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Abstract

Recent empirical studies suggests that affine models, a popular framework to analyse term structures of interest rates, are misspecified. This evidence is mainly based on time series properties of the data. This article re-examines this controversy, by investigating both cross-sectional and dynamic properties of affine models. To do so, it applies robust non-parametric techniques to two different sets of financial data, which contain information on the UK and US yield curve. The analysis shows the strong non-linearity in the relationship of yields to the US and UK short rate. The non-linear pattern is concave in the state variable, and increasing with respect to the maturity, for both countries. Linear and non-linear specifications are then compared by means of a formal statistical criterion, the Generalised Likelihood-Ratio test statistics, which confirms evidence against the linear specification.

KEY WORDS: interest rates; term structure; affine models; non-linearity; non-parametric regression.

A key area of enquiry for economists is what determines interest rates. This is important to solve real problems, such as, understanding the effects of monetary policy and, in financial theory, determining prices for fixed income derivatives. In particular, the *term structure* (TS) of interest rates has been the focus of much research (Vasicek, 1977; Cox et al., 1985; Dai and Singleton, 2000).

The TS (also known as the yield curve) captures the relation between yields and terms of risk-free securities, such as zero-coupon bonds. Yields are a fundamental determinant of asset pricing and economic decisions. They measure risk-free interest rates, because they determine the present value of a risk-free nominal sum payed at any future date. The relation between short and long-term yields, a key aspect of TS behaviour, is valuable to understand how the economy evolves. *Expectation theory* says that the long-short yield spread reflects shifts in the market's expectations on the future rate of interest (Campbell, 1995). The relation of yields to the short rate helps to understand the effects of the

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monetary policy: central banks target the short rate of interest, but there is evidence that economic decisions depend mainly on longer yields (Bernanke and Gertler, 1995). This paper takes this into account. It focuses not only on the short-end, but also on the relation between the short and long-end of the yield curve.

To understand TS behaviour, researchers resort to *affine models*. Such models derive yield curves and price of derivatives from a continuous-time stochastic process (diffusion) of the short rate of interest. As a result, yields are linear (affine) functions of the short rate and other underlying state variables. Due to their tractability, affine models have quickly become a paradigm widely used by academic and practitioners alike, but they rely on strong parametric assumptions. Using non-parametric methods, several studies have questioned the adequacy of linear specifications of the short rate diffusion (Aït-Sahalia, 1996a; Jiang and Knight, 1997; Arapis and Gao, 2006). Non-parametric estimation is robust to specification errors and discretisation bias, but is complicated by finite sample problems. When applied to financial data, non-parametric estimation techniques have been criticised because they fail to take into account the time series nature of the observations, and because they produce evidence of non-linearity at boundary regions (Chapman and Pearson, 2000; Pritsker, 1998). Furthermore, non-parametric TS analysis has so far focused on the short-end of the yield curve. (A notable exception to this is the study of Ahn and Gao, 1999.) The non-linear evidence, as well as the ability of affine models to capture real data features, have remained controversial. This controversy motivates the analysis performed in this article.

This paper re-examines these issues, and improves the analysis of the TS. The paper proposes a simple and intuitive testing procedure of affine models based on historical time series data. The idea is to focus on the cross-section properties of single-factor TS models, which describe yields as functions of the short rate, to avoid the drawbacks of diffusion analysis. Affine yields equations are estimated applying a set of robust non-parametric techniques which do not impose functional forms on the data and allow detection of departures from linearity.

The paper is organised as follows. Section 1 briefly reviews affine models. Section 2 details the techniques used to perform the analysis. Section 3 presents the data and discusses measurement issues that arise when yields are taken as synonymous with interest rates — as it is done in this article. Section 4 uses zero-coupon bond yields data from the United Kingdom (UK) and the United States (US) to estimate diffusion and affine yields equations for both countries. The problem of checking the adequacy of the linear representation of the yield-short rate relation is addressed in Section 5 through the implementation of a non-parametric likelihood-ratio test procedure. The test provides strong evidence against the linear specification of the relation of interest. Section 6 discusses the empirical evidence in the light of the debate on time series properties of the interest rates. Finally, Section 7 draws the conclusions.

1 Background

Affine models of the TS (ATSs) (Dai and Singleton, 2000) are the mainstream analysis framework for understanding movements of market yields in theoretical and empirical studies, because they are computationally tractable. Such models represent yields as linear (affine) functions of the underlying state variables (or factors).

For a state variable x, and coefficients A and B, which depend on the time to maturity τ, these models represents the yield y of a τ period discount bond as:

$$y_t^{(\tau)} = A(\tau) + B(\tau)x_t; \tag{1}$$

The coefficients A and B make the yield equations consistent with each other and with the state dynamics, represented by a stochastic differential equation (or diffusion) of the form:[1]

$$dx_t = (\alpha + \beta x_t)dt + \sigma x^{\gamma} dw_t; \tag{2}$$

Here, the functions $\alpha + \beta x$ and σx^{γ} represent, respectively, drift and volatility functions of the process; dw is the increment of a Wiener process. Equation 1 summarises the cross-sectional properties of the model, and gives a deterministic – *linear* – relationship between yields to maturity and the state variable x. (Yield curves are easily derived from equation 1, by fixing t and varying τ.) The diffusion equation 2, named Constant Elasticity of Variance (CEV) by Chan et al. (1992), tells how yields evolve over time, and represents the time series properties of the model.

Diffusion specifications encountered in the affine literature are special cases of equation 2. The models of Vasicek (1977), and Cox et al. (1985)(CIR) are important examples of affine models. Vasicek (1977) proposed a diffusion model for the interest rate as follows:

$$dr = k(\theta - r)dt + \sigma dw; \tag{3}$$

For $k > 0$, this represents a mean-reverting process: the interest rate is pulled towards a long-run mean, θ, at a speed determined by k and proportional to the deviation of the process to its mean. The stochastic component, σdw, with constant instantaneous variance σ^2, causes the process to fluctuate around θ. Vasicek's model has the considerable advantage of being simple; the mean-reverting specification for the drift reflects the fact that historical values of interest rates have an upper and lower bound. However, interest rate can turn negative, although with a low probability. This limitation has been addressed by the CIR model, based on the diffusion:

$$dr = k(\theta - r)dt + \sigma r^{\frac{1}{2}} dw, \tag{4}$$

This process is mean-reverting, as Vasicek's model, and its variance increases linearly in the interest rate (square-root diffusion). This specification aims to capture the fact that

[1] This is formally proven in Duffie and Kan (1996). One can also see Bolder (2001), who provides a clear illustration of discount bonds'pricing computation from the diffusion specification.

higher interest rates seem to be more volatile than lower rates, and prevents the interest rate from falling below zero.[2]

Single-factor models, such as those reviewed above, have several limitations. First of all, they have been criticised for their inability to explain the observed variability of the yield curve through time and across maturities. This because, it is argued, the TS dynamics is too complex to be summarised by a single-source of uncertainty. To address this, single-factor models have been extended to include several state variables. Both short and long-term yields, and often the long-short yield spread, have been used as factors. Examples of multi-factor models can be found in Schaefer and Schwartz (1984), Longstaff and Schwartz (1992), and, more recently in Cochrane and Piazzesi (2009).[3]

Another criticism to ATSs concerns the parametric assumptions underlying equations 1 and 2: using non-parametric estimators, several empirical studies have found evidence against the linear specification of the state variables' dynamics. The following problems were identified in the diffusion process: a) the affine diffusion is not correctly specified, due to non-linearities in the variance term (Aït-Sahalia, 1996a); b) the drift is also non-linear, and there is no evidence in favour of the overall mean-reverting property in the drift (Aït-Sahalia, 1996b; Jiang and Knight, 1997).

These findings have been challenged by several authors, who argued that non-parametric diffusion estimation suffers severe finite sample biases. Pritsker (1998) was first in pointing out that Aït-Sahalia (1996b) treats observations as identically independently distributed (iid), whereas interest rates are typically dependent and highly persistent. (As a consequence, the asymptotic distribution of the non-parametric kernel density estimator is a poor approximation of its finite sample counterpart, which causes estimation and testing procedures to perform badly in smaller samples.) In practice, non-parametric techniques require data over long period of time or very high frequency to deliver estimators with good asymptotic properties, namely consistency. This issue has been addressed by Bandi and Phillips (2003), who proposed a non-parametric estimator for diffusions valid under a notion of temporal dependence that is milder than stationarity ("recurrence"). More recently, Arapis and Gao (2006) proposed consistent non-parametric estimators for drift and diffusions, based on Jiang and Knight (1997) estimators, and obtained results comparable to those reported by previous studies, confirming the non-linearity of drift and diffusions.

The studies reviewed above have all focused on diffusion estimation. In contrast, Ahn and Gao (1999) looked at the affine-yield equation, they found evidence of increasingly monotone and concave non-linearity. The study of Ahn and Gao (1999), however, has several limitations: 1) the estimated yield equations display non-linearities at high interest rates, which may reflect the well-known kernel estimators' bias near the sample's boundaries ("boundary effect"); 2) the authors use a standard parametric approach to test for linearity, by fitting higher-order terms of the interest rate, followed by a test on the significance of the associated coefficient; 3) yield maturities longer than five years are not considered.

[2]The conditional probability density of the interest rate is given by a Gaussian distribution in Vasicek's model, and by a non-central χ^2 in CIR.

[3]The extension of ATSs to multi-factor models does not alter the functional form and parametric specification of equations 1 and 2.

The models of Vasicek (1977) and Cox et al. (1985)(CIR), and their multi-factor counterparts, have been used extensively by researchers and practitioners. One can see that the main advantage of such models is their tractability: the linearity assumption in the state variable dynamics yields an analytical representation of the term structure, and bond price formulas which are easy to interpret and well suited to empirical testing. Techniques are available to produce estimates of the model's parameters (see Piazzesi, 2005, Section 6). For these reasons, the affine class of models has been the work-horse of theoretical and empirical studies aiming at understanding yields' dynamics. To be sound, however, a model needs to be realistic as well as tractable: so, it is important to assess the ability of ATSs to capture real data features. Non-parametric studies have highlighted the inability of ATSs to detect non-linearities; in turn, it is not clear the extent to which these non-linearities are true feature of the data rather than spurious findings. More fundamentally, as pointed out in Pritsker (1998) and Chapman and Pearson (2000, p. 387), methods based on time series alone do not seem capable of giving a definite answer to the problem of ATSs' adequacy.

For this reason, this article focuses on cross-section properties of single-factor TS models, in the spirit of Ahn and Gao (1999) study, and improves the non-parametric analysis of the term structure while addressing the limitations of their investigation. The following section illustrates the techniques used in this analysis.

2 Methodology

This paper aims to fully characterise the behaviour of market interest rates, by analysing both time series and cross-section properties of a non-parametric term structure model. To do so, the dynamics of the short rate is specified as a non-parametric diffusion, whose drift and volatility are estimated using the method developed by Jiang and Knight (1997). This avoids misspecification and permits to capture non-linearities in the time series process of the data. Robust non-parametric regression methods and a formal test of linearity are used to test the fundamental implication of affine diffusion models: the linear representation of the yields-short rate relationship.

This section illustrates in some detail two important techniques used in this analysis: 1) Jiang and Knight (1997) method and 2) the Generalised Likelihood Ratio (GLR) test for linearity.

2.1 The estimation of the diffusion

In this paper, the dynamics of the interest rate is given by the following diffusion:

$$dr_t = \mu(r_t)dt + \sigma(r_t)dw_t; \tag{5}$$

Here, μ and σ represent, respectively, the drift and instantaneous variance (volatility) of the process; dw is the usual increment of a Wiener process. Note that the drift and diffusion functions depend only on the value in t of the interest rate r. In contrast to the affine model of equation 2, the functional forms of drift and variance are not specified, in order to allow departures from linearity.

Jiang and Knight (1997) developed a fully non-parametric procedure to estimate the functions μ and σ with the available data.[4] The main idea is to identify drift and volatility functions by forcing them to be consistent with the observed distribution of the data (i.e. the marginal density of the interest rate). In practice, an estimator for the variance is specified, and its properties assessed. Then, variance and density estimators are plugged-into the drift estimator implied by the solution to the SDE diffusion of equation 5.[5] This method uses Kernel density estimation techniques (see, for example, Silverman, 1986). The estimation procedure is as follows:

[4] Aït-Sahalia (This extended 1996a, who proposed a semi-parametric procedure for the variance based on a linear mean-reverting drift specification.)

[5] Under certain condition, the process in equation 5 has a unique solution, and it is fully characterised by the two coefficients:

$$\mu(r) = \frac{1}{2p(r)}\frac{d}{dr}[\sigma^2(r)p(r)], \tag{6}$$

$$\sigma^2(r) = \frac{2}{p(r)}\int_0^r \mu(u)p(u)du; \tag{7}$$

This shows that, with any specification for either drift or diffusion term, the other term will be specified, given the marginal density p of the diffusion process.

1. A kernel estimator of the variance function at r is specified as:

$$\hat{\sigma}^2(r) = \frac{1}{\Delta_n} \frac{\sum_{t=1}^{n-1} K_h(r_t - r)[r_{t+1} - r_t]^2}{\sum_{t=1}^{n} K_h(r_t - r)}, \tag{8}$$

where K is the Kernel function, and h the bandwidth. It is assumed that the sample $\{r_t; t = 1, 2, .., n\}$ is made of n observations equally spaced on the time interval $[0, T]$; thus, the sampling interval is given by $\Delta_n = \frac{T}{n}$. The numerator gives a weighted estimator for the second moment of the data, and the denominator represents the (local) kernel density.

2. Given the information in the variance, the estimator of the drift is given by:

$$\hat{\mu}(r) = \frac{1}{2}[\frac{d\hat{\sigma}^2(r)}{dr} + \hat{\sigma}^2(r) \frac{1}{\Delta_n} \frac{\sum_{t=1}^{n} K_h'(r_t - r)}{\sum_{t=1}^{n} K_h(r_t - r)}]; \tag{9}$$

here, the expression $K_h'(r)$ represents the kernel estimator of the marginal density's first derivative.

To obtain asymptotic properties of the estimators — namely, consistency and mixture normality — it is assumed that data are sampled more and more frequently over a fixed time interval, so that the interval between two subsequent observations shrinks (*in-fill* asymptotics). [6]

The method of Jiang and Knight (1997) has two advantages: 1) it does not place the parametric restrictions that are imposed by affine models on the diffusion process, and 2) it does not replace the continuous-time model by a discrete-time approximation (see, for example, Chan et al., 1992). Thus, the diffusion process is estimated using only the data available. So, the method avoids the linearity assumptions made by many studies of the TS, and offers a way of assessing affine specifications. I use Jiang and Knight (1997) in place of Bandi and Phillips (2003)'s estimator mainly for two reasons: 1) it is more intuitive and easier to implement; 2) as far as I am aware there is no practical rule to date for choosing the bandwidth for the Bandi-Phillips estimator.

2.2 The Generalised Likelihood Ratio test

This section presents the test used in the paper to check the linearity of the yield equations in the state variables. The set-up is as follows. Suppose we want to test the validity of a linear regression model against a non-parametric alternative, as follows:

$$H_0 : y_i = \alpha + \beta x_i + u_i \quad vs. \quad H_1 : y_i = m(x_i) + v_i \quad i = 1, 2.., n; \tag{10}$$

Here, α and β denote the unknown parameters, and u the random term for the parametric model (the parameters of the linear regression are assumed to be estimated by OLS

[6] In contrast, the approach proposed by Aït-Sahalia (1996a) requires the sample size to increase by prolonging the observation period, by adding more and more observations.

techniques); m is an unknown smooth function, and v is the random term for the non-parametric model. (Random terms are assumed *iid*, but this assumption may be relaxed.)

Several test statistics have been proposed in the statistical literature for problems such as the one above. Examples are the F-statistic of Azzalini et al. (1989), and the L-distance estimator of Härdle and Mammen (1993). Tests based on measures of distance, however, have a major problem: the null distribution of the test statistic is unknown and depends critically on the nuisance smooth function m (as shown in Fan et al., 2001). On the other hand, the test of Azzalini et al. (1989) is based on the restrictive assumption that linear and non-linear models are nested.

This paper uses a Generalised Likelihood Ratio (GLR) test (Fan et al., 2001): this is an applicable approach based on a generalised version of the Likelihood Ratio (LR) principle, for which an asymptotic null distribution result is available. The extension of the LR principle to the non-parametric context is motivated by a fundamental property of LR tests: the asymptotic null distribution is independent of nuisance parameters (known as the "Wilks phenomenon"). In particular, Fan et al. (2001) demonstrated that a class of GLR statistic, based on some appropriate non-parametric estimator, follows asymptotically a χ^2 distribution, under the null hypothesis. They showed how this result can be applied to a number of useful models, such as testing the linear regression model against a non-parametric alternative. Consider again the testing problem given in 10. The GLR statistic for such problem is given by:

$$\lambda_n(h) = \log(H_1) - \log(H_0) = \frac{n}{2} \log \frac{RSS_0}{RSS_1}; \tag{11}$$

Here $log(H_1)$ and $log(H_0)$ denote the log-likelihood estimators for the alternative and the null model, RSS_0 and RSS_1 the residual sum of squares (RSS) for the null and alternative model. In practice, the test works by substituting the maximum likelihood estimator of the alternative model by a reasonable non-parametric regression estimator, namely the local linear method. It is shown that

$$r_K \lambda_n(h) \overset{a}{\sim} \chi^2(d_n(h)), \tag{12}$$

where h is the smoothing parameter. The degrees of freedom can be calculated using the formula $d_n(h) = r_K c_K \left|\Omega\right| h^{-1}$, where Ω denotes the length of support of the covariate x. Intuitively, degrees of freedom depend on the amount of smoothing performed, through the term $\left|\Omega\right| h^{-1}$. Once the degrees of freedom are calculated, critical values can easily be found based on the known null distribution. (Values of the constants r_K and c_K for different Kernels are tabulated in Fan et al., 2001, table 2, p.170.)

Fan and Zhang (2003) applied the GLR statistic to test the linearity of the short rate diffusion process. This was done by approximating the continuous-time diffusion by a discrete model. (Parameters of the approximated diffusion are estimated using the local-linear estimator.) The test is as follows:

$$H_0 : dr_t = (a + br_t)dt + \sigma r_t^\gamma dW_t \quad vs. \quad H_1 : dr_t = \mu(r_t)dt + \sigma(r_t)dW_t. \tag{13}$$

Simulation results show that the GLR test is powerful and gives the correct test size. Evidence against the linear drift is weak, whereas evidence against the parametric specification for the volatility function is strong.

3 Data

This paper applies non-parametric techniques to two different sets of financial data, which contain information on the UK and US Term Structure.

For the US, the analysis uses an updated version of the yield data compiled by McCulloch and Kwon (1990), which have been extensively used in the empirical literature on the TS. The sample ranges from April 1953 to July 2000, which gives a total of 568 observations, and consists of monthly time series of Treasury discount bond yields from 0 to the longest maturity date available in the market. The short rate of interest is measured by the 3-month Treasury bill. Observations are at monthly frequency for both countries.[7]

An important problem with this data involves the measurement of the short-end of the TS. The instantaneous rate of interest (or short rate) plays the role of state variables in single-factor TS models (see Section 1); it is, however, unobservable, which makes it necessary to choose a suitable proxy for empirical analysis. Treasury bill rates are risk-free zero-coupon securities, so are often taken as short rate's proxies in place of yields with the shortest maturities, such as overnight rates or the zero-maturity rates in the McCulloch-Kwon data. This because shortest yields and rates contain large measurement errors and are subjects to spurious microstructure effects, reflected in high volatility. (An extensive discussion of short rate measurement issues, and a quantification of the "proxy error", can be found in Chapman et al., 1999, .) To overcome this problem, Ahn and Gao (1999) used the 1-month yield computed by McCulloch and Kwon (1990). Studies of the short rate dynamics used the 3-month Treasury bill (Jiang and Knight, 1997; Jiang, 1998; Arapis and Gao, 2006) and the 7-day eurodollar deposit rate (Aït-Sahalia, 1996a). Here, the choice of the 3-month Treasury bill as the short rate proxy is justified by data availability (the McCulloch-Kwon 1-month yield is not available in the updated dataset); it also allow us to compare results in this paper to those presented in previous literature.[8]

[7] The update of the long end of the original McCulloch-Kwon dataset has been compiled by Gong and Remolona (1997) and researchers at the Federal Reserve Bank of New York. This dataset has also been analysed, using a different framework, by Spencer (2004). I am grateful to Peter Spencer for supplying a copy of this dataset. Treasury bills series are available at http://research.stlouisfed.org/fred2/.

[8] Gurkaynak et al. (2007) present US daily yield curve data since 1961 to date. I do not use this dataset here for two reasons: (i) it does not include the shortest maturities (the authors argue that short-end

Figure 1 (left panel) presents time series of two important yields at the short and long end of the TS: the 3-month Treasury bill and the 10-year yield (in red). The 10-year yield follows closely the short rate, even though it displays less variability. Table 1 reports descriptive statistics, as well as values of the Shapiro-Wilk test for normality and the Dickey-Fuller test for a unit-root, for these time series. Normality and stationarity are rejected for all series.

The UK sample consists of monthly observations of zero yields on government bonds (*gilts*), which were obtained from the Bank of England database. This sample goes from January 1972 to July 2000. Once again, I measure the short rate using the 3-month Treasury bill yield.[9] Figure 2 (left panel) presents time series of equivalent yields on 3-month Treasury bill and 10-year gilt. Once again, the longer yield looks less volatile than the shorter yield, but the close correlation between the two yields observed in the US data is no longer present in the UK sample. Table 2 reports summary statistics for these series. One can see that, while UK maxima are comparable to those recorded in the US sample, minima and mean values are higher for the UK than the US, due the different (shorter) time period covered by the UK sample. Normality is rejected for all series, and the ADF test does not reject the null hypothesis of a unit-root for all series.[10]

Non-normality in the data is also apparent in the kernel density estimates of the UK and US short rate, which are presented in figures 1 and 2 (right panels). Key features of these estimates are the positive skewness, which indicates that interest rates are more likely to stay low, and the fat and long right tail. In contrast to the US sample, the short rate distribution for the UK looks bimodal. [11]

estimates based on overnight federal funds rate maybe preferable to those based on Treasury bills, due to market segmentation); (ii) it is compiled using a parametric methodology that differs from the one — spline based — used by McCulloch and Kwon. McCulloch-Kwon data are also a better match to the UK yield curve, compiled using a similar method (see also Anderson and Sleath, 1999).

[9] The 3-month Treasury bill rate series is compiled by the Office of National Statistics (ONS), series code: AJRP. This is used here for the following reason: the Bank of England uses repo rates to compute the yield curve at the shortest maturities (as discussed in Anderson and Sleath, 1999), but these data are available only from 1997. Furthermore, a comparable series is not available for the US.

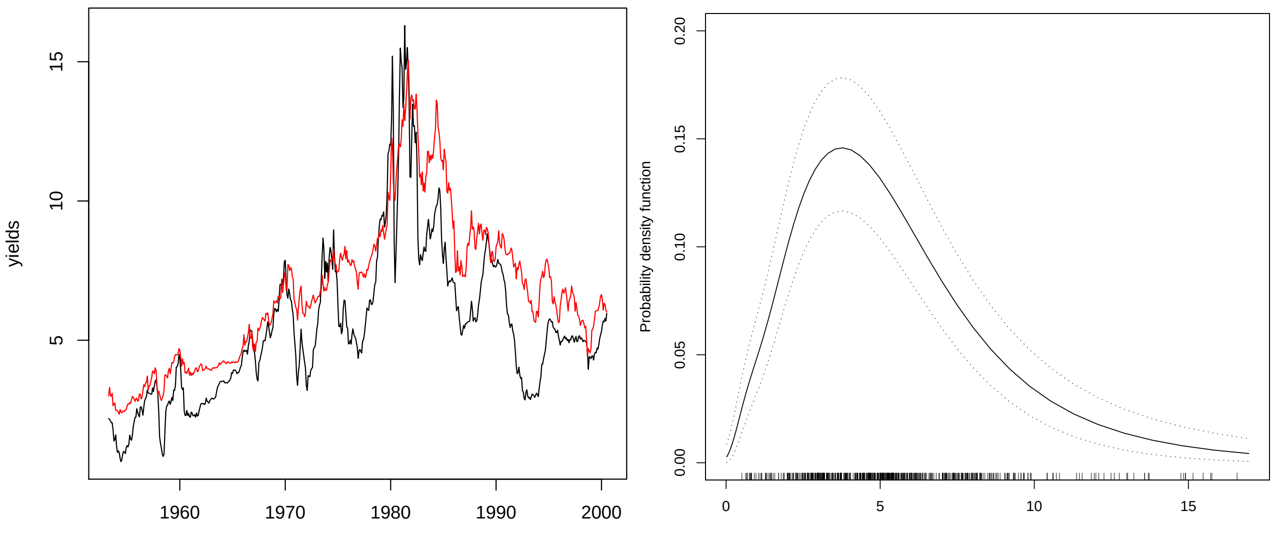
[10] Yield data are available on the Bank of England website. (http://www.bankofengland.co.uk/Statistics/yieldcurve/index.html.) For the UK, the 20-year yield is not reported as observations are sparse.

[11] US density estimates computed for the same time period as for the UK do not feature bimodality, although have accentuated skewness and a fatter right tail.

Table 1: *US Yield Data: Summary Statistics*

Statistics	r	$y1$	$y5$	$y10$	$y15$	$y20$
Mean	5.45	6.01	6.55	6.75	6.89	6.92
Median	5.05	5.63	6.31	6.56	6.84	6.95
Std. dev.	2.79	2.86	2.72	2.68	2.77	2.76
Minimum	0.64	0.85	1.77	2.34	2.47	2.53
Maximum	16.30	16.34	15.69	15.06	15.13	14.41
Kurtosis	4.72	3.83	3.48	3.08	2.69	2.44
Skewness	1.11	0.86	0.76	0.62	0.48	0.37
SW stat	7.88	7.22	6.79	6.66	6.45	6.51
ADF stat	-2.29	-2.47	-2.30	-2.08	-1.88	-1.73

`Legend:` r `denotes 3-month Treasury bill rate;` $y1$-$y20$ `are zero yields on` 1 `to` 20 `years maturity Treasury bond;` SW `is the Shapiro-Wilk test for normality, and` ADF `is the Augmented Dickey-Fuller test for unit root (`5% `cv` $= -2.860$`).`



(a) (b)

Figure 1: US sample: (a) time series of **3-month T-bill** (in black) and **10-year** yields; (b) short-rate density estimate.(The rug-plot at the basis of the density plot represents the frequency of the observations. Dotted lines are confidence bands.)

Table 2: *UK Yield Data: Summary Statistics*

Statistics	r	$y1$	$y5$	$y10$	$y15$
Mean	9.60	9.07	9.62	9.94	10.01
Median	9.67	9.41	9.99	9.91	9.50
Std. dev.	3.23	2.65	2.42	2.54	2.74
Minimum	4.31	4.14	4.20	4.12	4.25
Maximum	16.97	14.96	15.54	15.93	18.64
Kurtosis	2.08	2.03	2.29	2.37	2.80
Skewness	0.22	0.10	−0.04	−0.09	0.19
SW stat	5.26	6.53	6.28	6.62	6.73
ADF stat	−2.40	−1.86	−1.42	−1.02	−0.96

Legend: r denotes the 3-month Treasury bill rate; $y1$-$y15$ are yields on 1 to 15 years Treasury bond; SW is the Shapiro-Wilk test for normality, and ADF is the Augmented Dickey-Fuller test for unit root.

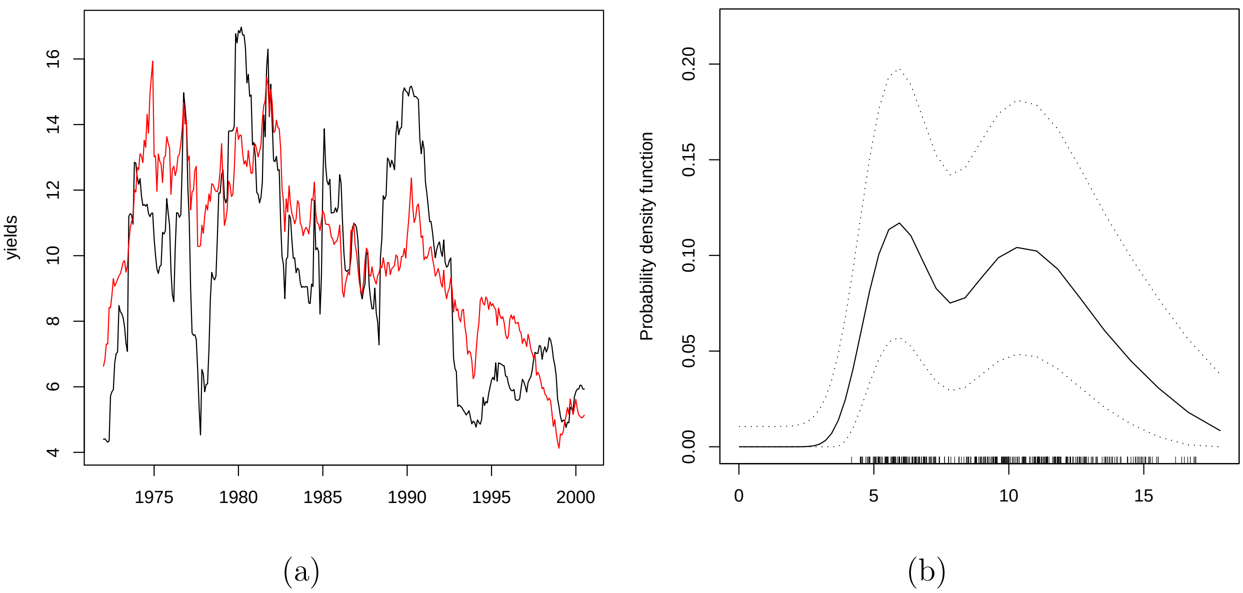


Figure 2: UK sample: (a) time series of **3-month T-bill** (in black) and **10-year** yields; (b) short-rate density estimate. (The rug-plot at the basis of the density plot represents the frequency of the observations. Dotted lines are confidence bands.)

4 Empirical results

This section presents results from the empirical investigation of the US and UK data, which focuses on the relation between the short rate and discount bond yields. The investigation uses non-parametric techniques to study the dynamics of the interest rate, as well as the yield curve. Firstly, a non-parametric model of the short rate is estimated employing the method proposed by Jiang and Knight (1997), which was presented in Section 2. Estimation results for the UK are compared to those available for the US, and to those in previous studies. Then, the short rate is used as the underlying variable of the TS model, and non-parametric regression techniques estimate yield curves for various maturities.

4.0.1 The interest rate diffusion

Recall that, in what follows, the short-rate dynamics is assumed to follow the non-parametric diffusion process:

$$dr_t = \mu(r_t)dt + \sigma(r_t)dW_t, \tag{14}$$

Where μ and σ denotes drift and instantaneous volatility, whose functional form is unspecified, and W is a Wiener process. Figure 3 presents estimates of the stochastic process for the US short-rate, as measured by the 3-month Treasury-bill.[12] The frequency of the observations is monthly. (This frequency is dictated by the frequency of longer-term yields data in the McCullock-Kwon dataset.)[13]

Diffusion and drift functions display noticeable variations from low to high values of the interest rate. The diffusion function is non-linear, and an overall increasing function of the interest rate (right panel). This confirms the "level-effect" conjecture that high interest rates should be more volatile than low interest rates. Yet, this "level-effect" is non-linear, and has several regimes: 1) the diffusion is very close to zero at low interest rates, 2) it is increasing, at an increasing rate, at medium and high values of the short rate; 3) as the interest rate approaches its maximum observed values, it slightly decreases.

The drift function displays non-linear features (left panel). It is zero for low and medium levels of the interest rate, but takes large negative values at high rates. Thus, for interest rate lower than 10% the process is mainly driven by the stochastic term in the diffusion. This is usually interpreted as evidence of mean reversion: the interest rate behaves like a random walk at low and medium levels; the decline in the drift, at the upper bound of the sample range, prevents the interest rate from exploding towards infinity, by "drifting" it back to its long-term mean.

[12]Diffusion estimates are computed using a Gaussian kernel; the bandwidth choice is guided by the discussion in Arapis and Gao (2006), who looked at the same data. So, the bandwidth is proportional to the optimal rate $T^{-1/5}$, where T is the sample size. (The use of this rule for time series data analysis is discussed Hall et al., 1995b, .)

[13]This monthly frequency also avoids the noise and the computational high costs that are involved in higher frequency samples. It is supported by empirical results in Pritsker (1998) and Chapman and Pearson (2000), who argue that the bandwidth choice is affected by the memory properties of the series rather than the frequency of the observations.

The above results should be interpreted with care at high level of the interest rate, due to the lack of observations in the region, which makes the estimates not very reliable. Previous literature, however, reports similar results, working with different interest rate measures and sample periods. Aït-Sahalia (1996a) used daily observations on the seven-day Eurodollar deposit rate, from 1973 to 1995; the estimated diffusion was increasing, with a peak at 17%, then decreasing, whereas the drift was assumed linear from the start. The drift dynamics in Jiang (1998), and Jiang and Knight (1997) resembles the general features of our estimate. Jiang (1998) used daily values of the secondary market yields on the 3-month US Treasury bill, from 1962 to 1996. Jiang and Knight (1997) used daily data of the Canadian 3-month Treasury bill, form 1982 to 1995. Comparable results for the diffusion can be found in Jiang (1998), and Bandi (2002), who re-examined the data presented in Aït-Sahalia (1996a) using a different method.

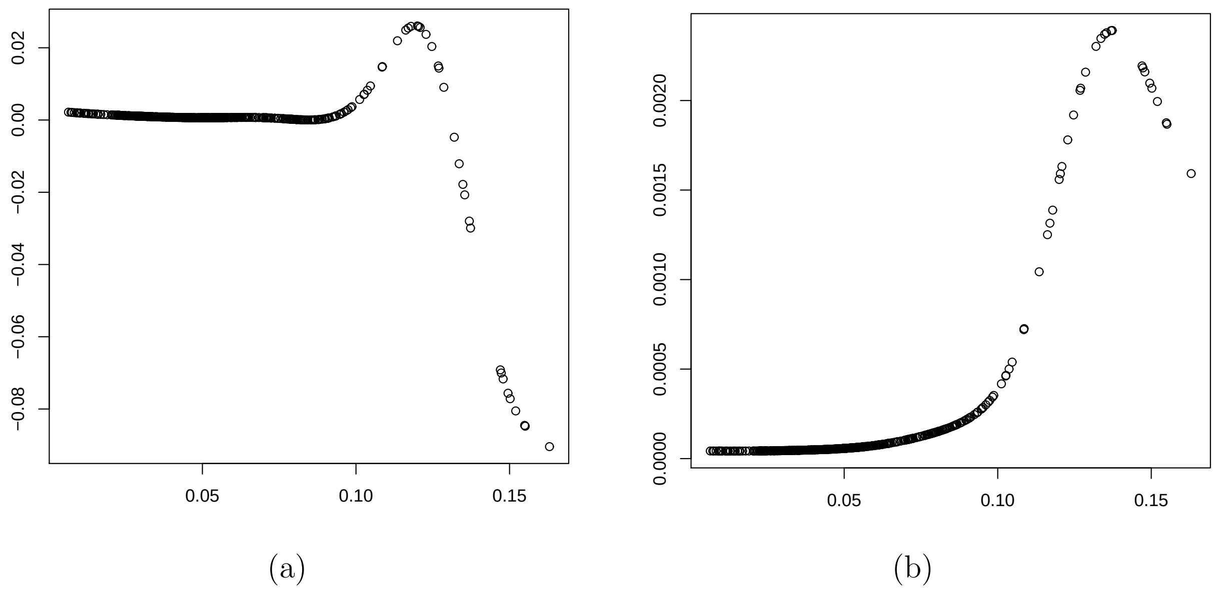
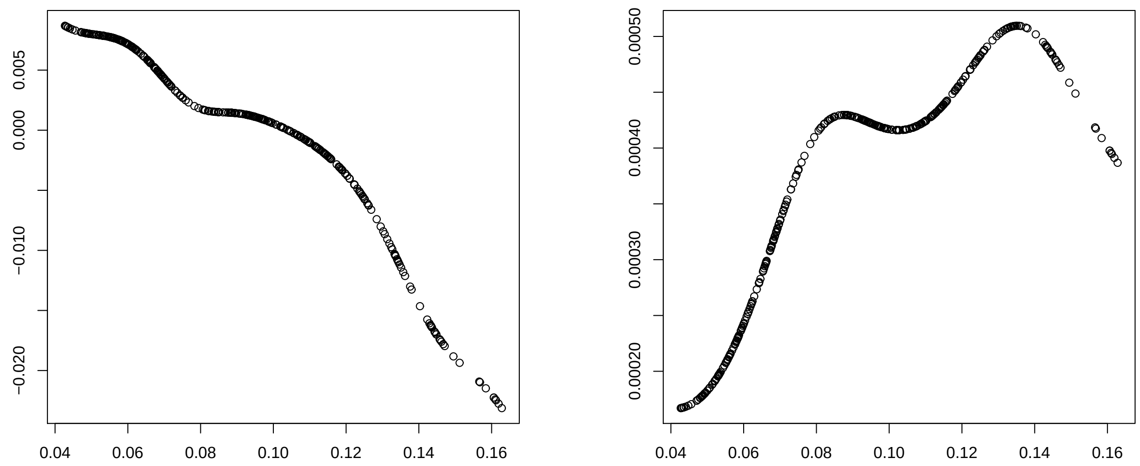


Figure 3: US data: non-parametric estimates of the **drift** (a) and **diffusion** (b) functions.

Figure 4 presents the non-parametric estimates of drift and diffusion for the UK interest rate, as measured by the 3-months Treasury bill rate (this is sampled between January 1972 and July 2000). The diffusion function (right panel) is increasing in the interest rate level. Once again, the "level-effect" is non-linear. The diffusion increases faster for medium-range values of the interest rate (from 6% to 9%), flattens, then increases again; finally, above 14%, it decreases. (We found a similar result for the US). The drift (left panel) is also non-linear. Interestingly, the drift is above zero and decreasing for low and mid-range values of the short rate, which offers evidence of mean-reversion for levels of the interest rate others than very high ones (usually, this is not found in US data). As the short rate increases, the drift plunges below zero and displays a strongly decreasing pattern at high rates, similar to the one estimated for the US sample.[14]



(a) (b)

Figure 4: UK interest rate: non-parametric estimates of **drift** (a) and **diffusion** (b) functions.

Estimation results for equation 14, such as those reported in this section, prompted authors to argue that affine interest rate models are misspecified. Here, this is strengthened by the UK evidence. There are, however, problems with this analysis. The evidence of non-linearities in the drift comes at high levels of the interest rate, in a region were only few observations are available; this makes the estimates not very reliable. The problem of the lack of observations has been mentioned several times in the literature (see Aït-Sahalia, 1996a; Chapman and Pearson, 2000; Bandi, 2002). This is a non-amendable problem, and a very common one indeed: few observations are available in areas of most interest to apply researchers, i.e. those areas in which variables of interest assume very high or very low values.[15] Furthermore, the evidence of non-linearity is re-inforced by the similarity of results in the two countries analysed. Another important result here is the evidence of strong non-linearity in the diffusion term, which has often been overlooked in previous literature, more focused on the issue of non-linearity in the drift.

The difficulties of non-parametric diffusion estimation motivate the analysis of the following section, which focuses on the TS generated by the non-parametric model. There, we base our test of affine specifications on the yield equation; this because the linearity of yields in the short rate is a fundamental implication of affine models, and we see no reason

[14]The similarity of results for the UK and US is not affected by the shorter time period covered by the UK sample. A similar exercise, conducted by cutting the US sample to cover the same period as the UK sample, delivers similar results.

[15]It would be useful to report bootstrap variability bands together with the estimation, as suggested in Jiang (1998); however, this is difficult in the present context and is left for future research.

why a test of affine models based on the static yield representation should not be as valid as a test based on the dynamics specification of the underlying diffusion.

4.1 The yield equation

ATS models are made up of an underlying variable's dynamics and a cross-section relationship between yields to maturities and the state variable. Thus, testing affine specifications involves also investigating features of the yield curve as well as the underlying diffusion process. So, this section focuses on the yield curve, and investigates the yield-spot rate relationship using non-parametric techniques. This follows the spirit of Ahn and Gao (1999) study, and extends and addresses limitations of their analysis in the following sense: first, I consider maturity longer than 5 years and a longer sample; second, I use kernel-based regression methods with better properties than the Nadaraya-Watson estimator of Ahn and Gao (1999); third, I choose the bandwidth using a simple selection method that takes into account the time series nature of the observations.

The following estimates a non-parametric yield curve for the US and the UK. Data are zero-coupon Treasury bond yields (*gilts* for the UK), from zero to the longest maturity on the market. The relationship, at time t, between yields (y) to maturity τ and the short rate (r) is specified as follows:

$$y_t^{(\tau)} = m(r_t) + \epsilon_t; \tag{15}$$

(Here, ϵ is a standard *iid* error term.) The non-parametric function m is estimated using a *local-linear* method (Fan, 1992). This method is chosen because it reduces the bias of the Nadaraya-Watson estimator, used by Ahn and Gao, especially large when estimating a curve at a boundary of the support of the data. (This propety of the local linear estimator is discussed in Fan and Gijbels, 1996, Section 3.4.) The "boundary bias" is especially relevant here, because the evidence of non-linearity comes mainly at boundary regions.[16] The bandwidth is chosen in accordance to Altman (1990)'s algorithm, to take into account the time series nature of the observations.[17]

Figures 5 to 6 present non-parametric estimates of the US yield curves. Yield curves are non-linear and the non-linear pattern becomes more pronounced as time to maturity increases. In particular, the relationship of yields to the short rate is concave, the concavity effect becoming stronger for maturities of 15 years and over. (The 5 year yield vs. short

[16]The local-linear estimator for the regression model $y = m(x) + \varepsilon$ has the form:

$$\widehat{m(x)} = \frac{\sum_i \omega_i y_i}{\sum_i \omega_i},$$

where $\omega_i = K_h(x_i - x)\{(S_{n,2} - (x_i - x)S_{n,1})\}$, with $S_{n,j} = \sum_1^n K_h(x_i - x)(x_i - x)^j$. The kernel K is Gaussian.

[17]Altman (1990)'s algorithm is an adjusted cross-validation criterion based on a parametric estimate of the correlation function. This method has the considerable advantage of being simpler than its alternatives proposed in the statistical literature, such as Hall et al. (1995a)'s. Furthermore, simulation experiments show that, even when the parametric part is misspecified, this method provides a substantial improvement over rules developed under the independence assumption.

rate regression is reported here for comparison with Ahn and Gao (1999)'s study.) So, our estimates confirm Ahn and Gao (1999) findings, in terms of the curvature and monotonic increasing properties of the non-linear pattern.[18]

Figures 7 and 8 present the non-parametric estimates of the relationship between the short-term interest rate and the Treasury zero yields for the UK. We consider three different yields to maturity, from 5 to 15 years. The 20-years yield is not considered, as too few observations are available for this maturity.) There is some evidence of non-linearities in the relationship between yields and short interest rate. The non-linear pattern is concave, and similar to the one found for the US TS, even though somewhat less pronounced.

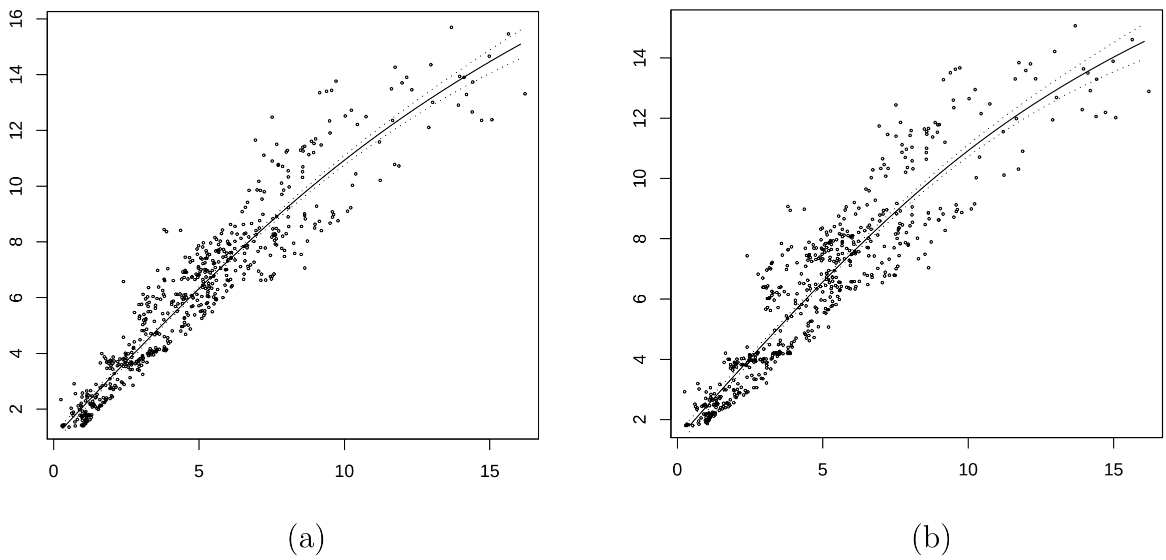


Figure 5: Scatter-plots and regression estimates of 5-year (a) and 10-year (b) yield vs. short term interest rate. (Dotted lines represent variability bands.)

[18]For robustness, one should note that the same non-linear results are obtained when using different measures of the short rate, such as the overnight Federal Fund rate, as suggested in Gurkaynak et al. (2007). The curves are not reported here for reasons of space, but can be obtained by the author.

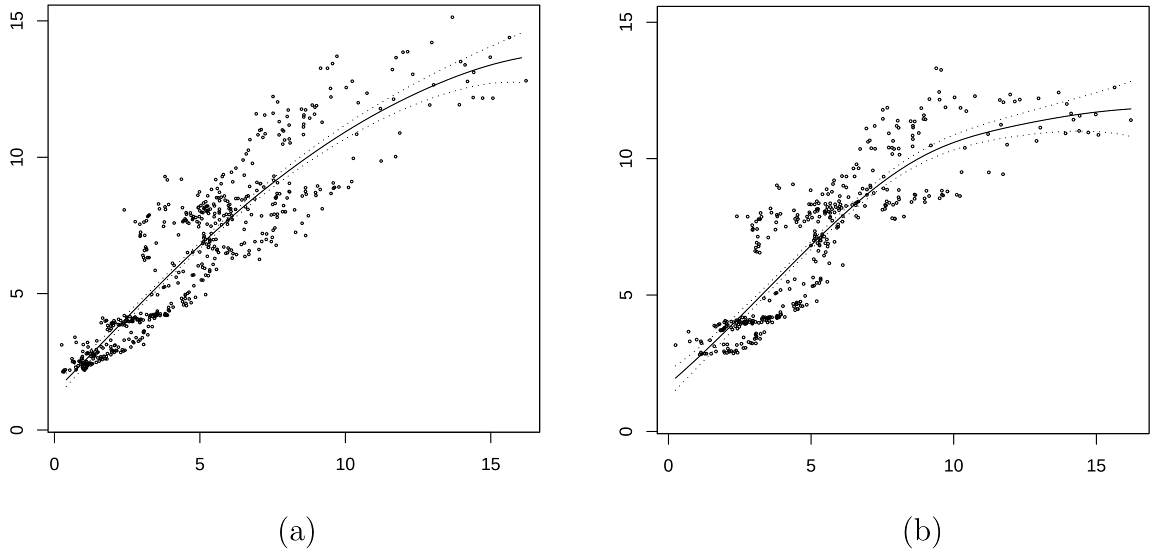


Figure 6: Scatter-plots and regression estimates of 15-year (a) and 20-year (b) yield vs. short term interest rate.

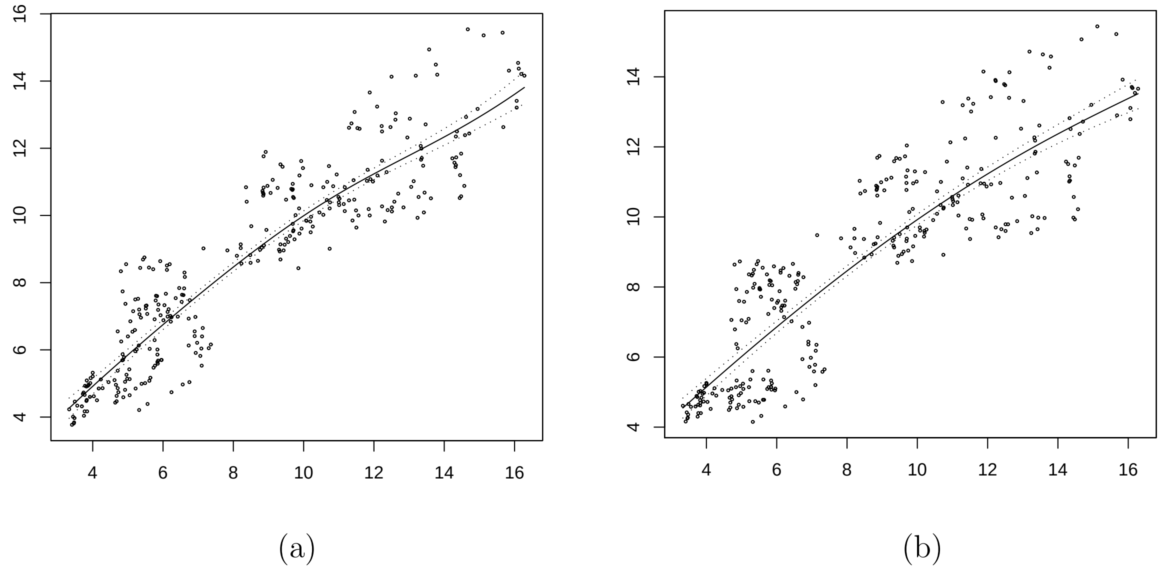


Figure 7: United Kingdom: scatter-plots and regression estimates of 5-year (a) and 10-year (b) yield vs. short term interest rate.

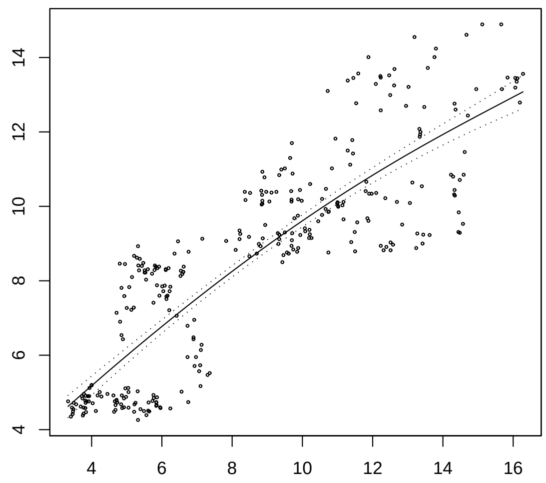


Figure 8: United Kingdom: 15-year yield vs. short term interest rate.

The analysis of this section shows that patterns found in UK data are very similar to those in US data. The diffusion functions are non-linear in both samples. Estimation results confirm the existence of a "level-effect", meaning that the volatility is increasing in the level of the interest rate. Comparable results can be found in previous studies, even though these worked with different series. The drift terms appear to be non-linear, as opposed to most conventional parametric specifications, and exhibit strong mean-reversion at the upper bound of the samples range. This is a common feature of the non-parametric literature, and substantially confirms the result in Aït-Sahalia (1996b). The misspecification of linear model of the TS is also confirmed by non-linearities in the fundamental implication of affine models: the relationship between yields to maturity and the short rate. Yield curves are increasingly concave, and this pattern is more pronounced for longer maturities.

The following applies a formal non-parametric testing procedure to the yield equation to help determine whether the observed non-linearities are indeed real.

5 Testing linearity of the term structure

Non-parametric regression can be used to test the validity of a parametric linear model: this can be done by adopting an informal approach, based on graphical methods, or by formal testing procedures. In the previous section, the graphical approach turned out to be useful to extrapolate characteristics of the data. The exploratory analysis of US and UK financial data highlighted the presence of non-linearities in the relation of discount bond yields to the short rate of interest, providing evidence that mainstream ATSs are misspecified. However, that approach was informal, as there was no formal assessment of the significance of the distance between the non-linear model and the affine one.

This section addresses the problem of checking the linear representation of the yield-short rate relationship by implementing of a formal test procedure for the linearity of the regression function. The test is non-parametric, as it compares the relative fits of the parametric and the non-parametric model over the sample points.

Recall the single-factor yield equation (15):

$$y_t^{(\tau)} = m(r_t) + \epsilon_t \quad t = 1, ..., T;$$

where m is the unknown regression function. We wish to test the affine specification of yields against the non-parametric alternative above using a formal approach, as follows:

$$H_0 : m(r) = \alpha + \beta r \quad vs. \quad H_1 : m(r) \neq \alpha + \beta r; \tag{16}$$

(One can see that this amounts to test the linearity of a regression model against a non-linear non-parametric alternative.) Fan et al. (2001)'s GLR test statistic for the linearity of the yield equation is given by:

$$\lambda_n(h) = \log(H_1) - \log(H_0) = \frac{n}{2} \log \frac{RSS_0}{RSS_1}; \tag{17}$$

Here, $RSS_0 = \sum_{i=1}^{T}(y_i - \widehat{\alpha} - \widehat{\beta} r_i)^2$, and $RSS_1 = \sum_{i=1}^{T}(y_i - \widehat{m}_h(r_i))^2$; $\widehat{\alpha}$ and $\widehat{\beta}$ are the estimates of the parameters of the null — linear — model, $\widehat{m}$ is the estimate of the non-parametric regression function, obtained by local linear fitting. Under the alternative, the GLR statistic will tend to take large positive values. So, we reject the null when the test statistic $\lambda_n(h)$ takes a large value. The null distribution of reference is the χ^2 (see section 2.2).

Tables 3 and 4 present values of the GLR statistic, and corresponding degrees of freedom, for, respectively, the US and UK yield equations estimated in section 4.1. (Yield equations are denoted by the response variable, ie the corresponding yield to maturity.) One can see that the linear models are strongly rejected at conventional confidence levels.

The distribution of the GLR statistic, however, is asymptotic, which may not be a good approximation when applied to small samples. So, se use a bootstrap procedure (as proposed in Fan and Zhang, 2003) in order to approximate the empirical null distribution of the test statistic, and to obtain p-values for the linearity test. The procedure comprises the following steps:

1. for the original data, compute the residuals $\{\widehat{u}\}_i$ and $\{\widehat{v}\}_i$ from the linear and non-parametric fit using a local linear approach with bandwidth h, and compute the observed value of the test statistic $\lambda_{n,obs}(h)$;

2. obtain bootstrap residuals $\{\widehat{u}^b\}_i$, by sampling randomly with replacement from $\{\widehat{v}\}_i$, and define the bootstrap responses $y_i^b = \widehat{\alpha} + \widehat{\beta} r_i + \widehat{u}_i^b$. The bootstrap sample is used to calculate the bootstrap statistic $\lambda_n^b(h)$;

3. repeat step (2) 1000 times and compute the proportion of time that $\lambda_n^b(h)$ exceeds the observed statistic $\lambda_{n,obs}(h)$; this yields the p-value of the observed GLR statistic.

The computed p-values are shown in table 5. Results from the bootstrap simulations indicates that, once again, the linear model cannot be validated by the GLR test.

Table 3: *US Yield Curve: linearity Test*

Model	*GLR* stat.	*df*
1-year yield	91.12	41.65
5-year yield	117.62	41.65
10-year yield	115.39	41.65
15-year yield	27.19	34.98
20-year yield	25.05	34.98
25-year yield	46.05	31.61

Table 4: *UK Yield Curve: linearity Test*

Model	*GLR* stat.	*df*
1-year yield	103.98	25.39
5-year yield	53.33	25.39
10-year yield	50.22	25.39
15-year yield	34.62	25.39

Legend: GLR is the observed value of the *GLR* statistic, and *df* are corresponding degrees of freedom.

6 Memory properties of the interest rate

This section examines the implication of the evidence presented in this article for the memory properties of the interest rate process.

The long term properties of nominal interest rate series have long remained a contentious issue. In a seminal paper, Nelson and Plosser (1982) tested the presence of unit roots in several key macroeconomic variables and failed to reject the null of a unit-root in the interest rate. So did many others. (This explains why interest rate series are often modelled as non-stationary processes in macroeconomics.) The stationarity of interest rates was checked in this paper using standard statistical tests. An ADF test statistic was computed, and results reported in tables 1 and 2.[19] The tests failed to reject the null hypothesis of a unit-root in all variables.

This result, however, is counter-intuitive: a unit-root process imposes no bounds on interest rates, while in practice yields cannot be negative, or tend to infinitive. Thus, on the basis of historically observed values of the interest rate, we would expect unit-root tests to support the stationarity of yields. Indeed, several recent studies on long-memory features of economic time series support the stationarity of interest rates.

The failure of classical ADF unit-root tests to reject the non-stationary hypothesis for the interest rate has prompted some authors to suggest that the stationary alternative is

[19] A regression equation which incorporates a constant and lagged values of the variable tested is used; the ADF statistics is the t-statistic for a first order autoregressive coefficient equal to 1.

Table 5: *linearity Test: bootstrap p-values.*

Model	**p-value** (US data)	**p-value** (UK data)
1-year yield	0	0
5-year yield	0	0
10-year yield	0	0
15-year yield	0.001	0
20-year yield	0.007	n.a.
25-year yield	0	n.a.

too stringent for unit-root tests (Kwiatkowski et al., 1992). A stream of literature has focused on fractionally-integrated models because of their ability to characterise persistent processes, while nesting the unit root hypothesis as a special case. This idea is appealing: if the interest rate were fractionally integrated, that would provide an explanation for the inability of empirical studies to reject the unit root hypothesis.[20] Among these studies, Gil-Alana and Robinson (1997) found that the bond yield behaviour is close to stationarity; they tested the null hypothesis of a unit root against the alternative of fractional integration (this substituted the alternative AR dynamics in the ADF formulation). Bierens (1997) proposed tests of the unit-root hypothesis against general trend stationary alternatives, based on the classic DF regression with linear and non-linear time trends, and found that the interest rate is a non-linear trend stationary process. This also poses the question of linearity.

In general, DF regressions are based on linear AR models, so their results could be misleading if the true dynamics is non-linear. This issue has emerged in the finance literature, in the context of diffusion estimation.

ATSs, which model interest rate/yields dynamics as continuous-time diffusion processes, assume stationarity of interest rates. (These models, such as those reviewed in section 1, specify a linear mean-reverting drift.) In contrast, the evidence of non-linearity in the drift of the diffusion, reported in this study and previous literature, suggests that the interest rate has a richer dynamics. For low and medium levels of the interest rate, the drift is flat and close to zero, and the interest rate behaves like a random walk. As the interest rate increases, the drift turns negative, pulling the process back towards its mean, determining global stationarity. This pattern has been interpreted as evidence of (non-linear) mean reversion in the interest rate behaviour: Aït-Sahalia (1996b, p. 387) suggests that the "non-linear mean reversion effectively makes the [interest rate] process stationary, even though it is locally non-stationary on most of its support".[21] So, the non-linearity in the

[20] Diebold and Rudebusch (1991) gave an important result: they showed that DF type tests have low power against fractionally integrated alternatives.

[21] An alternative interpretation can be found in Bandi (2002), who argue that the misspecification of parametric affine interest rate models is due to the martingale nature of the process.

diffusion process may explain the failure of standard statistical tests for unit-roots, based on less flexible parametric specifications. At the same time, a non-linear model seems capable to better capture memory features of the interest rates than linear frameworks. This also calls for a better integration of financial studies and studies of the macroeconomy.

7 Conclusions

Term structure affine models are tractable, but may be misleading. This is because the linearity assumption underlying such models is seldom observed in real economic and financial data. This paper developed a strategy to test the linearity assumption in affine models, focusing on the yield equation. This was illustrated with real data examples, based on zero-coupon yield data for the US and the UK. The strategy involved using non-parametric techniques to study both the dynamics of the interest rate and the yield curve.

The estimation of the short rate dynamics showed that both drift and diffusion functions are non-linear. The non-parametric estimation of yield equations produced an easily interpretable graphical output, depicting the relationship between yields to maturity and the short-end of the yield curve. Yield equations featured a monotonically increasing and concave pattern, which is more pronounced for longer maturities.

In addition, this paper implemented a formal test to determine whether the observed non-linearities in the yield equations are real. The adequacy of the linear representation of the yield-short rate relation was checked with a non-parametric Likelihood-Ratio test procedure for the linearity of the regression function (GLR). The test provided strong evidence against the linear specification of the relationships of interest.

In summary, the analysis of this paper showed that main-stream parametric TS models are misspecified. Although this is a confirmation of previous studies, the novelty here is that unlike most previous works, which focus on testing the short rate dynamics, here the analysis looked at the cross-section of yields. Furthermore, the results presented here were obtained by using robust non-parametric regression techniques.

This analysis, however, has some limitations. First, although non-linearity may be a true feature of the data, the lack of observations is well known to produce estimates that are not very precise. In the US case, the estimation resulted in non-linear drift at high interest rate values. Such values are seldom observed. It would be useful, therefore, to compute confidence bands for the estimations based on re-sampling methods (i.e. the bootstrap), which could provide valuable information on the precision of the estimates while taking into account time series features. This is left for future research. Second, this paper was confined to the analysis of single-factor models of the TS, in which the yield curve is driven by the short rate of interest. The relationship of yields to the short rate is a very important one. It is well known, however, that yields' dynamics are more complex and possibly reflect several state variables. Future work should study the applicability of the approach followed in this paper to the analysis of multi-factor yield models.

Computations. The non-parametric analysis of this paper have been carried out using the open-source statistical software `R` (R Development Core Team, 2007).

References

Ahn, D. Y. and Gao, B. (1999). A parametric nonlinear model of term structure dynamics. *The Review of Financial Studies*, 12:761–762.

Aït-Sahalia, Y. (1996a). Non-parametric pricing of interest rate derivative securities. *Econometrica*, 64:527–560.

Aït-Sahalia, Y. (1996b). Testing continuous-time models of the spot interest rate. *Review of Financial Studies*, 9:385–426.

Altman, N. S. (1990). Kernel smoothing of data with correlated errors. *Journal of the American Statistical Association*, 85:749–759.

Anderson, N. and Sleath, J. (1999). New estimates of the uk real and nominal yield curves. Bank of England Quarterly Bulletin.

Arapis, M. and Gao, J. (2006). Empirical comparisons in short-term interest rate models using nonparametric methods. *Journal of Financial Econometrics*, 4:310–45.

Azzalini, A., Bowman, A., and Härdle, W. (1989). On the use of nonparametric regression for model checking. *Biometrika*, 76:1–11.

Bandi, F. M. (2002). Short-term interest rate dynamics: a spatial approach. *Journal of Financial Economics*, 65:73–110.

Bandi, F. M. and Phillips, P. C. B. (2003). Fully non-parametric estimation of scalar diffusion models. *Econometrica*, 71:241–83.

Bernanke, B. and Gertler, M. (1995). Inside the black box: the credit channel of monetary policy transmission. *Journal of Economic Perspective*, 9:27–48.

Bierens, H. J. (1997). Testing the unit root with drift hypothesis against non-linear trend stationarity, with an application to the us price level and the interest rate. *Journal of Econometrics*, 81:29–64.

Bolder, D. J. (2001). Affine term-structure models: theory and implementation. Bank of Canada WP 2001-15.

Campbell, J. Y. (1995). Some lessons from the yield curve. *Journal of Economic Perspective*, 9:129–152.

Chan, K. C., Karoly, G. A., Longstaff, F. A., and Sanders, A. B. (1992). An empirical comparison of alternative models of short-term interest rate. *The Journal of Finance*, 47:1209–27.

Chapman, D. A., Long, J., and Pearson, N. D. (1999). Using proxies for the short rate: When are three months like an instant? *Review of Financial Studies*, 12:763–806.

Chapman, D. A. and Pearson, N. D. (2000). Is the short rate drift actually nonlinear? *The Journal of Finance*, LV:355–388.

Cochrane, J. and Piazzesi, M. (2009). Decomposing the yield curve. Available at SSRN: http://ssrn.com/abstract=1333274.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interets rates. *Econometrica*, 53:385–407.

Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *The Journal of Finance*, LV:1943–1978.

Diebold, F. X. and Rudebusch, G. L. (1991). On the power of dickey-fuller test against fractional alternatives. *Economic Letters*, 35:155–60.

Duffie, D. and Kan, R. (1996). A yield factor model of interest rates. *Mathematical Finance*, 6:379–408.

Fan, J. (1992). Design-adaptive nonparametric regression. *Journal of the American Statistical Association*, 87:998–1004.

Fan, J. and Gijbels, I. (1996). *Local polynomial modelling and its applications*. Chapman and Hall.

Fan, J. and Zhang, C. (2003). A reexamination of diffusion estimators with application to financial model validation. *The Journal of the American Statistical Association*, 98(118-134).

Fan, J., Zhang, C., and J.Zhang (2001). Generalized likelihood ratio statistics and wilks phenomenon. *The Annals of Statistics*, 23:153–93.

Gil-Alana, L. A. and Robinson, P. M. (1997). Testing of unit root and other nonstationary hypotheses in macroeconomic time series. *Journal of Econometrics*, 80:241–268.

Gong, F. and Remolona, E. (1997). Two factors along the yield curve. *The Manchester School*, 66:1–31.

Gurkaynak, R., Sack, B., and Wright, J. (2007). The us treasury yield curve: 1961 to present. *Journal of Monetary Economics*, 54:2291–2304.

Hall, P., Lahiri, S. N., and Polzehl, J. (1995a). On bandwidth choice in nonparametric regression with both short and long-range dependent errors. *The Annals of Statistics*, 23:1921–36.

Hall, P., Lahiri, S. N., and Truong, Y. K. (1995b). On bandwidth choice for density estimation with dependent data. *The Annals of Statistics*, 23(6):2241–2263.

Härdle, W. and Mammen, E. (1993). Comparing nonparametric versus parametric regression fits. *The Annals of Statistics*, 21:1926–47.

Jiang, G. J. (1998). Nonparametric modeling of u.s. interest rate term structure dynamics and implications on the price of derivative securities. *Journal of Financial and Quantitative Analysis*, 33:465–497.

Jiang, G. J. and Knight, J. L. (1997). A nonparametric approach to the estimation of diffusion processes, with an application to a short-term interest rate model. *Econometric Theory*, 13:615–645.

Kwiatkowski, D., Phillips, P. C. B., Schmidt, P., and Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics*, 54:159–178.

Longstaff, F. and Schwartz, E. S. (1992). Interest rate volatility and the term structure: a two-factor general equilibrium model. *The Journal of Finance*, 57:1259–1282.

McCulloch, J. H. and Kwon, H. (1990). volume 1 of *Handbook of Monetary Economics*, chapter U.S. Term Structure Data, 1946-87. North-Holland.

Nelson, C. R. and Plosser, C. I. (1982). Trends and random walk in economic time series. *Journal of Monetary Economics*, 10:139–162.

Piazzesi, M. (2005). *Handbook of Financial Econometrics*, chapter :Affine term structure models. North Holland.

Pritsker, M. (1998). Nonparametric density estimation and tests of continuous time interest rate models. *The Review of Financial Studies*, 11:449–487.

R Development Core Team (2007). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0.

Schaefer, S. M. and Schwartz, E. S. (1984). A two-factor model of the term structure: an approximate analytical solution. *Journal of Financial and Quantitative Analysis*, 19:413–424.

Silverman, B. W. (1986). *Density estimation for statistics and data analysis*. Chapman and Hall.

Spencer, P. (2004). Affine macroeconomic models of the term structure of the interest rates: the u.s. treasury market 1961–99. Technical Report 2004-16, The University of York. Discussion Papers in Economics.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5:177–88.