Risk Management Framework for Hedge Funds: Role of Funding and Redemption Options on Leverage

Dai, John and Sundaresan, Suresh

Capula Investment Management, 8 Lancelot Place, London SW7 1DR, Columbia Business School, 3022 Broadway, New York, NY 10027

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By

John Dai[1]

Suresh Sundaresan[2]

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ABSTRACT

We develop a model of hedge fund returns, which reflect the contractual relationships between a hedge fund, its investors and its prime brokers. These relationships are modelled as short option positions held by the hedge fund, wherein the "*funding option*" reflects the short option position with prime brokers and the "*redemption option*" reflects the short option position with the investors. Given an alpha producing human capital, the hedge fund's ability to deploy leverage to magnify its alpha is shown to be sharply constrained by the presence of these short options, which have a high probability of being exercised in "bad states" of the world, either due to poor performance or due to macroeconomic developments that are performance-independent. We show that the hedge funds typically have an optimal level of leverage that trades off rationally the ability to increase alpha with the risk of early exercise of short options, which may precipitate the liquidation of the fund. Optimal leverage is shown to differ across hedge funds reflecting their de-levering costs, Sharpe ratios, correlation of assets, secondary market liquidity of their assets, and the volatility of the assets. Using a minimum level of unencumbered cash level as a risk limit, we show how a hedge fund can optimally choose aggregate risk capital and then allocate its risk capital across different risk-taking units to maximize alpha in the presence of these short option positions. Implications of our analysis for hedge fund investors and policy makers are summarized. Our framework can be easily modified to study portfolio selection problem facing any fund, which has granted redemption rights to its investors (money market funds, long-only funds, etc).

[1] Chief Risk Officer, Capula Investment Management, London.
[2] Director of Research, Capula Investment Management, London, and Chase Manhattan Bank Professor of Economics and Finance, Columbia University, New York, (on leave).

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1. Introduction and Overview

Risk management issues relating to Hedge funds, which received increased scrutiny following the collapse of Long Term Capital Management (LTCM) in 1998, have once again been receiving renewed attention following the onset of credit crisis in mid-2007. This increased focus on risk management of hedge funds has arisen from two considerations: first, hedge funds themselves have come to more fully appreciate the risks associated with funding by prime brokers and investor redemptions. Second, the fact that hedge funds are counterparties to prime brokers, who are often international banks engaged in corporate and consumer lending has raised the spectre of propagation of systemic risk through hedge funds. The potential for hedge funds to transmit systemic risk (through their deleveraging processes) to the banking system has become a matter of concern as evidenced by Kambhu, Schuermann, and Stiroh (2007), Hildebrand (2007), Lo (2008) and Papademos (2007). President's working group (1999) has also emphasized this aspect.

In 2008 alone a record number of hedge funds have failed and the market positions of the hedge funds industry have shrunk dramatically. Lo (2008) reports that the estimated assets in the hedge fund industry grew from \$38 billion in 1990 to \$1.87 trillion in 2007. The estimated assets in the last quarter of 2008 stood at \$1.60 trillion. The market positions of hedge funds fell from \$5.23 trillion in 2007 to \$3.68 trillion as of the last quarter of 2008. This drop in market positions amounting to a little over \$1.5 trillion is the reduction in the overall leverage in the hedge fund industry, which is a result of extensive voluntary and involuntary deleveraging undertaken by many hedge funds.

Several important risk management lessons have emerged from the manner in which the credit crisis has impacted the hedge fund business. Two of these deserve special mention and form the focus of our study. First, hedge funds have realized that the prime brokers and counterparties[3] can either significantly increase margin

[3] For ease of exposition, we will refer to all counterparties that have a funding/margining relationship as "PBs" for the rest of the paper.

requirements and/or potentially withdraw their credit lines in periods of crisis. Such actions dramatically increase the funding costs of hedge funds and in some cases impair their ability to maintain (potentially profitable) risky positions. This risk then forces hedge funds to de-lever in bad states of the world thereby imposing losses, and threatening their survival. Second, hedge fund investors, who become liquidity-constrained as a result of the credit crisis, tend to withdraw their capital under precisely the same circumstances thereby increasing the risk of large-scale redemptions. These two sources of risk may be relevant even to a hedge fund that has been performing well by any benchmark of performance prior to the onset of (or even during) the credit crisis.

1.1 Funding and Redemption Options:

We can think of these two risk factors in options parlance as the hedge fund being *short* in two types of very valuable options:

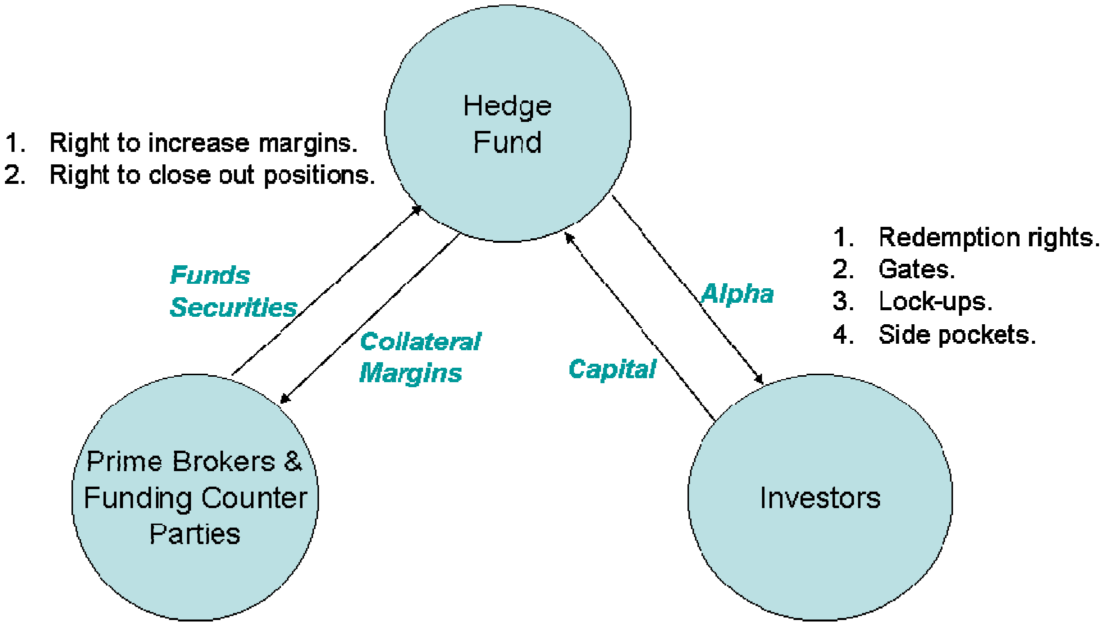
The ability and the willingness of prime brokers to withdraw credit lines in bad states of the world is equivalent to the hedge fund being short an option to reduce leverage in bad states of the world. By virtue of this short option position, the fund agrees or commits to reduce leverage in bad states of the world.

The willingness of investors to redeem their partnership shares in bad states of the world is equivalent to the hedge fund being short in redemption option, which obliges the fund to agree to provide its investors liquidity precisely when it is needed most for the fund to protect its continuing investors and enhance its chances of survival.

We will refer to the option held by the prime brokers as "*funding option*" and the option held by the investors as "*redemption option*". Figure 1 highlights the nature of these contractual agreements that a typical hedge fund will have with its funding counterparties (PBs) and its investors. To focus attention on our principal questions, we abstract from other contractual issues, including the performance and management fees that the fund manager negotiates with investors.

Figure 1

Contractual Relationships



These options operate through different channels. The funding options sold to prime brokers are exercised through increased margins and/or reduced credit lines in bad states of the world, which can lead to involuntary deleveraging if the fund has not placed risk limits anticipating such a possibility. Table 1 illustrates vividly the manner in which the "haircuts" were increased against different classes of collateral in August 2008. In the case of certain asset classes such as CDOs, prime brokers simply refused to accept them. In such instances as well as in those where the margin got multiplied by a factor of more than 10 (as in ABS in Table 1, for example), hedge funds found themselves in need of de-levering, often involuntarily.

Table 1

Typical "Haircut" or initial margin

In percent[4]

Collateral	April 2007	August 2008
U.S. Treasuries	0.25	3
Investment-grade bonds	0-3	8-12
High-yield bonds	10-15	25-40
Equities	15	20
Investment grade corporate CDS	1	5
Senior leveraged loans	10-12	15-20
Mezzanine leveraged loans	18-25	35+
AAA	2-4	95

[4] Source: Global Financial Stability Report, International Monetary Fund, October 2008.

ABS CDO:	AA	4-7	95
	A	8-15	95
	BBB	10-20	95
	Equity	50	100
AAA CLO		4	10-20
Prime MBS		2-4	10-20
ABS		3-5	50-60

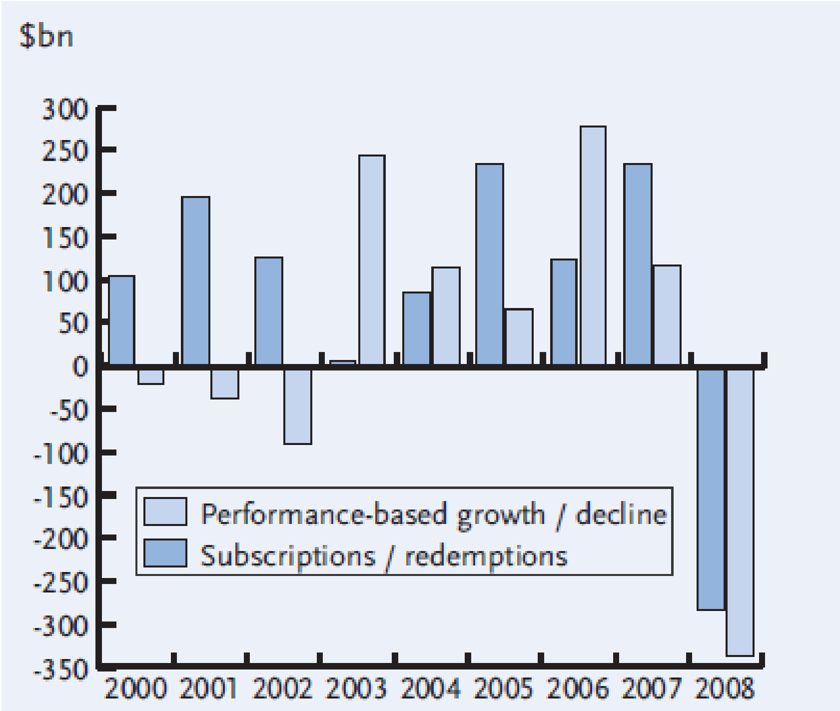
If the cost of involuntary deleveraging is very high, the funds may find themselves in a downward spiral threatening their survival. Moreover, prime brokers may also specify a *NAV* (net asset value) trigger for periodic (yearly, for example) declines below which they may terminate funding[5]. The funding option is especially potent, given the fact that most hedge funds (unlike banks) do not have access to equity or other capital markets for financing. Unlike banks, hedge funds cannot count on central bank facilities for emergency funding either.

On the other hand, the redemption options operate via reduction of assets under management (*AUM*) which can lead to some or all risk limits to become binding. A flurry of major redemptions or draw-downs may cause the hedge fund to breach the *NAV* trigger with its prime brokers. A deluge of redemptions forces the hedge fund into involuntary deleveraging, with significant impact on realized returns for both exiting and remaining investors. *Despite the presence of gates, lockup periods, and notice periods, the hedge fund industry returned nearly $400 billion of capital in 2008 to meet the requests for redemptions by investors*[6]. Figure 2 below estimates the redemptions at close to $300 billion. Note that in the year 2008, both poor performance and investor redemptions contributed to the massive declines in assets under management.

Figure 2[7]

Net asset flows to hedge funds

[5] The *NAV* trigger tends to be much lower than investor's redemption trigger and is less likely to be normally activated than investor's redemption trigger.
[6] "Hedge Fund Liquidations", New York Law Journal, March 2, 2009.
[7] Source: IFSL Research, Hedge Funds 2009, www.ifsl.org.uk, April 2009.



Since both options are likely to be triggered in the states of world where the cost of deleveraging is high, they can have a significant impact on the hedge fund performance. It follows that a proper evaluation of the expected hedge fund return should take into account of the likelihood that these options may be triggered and the likely impact on the fund's performance in the event that they are triggered. *One of the main points that we wish to get across (to be elaborated in much greater detail later) is the fact that the expected return of a hedge fund, as well as the associated risk, depends not only on the ex ante evaluation of the hedge fund portfolio strategies per se, but also on how effectively the hedge fund manages the funding and redemption options*.

Prudent risk management practice at hedge funds requires first and foremost that these options are formally recognized and correctly understood by the managers, transparently communicated to all relevant counterparties, properly priced *ex-ante*, and actively managed *ex-post*. The fact that a hedge fund is short in these options informs on the level of leverage and unencumbered cash levels it should deploy under normal circumstances and how it should manage them over time as states of the world move from "normal states" to "abnormal states"[8]. Finally, since these options tend to

[8] The concept of "unencumbered cash" is discussed in detail in section 2.1.

get exercised in bad states of the world, worst case losses must be estimated through appropriate stress scenario analysis, and incorporated ex ante in risk budgeting[9]. Our paper will focus on these important dimensions of risk management.

Hedge funds with good risk management practices attempt to deal with the "funding option" by having an ongoing and healthy (i.e., open and transparent) relationship with multiple prime brokers, and through these relationships building ample excess capacity in available credit lines. The availability of excess funding capacity effectively makes the funding option farther out of the money, which in turn provides reassurances to credit committees of relevant prime brokers. Having multiple prime brokers not only reduces the chance that the funding option will be exercised (it is much less likely that multiple prime brokers will pull credit lines at the same time – unless it is performance driven in which case all bets are off), but also diversifies and therefore reduces counterparty risk.[10] An "ever-green" facility is clearly desirable, but the price is usually prohibitive.

The "redemption option" held by the investors is usually dealt with through carefully articulated and investor-approved contractual provisions such as a reasonably long redemption cycle (say quarterly) and with a reasonable notification period (say 45 to 90 days), lock-up periods (can be hard or soft), early redemption penalties, investor-level or fund-level gates, etc. These contractual provisions are typically proposed to the investors and approved by the investors, *ex-ante*, so that all investors understand that liquidity in bad states of the world may only come at a price (effectively paid to those investors who are more patient and providing better liquidity to the fund). In designing such contracts, hedge funds must protect the interests of "long-term" investors, but agree to provide liquidity to "short-term" investors in bad states at a fair price. Designing different share classes is yet another way to address this short option position. Avoiding the co-mingled positions by short-term and long-

[9] Normally stress tests tend to focus on extreme scenarios of market and counterparty credit risk. Such scenarios may also trigger the funding and redemption options. Unanticipated surges in future volatilities are not easy to model, and hence stress-scenario analysis are a way in which one can get a better understanding of how the potential exercise of these options should limit current risk-taking.

[10] Hildebrand (2007) argues for the need of the prime broker to have a "complete risk metric" of each hedge fund that the prime broker is exposed to. In fact he argues that the prime broker should be aware of the margining terms agreed by their hedge fund clients with other counterparties and clients. Hildebrand ignores the potential for prime brokers to engage in predatory behaviour.

term investors through contractual provisions helps to manage the liquidity profile better. The generation of alpha might require a minimum investment time horizon, and hence there is a need to match this time horizon with the desire of the investors to have access to liquidity at frequent intervals. One of the purposes of the contractual provisions discussed is to try and minimize the gap between investment time horizon and the redemption cycles desired by investors.

While the ex-ante pricing of these two options is a topic of independent interest, the main thrust of our paper is the manner in which these options influence optimal leverage, risk budgeting, and the active management of hedge fund risk. We examine the questions primarily from the perspective of a hedge fund risk manager. The framework, however, should be of interest to hedge fund investors and regulators of financial markets for reasons that we discuss later in the paper.

1.2 Placing the paper in the hedge fund literature

Academic research on hedge funds is extensive. Lo (2008) in his written testimony to the United States Congress provides a detailed treatment of hedge funds in terms of their potential contribution to exacerbating systemic risk in the economy[11]. Our paper's contribution in this context is to show that funds may use conservative levels of leverage if they properly recognize the short option positions that are implied by their contractual arrangements with investors and prime brokers. The current crisis might have served to sharpen their focus on these options. One strand of literature has been focussed on the presence of nonlinearities in hedge fund returns. Some of the papers that have addressed this question include, Fung and Hsieh (1997a), Agarwal and Naik (2004), and Brunnermeier and Nagel (2004). Cheny, Getmansky, Shane and Lo (2005) proposes a specification in which there can be "phase-locking" behaviour in hedge fund returns, when with a small probability all hedge fund returns become exposed to common market-wide factors. Our paper identifies important sources of nonlinearities that are inherent in the way in which hedge funds contract with their investors and their funding counterparties. This is very distinct from the nonlinearities that arise from the portfolio strategies followed by hedge funds, which has been the

[11] Lo (2008) also contains a detailed list of papers and books that deal with systemic risk related issues pertaining to hedge funds.

focus of these papers[12]. In fact, our paper suggests that nonlinearities can arise in hedge fund returns, due to the rational behaviour of hedge funds in managing their short option positions even if their portfolio strategies did not involve option-like positions. Another strand of literature has examined the presence of survivorship bias, selection bias, and back-filling bias in hedge funds databases. They include papers by Brown, Goetzmann, Ibbotson, and Ross (1992, 1997), Fung and Hsieh (1997b, 2000), and Brown, Goetzmann, and Ibbotson (1999). This is an important empirical question and does not pertain directly to our paper. The role of managerial contracts, high-water marks, lockups and gates have been addressed by Goetzmann, Ingersoll, and Ross (2003), Stavros, and Westerfield (2009), Brown, Goetzmann, Park (2001). Such performance contracts and high water marks may induce the hedge fund to alter endogenously its risk-taking behaviour[13]. Brown, Goetzmann, Park (2001) argue that career concerns may moderate excessive risk-taking even in the presence of such contracts. Ang and Bollen (2009) have examined the role of lockups assuming exogenous arrival rates of failures. The presence of lockups and gates will serve to lower the value of redemption options, *cetaris paribus*. Aragon (2004) uses monthly data to document a positive, concave relationship between a fund's excess returns and its redemption notice period and minimum investment size. Hombert and Thesmar (2009) show empirically that funds with lockups outperform funds with no lockups, conditional on past bad performance. We can, in principle, model the gates and lockups by treating the redemption option as Bermudian, with restricted set of exercise dates and restricted sequential exercise. In our model, there is a parameter that allows us to examine the role of gates on hedge fund risk-return trade-offs: in the presence of gates, *AUM* grows relatively faster, on average, and the manager can use the unencumbered cash (which is a fraction of the *AUM*) as a risk management tool to choose the optimal leverage level. This in turn will determine *endogenously* the likelihood of liquidation. The relationship between gates and optimal leverage will be examined in the paper. Duffie, Wang and Wang (2008) come closest to the spirit of our paper in that they study the optimal use of leverage by a fund which trades off the costs of adjusting leverage with expected benefits as captured by the present values of

[12] Astute hedge funds often incorporate "macro-hedging" strategies, which often involve long positions in option-like instruments such as credit default swaps. Such long positions have the effect of at least partially offsetting the inherent short positions held by the hedge fund. A macro credit crisis that has the effect of pushing funding option in the money will also increase CDS spreads, for example.

[13] Such potential endogenous changes in risk-taking will be relevant to the questions that we address.

the fees earned. Their paper focuses on conditions under which a constant proportion of assets under management are an incentive compatible fee structure and examines the implications of regulations on leverage. In contrast, our paper focuses on how short option positions held by prime brokers and investors influence the leverage decision and the allocation of risk capital across different units within a hedge fund.

1.2 Roadmap and a summary of results

In section 2 of the paper, we will lay out a hedge fund model which characterizes the short options with prime brokers and investors and study the interplay between leverage and these options. Section 2 also motivates the role of unencumbered cash level in managing the insolvency risk, and how that enters into the hedge fund returns process through the options held by the prime brokers against the hedge fund. Our model will highlight some key elements of a sound risk management and operational framework, including how to set appropriate risk limits taking into account of such optionality. This section also details the link between counterparty exposure, margining efficiency and unencumbered cash levels. Section 3 works out the optimal leverage (risk capital) that a hedge fund should employ in order to maximize alpha[14]. This optimization problem facing the hedge fund explicitly accounts for short option position arising from its contractual commitments. Section 4 uses the framework developed in section 2 and the results in section 3 to characterize the risk budgeting problem when a central planner (the risk manager of the fund) assigns risk capital to many risk-taking units within a hedge fund by maximizing the overall fund alpha subject to a) short option positions, and b) aggregate risk constraints specified through a restriction on the minimum level of unencumbered cash that the fund must always hold to mitigate the risk of insolvency. Section 5 outlines some implications for hedge fund investors and policy makers. Section 6 concludes.

[14] Risk capital is defined as the sum of all volatility adjusted leverage ratios $1'(L_t * \sigma_t)$, where L_t is the vector of leverage ratios is employed by risk-taking units, σ_t is the vector of volatilities of assets used in the strategy and 1 is the (Nx1) vector of unities. We denote by the quantity $(L_t * \sigma_t)'$ the 1xN vector $[L_1\sigma_1, L_2\sigma_2, ...L_N\sigma_N]$.

Our main results cut to two important questions. First, we show that the hedge fund as a whole has an interior optimal level of aggregate risk capital, which is derived as follows. For each level of aggregate risk capital, which in the context of our model is the dot product of leverage and volatility of all risk-taking units, we maximize the overall alpha of the fund to determine the risk-capital allocation. As the aggregate risk-capital level increases from zero, the optimal alpha increases until the effects of the short option positions drive the optimal alpha down. This allows us to uniquely identify the optimal level of aggregate risk-capital. Second, we show that each risk-taking unit will get a finite risk budget, which reflects the shadow cost of the aggregate constraint on unencumbered cash and the presence of short options[15]. In particular, it is no longer the case that the relationship between the excess return and risk is linear. This is due to the fact that at high enough leverage and volatility the short options go deep in the money and recognizing this, the desks are allocated less risk capital, ex-ante. After characterizing these results, we show how the Sharpe ratios of each risk-taking unit, the correlation of excess returns across risk-taking units, short option positions, and the deleveraging costs associated with meeting unencumbered cash levels influence the allocation of risk capital.

2. *A Simple Model of Hedge Fund Returns*

To lay the ground work, we consider a generic return generating process in the absence of any leverage:

$$R_{t+1} = \alpha_t + \sigma_t \varepsilon_{t+1} \tag{1}$$

This return generating process assumes a "market-neutral" stance so that there are no market-wide risk factors that are part of the return generating process.[16] We can therefore interpret α_t as the excess return that can be earned by the hedge fund in the

[15] Even in the absence of an aggregate constraint on unencumbered cash, the risk capital allotted to each risk-taking unit is finite due to short option positions and de-levering costs.
[16] For ease of exposition, the return generating process abstract away from specific details of different styles in hedge fund strategy, such as the conditional directionality of a macro strategy or a CTA strategy. Such extensions should be immediate but distract away from the central themes. We also assume that this alpha generating process is fully scalable.

absence of any leverage[17] (or a real-money investor for that matter). To make the problem non-trivial, we can assume that $\alpha_t = s \times \sigma_t$, where the Sharpe ratio $s > 0$ so that it can be properly considered as a pure "alpha" strategy. The ability to identify such investment opportunities is the pre-requisite of a good hedge fund manager.[18]

We take for granted that the manner in which a hedge fund deploys the alpha strategy for investors will necessarily involve some degree of leverage. This is because in its pure form, an alpha strategy is "cash-neutral" or "self-financing". The "native" level of leverage, conventionally defined as the notional size of the risky position divided by the initial cash outlay, is not infinite only because, in practice, a hedge fund must post initial margin to its PBs (more generally any clearing and/or trading counterparties) in order to mitigate its credit exposure to the PBs[19]. The overall leverage level of a hedge fund (henceforth L_t, defined as the notional size of the risky position as a multiple of the AUM) will be lower if the risk margin posted is only a small portion of the overall AUM.

The presence of the funding and redemption options means that, in general, the expected hedge fund return may be linear in L_t only if L_t is sufficiently low. Heuristically, the higher is the leverage, the more volatile is the levered return, and the more likely that the funding and redemption options are in the money. Since the hedge fund is short the options, its expected return will be negatively affected by the expected impact (in the form of expected deleveraging costs) in the event that one or both options are exercised. At sufficiently (or excessively) high leverage levels, the nonlinearity associated with the funding and redemption options can indeed dominate. We formalize this intuition in section 3. It is clear that given any alpha strategy, a hedge fund deploying such a strategy can be either a good or a bad proposition for investors. It can be a good proposition if the leverage level is appropriate. It can be a bad proposition if the leverage level is excessive. Is there such a thing as an "optimal leverage level"? How can the leverage level be objectively measured, judiciously

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[17] We can allow σ to change over time.

[18] Our paper therefore does not speak to hedge fund managers who generate return either by leveraging beta or by writing options.

[19] In addition, the fund may set aside additional cash to meet variation margin calls resulting from daily marking to market.

managed, and clearly communicated? What is the manner in which the overall risk-capital should be allocated across different risk-taking units within a fund? We will address these questions below.

To formally address these questions, we consider a hedge fund manager who aims at deploying the strategy characterized by (1) cognizant of the fund's short option positions with PBs and investors. At the outset, the hedge fund has set out some contractual terms so that effectively its investors have the option to redeem a fraction $e^{-\varphi}$ of the AUM whenever the annualized return drops below a "redemption trigger" $\underline{R}$ [20]. Furthermore, the hedge fund has entered contractual terms with its prime brokers whereby the prime brokers have the option to withdraw the funding required to support the leverage level whenever the annualized return drops below a "funding trigger" $\underline{PB}$ [21].

Based on our discussions earlier, and industry experience, it is reasonable to assume that $|\underline{R}| < |\underline{PB}|$. For simplicity, we will treat these triggers as being exogenous and solely determined by the performance of the fund. It should be clear that the triggers depend on contractual terms: for example, we may expect that $|\underline{R}|$ will tend to be higher with gates than without.

The first option can be captured by specifying how the number of shares N_t in the fund evolves over time:

$$N_{t+1} = N_t e^{-\varphi 1_{\{L_t \times R_{t+1} < \underline{R}\}}}. \tag{2}$$

Where for simplicity we assume that there is no subscription or redemption other than possible redemptions in the event of extremely negative performance. We use the indicator functions (defined as $1_A = 1$ if A is true and 0 otherwise) to represent the redemption options held by investors.

[20] We will assume that $\underline{R} < 0$.

[21] We will assume that $\underline{PB} < 0$.

Note that the parameter φ may be interpreted as representing the "gate": if $\varphi = 0$, then the shares cannot be redeemed even after poor performance. On the other hand as $\varphi \to \infty$, the shares can be fully redeemed following poor performance, and the hedge fund essentially will have to liquidate following a period of poor performance[22].

The second option can be captured by specifying how the margin requirement demanded by PBs evolves over time:

$$\lambda_{t+1} = \lambda_t e^{\eta 1_{\{L_t \times R_{t+1} < \underline{PB}\}}} \tag{3}$$

Where λ_t is the "margin multiplier" or the amount by which margins may be increased by the PBs following a poor performance. The concept of margin multiplier reflects three economic considerations.

First, it contains the notion of a "credit multiplier" that each PB assigns to the hedge fund as part of the margin agreement. To the extent that the hedge fund is perceived as a prudent risk taker and therefore a safer credit, the PBs can agree to a lower credit multiplier. This component may be attributed to the informational differences between the fund and its PBs.

Second, if the hedge fund has multiple PBs, its positions are likely to be distributed across different PBs and therefore there is a loss of margin efficiency in that *risk across PBs can not be netted*. The aggregate risk perceived by each of the PBs will therefore be higher than the portfolio risk (namely $1^{'}(L_t * \sigma_t)$). As a result, λ_t will typically be much higher than the credit multiplier that each PB assigns to the hedge fund.

Finally, the margin requirement by each individual PB is almost always linked to the market (systematic) risk exposure of the positions held at the PB. To the extent that

[22] It is relatively straightforward to model redemption option exercise which may be non-performance related. For example, we could specify $N_{t+1} = N_t e^{-\varphi 1_{\{CDS_{t+1} > \bar{C}\}}}$. In this case, when the CDS spreads of (say) financial services index exceeds a high enough threshold level, hedge fund investors begin to redeem: this is not unlike the redemptions that happened during the credit crisis.

the hedge fund does not have systematic risk exposure but have such exposure with individual PBs, there is inefficiency: the hedge fund is paying extra margin (in fact twice) for risk exposures that are offset between two PBs. Therefore, it is meaningful to talk about the "optimality" of λ_t : the optimal margin multiplier is one when the hedge fund has no systematic risk exposure to each individual PB.[23]

When the funding trigger is breached, the PBs can increase the margin requirement by a factor of e^{η}. The dollar amount of the margin requirement is assumed to be a risk measure such as *VaR* (effectively the levered volatility) multiplied by λ_t. This becomes clear if we assume that, by contractual agreement, the total margin requirement is simply $\lambda_t L_t \sigma_t AUM_t$. Note that the parameter η may be loosely interpreted as representing the "funding friendliness": if $\eta = 0$, then the margin multiplier remains the same even after poor performance, capturing low risk premium in credit markets. On the other hand as $\eta \to \infty$, the funding is effectively withdrawn following poor performance, capturing extreme risk aversion in credit markets. In other words, $\lambda_t L_t \sigma_t AUM_t$ captures the willingness or the ability of banking sector to extend credit lines and is a measure of the stage in credit cycles. Higher the leverage, higher the volatility of assets, and higher is the margin multiplier, then greater will be the margin demanded of the hedge funds by the banking sector. It should therefore be clear that $|\underline{PB}|$ is likely to decrease with increases in volatility and credit spreads.

2.1 Unencumbered Cash

At this juncture, it is useful to introduce the notion of "unencumbered cash level" denoted by U_t, which is simply the *fraction of AUM not posted as margin*. This concept is closely related to the funding option.

Thus,

$$u_t = 1 - \lambda_t L_t \sigma_t .$$

[23] Clearly the margin multiplier is trivially optimized in this sense if a single PB is used. In reality, multiple PBs are preferred due to considerations of diversity of counterparty risk and funding capacity. The bankruptcy of Lehman and its prime broking division has also highlighted the need to have more than one prime broker for hedge funds.

(4)

The “unencumbered cash” is defined as the portion of the investor asset under independent custodian (i.e., unencumbered by counterparty obligations). Mathematically, the unencumbered cash is simply the compliment of the margin posted with the counterparties. In order for this “surplus” portion of the cash to be completely unencumbered, however, it is important that a legal structure is put in place. The economic significance of the “unencumbered cash” is that this is the minimum amount that investors can get back if *all* counterparties were default on their obligations and all margin postings are lost (or nearly lost as the Lehman bankruptcy has demonstrated).

In our view, the use of the unencumbered cash as a risk management tool has not been sufficiently emphasized. Indeed, we would argue that unencumbered cash is probably the most important risk management tool at the disposal of a hedge fund. There are several reasons why.

First, even though hedge fund investors are, by self selection, comfortable with the lack of “principal protection”, a commitment to a high level of unencumbered cash is the best that a hedge fund manager can do in providing some form of “principal protection” and ought to give comfort to most hedge fund investors. With a sufficiently high level of “principal protection”, the value of “redemption option” can be significantly mitigated.

Second, the unencumbered cash is not only a function of the portfolio risk perceived by the investment manager it is crucially also a function of the risk perceived by the counterparties. Since in most PB platforms the margin requirements are risk-based and margining terms are vigorously scrutinized and negotiated by both the hedge fund manager and the counterparties, the unencumbered cash gives a very objective measure of portfolio risk and there is usually a direct relationship with the amount of leverage that a hedge fund deploys.

Third, to the extent that lack of operational efficiency, or the lack of informational accuracy, or the sub-optimality of counterparty exposure tend to reduce unencumbered cash levels, a risk management framework centred on unencumbered

cash provides a natural and measurable objective function for many crucial aspects of the operational platform of a hedge fund.
The dynamic relationship between unencumbered cash and other key underlying variables are captured by equation (4). This simple specification suggests that there is an immediate response in unencumbered cash with changes in a) leverage, b) volatility of fund's assets, c) the trajectory of assets under management from one period to the next, and d) the inefficiencies associated with the funding arrangements with the prime brokers[24]. A risk management framework anchored on unencumbered cash levels will therefore respond much more quickly to potential movements in the moneyness of the "funding" and "redemption" options than traditional measures such as *VaR*.[25]

Note that the unencumbered cash level decreases with leverage as more risk capital must be allocated by the fund, reducing cash available. The unencumbered cash level is also decreasing in the volatility of fund's return generating process. As the AUM decreases over time (either due to losses or due to withdrawals) the unencumbered cash level goes down. Finally, the parameter λ captures the inefficiency in funding arrangements due the ability and market conventions that the prime brokers use in setting funding parameters. Greater the inefficiency, higher will be the parameter λ and hence lower will be the unencumbered cash level.

We illustrate below the manner in which unencumbered cash level evolve over time. Note that if the options are triggered, the un-encumbered cash will respond automatically:

[24] The inefficiencies come in several forms, some avoidable and some not. An unavoidable form of inefficiency is introduced when multiple prime brokers are used and therefore different legs of a tightly hedged position may scatter across different prime brokers. Avoidable inefficiencies are unnecessarily large market risk exposure with counterparties or unrealistic margin parameters imposed by counterparties.
[25] In an ideal world with a single prime broker, there is a close relationship between unencumbered cash and *VaR*, since margin requirement is usually computed based on *VaR* and a so-called credit multiplier and latter is relatively static. The disadvantage of *VaR* compared to unencumbered cash are two fold: (1) it is ultimately a theoretical construct and model dependent; an internal *VaR* model is not as objective as a margin agreement even if the latter is also *VaR* based; (2) it abstracts away from the real-world inefficiencies (as captured by λ) which can be a significant source of risk in the context of the "funding" and "redemption" options.

$$u_{t+1}^{-} = 1 - \frac{\lambda_{t+1} L_t \sigma_t AUM_t}{AUM_{t+1}} < u_{t+1} = 1 - \lambda_{t+1} L_{t+1} \sigma_{t+1} \tag{4a}$$

Where u_{t+1}^{-} is the unencumbered cash level *before* the hedge fund is is able to adjust the leverage ratio from L_t to L_{t+1}. One possible risk management policy (again for illustrative purposes) is to maintain the same level of unencumbered cash through time by dynamically changing the leverage ratio as follows.

$$L_{t+1} = L_t \times \frac{\sigma_t}{\sigma_{t+1}} \times \frac{\lambda_t}{\lambda_{t+1}} \times \frac{AUM_{t+1}}{AUM_t}$$

That is, under the risk management policy that the unencumbered cash is kept at certain level, the leverage level needs to be adjusted down when (i) volatility increases; and/or (ii) the margin multiplier increases; and/or (iii) the AUM decreases. If the adjustment of the leverage level is voluntary and/or pre-emptive, the cost of deleveraging may be negligible. If the deleveraging is involuntary, triggered by the exercise of the funding and/or redemption options, the cost of deleveraging can be substantial, and in some cases devastating.

The concept of a risk limit on unencumbered cash level will be integrated into the choice of optimal risk capital in section 4.

2.2 Dynamics of NAV and AUM

We can capture the dynamics of net asset value (*NAV*), or *AUM* per share, as follows:

$$NAV_{t+1} = NAV_t \times e^{L_t \times R_{t+1} - a \times (L_t - L_{t+1}) \sigma_t 1_{\{L_t \times R_{t+1} < \underline{R}\}} - b \times (L_t - L_{t+1}) \sigma_t 1_{\{L_t \times R_{t+1} < \underline{PB}\}}} \tag{5}$$

Where the indicator functions represent the two options and the deleveraging costs associated with the triggering of the options are represented by the parameters "*a*" and "*b*", respectively. We have taken for granted that the deleveraging costs tend to be higher if market volatility is higher. In general, $b > a$, as the asset sale after PB seizes control tends to be done at "fire-sale" prices, whereas de-leveraging in response to investor redemption can often be done at somewhat more orderly fashion and reasonable price levels, especially if the hedge fund is endowed with reasonable investor liquidity terms such as a reasonable notice period and a reasonable gate policy. On the other hand, the trigger level for funding option tends to be much lower than the redemption trigger level.

It follows from the identity $AUM_t = N_t \times NAV_t$ that:

$$AUM_{t+1} = AUM_t \times e^{L_t \times R_{t+1} - (\varphi + a \times (L_t - L_{t+1})\sigma_t) 1_{\{L_t \times R_{t+1} < \underline{R}\}} - b \times (L_t - L_{t+1})\sigma_t 1_{\{L_t \times R_{t+1} < \underline{PB}\}}} \tag{6}$$

Taking the logarithm of (5), we have:[26]

$$h_{t+1} \equiv \ln\left(\frac{NAV_{t+1}}{NAV_t}\right) = L_t \sigma_t \left[\frac{R_{t-1}}{\sigma_t} - a\left(1 - \frac{L_{t+1}}{L_t}\right) 1_{\left\{\frac{R_{t+1}}{\sigma_t} < \frac{\underline{R}}{L_t \sigma_t}\right\}} - b\left(1 - \frac{L_{t+1}}{L_t}\right) 1_{\left\{\frac{R_{t+1}}{\sigma_t} < \frac{\underline{PB}}{L_t \sigma_t}\right\}} \right] \tag{7}$$

Equation (7) immediately implies that, in general, the expected excess return of a hedge fund is less than the alpha of the underlying strategy multiplied by the leverage ratio. If the leverage ratio is too high, the funding and redemption options will have a significant probability of being exercised and the expected excess return can be negative. *Excessively levered hedge funds can have a return profile similar to shorting options directly as part of its investment strategies*.

The second conclusion that can be drawn from the above equation is that leverage ratio is inherently a dynamic concept. What determines the moneyness of the options is the "levered volatility" $L_t \sigma_t$ relative to the trigger levels. It follows that the hedge fund should adjust the leverage ratio commensurate with the changing volatility. It also implies that different hedge funds deploying strategies with different volatilities should naturally have different leverage levels.

Finally, it can be easily demonstrated that there is an "optimal" level of leverage ratio or risk capital, characterized by $L_t^* \sigma_t = g(s, a, b, \varphi, \eta)$, in the sense that at this level, the expected hedge fund return is maximized. One problem with adopting this as the

[26] We can interpret e^{-a} can be properly interpreted as the "cost of deleveraging" (which may be partially offset by early-redemption penalty fees) in meeting redemption requests.

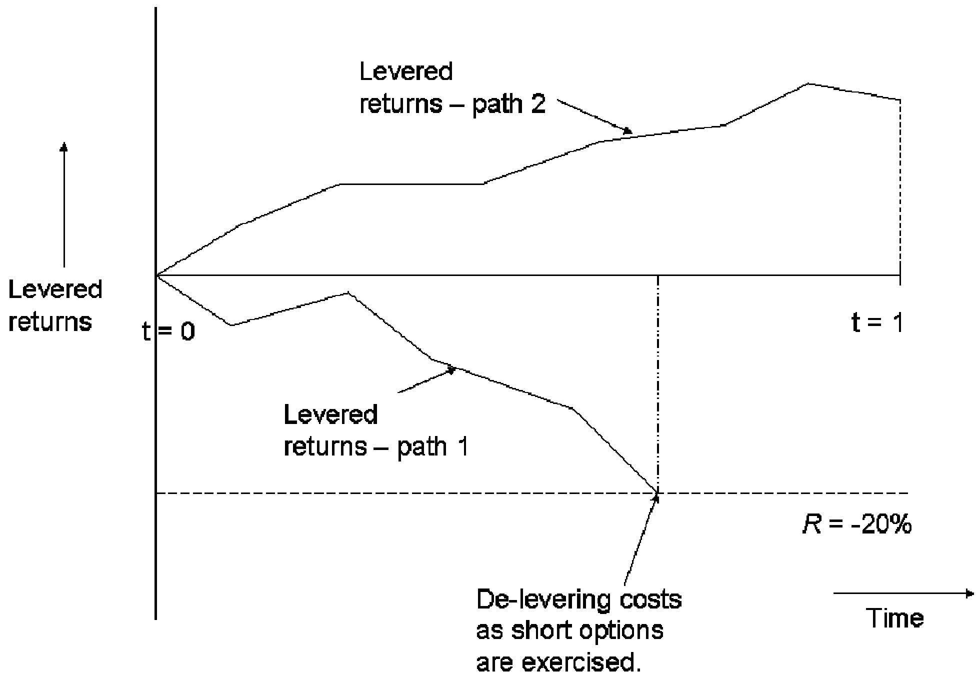
optimal leverage policy is that the implied unencumbered cash level $u_t^* = 1 - \lambda_t L_t^* \sigma_t$ may be too low either due to an unanticipated surge in volatility or redemptions or both. In order to avoid premature liquidation by not having enough unencumbered cash, hedge funds may rationally choose optimal leverage by imposing the restriction that the unencumbered cash level may not fall below a chosen (lower) risk limit. The concept of optimal leverage is addressed in section 3 and the role of unencumbered cash in constraining the leverage further to enhance continued survival of the fund is analyzed in section 4.

3. Optimal Level of Risk Capital (Leverage)

We will derive in this section the optimal level of risk capital that the hedge fund will seek to deploy. Traditional approach to risk budgeting when the portfolio manager has no short option positions is to solve a volatility or *VaR* minimization problem subject to an expected return objective. In the new framework that we have proposed, it is necessary to solve an optimization problem in which the expected return is maximized subject to the fact that the short option positions may be exercised with a high probability when the risk capital (leverage) is high.

In order to derive analytical results, we make the following simplifying assumptions. First, we consider a simple one period setting with two dates, used in mean-variance portfolio optimization models. We can regard the one period to be one year to aid intuition. The fund must choose its optimal risk capital at date 0, knowing that the short options may be exercised by either investors or prime brokers before date 1. The time-line of the model is shown in Figure 3 below.

Figure 3



In path 1, the levered returns reach the barrier (assumed to be -20%) before $t = 1$ and the option is exercised by investors leading to de-levering costs. In path 2, the fund produces positive returns and the options are left unexercised, and the fund reaches $t = 1$ without any redemptions.

We rewrite the generic return generating process (1) to reflect that there are many portfolio managers ("risk-taking" units) each of whom has an alpha generating mechanism. It is by aggregating across all those processes that we get the overall return generating process for the hedge fund (excluding the short option positions) as shown below:

$$R_{t+1} = \underline{s} \circ \underline{\sigma_t} + \underline{\sigma_t} \circ \underline{\varepsilon}_{t+1} \tag{8}$$

The notation $\underline{s} \circ \underline{\sigma_t} = \sum_{i=1}^{N} s_i \sigma_{it}$ represents the element by element multiplication of two vectors $\underline{s}$ and $\underline{\sigma_t}$. In equation (8), we denote the vector of Sharpe ratios by $\underline{s}$ and the vector of volatilities by $\underline{\sigma_t}$.

Given equation (8), the return generating process with the short option position when the managers employ their idiosyncratic leverage levels can be specified next. We first define the (Nx1) vector of risk capital as $x \equiv \underline{L_t} * \underline{\sigma_t}$ each element of which contains the product of leverage and volatility of a risk-taking unit. Then the return generating process of the hedge fund may be written as follows.

$$h_{t+1} = x^{'}\underline{s} + x^{'}\underline{\varepsilon}_{t+1} - a\sqrt{x^{'}\rho x}\,1_{\{(\underline{L_t} \circ \underline{R_{t+1}} \le \underline{R})\}} \tag{9}$$

The return process for the fund is obtained by aggregating across all risk-taking units, taking into account the following features: first, each risk-taking unit of the fund must be assigned some risk capital ($x \equiv \underline{L_t} * \underline{\sigma_t}$) that they should deploy. Second, in making this decision, they must take into account the extent to which their decentralized risk capital deployment will increase/decrease the likelihood that the short position in the option will be exercised. Third, different risk-taking units may subject the firm to deleveraging costs if and when the short option position is exercised. This is reflected through the term $a\sqrt{x^{'}\rho x}$. This term shows that the contribution of all risk-taking units to the overall deleveraging costs of the firm is related to funding liquidity as measured by the leverage factor, the secondary market liquidity as measured by the volatility factor, and the macroeconomic circumstances as reflected by the parameter a and the correlation matrix ρ . In a period of aggregate liquidity shock, the parameter a will increase and ρ tends to a diagonal matrix of unities. Finally, the cross-correlation benefits of different risk-taking units engaging in different (and hopefully less-correlated) risk-taking activities under normal market conditions must be reflected in the decentralized decisions.

The optimality problem can be presented in the classical portfolio optimization framework with the explicit recognition of the short option position as follows. Taking expectations of equation (9) we get,[27]

[27] In deriving Equation (10), we have made a few additional assumptions. First, we have assumed that $\underline{\sigma_t}$ is constant. Second, we are treating Equation (1) as the discrete-time representation of a continuous-time return process and the option is triggered whenever the underlying continuous-time process breaches the trigger level for the first time within the next year.

$$E(h_{t+1}) \equiv g\left(\underline{x}^{'}s, \underline{x}^{'}\rho\underline{x}\right) \equiv g(y,z) = y - a\sqrt{z} \times \left\{ N\left[\frac{\underline{R} - y}{\sqrt{z}}\right] + e^{\frac{2y\underline{R}}{z}} N\left[\frac{\underline{R} + y}{\sqrt{z}}\right] \right\}$$

(10)

In equation (10) we denote by $N(\bullet)$ the normal distribution function[28]. Our approach to modelling hedge fund risk-return tradeoffs is captured by equation (10), which looks similar to the classic mean-variance portfolio optimization problem, except that the "implied risk aversion" itself is endogenous. Equation (10) also reflects one of the key contributions of our paper: the short option position that is faced by fund managers is an explicit part of their optimization program. As such the term within the curly bracket in equation (10) reflects the probability of the levered returns hitting the barrier before date $t = 1$[29]. Intuitively, we may think of the quantity inside the curly brackets of the second term as the value of a digital option that pays $1 if and when the realized returns of the fund breaches the barrier $\underline{R}$ the first time before date $t = 1$. Then the payoff of the digital is amplified by the deleveraging costs faced by the fund, which are assumed to be increasing in the risk-factor $\sqrt{z}$ and the parameter a. For the sake of simplicity, we will model just one option: the interpretation of our results will naturally depend on whether the option considered is the funding option or the redemption option. The combined treatment of both options is straightforward, and the qualitative results are likely to be similar to the ones that we report here[30].

It is useful to note the following properties of the objective function. First, note that the partial derivative with respect to the first variable (g_y) is positive as shown below:

$$g_y(y,z) = 1 - \frac{2a\underline{R}}{\sqrt{z}} \times \left\{ e^{\frac{2y\underline{R}}{z}} N\left[\frac{\underline{R} + y}{\sqrt{z}}\right] \right\} > 0$$

Next, the partial derivative with respect to the second variable (g_z) is negative as shown next.

---

[28] We use the following notions, going forward: $y = x^{'}s$ and $z = x^{'}\rho x$.

[29] This follows from the results presented in Harrison (1985).

[30] One interesting interaction between the two options is the following. The exercise of redemption options may cause the funding option to go deeper in the money. This interaction is not studied in the paper.

$$g_z(y,z)=\frac{g-y}{2z}+\frac{aR}{2z}n\left(\frac{\underline{R}-y}{\sqrt{z}}\right)+\frac{aR}{2z}n\left(\frac{\underline{R}+y}{\sqrt{z}}\right)e^{\frac{2yR}{z}}+\frac{2ayR}{z^{\frac{3}{2}}}\times\left\{e^{\frac{2yR}{z}}N\left[\frac{\underline{R}+y}{\sqrt{z}}\right]\right\}<0$$

These properties ensure that the risk-capital $\underline{x}$ has a finite optimal level. The optimal risk capital for unconstrained problem can be written as follows.

$$x=-\frac{g_y}{2g_z}\rho^{-1}s \tag{11}$$

Although we have squarely formulated the problem as one facing the hedge fund, and as one in which optimal allocation of risk-capital is the variable of interest, it is very easy to see the generality of our approach: consider for example, a “long-only” fund with no leverage. Namely, $L=1$.

For this fund, investors may still have the option to redeem, and this short option is precisely the digital option represented in equation (10) with the modification that the option is now triggered by investors when there is either a flight to quality or when the fund posts sub-par performances relative to its peers over a threshold period of time. The choice variable facing such a long-only fund is obviously not the leverage (risk-capital) level, but its asset allocation and hence the choice of its beta, as equation (1) for the long-only fund will have a systematic (beta) component. The barrier level for the long-only fund is a threshold level of poor returns history. The formulation is therefore very general and can be applied to a broad range of optimal portfolio selection problems in which the funds are faced with varying degrees of short option positions with their investors.

Substituting for g_y and g_z in equation (11) and simplifying, we get the following closed-form expression for the optimal level of risk capital $x_t=\left(\underline{L_t}*\underline{\sigma_t}\right)'$:

$$x = \frac{\dfrac{\rho^{-1}s}{a}\left[1-2aN\left(\dfrac{R+y}{\sqrt{z}}\right)e^{\frac{2yR}{z}}\dfrac{R}{\sqrt{z}}\right]}{z^{-\frac{1}{2}}\left[N\left(\dfrac{R-y}{\sqrt{z}}\right)+e^{\frac{2yR}{z}}N\left(\dfrac{R+y}{\sqrt{z}}\right)\right]-2z^{-1}Rn\left(\dfrac{R-y}{\sqrt{z}}\right)-e^{\frac{2yR}{z}}z^{-\frac{3}{2}}4RyN\left(\dfrac{R+y}{\sqrt{z}}\right)} \tag{12}$$

Where $n(.)$ is the normal density function. Equation (12) shows the optimal leverage employed by each risk taking unit. The relationship between the aggregate risk, z, and the underlying parameters is highly nonlinear.

We can compute $z = x^{'}\rho^{-1}x$ using (12) to get the following expression for aggregate risk z.

$$\sqrt{z} = \frac{\dfrac{\sqrt{s^{'}\rho^{-1}s}}{a}\left[1-2aN\left(\dfrac{R+y}{\sqrt{z}}\right)e^{\frac{2yR}{z}}\dfrac{R}{\sqrt{z}}\right]}{z^{-\frac{1}{2}}\left[N\left(\dfrac{R-y}{\sqrt{z}}\right)+e^{\frac{2yR}{z}}N\left(\dfrac{R+y}{\sqrt{z}}\right)\right]-2z^{-1}Rn\left(\dfrac{R-y}{\sqrt{z}}\right)-e^{\frac{2yR}{z}}z^{-\frac{3}{2}}4RyN\left(\dfrac{R+y}{\sqrt{z}}\right)} \tag{13}$$

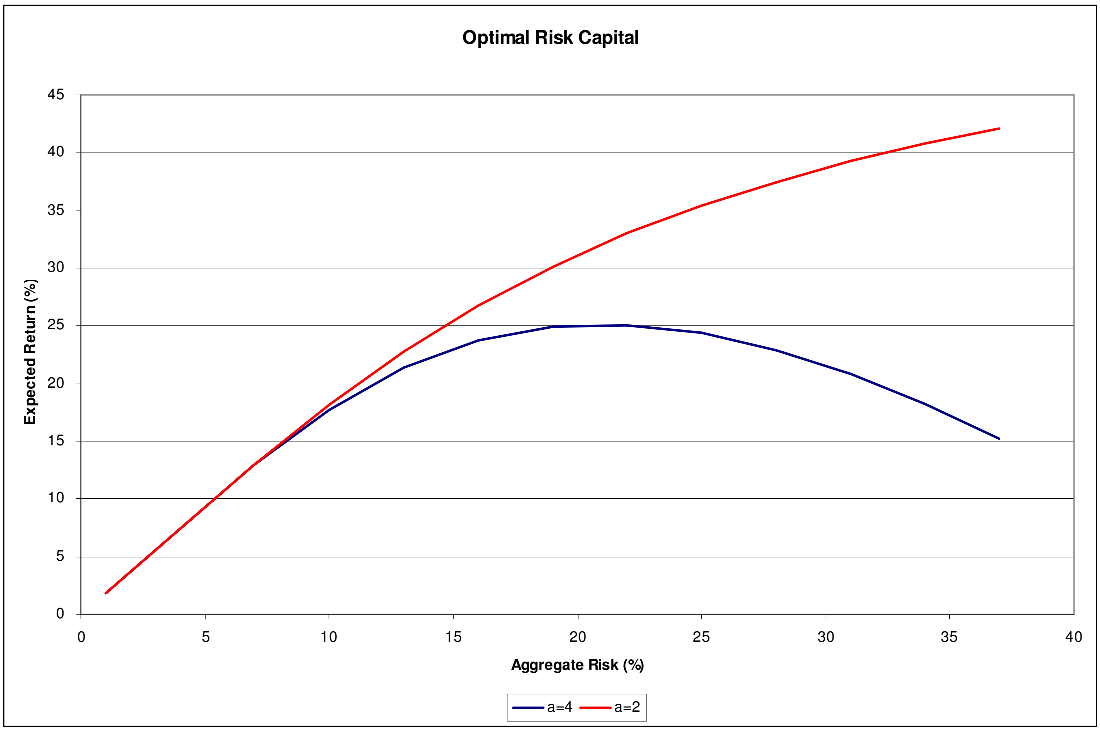
Note that (13) provides an explicit way to compute the aggregate risk, given the underlying contractual structure and the fund parameters: a) trigger level for the option, b) Sharpe ratios, c) correlations, and d) de-leveraging costs. We can combine (12) and (13) to get the risk-budget as follows.

$$x = \frac{\rho^{-1}s\sqrt{z}}{\sqrt{s^{'}\rho^{-1}s}} \tag{14}$$

Given the aggregate risk capital z, we can determine the allocations *independent* of de-leveraging costs and option triggers. *But the aggregate risk capital itself is endogenous and depends on these parameters as shown in (13).*

Note from figure 4 that the optimal aggregate risk capital is decreasing in the parameter a, which reflects the costs of de-leveraging due to the presence of short option positions. *Thus the absolute level of risk capital (leverage) goes down as the de-leveraging costs associated with the short option positions increase*. This is illustrated in figure 4 for the case of two risk-taking units. We have assumed the following parameters: $R = -10\%, \rho = 0, s_1 = 1.5,$ and $s_2 = 1.1.$

Figure 4



This result has an important risk management message: hedge funds will be well advised to factor their contractual relationships with prime brokers and investors in determining their aggregate risk capital.

3.1 Implied Risk Aversion and Distance to Default

Equation (10) can be used to distinguish our formulation from the classic portfolio optimization problem. In the absence of any short options, hedge fund can allocate as much risk-capital as it wishes: such a strategy will increase the expected return of the portfolio without affecting the Sharpe ratio, assuming that the strategies are scalable. The presence of short options induces risk aversion as is clear from (10). This will in turn introduce a non-linearity in the relationship between expected returns and risk. To see this clearly, let us consider the mean-standard deviation optimization problem of a *risk-averse hedge fund* which maximizes the objective function shown below.

$$g = y - \frac{\lambda}{2} z \tag{15}$$

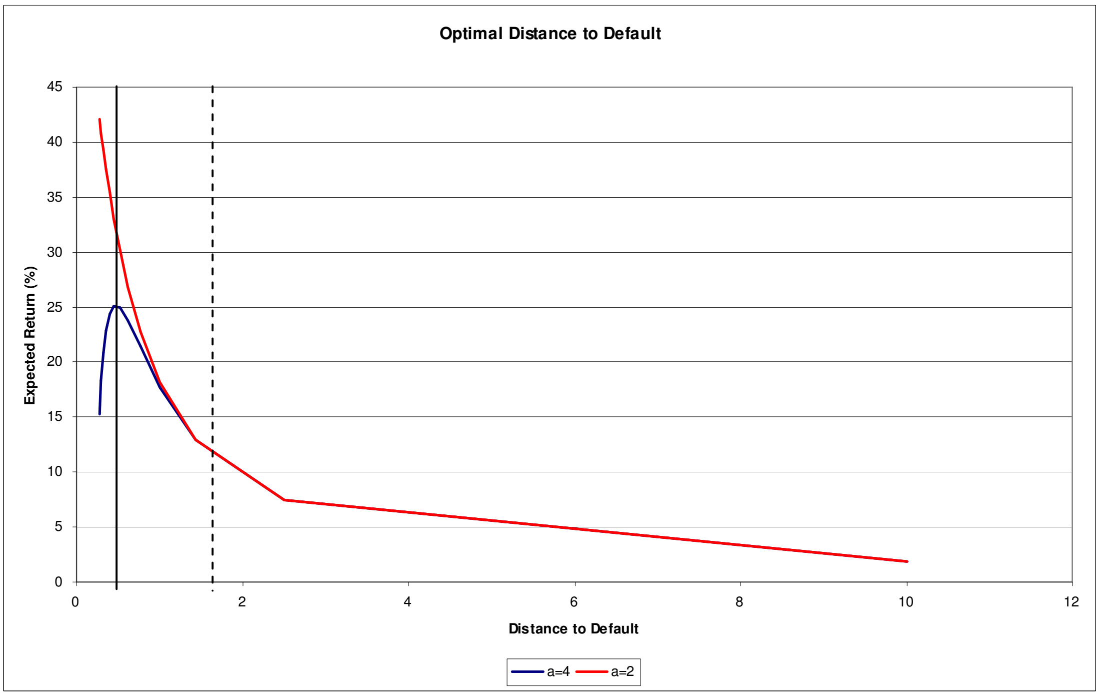
Comparing (10) and (15), we can see that the risk-aversion arises from the costs associated with the potential exercise of short option positions. Note from equation

(14) that $y = x^{'}s = \sqrt{z}\sqrt{s^{'}\rho^{-1}s}$. This implies that $\frac{y}{\sqrt{z}} = \sqrt{s^{'}\rho^{-1}s}$. For the funding option, we may regard the quantity $\frac{|R|}{\sqrt{z}} \equiv DD$ as essentially the distance to default or distance to liquidation. Given a set of Sharpe ratios, correlations, option triggers, and de-leveraging costs, the distance to default for the fund is *endogenously* chosen. This is seen by writing (13) as follows.

$$\sqrt{s^{'}\rho^{-1}s} = \frac{a\left[N\left(-DD-\sqrt{s^{'}\rho^{-1}s}\right)+e^{-2DD\sqrt{s^{'}\rho^{-1}s}}N\left(-DD+\sqrt{s^{'}\rho^{-1}s}\right)\right]+2aDDn\left(-DD-\sqrt{s^{'}\rho^{-1}s}\right)+4ae^{-2DD\sqrt{s^{'}\rho^{-1}s}}DD\sqrt{s^{'}\rho^{-1}s}N\left(-DD+\sqrt{s^{'}\rho^{-1}s}\right)}{\left[1+2aN\left(-DD+\sqrt{s^{'}\rho^{-1}s}\right)e^{-2DD\sqrt{s^{'}\rho^{-1}s}}DD\right]} \tag{16}$$

This concept of distance to default is a convenient way to think about risk limits: the optimal distance to default that emerges from (16) could be used as a guideline by the risk managers to set more conservative distance to default to cover unanticipated surges in volatility or unforeseen changes in triggers by investors and prime brokers. This more conservative level may give the necessary cushion for the funds to perform voluntary de-leveraging in the face of an unanticipated crisis. This is best understood by recasting figure 4 in terms of distance to default as shown below.

Figure 5

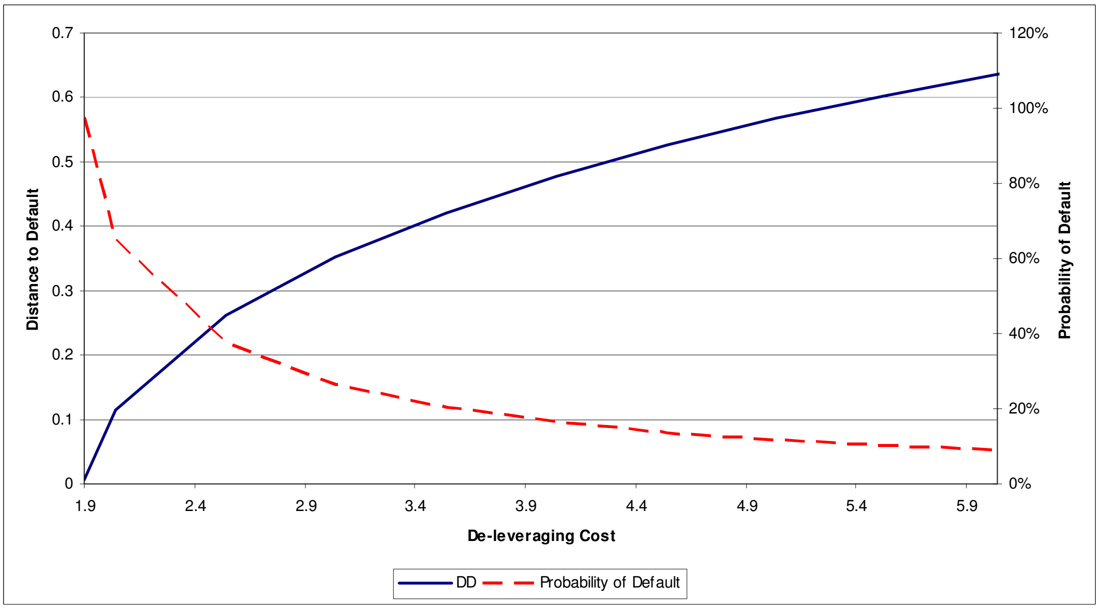


In Figure 5, note that the optimal distance to default is 0.45 when a = 4. One risk management policy might be to choose a more conservative level denoted by the dashed vertical line to the right of the unconstrained optimum. This choice sacrifices some expected returns. The precise location of the more conservative policy is dictated by the fund's desired level of DD, which is a function of the amount of flexibility it wants to be able to perform voluntary de-levering when there is an unanticipated shock to volatility and/or funding conditions with PBs.

Equation (16) places an important restriction on optimal risk-capital, and the implied distance to default for hedge funds. Given a distribution of Sharpe ratios (which capture the volatility of assets deployed by the fund and the correlation across different risk-taking units), and de-levering costs (which captures the secondary market liquidity of assets) the fund's distance to default is pre-determined at the optimally chosen risk-capital.

This is an important prediction: if the Sharpe ratios are the same for two funds, with one facing higher de-levering costs, then that fund must reduce its aggregate risk-capital and increase its distance to default. This is presented in figure 6 below.

Figure 6



Even if the fund negotiates a sufficiently low trigger level with investors and prime brokers, its endogenous (unconstrained) risk capital will leave the DD the same.

Equation (16) also sets an upper bound on aggregate risk: the aggregate risk $\bar{z}$ upper bound that causes the distance to default to go to zero is at the point where

$\sqrt{s^{'}\rho^{-1}s} = a.$ This is the point where the Sharpe ratio of the fund is exactly equal to the de-leveraging costs.

The probability of exercise of the option is $\left\{ N\left[\frac{R-y}{\sqrt{z}}\right] + e^{\frac{2yR}{z}} N\left[\frac{R+y}{\sqrt{z}}\right] \right\}$ and is decreasing with the deleveraging cost a. This result arises from the fact that a fund with lower deleveraging cost for its short option positions as others will rationally choose a higher risk-capital.

3.2 Risk capital allocation (unconstrained case)

The behaviour of optimal risk capital with respect to other underlying parameters is easy to understand with two risk-taking units. They are illustrated below.

First, we examine the relationship between the barrier levels, expected returns and the aggregate risk capital employed by the hedge fund. This is illustrated in figure 7 below.

Figure 7

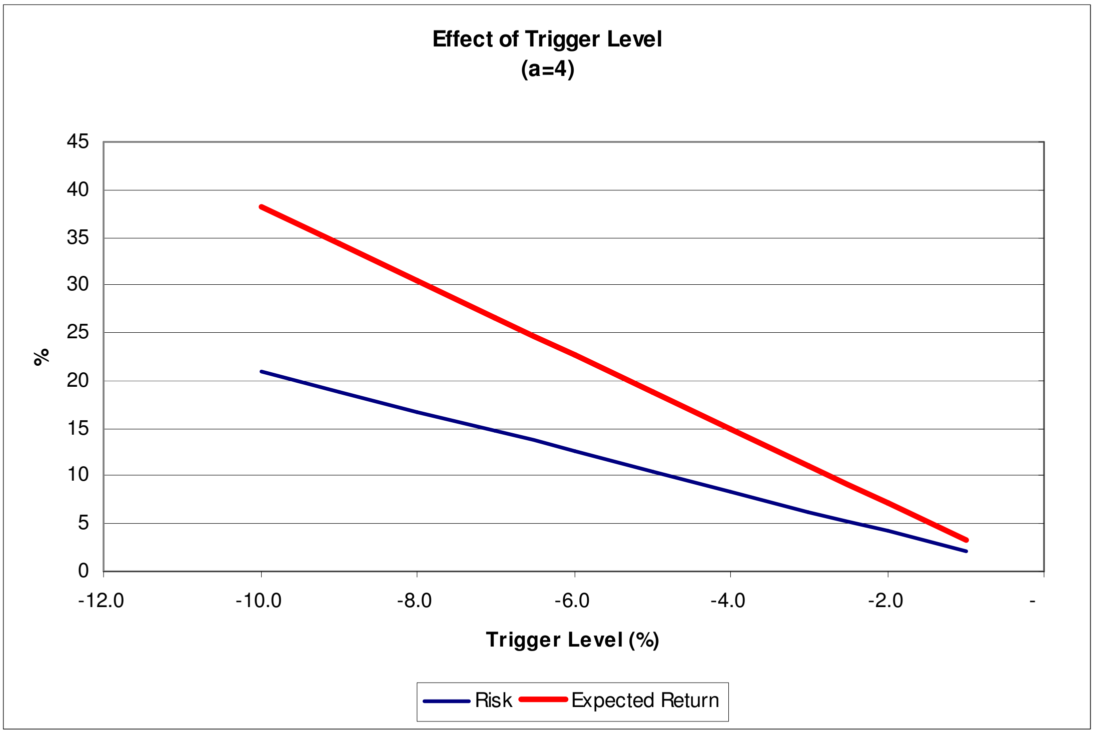
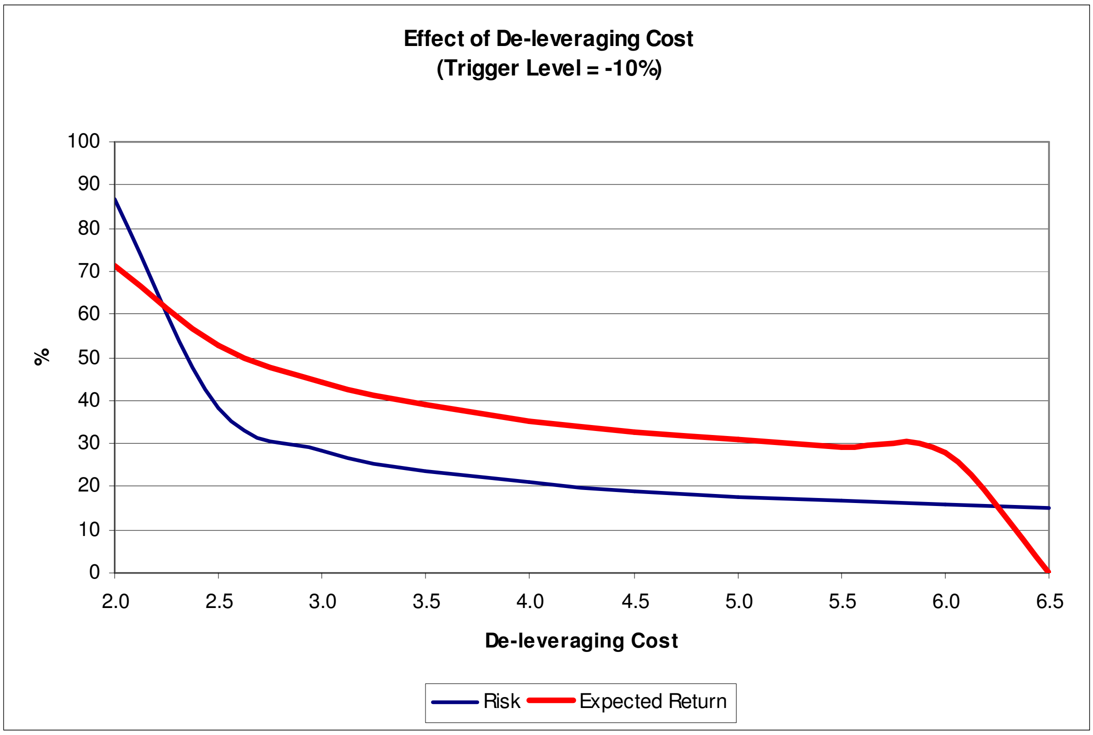


Figure 7 shows that the aggregate risk increases if the trigger level is set sufficiently low – prime brokers are prepared to wait until the performance becomes really intolerable. For example, if the trigger level is -10%, then the hedge fund can take a rather high level of risk at 21%. If the trigger is tightened to -2.5%, the aggregate risk reduced to 5.23%, which is a nearly a 4-fold reduction in risk. This result shows that a rational hedge fund will curtail its leverage in response to tightened market conditions for funding.

In a similar fashion, we can investigate the effect of de-leveraging costs (as parameterized by *a* on endogenous leverage levels chosen by the hedge fund and its resulting expected returns. Figure 8 plots this relationship.

Figure 8



Note the steep drop in risk capital, when the de-leveraging costs increase. Funds operating in illiquid secondary asset markets (such as structured credit) will naturally be careful about the level of leverage that they may wish to employ.

Throughout this section we have analyzed the unconstrained choice of aggregate risk-capital by a hedge fund. In reality, hedge funds are concerned about the potential for liquidation due to unexpected surge in volatility, credit market dislocations, and large-scale investor redemptions. While the short options framework addresses some of these concerns, prudent risk management may require explicit risk limits that further protect the hedge from insolvency by choosing a risk-capital which is *below* the unconstrained optimum that we have characterized in this section.

4. Risk Budgeting

We have shown that the optimal level of aggregate risk capital is smaller in the presence of short options positions. The *relative risk capital* defined as the ratio of risk capital employed across two risk taking units can be computed from (14) as follows for the case of two risk-taking units. Let us now denote by ρ simply the correlation coefficient between the risk capital of two risk-taking units. Then the ratio of risk capital is:

$$\frac{x_1}{x_2} = \frac{s_1 - \rho s_2}{s_2 - \rho s_1}. \tag{17}$$

Equation (17) shows that the relative risk capital depends on both Sharpe ratios and the correlation coefficient. When the two risk-taking units are completely uncorrelated, then the relative risk capital is simply the ratio of Sharpe ratios.

In the risk budgeting problem, the firm imposes a constraint on the aggregate risk limit on unencumbered cash, which bounds the amount of margin that can be used by the hedge fund in determining its overall leverage. As before, the expected return will also have the short option positions of the hedge fund. We will show that our formulation will enable us to exploit the powerful tools of traditional, mean-variance portfolio theory, notwithstanding the short option positions and the aggregate risk limit on unencumbered cash. We can now set up the optimization problem associated with the risk budgeting as follows.

$$\underset{\{x,\lambda\}}{Max} \left\{ g\left(x^{'}s, x^{'}\rho x\right) - \lambda\left(x^{'}\rho x - v^2\right) \right\} \tag{18}$$

Equation (18) says that the individual risk-taking units are simultaneously choosing their risk-levels bearing in mind the shadow costs to the fund arising from the likelihood of the fund-level risk limits from becoming binding.

We ended the preceding section by pointing out that the "optimal leverage ratio" derived from maximizing expected return may not be optimal after all if the implied unencumbered cash level is too low. A low unencumbered cash level is extremely dangerous for a hedge fund because an unexpected increase in margin requirement, an untimely redemption request, coupled with an untimely increase in market volatility can easily lead the hedge fund down the path of insolvency: its inability to meet margin calls puts itself at the mercy of a liquidation process where fire sale is the rule rather than the exception. To avoid such a fate, the hedge fund should adopt a policy that at no times the unencumbered cash should fall below an "insolvency" threshold or a more conservative threshold $\underline{u}$. Note that if both the funding and the redemption options are triggered, the unencumbered cash level will drop down to $1-(1-u_t)e^{+\varphi+\eta}$. Prudent risk management would call for the "stressed unencumbered cash level" to stay above the threshold $\underline{u}$. It follows that the optimal leverage ratio should be[31]:

$$L_t^{**}\sigma_t = \min\left[L_t^{*}\sigma_t, \frac{1-\underline{u}}{\lambda_t}\times e^{-\varphi-\eta}\right]. \tag{19}$$

The second term in the right hand of equation (19) captures three effects: (i) the higher is the unencumbered cash limit $\underline{U}$, the lower is the optimal leverage level; (ii) if the investor base of the hedge fund has a higher propensity to redeem at bad news (i.e., higher φ), or its PBs have a higher tendency to raise margin requirement (i.e., higher η), the optimal leverage ratio should be lower; this can be viewed an allowance for "worst-case scenarios"; and (iii) the higher is the margin multiplier λ_t, the lower the leverage ratio should be; this can be viewed as a penalty for "margin

[31] Note from equation (4a) (under the assumption that both options are exercised) that $u = 1-\frac{\lambda_{t+1}L_t\sigma_t AUM_t}{AUM_{t+1}} = 1-\frac{\lambda_{t+1}\lambda_t L_t\sigma_t AUM_t}{\lambda_t AUM_{t+1}} = 1-(1-u_t)e^{+\varphi+\eta}$. In addition, note that we want this level to be greater than the risk limit. Or, we want $1-(1-u_t)e^{+\varphi+\eta} \geq \underline{u}$. This in conjunction with the definition of unencumbered cash in equation (4) leads to equation (19).

inefficiency”. If the unencumbered cash level is chosen conservatively than the risk limit in (19) will be binding, and we get

$$v = \frac{1-\underline{u}}{\lambda_t} e^{-\varphi-\eta} < v^* = L^* \sigma_t .$$

(20)

In equation (20), we represent the unconstrained optimal risk-capital level (found in section 3 earlier) as v^*. This is an important restriction, as the unconstrained optimal risk capital was computed under the following key assumptions:

a) There are no unforeseen surges in volatility in the underlying assets, and

b) There are no aggregate shocks in the banking system, which may cut off funding.

In order to manage these risks, we set the risk limit to be less than the level implied by unconstrained optimal risk-capital. When the constraint is binding, we simply set $x^{'} \rho x^{'} = v^2$ and solve for the optimal risk level as follows.

$$x = \frac{v \rho^{-1} s}{\sqrt{s^{'} \rho^{-1} s}}$$

(21)

The individual risk-taking units maximize their expected returns (with the short option positions) subject to an overall risk limit imposed by the risk management philosophy of the fund. This risk limit is explicitly driven by the minimum level of unencumbered cash level that the firm must have under all circumstances as specified in equation (4). Note from (4) that the imposition of such a constraint on unencumbered cash translates to a natural constraint on leverage and risk capital. Such a limit will precipitate voluntary deleveraging activity, and will serve to increase the expected return of the fund over long horizon.

Note that equation (20) captures in a succinct manner the effect of gates and funding environments. Recall from (4) that we may write unencumbered cash level as $u = 1 - \lambda\sqrt{z}$. Hence, setting a risk limit on unencumbered cash level at $\underline{u}$ implies a limit on risk-capital itself. In other words,

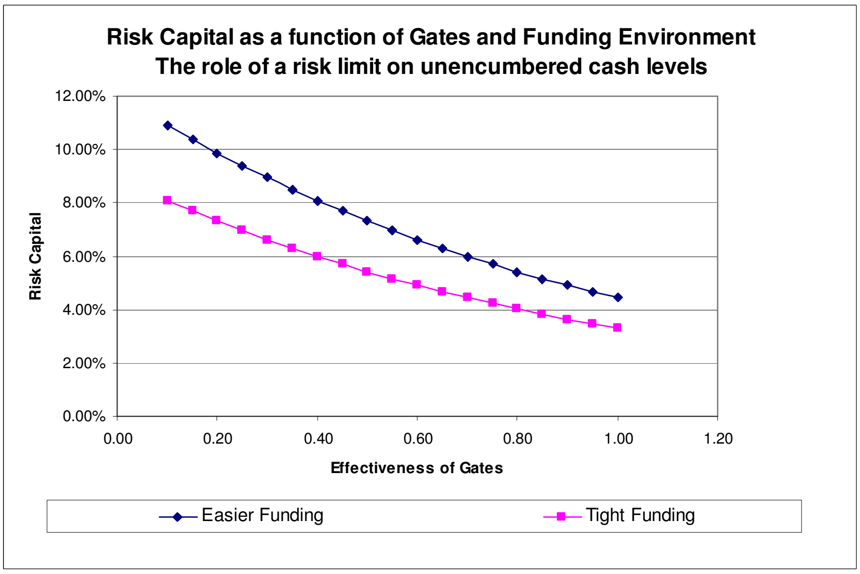
$$\underline{u} = 1 - \lambda\sqrt{\underline{z}}.$$

Substituting this into (20) we find the feedback effects of a risk management policy, which imposes a lower limit on unencumbered cash level as follows.

$$v = \sqrt{\underline{z}}e^{-\varphi-\eta}.$$

We use this as an illustrative risk limit to motivate the optimization problem facing the hedge fund. Hedge funds differ in terms of their respective margin multipliers, and the parameters φ and η which capture the uncertainties associated with investor redemptions and prime broker behaviour. To accomplish the same risk limit, hedge funds may have to impose different standards on the levels of unencumbered cash. For example, a fund in which investors have no redemption possibilities until t = 1, $(\varphi=0)$ the risk-capital can be higher in order to maintain the same level of unencumbered cash risk limit. On the other hand, if the fund has provided ample redemption provisions, then to keep the same level of unencumbered cash risk limit, it is necessary to lower the optimal level of risk-capital. These trade-offs are shown in Figure 9. Another point that is borne out by figure 9 is the following: funds which have worked out abundant funding facilities with their prime brokers may be able to maintain a higher risk-capital at every level of redemption frequencies.

Figure 9



The first order conditions of optimality can now be derived from equation (18) as follows. The optimality condition corresponding to the choice of the risk-capital choice is shown next[32].

$$g_y \underline{s} + 2g_z \rho x - 2\lambda \rho x = 0 \Rightarrow$$

$$x = \frac{1}{2}\frac{g_y}{\lambda - g_z}\rho^{-1}\underline{s}$$

(22)

The shadow cost condition with respect to the variable λ is shown next.

$$x^{'}\rho x = v^2$$

(23)

Combining (22) and (23) we get the expression for the shadow cost as follows.

$$\lambda = g_z + \frac{g_y}{2v}\sqrt{\underline{s}^{'}\rho^{-1}\underline{s}}$$

(24)

Given a risk limit, we need to solve (18) and (20) for the optimal levels of risk-capital and the shadow cost, given by $\left(x^*, \lambda^*\right)$.

4.1 Shadow cost of risk-capital

Consider a simple case in which the fund has two desks and the fund must decide on the risk-capital allocation problem. For simplicity we will suppose that the Sharpe ratios and the correlation coefficients are given as shown below.

$$s = \begin{bmatrix} 1.5 \\ 1.1 \end{bmatrix} \qquad \rho = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let us assume that the returns threshold (barrier) is set at -1.0%. The deleveraging costs are set at 2. For various risk limits on unencumbered cash level, how will the risk budget be allocated between these two risk-taking units?

[32] We denote by g_y the partial derivative with respect to *y*, and by g_z the partial derivative with respect to *z*.

In Table 2 below, we present the risk budgeting and how it is informed by the short option position held by the hedge fund.

First, it is useful to recall from equation (20) that the risk limit ν decreases as the fund sets a high level of minimum unencumbered cash level $\underline{u}$ as its risk management policy. Note also from Table 2 that when the risk limits are tight, the shadow costs as captured by λ are high, and at this low level of capital allocated, the short option is out of the money. This is reflected by the low cost of short option position. As we increase the risk capital, (either through high leverage, or ex-ante volatility, or both) maximum expected return achievable increases as the increase in short option value is more than offset by the increase in the expected return that would accrue to the fund, ignoring the short option. The unconstrained optimal risk-capital is 8.66 when the shadow costs go to zero.

Table 2

Risk capital to desk 1	Risk capital to desk 2	Total risk capital LIMIT	Expected Returns (no short options)	Cost of short options	Net Expected Returns	Lambda Shadow Cost
1.803	1.322	2.24	4.1593	0.8274	3.3319	0.1922
2.550	1.870	3.16	5.8822	1.9242	3.9580	0.0827
3.123	2.290	3.87	7.2042	2.9347	4.2695	0.0469
3.606	2.645	4.47	8.3187	3.8612	4.4575	0.0300
4.032	2.957	5.00	9.3005	4.7186	4.5819	0.0205
5.702	4.182	7.07	13.1529	8.3191	4.8338	0.0041
6.984	5.121	8.66	16.1090	11.2326	4.8764	0.0000

A fund which requires a high level of unencumbered cash limit will choose a very tight risk limit, and hence will lower the cost of the short option positions. In the process, it might end up giving up some upside potential. This is the trade-off that prudent risk management will have to evaluate.

A feature of the risk budgeting in Table 2 is that the desk with the higher Sharpe ratio tends to get higher risk capital. A less obvious implication is that the ratio of the capital allocated to the desk with the high Sharpe ratio to the allocation for the other desk is always a constant. In the context of the example in Table 1 this constant is 1.3636. This is irrespective of the aggregate risk limit. This implication follows from our result in section 3 in equation (13). Note that the absolute level of risk budget allotted to each desk is a function of the de-levering costs as explained in section 3.

What is the effect of correlations between trading desk on the level of risk capital and on its allocation across each desk? Table 3 provides some economic intuition behind the relationship between aggregate risk capital choice and correlation. Table 3 shows that as the correlation decreases, the optimal overall risk limit increases. This is intuitive. For the parameter values chosen, the optimal risk capital is around 13 for the case when the correlation is zero. For the case when the correlation is 0.4, the optimal risk limit is around 7. These results are sensitive to the high costs of de-levering. If we relax the risk limit imposed on the level of unencumbered cash levels, then the alpha will start to go down as excess risk capital gets allocated.

Table 3

Risk-capital	rho =0	Risk-capital	rho=0.4
10.00	8.692	3.16	4.228
10.49	8.745	4.47	4.975
10.95	8.785	5.48	5.262
11.40	8.814	6.32	5.370
11.83	8.834	**7.07**	**5.388**
12.25	8.847	7.75	5.357
12.65	8.854	8.37	5.294
13.04	**8.855**	8.94	5.212
13.42	8.852	9.49	5.117
13.78	8.846	10.00	5.012
14.14	8.836	10.49	4.901
14.49	8.823	10.95	4.786
14.83	8.808	11.40	4.668

What is the allocation of risk capital across the two desks as the correlation changes? Predictably, as the correlation increases, the desk with the higher Sharpe ratio tends to get the lion's share of risk capital. When the correlation is zero, the desk with high Sharpe ratio gets 1.36 times the risk capital allotted to the other desk. As the correlation increases, this ratio begins to increase. Finally, when the two desks are perfectly correlated, the desk with lower Sharpe ratio gets negative risk capital – it is used to hedge the risk capital in the desk which has a higher Sharpe ratio. This implication of the model is intuitive: with high correlation, the desk with higher Sharpe ratio begins to strictly dominate the one with a lower Sharpe ratio.

5. Implications for Investors and Policy Makers

Our analysis has some implications for hedge fund investors and policy makers. The extent to which due diligence is paid by hedge funds in managing their short option positions with investors and prime brokers should be of great interest to both hedge fund investors and policy makers. As we noted at the outset, these options should be fully understood by all parties ex-ante and vigorously managed, ex-post. Such a policy promotes transparency and helps to properly evaluate the economics of investing in hedge funds. *From the investor's perspective a careful evaluation of hedge fund's risk management policy in the management of these short option positions is at least as important as understanding the fund's investment philosophy, market risk and counterparty risk*. Such an understanding can help make investors to make more informed decisions, and minimizes surprises ex-post. From a regulatory perspective, policy makers may benefit by focusing attention on these short option positions: in the event of a banking crisis, all prime brokers have tendency to withdraw their credit lines or increase significantly their margin requirements as we have documented. In some cases, prime brokers have been aggressive in exercising their funding options to terminate the hedge funds. In such a case, most hedge funds are obliged to de-lever simultaneously precipitating secular declines in asset prices. To the extent that hedge fund risk management policies already set prudent risk limits anticipating the potential exercise of short options, such de-leveraging is more likely to be planned and voluntary, as the funds would have allocated less risk-capital, ex-ante. As a tool for hedge fund risk management, regulators may be better off focusing more on unencumbered cash levels to judge how well the funds are managing their risks, and how well they are placed to voluntarily de-lever. It is our view that this measure is much more transparent, relatively model-independent and easy to verify, unlike measures such as *VaR*, which are often dependent on models and assumptions. In policy discussion of gates, lockup periods and notification periods, it is useful to remember that these are usually agreed upon by investors, ex-ante in a transparent relationship between the fund and its investors. Regulators should make every effort to ensure that these contractual provisions are transparent to all investors. Their presence in the contractual agreements serves to mitigate systemic risk when large-scale redemptions ensue due to unanticipated banking crisis or other macroeconomic developments. These provisions enable orderly and planned liquidations to meet the liquidity demands of exiting investors while protecting continuing investors and the fund to continue to function. There is presently no formal coordination mechanism for orderly liquidations and workouts in the hedge fund industry that we are aware of. In the absence of a provisions such as “automatic stay” (which are part of chapter 11 proceedings under the bankruptcy code) it is important that the hedge fund industry has well-articulated contractual provisions that enable the fund its prime brokers and its investors to have an orderly resolution of redemptions and settlement of claims in the event of distress. To the extent that contracts allow investors to redeem at different points in calendar time (investor-level gates as opposed to fund-level gates tend to accomplish this better) it may mitigate systemic effects of investor redemptions.

6. Conclusion

We make the observation that hedge funds, by construction of their funding arrangements and contractual arrangements with their investors are short in two very valuable options. We argue that these options introduce significant nonlinearities in their return generating process, quite independent of any portfolio strategies that the funds may choose to follow. In this sense, we depart from many of the papers in the hedge funds literature which focus on nonlinearities in hedge funds returns arising from portfolio strategies followed by hedge funds. An important consequence of these short options position is that there is typically a well defined leverage or optimal risk-capital for hedge funds. Our paper makes the argument that setting prudent risk limits on unencumbered cash allows the hedge fund to stay well within the prudent leverage levels without undue sacrifice in expected return. We show, through explicit examples, how to develop ex-ante leverage constraints through such risk limits. A contribution of our paper is the integration of contractual short option positions into the risk management principles for hedge funds. We also stress the important role played by unencumbered cash as a tool for risk management.

The paper did not explicitly model the compensation structure of hedge fund managers to examine how they might interact with risk-taking behaviour and hence on prudent risk management structure. Aspects of compensation structure such as the management fee, performance fee, high water marks, etc. represent an important avenue for further research in hedge fund’s risk management.

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