Another characterization of quasisupermodularity

Kukushkin, Nikolai S.

Russian Academy of Sciences, Dorodnicyn Computing Center

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Nikolai S. Kukushkin†

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Abstract

An ordering on a lattice is quasisupermodular if and only if inserting it into any parametric optimization problem with the single crossing property cannot destroy the monotonicity of the set of optima. More detailed conditions for the monotonicity of the set of optima in a parameter influencing the preference ordering are also obtained. *JEL* Classification Number: C 72.

Key words: Best response correspondence; increasing correspondence; single crossing; quasisupermodular ordering

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†Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru

1 Introduction

The concept of strategic complementarity was first developed in a cardinal form, around the notion of a supermodular function (Topkis, 1978, 1979; Veinott, 1989; Vives, 1990; Milgrom and Roberts, 1990). Milgrom and Shannon (1994) gave the idea an ordinal expression and obtained a neat characterization result, Theorem 4.

For our purposes here, that result is better perceived as two independent statements, related to "type B," respectively, "type A" problems in the terminology of Quah (2007). First (Milgrom and Shannon, 1994, Corollary 1), the set of optimal choices depends on the sublattice of available choices in a monotone way if and only if the utility function is quasisupermodular. Second, if a quasisupermodular utility function is perturbed, the set of optimal choices from every sublattice ascends if and only if the single crossing conditions hold.

In the standard model of a strategic game, where each player's utility depends on the choices of others but the strategy set does not, Milgrom and Shannon's necessity result does not mean that we cannot have increasing best responses without the quasisupermodularity of preferences. Indeed, there are weaker sufficient conditions in the literature (Kukushkin et al., 2005, Lemma 3.1).

This paper has resulted from an attempt to develop an analog of Milgrom and Shannon's Theorem 4 for situations where no comparison between optimal choices from different sets is ever made. In a sense, the attempt was successful, but necessity has to be interpreted in a much broader sense: the immersion of a given optimization problem into various parametric settings has to be considered. Quasisupermodularity is partitioned into four independent constituent parts, each of which is necessary and sufficient (in that sophisticated sense) for a kind (actually, two kinds) of the monotonicity of optima.

Section 2 reproduces the standard notion of a choice function generated by the maximization of a binary relation; in Section 3, we consider a number of extensions of an order from points to subsets, which provide various ways to define an increasing correspondence. In Section 4, a range of single crossing conditions is formulated and their (rather straightforward) connections with the monotonicity of optima on chains are described. Section 5 contains the definitions of four partial versions of quasisupermodularity and the proofs of their basic properties. The central results of the paper are collected in Section 6; concluding remarks, in Section 7.

2 Preferences and choice

Throughout the paper, we assume a set A of alternatives given. There is an agent whose preferences over the alternatives are expressed by a binary relation $\succ$ on A, which is assumed to be an *ordering*, i.e., *irreflexive*, *transitive*, and *negatively transitive* ($z \not\succ y \not\succ x \Rightarrow z \not\succ x$). Then the "non-strict preference" relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total.

Orderings can also be defined in terms of representations in chains: $\succ$ is an ordering if and only if there is a chain $\mathcal{C}$ and a mapping $u\colon A \to \mathcal{C}$ such that $y \succ x \iff u(y) > u(x)$ for all $x, y \in A$ (then $y \succeq x \iff u(y) \geq u(x)$). The most usual assumption in game theory is that the preferences of a player are described by a *utility function* $u\colon A \to \mathbb{R}$. Here we work in a purely ordinal framework, so it is natural to replace $\mathbb{R}$ with an arbitrary chain.

As is usual in decision theory, we allow for the possibility that only a subset $X \subseteq A$ may be available for choice. The set of all subsets of A is denoted $\mathfrak{B}_A$. Given $X \in \mathfrak{B}_A$, we define

$$M(X, \succ) := \{x \in X \mid \nexists\, y \in X\, [y \succ x]\} = \{x \in X \mid \forall\, y \in X\, [x \succeq y]\}, \tag{1}$$

the set of *maximizers* of $\succ$ on X. Clearly, $M(X, \succ) \neq \emptyset$ whenever X is a finite nonempty subset of A. We do not restrict ourselves to finite subsets here; nor do we study more general conditions for the existence of maximizers. Very formally speaking, we view an empty set $M(X, \succ)$ as no worse than a nonempty one. A rationalization for this attitude is given in Section 3 below. A very helpful observation is that $y \succ x$ whenever $\succ$ is an ordering, $x, y \in X$, and $x \notin M(X, \succ) \ni y$ ("revealed preference").

With game-theoretic applications in mind, we consider *parametric families* $\langle \succ^t \rangle_{t \in T}$ of orderings on A; the parameter t may be interpreted as (an aggregate of) the choice(s) of other agent(s). Given a parametric family and $X \in \mathfrak{B}_A$, the *best response correspondence* $\mathcal{R}^X : T \to \mathfrak{B}_X \subseteq \mathfrak{B}_A$ is defined in the usual way:

$$\mathcal{R}^X(t) := M(X, \succ^t)\,. \tag{2}$$

Admittedly, the preferences of a player in the standard strategic game model are described by an ordering on $X \times T$ rather than by a parametric family of orderings on X. It is impossible to study, say, strong equilibria or the (in)efficiency of Nash equilibria in the latter framework. On the other hand, a parametric family of orderings is adequate when the subject is the existence of a Nash equilibrium or the behavior of individual adaptive dynamics. Some twenty five years ago, Olga Bondareva argued that the proper definition of a non-cooperative game must stipulate that each player is only able to compare strategy profiles differing in her own choice. Although one does not have to accept this, rather extreme, view, there is something in it.

3 Monotonicity

We always assume A to be a *partially ordered set* (a *poset*). Most often, it is a *lattice*, in which case $\mathfrak{L}_A$ denotes the set of all sublattices of A. The exact definitions are assumed commonly known. Given a lattice A and $x, y \in A$, we denote $L(x, y) := \{x, y, x \vee y, x \wedge y\}$, the minimal sublattice of A containing both x and y; clearly, $\#L(x, y) \in \{1, 2, 4\}$.

The reversal of an order ($y < x \rightleftharpoons x > y$) produces an order again; moreover, a lattice remains a lattice. Having proved a theorem, we can replace, in all assumptions and the statement itself, the relations and operations $>$, $\geq$, $\vee$, etc. with $<$, $\leq$, $\wedge$, etc., and obtain another valid theorem. The use of this simple observation (referred to as "duality") leads to considerable economy in the total length of proofs.

When considering a parametric family of preference relations, we assume that T is also a poset. A mapping $r\colon T \to A$ is *increasing* if $r(t') \geq r(t)$ whenever $t' > t$; when it comes to correspondences $R\colon T \to \mathfrak{B}_A$, we have to extend the order from A to $\mathfrak{B}_A$. Following Veinott (1989), we consider a few ways to do so for a lattice A:

$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \wedge x \in X]; \tag{3a}$$

$$Y \geq^{\vee} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \vee x \in Y]; \tag{3b}$$

$$Y \geq^{\mathrm{Vt}} X \rightleftharpoons [Y \geq^{\vee} X \;\&\; Y \geq^{\wedge} X]; \tag{3c}$$

$$Y \geq^{\mathrm{wV}} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \vee x \in Y \;\text{ or }\; y \wedge x \in X]. \tag{3d}$$

Another relation can be defined on any poset:

$$Y \ggg X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \geq x]. \tag{3e}$$

Clearly, $Y \ggg X$ implies every other relation (3). None of the relations is an order, even on nonempty subsets: $\ggg$ and $\geq^{\mathrm{Vt}}$ are antisymmetric and transitive, but generally not reflexive; neither $\geq^{\mathrm{wV}}$, nor $\geq^{\wedge}$ or $\geq^{\vee}$ need even be transitive.

Let $\geq^{*}$ denote one of the relations (3) and T be a poset. A correspondences $R\colon T \to \mathfrak{B}_A$ is *increasing w.r.t.* $\geq^{*}$ if $R(t') \geq^{*} R(t)$ whenever $t' > t$. Veinott (1989) called correspondences increasing w.r.t. $\geq^{\mathrm{Vt}}$ ($\geq^{\mathrm{wV}}$) in this sense (weakly) ascending.

Such monotonicity is closely related to the existence of *monotone selections* from R [i.e., increasing mappings $r\colon T \to A$ such that $r(t) \in R(t)$ for every $t \in T$], provided $R(t) \neq \emptyset$ for all t. If a correspondence $R\colon T \to \mathfrak{B}_A$ is increasing w.r.t. $\ggg$, then every selection from R is increasing. If R is increasing w.r.t. $\geq^{\wedge}$, $\geq^{\vee}$, or $\geq^{\mathrm{wV}}$, then an increasing selection exists under a completeness assumption about every value $R(t)$ (Veinott, 1989, Theorem 3.2; Kukushkin, 2009, Proposition 3.1 and Theorem 1); naturally, a stronger assumption is needed in the last case. If R is increasing w.r.t. $\geq^{\mathrm{Vt}}$, then no completeness assumption at all is needed provided A is a sublattice of the Cartesian product of a finite number of chains (Kukushkin, 2009, Theorem 2).

Remark. When the order on A is reversed, $Y \geq^{\wedge} X$ transforms into $X \geq^{\vee} Y$, $Y \geq^{\vee} X$ into $X \geq^{\wedge} Y$, and $Y \geq^{*} X$ into $X \geq^{*} Y$ for $\geq^{*}$ defined by (3c), (3d), or (3e).

We are interested in conditions on the preferences ensuring monotonicity, w.r.t. one or another of the relations (3), of correspondences $\mathcal{R}^X$ defined by (2). The monotonicity of a single correspondence $\mathcal{R}^X$ may happen just "by accident"; however, when a wide enough class of admissible subsets X is taken into account, necessity results become obtainable. Separation between existence and monotonicity is possible because each of the relations (3) holds trivially if either Y or X is empty (non-existence cannot spoil monotonicity).

An ordering $\succ$ on a poset A is (*strictly*) *increasing* if $y \succeq x$ ($y \succ x$) whenever $y > x$; dually, $\succ$ is (*strictly*) *decreasing* if $y \succeq x$ ($y \succ x$) whenever $x > y$. The well-known Szpilrajn theorem asserts the existence of a strictly increasing total order on every poset. When the preferences are increasing or decreasing, most of the following becomes trivial. Naturally, we are interested in less straightforward connections between preferences and order.

4 Single crossing

It is well known that the "single crossing" conditions of various kinds (Milgrom and Shannon, 1994) are important for the monotonicity of best responses. Those conditions are most conveniently presented with the help of a ternary relation on the set of binary relations on a given set: "$\rhd_1$ is closer to $\rhd_0$ than $\rhd_2$ is"; similar observations were made by Quah and Strulovici (2007) and Alexei Savvateev (a seminar presentation, 2007). In the following, the role of $\rhd_0$ is always played by the basic order on A, while $\rhd_1$ and $\rhd_2$ are (strict or non-strict) preference relations.

Let $\succ$ and $\succ'$ be orderings on a poset A. We consider four conditions:

$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succ' x\right]; \tag{4a}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succeq' x\right]; \tag{4b}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succ' x\right]; \tag{4c}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succeq' x\right]. \tag{4d}$$

Each condition defines a binary relation on the set of orderings on A. The first two are preorders. The third is transitive, but generally not reflexive. The last relation need not even be transitive.

Given a poset A and a parametric family $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ of orderings on A, we say that $\mathcal{U}$ satisfies the *lower single crossing* condition if (4a) holds for $\succ^t$ as $\succ$ and $\succ^{t'}$ as $\succ'$ whenever $t, t' \in T$ and $t' > t$. Similarly, $\mathcal{U}$ satisfies the *upper*, *strict*, or *weak single crossing* condition if (4b), (4c), or (4d) holds under the same circumstances. We say that $\mathcal{U}$ satisfies the *single crossing* condition if it satisfies both upper and lower single crossing conditions. Our terminology coincides with that of Milgrom and Shannon (1994) when $\mathcal{U}$ is represented by a utility function (they did not define the lower, upper, and weak single crossing conditions, though).

For more convenience in further referencing, we consider four "reversed" versions of conditions (4):

$$\forall x, y \in A \left[y < x \;\&\; y \succeq x \Rightarrow y \succeq' x\right]; \tag{5a}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succ x \Rightarrow y \succ' x\right]; \tag{5b}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succeq x \Rightarrow y \succ' x\right]; \tag{5c}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succ x \Rightarrow y \succeq' x \right]. \tag{5d}$$

It is easily checked that each condition (5) is equivalent to the corresponding condition (4) after the exchange of the roles of $\succ$ and $\not\succeq'$. Therefore, the (upper, lower, strict, or weak) single crossing conditions could be defined with references to (5) as well.

Remark. It is easy to see that conditions (5a), (5b), (5c), and (5d) are dual to (4b), (4a), (4c), and (4d), respectively.

Proposition 4.1. *Let A and T be posets, and $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on A. Then the following statements are equivalent.*

1. *$\mathcal{U}$ satisfies the lower single crossing condition.*
2. *There holds $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain with $\#X = 2$.*

Proof. Let Statement 1 hold, $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ be a chain. We have to show $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$; let $y \in \mathcal{R}^X(t')$ and $x \in \mathcal{R}^X(t)$. If $y \geq x$, we are home immediately; let $x > y$. If $y \in \mathcal{R}^X(t)$, we are home again. If $y \notin \mathcal{R}^X(t)$, then $x \succ^t y$, hence $x \succ^{t'} y$ by (4a), contradicting the assumption $y \in \mathcal{R}^X(t')$.

Let Statement 1 be violated: there are $t', t \in T$ and $x, y \in A$ such that $t' > t$, $y > x$, $y \succ^t x$, but $x \succeq^{t'} y$. Then we define $X := \{x, y\}$ and immediately obtain $x \in \mathcal{R}^X(t') \setminus \mathcal{R}^X(t)$ while $\mathcal{R}^X(t) = \{y\}$, hence $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$ does not hold, i.e., Statement 3 is invalid. $\square$

Proposition 4.2. *Let A and T be posets, and $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on A. Then the following statements are equivalent.*

1. *$\mathcal{U}$ satisfies the upper single crossing condition.*
2. *There holds $\mathcal{R}^X(t') \geq^\vee \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $\mathcal{R}^X(t') \geq^\vee \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain with $\#X = 2$.*

The proof is dual to that of Proposition 4.1.

Proposition 4.3 (Milgrom and Shannon, 1994). *Let A and T be posets, and $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on A. Then the following statements are equivalent.*

1. *$\mathcal{U}$ satisfies the single crossing condition.*
2. *There holds $\mathcal{R}^X(t') \geq^{\mathrm{Vt}} \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $\mathcal{R}^X(t') \geq^{\mathrm{Vt}} \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain with $\# X = 2$.*

The equivalence immediately follows from Propositions 4.1 and 4.2.

Proposition 4.4. *Let A and T be posets, and $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on A. Then the following statements are equivalent.*

1. *$\mathcal{U}$ satisfies the strict single crossing condition.*
2. *There holds $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain with $\# X = 2$.*

Proof. Let the strict single crossing condition hold, $t', t \in T$, and $t' > t$. We have to show $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$; let $y \in \mathcal{R}^X(t')$ and $x \in \mathcal{R}^X(t)$. If $y \geq x$, we are home; let $x > y$. We have $x \succeq^t y$ since $x \in \mathcal{R}^X(t)$; applying (4c), we obtain $x \succ^{t'} y$, which contradicts $y \in \mathcal{R}^X(t')$.

Let the strict single crossing condition be violated: there are $t', t \in T$ and $x, y \in A$ such that $t' > t$, $y > x$, $y \succeq^t x$, but $x \succeq^{t'} y$. Then we define $X := \{x, y\}$ and immediately obtain $y \in \mathcal{R}^X(t)$ while $x \in \mathcal{R}^X(t')$, hence $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ does not hold. $\square$

Proposition 4.5. *Let A and T be posets, and $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on A. Then the following statements are equivalent.*

1. *$\mathcal{U}$ satisfies the weak single crossing condition.*
2. *There holds $\mathcal{R}^X(t') \geq^{\mathrm{wV}} \mathcal{R}^X(t)$ whenever $t', t \in T$, $t' > t$, and $X \in \mathfrak{B}_A$ is a chain.*

3. *There holds* $\mathcal{R}^X(t') \geq^{\mathrm{wV}} \mathcal{R}^X(t)$ *whenever* $t', t \in T$, $t' > t$, *and* $X \in \mathfrak{B}_A$ *is a chain with* $\#X = 2$.

Proof. Let the weak single crossing condition hold, $t', t \in T$, and $t' > t$. We have to show $\mathcal{R}^X(t') \geq^{\mathrm{wV}} \mathcal{R}^X(t)$; let $y \in \mathcal{R}^X(t')$ and $x \in \mathcal{R}^X(t)$. If $y \geq x$, we are home; let $x > y$. We have to show that either $y \in \mathcal{R}^X(t)$ or $x \in \mathcal{R}^X(t')$. Since $x \in \mathcal{R}^X(t)$, we have $x \succeq^t y$. If $y \succeq^t x$ as well, we have $y \in \mathcal{R}^X(t)$; otherwise, we apply (4d), obtaining $x \succeq^{t'} y$, which implies $x \in \mathcal{R}^X(t')$.

Let the weak single crossing condition be violated: there are $t', t \in T$ and $x, y \in A$ such that $t' > t$, $y > x$, $y \succ^t x$, but $x \not\succeq^{t'} y$. Then we define $X := \{x, y\}$ and immediately obtain $\mathcal{R}^X(t) = \{y\}$ while $\mathcal{R}^X(t') = \{x\}$, hence $\mathcal{R}^X(t') \geq^{\mathrm{wV}} \mathcal{R}^X(t)$ does not hold. $\square$

5 Quasisupermodularity

Naturally, one does not have to be satisfied with maximization on chains, although scalar strategies are met in economics models most often. The *necessity* of single crossing conditions, obviously, holds on any class of admissible subsets that contains all finite chains (but not otherwise, see Quah and Strulovici, 2007). The sufficiency is less robust.

Example 5.1. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$, $T := \{0,1\}$, and a function $u\colon A \times T \to \mathbb{R}$ be defined by the following matrices (the axes are directed upwards and rightwards):

$$\begin{array}{cc} t=0 & t=1 \\ \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 5 & 4 \\ 0 & 6 \end{bmatrix} \end{array}$$

Clearly, u satisfies the strict single crossing condition, even the strictly increasing differences condition. However, $\mathcal{R}^A(0) = \{(0,1)\}$, while $\mathcal{R}^A(1) = \{(1,0)\}$. Therefore, $\mathcal{R}^A$ is not increasing w.r.t. any relation (3); there is no monotone selection either.

Milgrom and Shannon (1994) called a function u on a lattice A *quasisupermodular* if

$$\forall x, y \in A \left[\operatorname{sign}\bigl(u(x \vee y) - u(y)\bigr) \geq \operatorname{sign}\bigl(u(x) - u(x \wedge y)\bigr)\right]. \tag{6}$$

The condition is purely ordinal and can easily be reformulated in terms of a preference ordering (Alexei Savvateev, a seminar presentation, 2007):

$$\forall x, y \in A\, \big[x \succ y \wedge x \Rightarrow y \vee x \succ y\big]; \tag{7a}$$
$$\forall x, y \in A\, \big[x \succeq y \wedge x \Rightarrow y \vee x \succeq y\big]. \tag{7b}$$

We replace conditions (7) with a conjunction of four independent conditions:

$$\forall x, y \in A\, \big[x \succ y \wedge x \Rightarrow [(y \vee x \succ y) \text{ or } (y \vee x \succ x)]\big]; \tag{8a}$$
$$\forall x, y \in A\, \big[y \succeq y \vee x \Rightarrow [(y \wedge x \succeq x) \text{ or } (y \wedge x \succeq y)]\big]; \tag{8b}$$
$$\forall x, y \in A\, \big[x \succeq y \wedge x \Rightarrow [(y \vee x \succeq y) \text{ or } (y \vee x \succeq x)]\big]; \tag{8c}$$
$$\forall x, y \in A\, \big[y \succ y \vee x \Rightarrow [(y \wedge x \succ x) \text{ or } (y \wedge x \succ y)]\big]. \tag{8d}$$

Remark. Each condition (8) holds trivially when x and y are comparable in the basic order.

Proposition 5.2. *An ordering on a lattice satisfies both conditions* (7) *if and only if it satisfies all conditions* (8).

Proof. The necessity is obvious. To prove the sufficiency, we suppose the contrary. Let $x \succ y \wedge x$, but $y \succeq y \vee x$; then $y \vee x \succ x$ by (8a), hence $y \succ y \wedge x$ by transitivity, which contradicts (8b). The proof of the equivalence (7b) $\equiv$ [(8c) & (8d)] is dual. □

Example 5.3. Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; we consider four orderings on A represented by these matrices (the axes are directed upwards and rightwards):

$$\textbf{a}. \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad \textbf{b}. \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \qquad \textbf{c}. \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \qquad \textbf{d}. \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

The ordering represented by the matrix "a" satisfies all conditions (8) except (8a), and similarly with other matrices.

To obtain characterization results for preferences ensuring monotonicity of $\mathcal{R}^X$ for sublattices $X \in \mathfrak{L}_A$, we have to modify the problem itself. The sufficiency parts of Propositions 4.1–4.5 can be interpreted as the monotonicity of the correspondence $M(X, \cdot)$ w.r.t. relations (4) on the set of orderings when X is a chain. Here, each condition (8) is shown to be necessary and sufficient for a kind (actually, two kinds) of such monotonicity on sublattices.

Proposition 5.4. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (8a)*.*
2. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (4a) *holds on X.*
3. *There holds $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (4d) *holds on X.*
4. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (4a) *and* (4b) *hold on A.*
5. *There holds $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (4a) *holds on A.*

Proof. The implications Statement 2 $\Rightarrow$ Statement 4 and Statement 3 $\Rightarrow$ Statement 5 are obvious.

Statement 1 $\Rightarrow$ Statement 2. Let (8a) and (4a) hold. We have to show that $y \wedge x \in M(X, \succ)$ whenever $y \in M(X, \succ')$ and $x \in M(X, \succ)$. Supposing the contrary, we have $x \succ y \wedge x$, hence $y \vee x \succ y$ by (8a) and the optimality of x. Therefore, $y \vee x \succ' y$ by (4a), contradicting the optimality of y.

Statement 1 $\Rightarrow$ Statement 3. Let (8a) and (4d) hold. We have to show $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$; let $y \in M(X, \succ')$ and $x \in M(X, \succ)$. If $y \wedge x \in M(X, \succ)$, we are home; otherwise, $x \succ y \wedge x$, hence $y \vee x \succ y$ by (8a) and the optimality of x. Therefore, $y \vee x \succeq' y$ by (4d), hence $y \vee x \in M(X, \succ')$.

Statement 4 $\Rightarrow$ Statement 1. Let (8a) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$, but $y \succeq y \vee x$ and $x \succeq y \vee x$. Without restricting generality, $x \succeq y$. We define $X := L(x, y)$, so $y \wedge x \notin M(X, \succ) \ni x$, and $Y := \{z \in A \mid z \geq y\}$; our assumptions imply $x \notin Y$. Then we define an ordering $\succ'$ on A: it coincides with $\succ$ on $A \setminus Y$ and on Y, whereas $z' \succ' z$ whenever $z \notin Y \ni z'$. Both (4a) and (4b) are obvious: Whenever $z' > z$, $z' \succeq z$, and $z \in Y$, we have $z' \in Y$ as well. Meanwhile, $y \in M(X, \succ')$, hence $M(X, \succ') \geq^{\wedge} M(X, \succ)$ does not hold, i.e., Statement 4 is invalid.

Statement 5 $\Rightarrow$ Statement 1. Let (8a) be violated. We pick x and y as in the previous paragraph, define $X := L(x,y)$, so $y \wedge x \notin M(X,\succ) \ni x$ again, and then define $\succ'$ in the same manner, but with $Y := \{z \in A \mid z \succ y \;\&\; z > y\} \cup \{y\}$. Clearly, $M(X,\succ') = \{y\}$, hence $M(X,\succ') \geq^{\mathrm{w}\vee} M(X,\succ)$ does not hold. Since $\succ'$ and $\succ$ satisfy (4a), Statement 5 is invalid. $\square$

Proposition 5.5. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (8b)*.*
2. *There holds $M(X,\succ) \geq^{\wedge} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (5a) *holds on X.*
3. *There holds $M(X,\succ) \ggg M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (5c) *holds on X.*
4. *There holds $M(X,\succ) \geq^{\wedge} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (5c) *holds on A.*

Proof. Let (8b) hold, $x \in M(X,\succ')$ and $y \in M(X,\succ)$. Let us show that $y \wedge x \in M(X,\succ')$ if (5a) holds. Supposing the contrary, we have $x \succ' y \wedge x$, hence $x \succ y \wedge x$ by (5a). Since $y \succeq x$, (8b) implies that $y \vee x \succ y$, which contradicts the optimality of y. Let us show that $y \geq x$ if (5c) holds. Supposing the contrary, we have $x > y \wedge x$; since $y \succeq y \vee x$ and $y \succeq x$, we have $y \wedge x \succeq x$ by (8b). Therefore, $y \wedge x \succ' x$ by (5c), which contradicts the optimality of x.

Let (8b) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$ and $y \succ y \wedge x$, but $y \succeq y \vee x$. Without restricting generality, $y \succeq x$; we define $X := L(x,y)$, so $y \in M(X,\succ)$. Then we define an ordering $\succ'$ on A in the same manner as in the proof of Proposition 5.4, but with $Y := \{z \in A \mid z \leq x\}$. On every equivalence class E of $\succeq'$, we pick a strictly increasing total order $\gg_E$, existing by the Szpilrajn theorem. Then we define $\succ''$ as a lexicography: $z' \succ'' z$ if $z' \succ' z$, or if they belong to the same equivalence class E and $z \gg_E z'$. Clearly, $\succ''$ is a total order. Let $z' > z$ and $z' \succeq'' z$, hence $z' \succ'' z$. By the definition of $\succ''$, we have $z' \succ' z$. If $z' \in Y$, then $z \in Y$ as well, hence $z' \succ z$; if $z' \notin Y$, then $z \notin Y$ and $z' \succ z$ again. Therefore, (5c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X,\succ'') = \{x\}$, hence $M(X,\succ) \geq^{\wedge} M(X,\succ'')$ does not hold. $\square$

Proposition 5.6. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (8c).
2. *There holds $M(X,\succ') \geq^{\vee} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (4b) *holds on X.*
3. *There holds $M(X,\succ') \ggg M(X,\succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (4c) *holds on X.*
4. *There holds $M(X,\succ') \geq^{\vee} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (4c) *holds on A.*

The proof is dual to that of Proposition 5.5.

Proposition 5.7. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (8d).
2. *There holds $M(X,\succ) \geq^{\vee} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (5b) *holds on X.*
3. *There holds $M(X,\succ) \geq^{\mathrm{w}\vee} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (5d) *holds on X.*
4. *There holds $M(X,\succ) \geq^{\vee} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (5a) *and* (5b) *hold on A.*
5. *There holds $M(X,\succ) \geq^{\mathrm{w}\vee} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (5b) *holds on A.*

The proof is dual to that of Proposition 5.4.

Proposition 5.8. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *Both conditions* (8a) *and* (8c) *hold.*

2. *There holds* $M(X, \succ') \geq^{\mathrm{Vt}} M(X, \succ)$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (4a) *and* (4b) *hold on* X.

3. *There holds* $M(X, \succ') \geq^{\mathrm{Vt}} M(X, \succ)$ *whenever* $X \in \mathfrak{L}_A$, $\#X \leq 4$, *and* $\succ'$ *is an ordering on* A *such that* (4a) *and* (4b) *hold on* A.

The equivalence immediately follows from Propositions 5.4 and 5.6.

Proposition 5.9. *Let* A *be a lattice and* $\succ$ *be an ordering on* A. *Then the following statements are equivalent.*

1. *Both conditions* (8b) *and* (8d) *hold.*

2. *There holds* $M(X, \succ) \geq^{\mathrm{Vt}} M(X, \succ')$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (5a) *and* (5b) *hold on* X.

3. *There holds* $M(X, \succ) \geq^{\mathrm{Vt}} M(X, \succ')$ *whenever* $X \in \mathfrak{L}_A$, $\#X \leq 4$, *and* $\succ'$ *is an ordering on* A *such that* (5a) *and* (5b) *hold on* A.

The equivalence immediately follows from Propositions 5.5 and 5.7.

Remark. Agliardi (2000) called a function u on a lattice A *pseudosupermodular* if the ordering represented by u satisfies (8a) and (8c). In the light of Propositions 5.8 and 5.9, it might be appropriate to call an ordering *pseudosupermodular upwards* [*downwards*] if it satisfies (8a) and (8c) [(8b) and (8d)].

Sufficient conditions for monotonicity of the best responses (in one sense or another) are obtained as easy corollaries. Given a poset T, a *monotonic pseudopartition* of T consists of two subsets $T^{\uparrow}, T^{\downarrow} \subseteq S$ such that $\forall t', t \in T \left[t' > t \Rightarrow [t \in T^{\uparrow} \text{ or } t' \in T^{\downarrow}]\right]$. Clearly, any two points outside $T^{\uparrow} \cup T^{\downarrow}$ must be incomparable.

Proposition 5.10. *Let* $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ *be a parametric family of orderings on a lattice* A; *let* $\mathcal{U}$ *satisfy the strict single crossing condition. Let there be a monotonic pseudopartition* $\langle T^{\uparrow}, T^{\downarrow} \rangle$ *of* T *such that* $\succ^t$ *satisfies* (8c) *for* $t \in T^{\uparrow}$ *and* (8b) *for* $t \in T^{\downarrow}$. *Then every* $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ *is increasing w.r.t.* $\ggg$.

Proof. Let $t' > t$. If $t \in T^{\uparrow}$, then (8c) holds with $\succ^t$ as $\succ$ while (4c) holds with $\succ^{t'}$ as $\succ'$ and $\succ^t$ as $\succ$. Therefore, $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ for every $X \in \mathfrak{L}_A$ by Statement 3 of Proposition 5.6.

If $t' \in T^{\downarrow}$, then (8b) holds with $\succ^{t'}$ as $\succ$ while (5c) holds with with $\succ^{t}$ as $\succ^{\prime}$ and $\succ^{t'}$ as $\succ$. Therefore, $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ for every $X \in \mathfrak{L}_A$ by Statement 3 of Proposition 5.5. $\square$

Proposition 5.11. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the lower single crossing condition. Let there be a monotonic pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (8a) *for $t \in T^{\uparrow}$ and* (8b) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ ($X \in \mathfrak{L}_A$) is increasing w.r.t. $\geq^{\wedge}$.*

Proposition 5.12. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the upper single crossing condition. Let there be a monotonic pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (8c) *for $t \in T^{\uparrow}$ and* (8d) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ ($X \in \mathfrak{L}_A$) is increasing w.r.t. $\geq^{\vee}$.*

Proposition 5.13. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the single crossing condition. Let there be a monotonic pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (8a) *and* (8c) *for $t \in T^{\uparrow}$, and* (8b) *and* (8d) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ ($X \in \mathfrak{L}_A$) is increasing w.r.t. $\geq^{\mathrm{Vt}}$.*

Proposition 5.14. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the weak single crossing condition. Let there be a monotonic pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (8a) *for $t \in T^{\uparrow}$ and* (8d) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ ($X \in \mathfrak{L}_A$) is increasing w.r.t. $\geq^{\mathrm{wV}}$.*

Each proof is quite similar to that of Proposition 5.10.

Remark. Lemma 3.1 of Kukushkin et al. (2005) immediately follows from Proposition 5.13 ($T^{\uparrow} = T$).

6 Main characterization results

Theorem 1. *An ordering $\succ$ on a lattice A is quasisupermodular if and only if both following requirements are satisfied.*

1. *There holds $M(X, \succ^{\prime}) \geq^{\mathrm{Vt}} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ^{\prime}$ is an ordering on X, and* (4a) *and* (4b) *hold on X.*

2. *There holds* $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\mathrel{\succ\!\!\!\lrcorner})$ *whenever* $X \in \mathfrak{L}_A$, $\mathrel{\succ\!\!\!\lrcorner}$ *is an ordering on* X, *and* (5a) *and* (5b) *hold on* X.

Moreover, both requirements can be restricted to $X \in \mathfrak{L}_A$ *with* $\#X \leq 4$.

Proof. The equivalence immediately follows from Propositions 5.8 and 5.9. $\square$

Theorem 2. *An ordering* $\succ$ *on a lattice* A *is quasisupermodular if and only if all the following requirements are satisfied.*

1. *There holds* $M(X,\mathrel{\succ\!\!\!\lrcorner}) \ggg M(X,\succ)$ *whenever* $X \in \mathfrak{L}_A$, $\mathrel{\succ\!\!\!\lrcorner}$ *is an ordering on* X, *and* (4c) *holds on* X.

2. *There holds* $M(X,\succ) \ggg M(X,\mathrel{\succ\!\!\!\lrcorner})$ *whenever* $X \in \mathfrak{L}_A$, $\mathrel{\succ\!\!\!\lrcorner}$ *is an ordering on* X, *and* (5c) *holds on* X.

3. *There holds* $M(X,\mathrel{\succ\!\!\!\lrcorner}) \geq^{\mathrm{wV}} M(X,\succ)$ *whenever* $X \in \mathfrak{L}_A$, $\mathrel{\succ\!\!\!\lrcorner}$ *is an ordering on* X, *and* (4d) *holds on* X.

4. *There holds* $M(X,\succ) \geq^{\mathrm{wV}} M(X,\mathrel{\succ\!\!\!\lrcorner})$ *whenever* $X \in \mathfrak{L}_A$, $\mathrel{\succ\!\!\!\lrcorner}$ *is an ordering on* X, *and* (5d) *holds on* X.

Moreover, all requirements can be restricted to $X \in \mathfrak{L}_A$ *with* $\#X \leq 4$.

Proof. The equivalence immediately follows from Propositions 5.4–5.7. $\square$

To present our last characterization result, somewhat cumbersome terminology is needed. Let T and T' be two posets such that $T = T' \setminus \{t^0\}$ ($t^0 \in T'$); let $\mathcal{U}' = \langle \succ^t \rangle_{t\in T'}$ be a parametric family of orderings on a lattice A, and $\mathcal{U} := \langle \succ^t \rangle_{t\in T}$. We say that $\mathcal{U}'$ is an *extension of* $\mathcal{U}$ *with the single crossing property* if (4a) and (4b) hold for $\succ^{t^0}$ as $\succ$ and $\succ^{t}$ as $\mathrel{\succ\!\!\!\lrcorner}$ whenever $T \ni t > t^0$, whereas (4a) and (4b) hold for $\succ^{t}$ as $\succ$ and $\succ^{t^0}$ as $\mathrel{\succ\!\!\!\lrcorner}$ whenever $t^0 > t \in T$. Similarly, $\mathcal{U}'$ is an *extension of* $\mathcal{U}$ *with the strict* [*weak*] *single crossing property* if (4c) [(4d)] holds for $\succ^{t^0}$ as $\succ$ and $\succ^{t}$ as $\mathrel{\succ\!\!\!\lrcorner}$ whenever $T \ni t > t^0$, whereas (4c) [(4d)] holds for $\succ^{t}$ as $\succ$ and $\succ^{t^0}$ as $\mathrel{\succ\!\!\!\lrcorner}$ whenever $t^0 > t \in T$.

We say that an ordering $\succ$ on a lattice A *preserves ascendance* if, whenever $\mathcal{U}$ is a parametric family of orderings on A such that $\mathcal{R}^X$ defined by (2) is increasing w.r.t. $\geq^{\mathrm{Vt}}$ for every $X \in \mathfrak{L}_A$, and $\mathcal{U}'$ is an extension of $\mathcal{U}$ with the single crossing property such that $\succ^{t^0}$ coincides with $\succ$, every correspondence $\bar{\mathcal{R}}^X$ $(X \in \mathfrak{L}_A)$ defined by (2) for $\mathcal{U}'$ is increasing w.r.t. $\geq^{\mathrm{Vt}}$. Similarly, an ordering $\succ$ on a lattice A *preserves strong/weak ascendance* if, whenever $\mathcal{U}$ is a parametric family of orderings on A such that $\mathcal{R}^X$ defined by (2) is increasing w.r.t. $\ggg$, respectively $\geq^{\mathrm{wV}}$, for every $X \in \mathfrak{L}_A$, and $\mathcal{U}'$ is an extension of $\mathcal{U}$ with the strict/weak single crossing property such that $\succ^{t^0}$ coincides with $\succ$, every correspondence $\bar{\mathcal{R}}^X$ $(X \in \mathfrak{L}_A)$ defined by (2) for $\mathcal{U}'$ is increasing w.r.t. $\ggg$, respectively $\geq^{\mathrm{wV}}$.

Theorem 3. *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *$\succ$ is quasisupermodular.*
2. *$\succ$ preserves ascendance.*
3. *$\succ$ preserves both strong ascendance and weak ascendance.*

Proof. The equivalence immediately follows from Theorems 1 and 2. □

7 Concluding remarks

7.1. The description of preferences with an ordering may seem very general, but it may also seem not general enough. Leaving aside the abstract question of how much rationality in an agent's preferences it is right to assume, there is a mundane reason to go beyond orderings. Suppose the utility function $u(x,t)$ is bounded above in x for every t, but need not attain a maximum; then ε-optimization suggests itself strongly, and this means considering a preference relation

$$y \succ^t x \rightleftharpoons u(y,t) > u(x,t) + \varepsilon$$

(with $\varepsilon > 0$). $\mathcal{R}(t)$ consists of all ε-maxima of $u(\cdot,t)$. The relation $\succ^t$ is a *strongly acyclic semiorder*, but need not be an ordering. If u satisfies Topkis's (1978) *increasing differences* condition, then $\{\succ^t\}_{t\in T}$ satisfies the single crossing conditions; if u is supermodular in the first argument, then $\succ^t$ satisfies both (7). Nevertheless, none of the results of this paper is applicable even under so strong assumptions; actually, $\mathcal{R}$ need not be ascending. The existence of a monotone selection can be proven when both X and T are chains; the existence of an ε-Nash equilibrium, when every strategy set is a chain (Kukushkin, 2009, Theorems 3 and 4). However, there is no similar result of any kind for non-scalar sets X.

7.2. Five "order" relations (3) form a lattice (with the logical implication as order); five single crossing conditions [four (4) and the conjunction of (4a) and (4b) – single crossing proper] form an isomorphic lattice. Neither is a sublattice of the lattice of all binary relations on, respectively, $\mathfrak{B}_A$ or the set of orderings on A. Both will become sublattices if we add the disjunction (meet) of $\geq^{\wedge}$ and $\geq^{\vee}$, respectively, (4a) and (4b); even an analog of Propositions 4.1–4.5 will hold. However, this additional "order" seems to lead to no interesting result about, say, monotone selections.

7.3. Assuming $\succ$ represented with a mapping u from A to a chain, conditions (8) can be written in a "more algebraic" style, cf. Agliardi (2000) (or Veinott, 1989, for that matter).

$$\forall x, y \in A\left[u(y) \vee u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \wedge u(x)\right]; \tag{9a}$$

$$\forall x, y \in A\left[u(y) \vee u(x) \geq u(y \vee x) \Rightarrow u(y \wedge x) \geq u(y) \wedge u(x)\right]; \tag{9b}$$

$$\forall x, y \in A\left[u(y) \vee u(x) \geq u(y \wedge x) \Rightarrow u(y \vee x) \geq u(y) \wedge u(x)\right]; \tag{9c}$$

$$\forall x, y \in A\left[u(y) \vee u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \wedge u(x)\right]. \tag{9d}$$

Clearly, each condition (9) is equivalent to the corresponding condition (8).

7.4. It would seem natural to develop analogs of Milgrom and Shannon's Corollary to Theorem 4 for other relations (3). Unfortunately, nothing interesting emerges: the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{L}$ w.r.t. $\ggg$ holds for any $\succ$; the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{L}$ w.r.t. $\geq^{\vee}$ ($\geq^{\wedge}$) holds if and only if $\succ$ is in(de)creasing; the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{L}$ w.r.t. $\geq^{\mathrm{wV}}$ is only possible if the agent is indifferent between all outcomes. In each case, there is no connection to conditions for the monotonicity of optima in a parameter influencing the preference relation.

7.5. A preference ordering $\succ$ on a lattice A is called *strictly quasisupermodular* if $y \vee x \succ y$ whenever $x \succeq y \wedge x$ and x and y are incomparable in the basic order. Whereas strict single crossing is crucial for monotonicity w.r.t. $\ggg$, this property seems to play no role here (nor in "type B" problems, see the previous item).

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