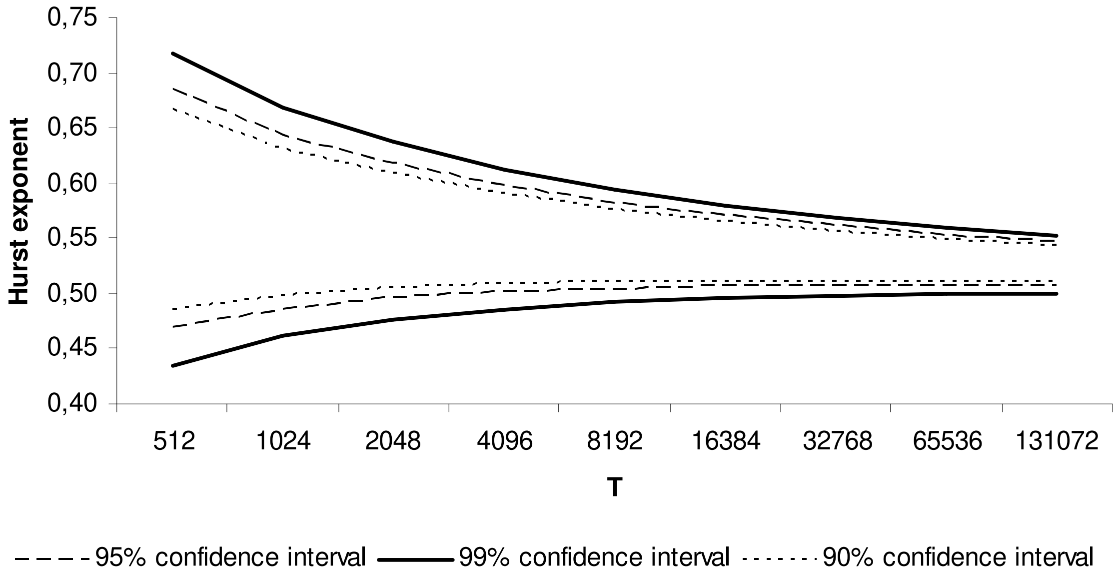


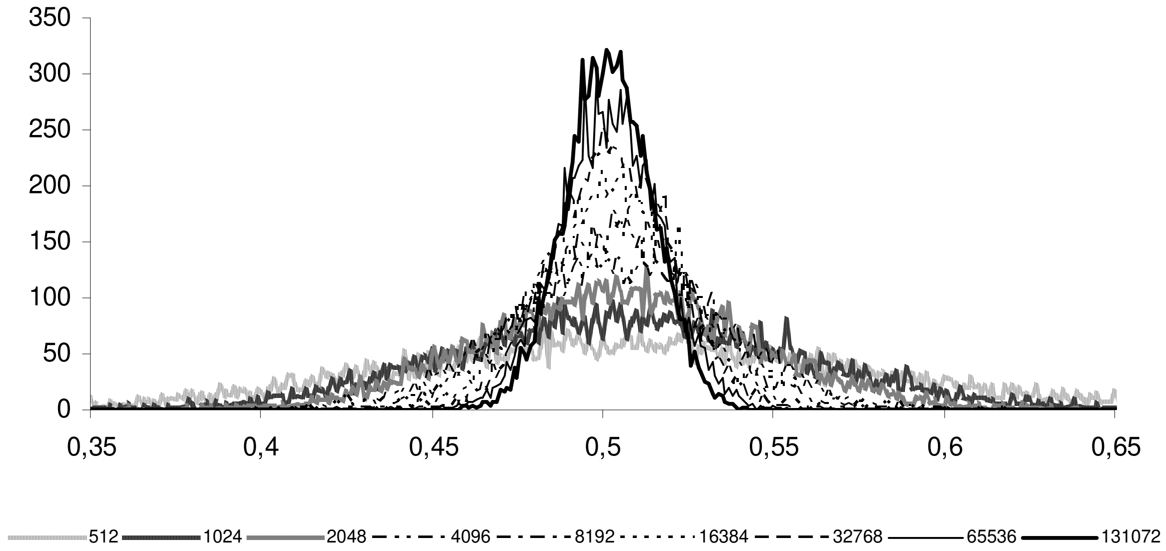
Chart 3: Confidence intervals are based on standard deviation of Monte Carlo simulations as the difference between these and corresponding percentiles are insignificant. The estimated confidence intervals are rather wide for short time series and narrow significantly for longer ones.

3.2. Detrended fluctuation analysis

DFA-1 was already shown to estimate Hurst exponent for random normal series with expected value close to 0.5 (Weron, 2002; and Grech & Mazur, 2005) so that there is no need for similar procedure as for rescaled range presented before. We present the results of simulations for DFA-1 with minimum scale of 16 observations and maximum scale of one quarter of the time series length as was the case for R/S. Chart 4 and Table 2 shows that expected values for DFA-1 are very close to asymptotic limit of 0.5 even for short time series. Normal distribution of the simulated Hurst exponents cannot be rejected with exception for two lowest scales. Therefore, we stick with use of standard deviations for estimation of confidence intervals. The standard deviation can be modeled as

$$\hat{\sigma}(T) = \frac{0.3912}{T^{0.3}}.$$

Chart 4 Histogram of Monte Carlo simulations (DFA-1)



The evolution of standard deviation for different time series lengths together with the fit are shown in Chart 5. The fit is again reliable with R^2 equal to 98.44%. Note that power values for both (3.3) and (3.4) are equal to 0.3 which might be the case of future research. The estimates for expected value of Hurst exponent are close to 0.5 so that we do not present any approximation for different time series lengths. Therefore, we propose to use 0.5 as the expected values and our approximation of standard deviation for construction of confidence intervals for different time series lengths than the ones we present.

Table 2 Monte Carlo simulations descriptive statistics (DFA-1)

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,5079	0,5062	0,504	0,5031	0,5025	0,5022	0,502	0,5015	0,5013
standard deviation	0,0687	0,05	0,0386	0,0304	0,0247	0,0202	0,0173	0,0149	0,0126
skewness	0,1189	0,063	0,043	-0,0069	0,0053	-0,0258	-0,0398	-0,0227	-0,0323
Excess kurtosis	-0,0205	-0,0512	-0,0796	-0,0711	-0,0795	-0,0739	-0,0051	0,0109	-0,0919
JB statistic	23,7407	7,7276	5,7584	2,2171	2,7205	3,4246	2,658	0,899	5,3017
p-value	0,0000	0,0210	0,0562	0,3300	0,2566	0,1804	0,2647	0,6379	0,0706

Even though the expected values are in hand with asymptotic limit, the constructed confidence intervals are still rather wide and rejection of hypothesis for short time series might be again quite problematic. Again, the confidence intervals are quite narrow for long time series. However, the most interesting results come if for a single time series, we estimate Hurst exponent with both R/S and DFA-1 and compare the results. We present the results in detail in the following section.

Chart 5 Standard deviations based on Monte Carlo simulations (DFA-1)

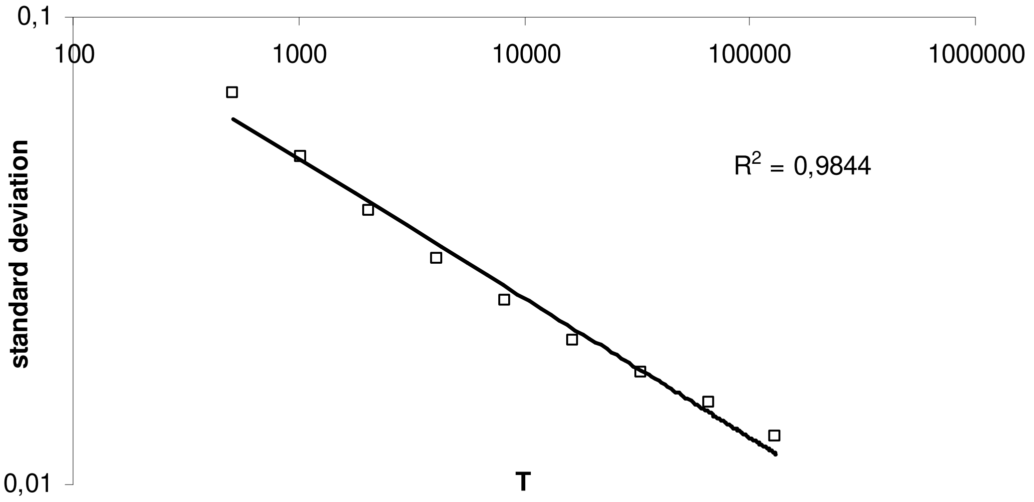


Chart 5: Similarly to R/S, standard deviations in double-logarithmic scale show decreasing trend with growing time series length. The evolution is well fitted with linear approximation with high squared R. The use of the approximation is suggested for time series lengths different from the ones we present.

Chart 6 Confidence intervals for DFA-1

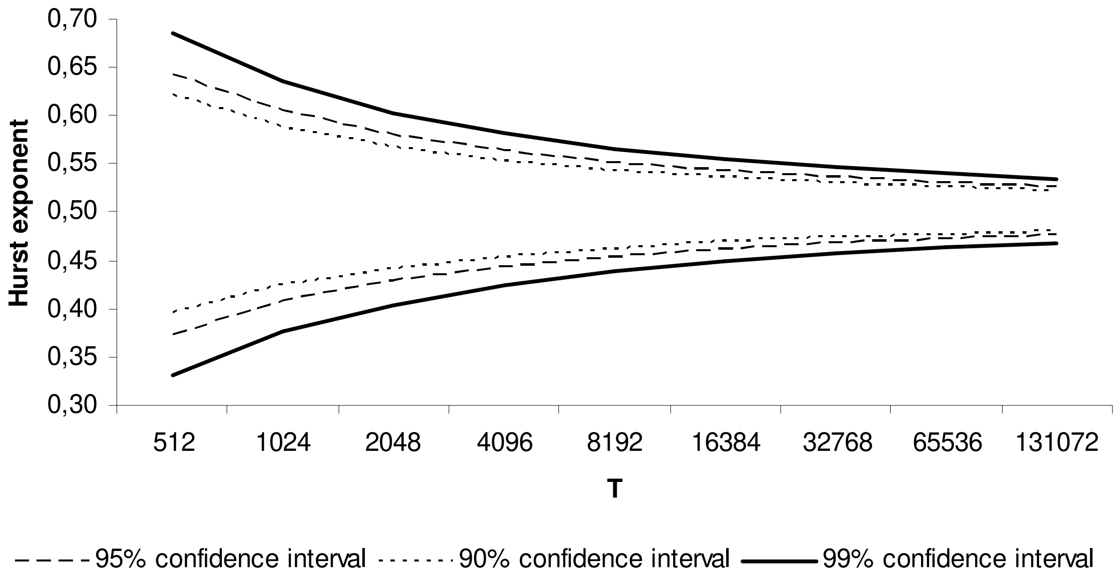


Chart 6: Confidence intervals are based on standard deviation of Monte Carlo simulations as the difference between these and corresponding percentiles are insignificant. The estimated confidence intervals are rather wide for short time series and narrow significantly for longer ones which is also the case for R/S.

4 Simultaneous finite sample properties

We again simulated 10000 random standardized normally distributed *N(0,1)* time series for each set length. This time, we estimated Hurst exponent based on both R/S and DFA-1 on each time series while estimating the results for the lengths from 256 to 131072 observations. Descriptive statistics for differences between estimates of R/S and DFA-1 are summed in Table 3. The results show that R/S on average overestimates Hurst exponent when compared to DFA-1 while the overestimation decreases with growing time series length. For illustration, we present Chart 7 which shows the estimates for both techniques for the time series lengths of 512 and 131072.

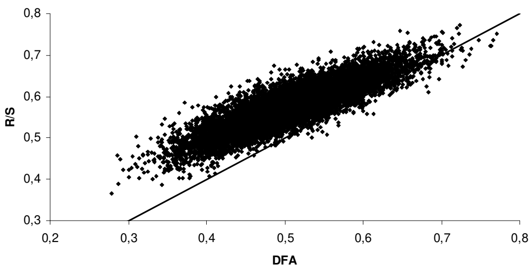
Table 3 Descriptive statistics for (R/S - DFA-1) estimates of Hurst exponent

	256	*512*	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,0783	0,0687	0,0598	0,0525	0,0458	0,0406	0,0358	0,0321	0,0285	0,0256
standard deviation	0,0573	0,0351	0,0239	0,0174	0,0136	0,011	0,0089	0,0075	0,0063	0,0054
max	0,3159	0,213	0,152	0,113	0,0989	0,0861	0,075	0,0624	0,06	0,0477
min	-0,1143	-0,0726	-0,032	-0,0073	-0,0057	-0,0059	0,0035	0,0081	0,0059	0,0052
97.5 percentile	0,1933	0,1394	0,1074	0,087	0,0734	0,0626	0,0541	0,0472	0,041	0,0366
2.5 percentile	-0,032	0,0012	0,014	0,0193	0,0202	0,0195	0,0189	0,0177	0,0167	0,0151
skewness	0,1114	0,0832	0,0962	0,0944	0,1539	0,0849	0,1523	0,1217	0,1263	0,1177
excess kurtosis	0,1187	0,0829	0,0192	-0,0332	0,0992	0,0947	0,103	0,0252	0,0417	0,1214
JB	26,5653	14,3937	15,5837	15,3193	43,5845	15,7368	43,0641	24,9547	27,2917	29,2214
p-value	0,0000	0,0007	0,0004	0,0005	0,0000	0,0004	0,0000	0,0000	0,0000	0,0000

From the chart, we can see that estimates are both strongly correlated and also that the relationship between both estimates is rather linear and not related in more complicated way. Moreover, the overestimation of Hurst exponent by R/S is evidently decreasing with the time series length. The proportion of estimates which are higher for R/S than for DFA-1 is illustrated in Chart 8a. From the time series of length of 4096, all of the estimates are higher for R/S.

Chart 7 Comparison of R/S and DFA-1 estimates

(a)



(b)

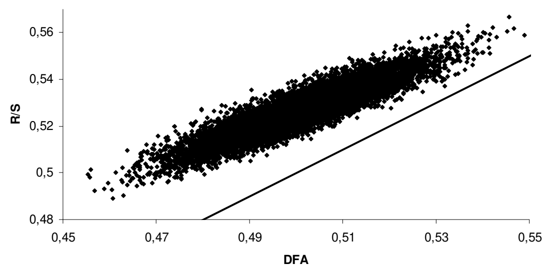


Chart 7: (a) estimates for time series length of 512, **(b)** estimates for time series length of 131072. The results show that majority of R/S estimates are higher than the ones of DFA-1. For better clarity, the solid line presents the case if the estimates were equal so that it has 45 degrees slope. For longer time series, all estimates of R/S are higher than those of DFA-1.

Chart 8 Comparison of R/S and DFA-1 estimates and corresponding correlations

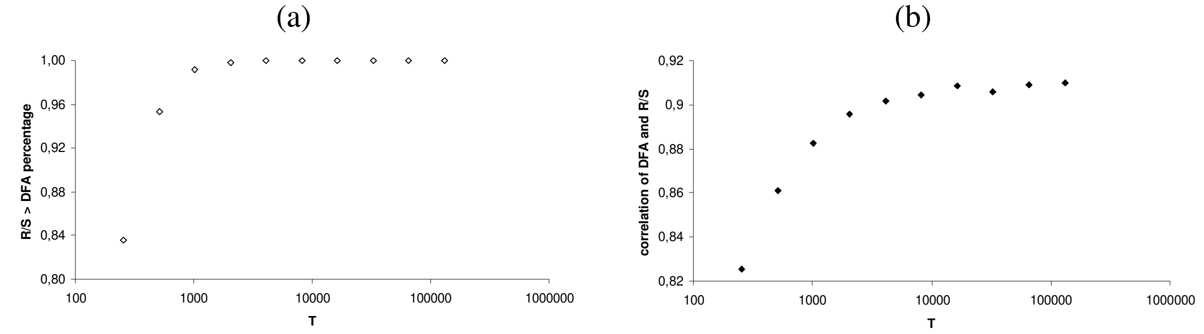


Chart 8: (a) percentage of situation when the estimate of R/S is higher than the one of DFA-1 for different time series lengths, **(b)** correlation of estimates of R/S and DFA-1 for different time series lengths. We can see that the estimates are becoming closer to each other with growing time series length.

Chart 8b shows the evolution of correlations between the estimates of the used methods for different time series lengths. We can see that correlations are quite high even for short time series and convergence above the value of 0.9 for time series with more than 2048 observations. Different aspects are shown in Chart 9. Percentiles (97.5 and 2.5%) show that the estimates can differ significantly for low scales. The difference can be as high as 0.32 for time series length of 256 observations. Nevertheless, the difference narrows significantly for longer time series.

Chart 9 Comparison of R/S and DFA-1 estimates and corresponding correlations

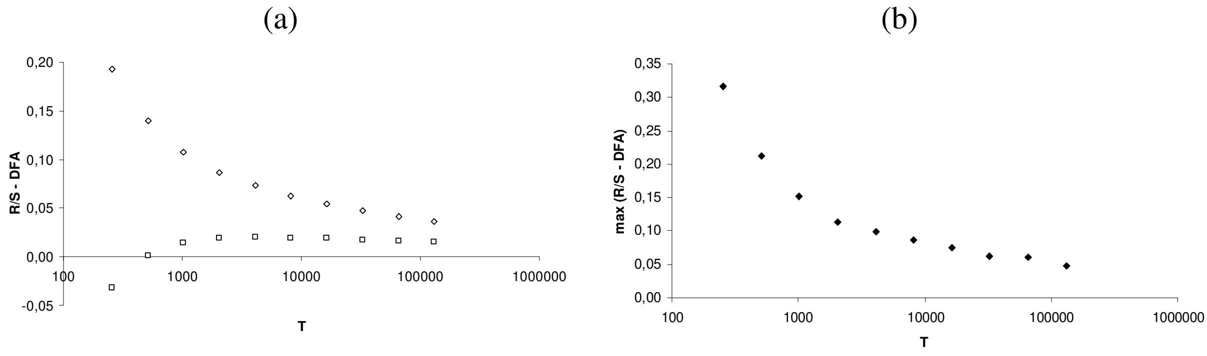
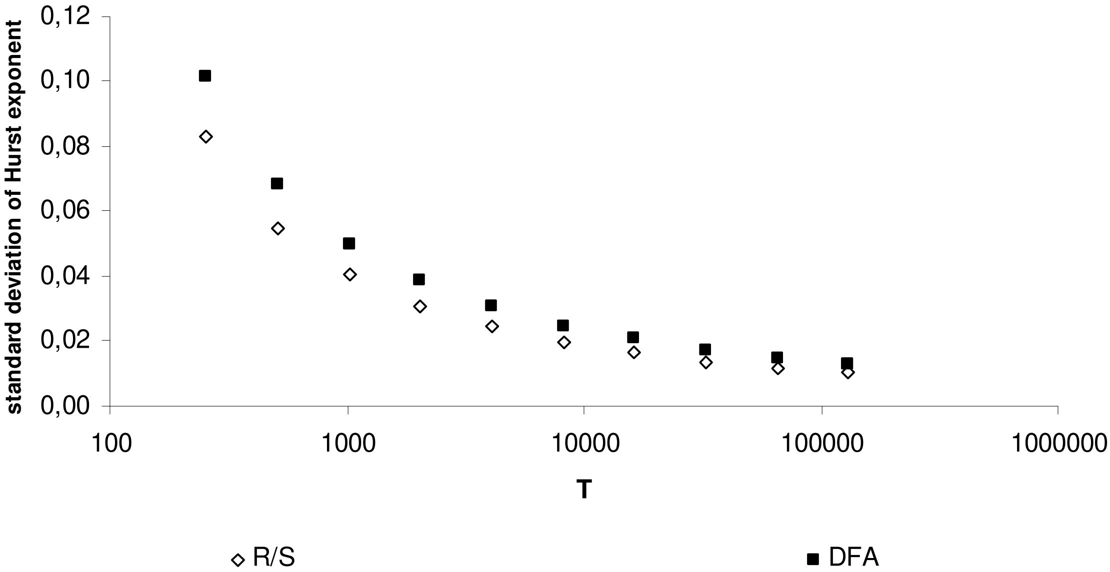


Chart 9: (a) 97.5 and 2.5 percentiles for difference between R/S and DFA-1 estimates for different time series length, **(b)** maximum difference between R/S and DFA-1 estimates for different time series length.

However, the most important findings, which contradict results in Weron (2002), are based on results of estimated standard deviations of Hurst exponents. R/S is generally considered as less the efficient method and is replaced by DFA in majority of recent applied papers (Grech & Mazur, 2004; Czarnecki, Grech & Pamula, 2008; and Alvarez-Ramirez *et al.*, 2008). Reasons for such replacement are usually stated as bias for non-stationary data and general overestimation of Hurst exponent of R/S. However, we have already shown that the overestimation is built in the procedure for finite samples (as was already shown in Weron, 2002; Couillard & Davison, 2005; and Peters, 1994). Moreover, non-stationarity is usually not the case for the financial time series while the statement is truer for daily data which are mostly examined (Cont, 2001). Further, as we show in Chart 10, standard deviations are lower for R/S than for DFA-1 for all examined time series lengths. Therefore, also confidence intervals are narrower for R/S which makes the independence better testable by this procedure. The values of standard deviations are more important than expected values of the Hurst exponent for the hypothesis testing. Nevertheless, we need to keep in mind that expected values for Hurst exponent based on R/S for finite samples are far from the asymptotic limit.

Chart 10 Comparison of standard deviations of R/S and DFA-1



5 Conclusions and discussion

We have shown that rescaled range analysis can still stand the test against new methods. Our comparison with detrended fluctuation analysis has supported the known fact that R/S overestimates Hurst exponent. However, the overestimation is in hand with estimates of Anis & Lloyd (1976) and thus is not unexpected. Importantly, the standard deviations of R/S are lower than those of DFA-1 which is crucial for construction of confidence intervals for hypothesis testing. The results are different from the ones of Weron (2002) who asserts that DFA-1 is "a clear winner" when compared to R/S. Such difference is caused by different choice of minimum and maximum scales for Hurst exponent estimation. Our results are based on recommendations of several other authors (Peters, 1994; Grech & Mazur, 2004; Matos *et al.*, 2008; Alvarez-Ramirez, Rodriguez & Echeverria, 2005; a Einstein, Wu & Gil, 2001) so that we use minimum scale of 16 observations with maximum scale equal to a quarter of time series length. The choice of scales is thus crucial for final results and its further research should be of future interest.

Nevertheless, we show that both methods show similar results which become closer as the time series becomes longer. We show that testing the hypothesis for short time series, especially with 256 and 512 observations, can be complicated as the confidence intervals are very broad.

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