A comparative analysis of correlation skew modeling techniques for CDO index tranches

Claudio, Ferrarese

King's College London

8 September 2006

A Comparative Analysis of Correlation Skew Modeling Techniques for CDO Index Tranches

by

Claudio Ferrarese

Department of Mathematics
King's College London
The Strand, London WC2R 2LS
United Kingdom

Email: claudio.ferrarese@gmail.com
Tel: +44 799 0620459
8 September 2006

Abstract

In this work we present an analysis of CDO pricing models with a focus on "correlation skew models". These models are extensions of the classic single factor Gaussian copula and may generate a skew. We consider examples with fat tailed distributions, stochastic and local correlation which generally provide a closer fit to market quotes. We present an additional variation of the stochastic correlation framework using normal inverse Gaussian distributions. The numerical analysis is carried out using a large homogeneous portfolio approximation.

Keywords and phrases: default risks, CDOs, index tranches, factor model, copula, correlation skew, stochastic correlation.

Contents

Chapter 1

Introduction

The credit derivatives market has grown quickly during the past ten years, reaching more than US$ 17 trillion notional value of contracts outstanding in 2006 (with a 105% growth during the last year) according to International Swaps and Derivatives Association (ISDA)[1]. The rapid development has brought market liquidity and improved the standardisation within the market. This market has evolved with collateralised debt obligation (CDOs) and recently, the synthetic CDOs market has seen the development of tranched index trades. CDOs have been typically priced under reduced form models, through the use of copula functions. The model generally regarded as a market standard, or the "Black-Scholes formula" for CDOs, is considered to be the Gaussian copula (as in the famous paper by Li [36]). Along with market growth, we have seen the evolution of pricing models and techniques used by practitioners and academics, to price structured credit derivatives. A general introduction on CDOs and pricing models is given in §2. In §3.1 we describe and attempt to explain the existence of the so called "correlation skew". Another standard market practice considered here is the base correlation. The core part of this work concentrates on the analysis of the extensions of the base model (i.e. single factor Gaussian copula under the large homogeneous portfolio approximation). These extensions have been proposed during the last few years and attempt to solve the correlation skew problem and thus produce a good fit for the market quotes. Amongst these model extensions, in §4 we use fat-tailed distributions (e.g. normal inverse Gaussian (NIG) and α-stable distributions), stochastic correlation and local correlation models to price the DJ iTraxx index. In addition to these approaches, we propose the "stochastic correlation NIG", which represents a new variation of the stochastic correlation model. In particular we use the normal inverse Gaus-

---

[1]Source: *Risk* May 2006, pg. 32.

sian distributions instead of standard Gaussian distributions to model the common factor and the idiosyncratic factor. We conclude our analysis in §5 with the numerical results and a comparison of the different models. All the models produce a better fit than the base model, although different levels of goodness have been registered. The best performance in term of fit has been assigned to our proposed model, i.e. the stochastic correlation NIG.

Chapter 2

CDOs pricing

In this chapter, after a quick introduction to CDOs and the standardised tranche indices actively traded in the market, we present the most famous approaches, such as copula functions and factor models, used to price structured credit derivatives.

2.1 CDO tranches and indices: iTraxx, CDX

CDOs allow, through a securitisation technique, the repackaging of a portfolio credit risk into tranches with varying seniority. During the life of the transaction the resulting losses affect first the so called "equity" piece and then, after the equity tranche as been exhausted, the mezzanine tranches. Further losses, due to credit events on a large number of reference entities, are supported by senior and super senior tranches. The difference between a cash and synthetic CDO relies on the underlying portfolios. While in the former we securitise a portfolio of bonds, asset-backed securities or loans, in the latter kind of deals the exposition is obtained synthetically, i.e. through credit default swaps (CDS) or other credit derivatives. The possibility of spread arbitrage[1], financial institutions' need to transfer credit risk to free up regulatory capital (according to Basel II capital requirements), and the opportunity to invest in portfolio risk tranches have boosted the market. In particular it has been very appealing for a wide range of investors who, having different risk-return profiles[2], can invest in a specific part of CDO capital

---

[1]An common arbitrage opportunity exists when the total amount of credit protection premiums received on a portfolio of, e.g., CDS is higher than the premiums required by the CDO tranches investors.

[2]From e.g. risk taking hedge funds, typically more interested in equity/mezzanine tranches to e.g pension funds willing to take exposure on the senior tranches.

structure. Clearly, CDO investors are exposed to a so called "default correlation risk" or more precisely to the co-dependence of the underlying reference entities' defaults[3]. This exposure varies amongst the different parts of the capital structure: e.g. while an equity investor would benefit from a higher correlation, the loss probability on the first loss piece being lower if correlation increases, a senior investor will be penalised if the correlation grows, as the probability of having extreme co-movements will be higher.

CDO tranches are defined using the definition of attachment K_A and detachment points K_D as, respectively, the lower and upper bound of a tranche. They are generally expressed as a percentage of the portfolio and determine the tranche size. The credit enhancement of each tranche is then given by the attachment point. For convenience we define the following variables:

- n- the number of reference entities included in the collateral portfolio (i.e. number of CDS in a synthetic CDO);

- A_i- the nominal amount for the $i-th$ reference entity;

- δ_i- the recovery rate for the $i-th$ reference entity;

- T- the maturity;

- N- the number of periods or payment dates (typically the number of quarters);

- $B(0,t)$- the price of a risk free discount bond maturing at t;

- τ_i- the default time for the $i-th$ reference entity;

- s_{BE}- the break-even premium in basis point per period (i.e. spread).

The aggregate loss at time t is given by:

$$L(t) = \sum_{i=1}^{n} A_i(1-\delta_i)\mathbf{1}_{\{\tau_i \leq t\}}, \tag{2.1}$$

assuming a fixed recovery rate δ, notional equal to 1 and the same exposure to each reference entity in the portfolio $A_i = A = \frac{1}{N}$, the loss can be written

---

[3]The concept of dependence is more extensive than the concept of correlation generally used by practitioners. As reported by e.g. Mashal and Zeevi [39], the dependence can be measured by correlation, and the two concepts can converge when we refer to elliptical distributions (i.e. using Gaussian copula to model the joint defaults distribution), but in general correlation is not enough to measure co-dependence, e.g. Student t-distribution requires the specification of the degrees of freedom.

as follows:

$$L(t) \;=\; \frac{1}{N}(1-\delta)\sum_{i=1}^{n}\mathbf{1}_{\{\tau_i \le t\}}, \qquad (2.2)$$

For every tranche, given the total portfolio loss in 2.2, the cumulative tranche loss is given by:

$$L_{[K_\mathrm{A},K_\mathrm{D}]}(t) = \max\{\min[L(t),K_\mathrm{D}] - K_\mathrm{A}, 0\}. \qquad (2.3)$$

The so called "protection leg", which covers the losses affecting the specific tranche and is paid out to the protection buyer, given the following payment dates

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

can be calculated taking the expectation with respect to the risk neutral probability measure:

$$\mathrm{ProtLeg}_{[K_\mathrm{A},K_\mathrm{D}]} = \mathbb{E}\left[\sum_{j=1}^{N}\left(L_{[K_\mathrm{A},K_\mathrm{D}]}(t_j) - L_{[K_\mathrm{A},K_\mathrm{D}]}(t_{j-1})\right)B(t_0,t_j)\right], \quad (2.4)$$

Similarly, assuming a continuous time payment, the protection leg can be written as follows:

$$\mathrm{ProtLeg}_{[K_\mathrm{A},K_\mathrm{D}]} = \mathbb{E}\left[\int_{t_0}^{T}B(t_0,s)dL_{[K_\mathrm{A},K_\mathrm{D}]}(s)\right]. \qquad (2.5)$$

On the other hand, the so called "premium leg" is generally paid quarterly in arrears[4] to the protection seller and can be expressed as follows:

$$\mathrm{PremLeg}_{[K_\mathrm{A},K_\mathrm{D}]} = \mathbb{E}\left[\sum_{j=1}^{N}s_{\mathrm{BE}}\times\Delta t_j\left(\min\left\{\max[K_\mathrm{D}-L(t_j),0],K_\mathrm{D}-K_\mathrm{A}\right\}\right)B(t_0,t_j)\right].$$
$$(2.6)$$

The fair s_{BE} can be easily calculated by:

$$s_{\mathrm{BE}} = \frac{\mathbb{E}\left[\int_{t_0}^{T}B(t_0,s)dL_{[K_\mathrm{A},K_\mathrm{D}]}(s)\right]}{\mathbb{E}\left[\sum_{j=1}^{N}\Delta t_j\left(\min\left\{\max[K_\mathrm{D}-L(t_j),0],K_\mathrm{D}-K_\mathrm{A}\right\}\right)B(t_0,t_j)\right]}.$$

---

[4]This is generally not true for junior tranches for which, together with a quarterly premium, an amount upfront can be paid.

In order to price a CDO it is then central to calculate the cumulative portfolio loss distribution necessary to evaluate the protection and premium legs. In the sequel, the loss distribution will be analysed comparing some of the most popular approaches proposed by academics and practitioners, having as a primary goal the fitting of the prices quoted by the market.

Traded indices

As already mentioned, one of the most effective innovations in this market has been the growth of standardised and now liquid index tranches such as DJ iTraxx and DJ CDX. These indices allow investors to long/short single tranches, bet on their view on correlation, hedge correlation risk in a book of credit risks and infer information from the market, computing the correlation implied by the quotes.

Tranche	Rating†	Premium (bp)
Super senior (22-100%)	Unrated	(Unquoted)
Super senior (junior) (12-22%)	AAA	4
Senior (9-12%)	AAA	9
Mezzanine (senior) (6-9%)	AAA	18
Mezzanine (junior) (3-6%)	BBB	63
Equity (0-3%)	N.A.	24%‡

Figure 2.1: DJ iTraxx standard capital structure and prices in basis pints (bp) at 13 April 2006.
Source: CDO-CDS Update: Nomura Fixed Income Research, 4/24/2006.
†These are not official ratings, but only an assessment of the creditworthiness of iTraxx index tranches provided by Fitch, for further information see http://www.fitchcdx.com.
‡ For the equity tranche an upfront premium plus 500 bp running spread is usually quoted.

These indices are build according to the following rules[5]:

- they are composed of 125 European (US for the DJ CDX IG) equally weighted investment grade firms;

- transparent trading mechanics and standard maturities;

---

[5]A complete list of the rules and criteria can be found at http://www.indexco.com.

- standardised credit events definitions and ISDA based documentation;

- cross sector index including entities from auto, consumer, energy, industrial, TMT and financial sectors;

- the most liquid entities with the highest CDS trading volume, as measured over the previous 6 months, are chosen for the index composition;

- the index composition is periodically revisited, every new series shares the majority of its names with the previous series.

2.2   Pricing Models overview

In credit models two main approaches have been used:

- *Structural models*, based on option theory, were introduced by Merton [41] and further developed by e.g. Black and Cox [11], Leland and Toft [35]. Under this approach a default occurs when the value of the firm's assets drops below the face value of the debt (i.e. strike price), thus considering the stock being a call option on the company's value.

- *Reduced form* approach attempts to model the time of the default itself through the use of an exogenously given intensity process (i.e. jump process), see e.g. Jarrow and Turnbull [32], Lando[34], Duffie [19], Hughston and Turnbull [28]. Within the reduced form models, the *incomplete information* approach , see Brody *et al.* [13] focuses on the modelling of the information available in the market.

In this work we use the intensity based approach. We briefly introduce the main assumptions on which the following is based. Given a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, we assume the existence of a pricing kernel and the absence of arbitrage to guarantee the existence of a unique risk-neutral probability measure $\mathbb{P}$. Under this framework, non-dividend paying assets (default-free) are martingales if discounted at the risk-free rate. The existence of such a risk neutral probability measure allows to price our bonds or credit derivatives correctly, without the stronger market completeness assumption which would be required in order to hedge the securities. In addition, in order to simplify the exposition and focus on the credit risk modelling, we assume a flat interest rate structure, fixed recovery rates and independence between default probabilities, interest rate curve and recovery rates.

In general, pricing a CDO requires us to model both the risk neutral default probabilities for each name in the portfolio and the joint default distribution of defaults. The risk neutral default probabilities can be calculated following the popular practice of bootstrapping from CDS premiums. Here we summarise the main steps. For a complete exposition we refer to, e.g., Bluhm *et al.*[12], Li [36, 37], Schönbucher[49] or Lando [34]. Given the standard probability space defined above, we consider the $\{\mathcal{F}_t\}$-stopping time τ_i to model the default of the $i-th$ obligor in a portfolio, the default probability distribution is given by $F_i(t) = \mathbb{P}\{\tau_i < t\}$ and the probability density distribution is $f_i(t)$. We define the "hazard rate" or "intensity" as follows:

$$\lambda_i(t) = \frac{f_i(t)}{1 - F_i(t)}. \tag{2.7}$$

From 2.7 the following O.D.E. can be obtained:

$$\lambda_i(t) = -\frac{d\ln(1 - F_i(t))}{dt}, \tag{2.8}$$

and, solving the O.D.E., an expression for the default probability distribution follows:

$$F_i(t) = 1 - e^{-\int_0^t \lambda_i(s)ds}. \tag{2.9}$$

We make a further step defining τ_i as the first jump of an inhomogeneous Poisson[6] process $N(t)$ with parameter $\Lambda(T) - \Lambda(t) = \int_t^T \lambda_i(s)ds$, then:

$$\mathbb{P}\{N_T - N_t = k\} = \frac{1}{k!} \left(\int_t^T \lambda_i(s)ds \right)^k e^{-\int_t^T \lambda_i(s)ds}. \tag{2.10}$$

In fact, using the fact that $\mathbb{P}\{N_t = 0\} = e^{-\int_t^T \lambda_i(s)ds}$, the survival probability for the $i-th$ obligor can be written, similarly to 2.9, as follows:

$$1 - F_i(t) = e^{-\int_t^T \lambda_i(s)ds}. \tag{2.11}$$

Once the hazard rate[7] $\lambda_i(t)$ has been defined, it is straightforward to bootstrap the default probabilities, e.g. a standard practice is to use a piecewise constant hazard rate curve and fit it with the CDS quotes.

In the sequel we assume that risk neutral default probabilities have been calculated for all names in the underlying reference portfolio.

---

[6]In addition an inhomogeneous Poisson process with $\lambda_i(t) > 0$ is characterised by independent increments and $N_0 = 0$.

[7]For the general case of a Cox process with stochastic intensity see [34] or [49] pg. 123.

2.2.1 Copulas

Copulas are a particularly useful class of function, since they provide a flexible way to study multivariate distributions. Li [36] was certainly amongst the first to introduce these functions in credit derivatives modelling because of the copulas' characteristic of linking the univariate marginal distributions to a full multivariate distribution. We present here the definition and an important result regarding copulas. For a full presentation of the argument please refer to e.g. Cherubini *et al.* [18].

Definition 1 (Copula) *A n-dimensional copula is a joint cdf* $C : [0,1]^n \to [0,1]$ *of a vector* $\mathbf{u}$ *of uniform* $U(0,1)$ *random variables:*

$$C(u_1, u_2, \ldots, u_n) = \mathbb{P}(U_1 < u_1, U_2 < u_2, \ldots, U_n < u_n), \qquad (2.12)$$

where $u_i \in [0,1], i = 1, 2, \ldots, n$.

Theorem 1 (Sklar theorem) *Given* $H(x_1, x_2, \ldots, x_n)$ *the joint distribution function with marginals* $F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)$, *then there exists a copula* $C(u_1, u_2, \ldots, u_n)$ *such that:*

$$H(x_1, x_2, \ldots, x_n) = C(f_{X_1}(x_1), f_{X_2}(x_2), \ldots, f_{X_n}(x_n)). \qquad (2.13)$$

Furthermore the copula C *is given by:*

$$C(u_1, u_2, \ldots, u_n) = H(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)). \qquad (2.14)$$

In particular, if the $F_i, i = 1, 2, \ldots, n$, *are continuous, then* C *is unique.*

One of the most used copulas in financial applications is certainly the standard Gaussian copula. We give here the definition and describe how it can be used to model joint default distributions.

Definition 2 (Standard Gaussian copula) *The standard Gaussian copula function is given by :*

$$C_\Sigma^G(u_1, u_2, \ldots, u_n) = \Phi_n^\Sigma(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n)), \qquad (2.15)$$

where Φ_n^Σ *is a n variate Gaussian joint distribution function,* Σ *is a correlation matrix and* Φ *in a standard Gaussian distribution function.*

Using the arguments in §2.2, for the $i - th, i = 1, 2, \ldots, n$ name in the portfolio, a default threshold[8] can be found:

$$\mathbb{P}\{1 - F_i(t) = e^{-\int_t^T \lambda_i(s)ds}\} < U_i. \tag{2.16}$$

U_i being a uniform $U(0, 1)$ random variable[9].

It is then possible to build a joint default time distribution from the marginal distributions $F_i(t) = \mathbb{P}\{\tau_i < t\}$ as follows:

$$\mathbb{P}\{\tau_1 \leq t, \tau_2 \leq t, \ldots, \tau_n \leq t\} = C_\Sigma^G(F_1(t), F_2(t), \ldots, F_n(t))$$

$$= \Phi_n^\Sigma \left(\Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), \ldots, \Phi^{-1}(F_n(t)) \right). \tag{2.18}$$

Using the results in 2.16 and 2.18, a Monte Carlo simulation algorithm can be used to find the joint default time distribution:

- sample a vector $\mathbf{z}$ of correlated Gaussian random variables with correlation matrix Σ[10];

- generate a vector of uniform random variable $\mathbf{u} = \Phi(\mathbf{z})$;

- for every $i = 1, 2, \ldots, n$ and time t, we have a default if $\tau_i = F_i^{-1}(u_i) < t$;

- evaluate the joint default distribution;

- repeat these steps for the required number of simulations.

Credit derivatives can now be priced since the cumulative portfolio loss distribution follows from the joint default distribution calculated according to the previous algorithm.

---

[8]Note that this approach is based on the simple idea of a default event modelled by a company process falling below a certain barrier as stated in the original Merton structural model [41]

[9]Using that $F(t)$ is a distribution function then

$$\mathbb{P}\{F(t) < t\} = \mathbb{P}\{F^{-1}(F(t)) < F^{-1}(t)\} = \mathbb{P}\{t < F^{-1}(t)\} = F(F^{-1}(t)) = t \tag{2.17}$$

it follows that $F(t)$ is a uniform $U(0, 1)$ random variable and so is $1 - F(t)$.

[10]Given a vector $\mathbf{v}$ of independent Gaussian random variables and using an appropriate decomposition, e.g. Cholesky, of $\Sigma = CC^T$, a vector of correlated Gaussian random variables is given by $\mathbf{z} = \mathbf{v}C$.

A number of alternative copulas such as, e.g. Student t, Archimedean, Marshall-Olkin copulas have been widely studied and applied to credit derivatives pricing with various results, see e.g. Galiani [24], Schönbucher[47, 49], Chaplin [17] for a complete description.

We conclude this section recalling that the algorithm described above can be particularly slow when applied to basket credit derivatives[11] pricing, hedging or calibration. In the following sections we describe another approach to overcome the main limitations described above.

2.2.2 Factor models

In this section we present a general framework for factor models. The idea behind factor models is to assume that all the names are influenced by the same sources of uncertainty. For simplicity we will use a so called single factor model, i.e. there is only one common variable influencing the dynamics of the security, the other influences are idiosyncratic. Factor models have widely been applied by many authors[12] to credit derivatives modelling for essentially two reasons:

- factor models represent an intuitive framework and allow fast calculation of the loss distribution function without the need to use a Monte Carlo simulation;

- there is no need to model the full correlation matrix, which represents a challenging issue, since default correlation is very difficult to estimate, i.e. joint defaults are particulary uncommon and there is generally a lack of data for a reliable statistic, and CDS spread correlation or equity correlation[13] can only be assumed as a proxy of default correlation, these quantities being influenced by other forces, e.g. liquidity.

Factor models can then be used to describe the dependency structure amongst credits using a so called "credit-vs-common factors" analysis rather than a pairwise analysis.

An example of factor model is given by the following expression:

$$V_i = \sqrt{\rho}Y + \sqrt{1 - \rho}\epsilon_i, \tag{2.19}$$

---

[11]Synthetic CDOs or indices have generally more than 100 reference entities and defaults are "rare events" especially when the portfolio is composed of investment grade obligors. A large number of simulation is then needed for reliable results.

[12]See e.g. Laurent and Gregory [?], Andersen *et al.* [3], Finger [22], Hull and White [30], Schönbucher [46] or Galiani *et al.* [25].

[13]Moreover the analysis is complicated by the fact that, given e.g. the index iTraxx Europe, it is necessary to estimate $N(N - 1)\frac{1}{2} = 7750$ pairwise correlations.

where V_i is a individual risk process and $Y, \epsilon_i, i = 1, 2, \ldots, n$ are i.i.d. $\Phi(0,1)$. We note that, conditioning on the systemic factor Y, the V_i are pairwise independent. Moreover, the random variables Y and ϵ_i being independent, it follows that V_i is $\Phi(0,1)$. It is then possible to calculate the correlation between each pair as follows:

$$
\begin{aligned}
\mathrm{Corr}\,[V_i, V_j] &= \mathbb{E}\,[V_i, V_j] = \mathbb{E}[\sqrt{\rho}Y + \sqrt{(1-\rho)}\epsilon_i, \sqrt{\rho}Y + \sqrt{(1-\rho)}\epsilon_j] \\
&= \rho\mathbb{E}[Y^2] = \rho.
\end{aligned}
$$

For each individual obligor we can calculate the probability of a default happening before maturity t, $\mathbb{P}\{\tau_i \leq t\}$, as the probability that the value of the company V_i falls below a certain threshold k_i:

$$
\begin{aligned}
\mathbb{P}\{V_i \leq k_i\} &= \mathbb{P}\left\{\sqrt{\rho}Y + \sqrt{1-\rho}\epsilon_i \leq k_i\right\} \\
&= \mathbb{P}\left\{\epsilon_i \leq \frac{k_i - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right\} \\
&= \Phi\left[\frac{k_i - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right].
\end{aligned}
$$

The value of the barrier k_i can be easily calculated following the theory presented in §2.2. Assuming that the time of default τ_i is the time of the first jump of an inhomogeneous Poisson process $N(t)$ and using the survival probability found in the equation 2.11, the probability of default for a single name is then given by $\mathbb{P}\{\tau_i \leq t\} = 1 - e^{\int_0^t \lambda_i(s)ds} = F_i(t) = \mathbb{P}\{V_i \leq k_i\} = \Phi(k_i)$. The value of the barrier[14] can then be written as follows: $k_i = \Phi^{-1}(F_i(t))$.

Using the tower property of conditional expectation we can calculate the probability of default with the following expression:

$$
\begin{aligned}
\mathbb{P}\{V_i \leq k_i\} &= \mathbb{E}[\mathbf{1}_{\{V_i \leq k_i\}}] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{V_i \leq k_i\}}|Y = y\right]\right] \\
&= \mathbb{E}\left[\mathbb{P}\left\{\epsilon_i \leq \frac{k_i - \sqrt{\rho}y}{\sqrt{1-\rho}}\Big|Y = y\right\}\right] \\
&= \mathbb{E}\left[\Phi\left[\frac{k_i - \sqrt{\rho}y}{\sqrt{1-\rho}}\right]\right] \\
&= \int_{-\infty}^{\infty} \Phi\left[\frac{k_i - \sqrt{\rho}y}{\sqrt{1-\rho}}\right] dF_Y(y). \tag{2.20}
\end{aligned}
$$

---

[14]In the sequel, for ease of notation, the time dependence in k_i is often omitted.

Furthermore, using that, conditioning on Y the defaults of the underlyings are independent, the loss distribution for a sequence of names is given by:

$$
\begin{aligned}
\mathbb{P}\{\tau_1 \leq t, \tau_2 \leq t, \ldots, \tau_n \leq t\} &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\tau_1 \leq t, \tau_2 \leq t, \ldots, \tau_n \leq t\}}|Y = y]] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\tau_1 \leq t\}}|Y = y]\mathbb{E}[\mathbf{1}_{\{\tau_2 \leq t\}}|Y = y] \ldots \mathbb{E}[\mathbf{1}_{\{\tau_n \leq t\}}|Y = y]] \\
&= \mathbb{E}\Big[\prod_{i=1}^{n} p_i(t|Y)\Big] \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{n} p_i(t|y)dF_Y(y).
\end{aligned}
\tag{2.21}
$$

where $p_i(t|y) = \Phi\left[\frac{k_i - \sqrt{\rho}y}{\sqrt{1-\rho}}\right]$.

The expression in 2.21, similarly to the definitions presented in §2.2.1 represents the so called "one-factor Gaussian copula":

$$
C_\rho^G = \int_{-\infty}^{\infty} \prod_{i=1}^{n} p_i(t|y)dF_Y(y).
\tag{2.22}
$$

2.2.3 Homogeneous portfolios and large homogeneous portfolios

The framework described in the previous sections can be easily used to price CDOs making further assumptions on the nature of the underlying obligors. We consider here two main approximations: *homogeneous portfolios(HP)*, i.e. the portfolio consists of identical obligors, and *large homogeneous portfolios(LHP)*, under which the number of obligors is very large $n \to \infty$. Under the homogeneous portfolios approximation we use same recovery rate[15] δ and default probability p for every name, the loss given default would also be identical for all the entities in the portfolio. Using the conditional default probability $p_i(t|y) = \Phi\left[\frac{k_i - \sqrt{\rho}y}{\sqrt{1-\rho}}\right]$ and the fact that, conditional upon the common factor Y the defaults in the portfolio are independent events, it is possible to write the following formula for $\mathbb{P}\{L = i(1-\delta)\}$:

$$
\mathbb{P}\{L = i(1-\delta)\} = \int_{-\infty}^{\infty} \binom{n}{i} p(t|y)^i (1 - p(t|y))^{n-i} dF_Y(y).
\tag{2.23}
$$

---

[15]Using constant recovery rate is a very strong assumption and implies that the maximum portfolio loss is $100(1-\delta)\%$. As will be clear in the sequel it has an impact particularly on the pricing of senior tranches. A way to overcome this problem is to use stochastic recovery rates, see e.g. [3], [45].

where the number of defaults (conditioning on Y) follows a binomial distribution $\mathcal{BIN}(n, p(t|y))$.

The equation 2.23 can be easily solved numerically using e.g. Simpson's rule or another numerical integration technique. An expression for the distribution of the aggregate loss G_L is then given by:

$$G_L(i(1-\delta)) = \mathbb{P}\{L \le i(1-\delta) = \sum_{m=0}^{i} \mathbb{P}\{L = m(1-\delta)\} \qquad (2.24)$$

The large homogeneous portfolios approximation allows us to express the loss distribution function in a closed form, given that for a large portfolio the latent variable has a strong influence. This result, firstly proved by Vasicek [52], states that as $n \longrightarrow \infty$, by the law of large numbers, the fraction of defaulted credits in the portfolio converges to the individual default probability:

$$\mathbb{P}\left\{\left|\frac{L}{n} - p(t|y)\right| > \varepsilon \middle| Y = y\right\} \to 0, \qquad (2.25)$$

$\forall \varepsilon > 0$ and $n \to \infty$.

Assuming for simplicity zero recoveries and that L, conditioning on Y, follows a Binomial distribution, we can immediately write down the mean and variance as follows:

$$\mathbb{E}[L|Y = y] = np(t|y),$$

and

$$\mathrm{Var}[L|Y = y] = np(t|y)(1 - p(t|y)).$$

It is then possible to calculate:

$$\mathbb{E}\left[\frac{L}{n}|Y = y\right] = p(t|y),$$

and

$$\mathrm{Var}\left[\frac{L}{n}|Y = y\right] = \frac{1}{n}p(t|y)(1 - p(t|y)).$$

Since, as $n \to \infty$, $\mathrm{Var}\left[\frac{L}{n}|Y = y\right] \to 0$, given $Y = y$, it follows that

$\frac{L}{n}$ converges to its mean.

Using this result, i.e. $\frac{L}{n(1-\delta)} \longrightarrow p(t|y)$, given any $x \in [0,1]$, an expression for the loss distribution function $G(x)$ can be calculated as follows:

$$
\begin{aligned}
G(x) &= \mathbb{P}\left\{p_i(t|Y) \leq x\right\} \\
&= \mathbb{P}\left\{\Phi\left[\frac{k_i - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right] \leq x\right\} \\
&= \mathbb{P}\left\{\frac{k_i - \sqrt{\rho}Y}{\sqrt{1-\rho}} \leq \Phi^{-1}(x)\right\} \\
&= \mathbb{P}\left\{Y > \frac{k_i - \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right\} \\
&= 1 - \mathbb{P}\left\{Y \leq \frac{k_i - \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right\} \\
&= \Phi\left[\frac{\sqrt{1-\rho}\Phi^{-1}(x) - k_i}{\sqrt{\rho}}\right]
\end{aligned}
\tag{2.26}
$$

2.2.4 Algorithms and fast Fourier transform for CDOs pricing

We conclude this chapter with a quick overview of two amongst the most popular techniques used to price CDOs when the idealised LHP or HP assumptions don't hold, i.e. the underlying obligors have different default probabilities and they are a finite number. For simplicity we will continue to assume the same weight for each underlying name in the portfolio (which is generally not a problem for index tranches) and the same recovery rates. We will focus on the Hull and White stile algorithm, see [30] and on the fast Fourier transform, see e.g. Laurent and Gregory [27] or Schönbucher [48] to solve the problem. The first method is based on a recursive approach and uses that, conditioning on $Y = y$, the default events are independent. We present here a popular version of this algorithm.

Denoting with $P^{(n)}(i)$ the probability[16] of having i defaults from the n names, and with p_i the default probability of the $i - th$ company, we can write:

$$
P^{(1)}(i) = \left\{
\begin{array}{ll}
1 - p_1 & \text{for} \quad i = 0, \\
p_1 & \text{for} \quad i = 1.
\end{array}
\right.
$$

---

[16]Where we are omitting, for ease of the notation, the dependence on time t and Y.

$$P^{(2)}(i) = \begin{cases} (1-p_1)(1-p_2) & \text{for} \quad i = 0, \\ p_1(1-p_2) + p_2(1-p_1) & \text{for} \quad i = 1, \\ p_1 p_2 & \text{for} \quad i = 2. \end{cases}$$

Observing that $P^{(2)}(i)$ can be written recursively:

$$P^{(2)}(i) = P^{(1)}(i)(1-p_2) + P^{(1)}(i-1)p_2,$$

and for $2 < j < n$,

$$P^{(j+1)}(i) = P^{(j)}(i)(1-p_{j+1}) + P^{(j)}(i-1)p_{j+1}.$$

Using the recursion it is then possible to find an expression for the conditional portfolio loss $P^{(n)}(i|y)$.

Analogously to the method described with formula 2.23, integrating out the latent variable Y we can recover the expression for the (unconditional) portfolio loss:

$$\mathbb{P}\left\{L = i(1-\delta)\right\} \quad = \quad \int_{-\infty}^{\infty} P^{(n)}(i|y) dF_Y(y). \tag{2.27}$$

We consider here a second approach to solve this problem with the fast Fourier transform(FFT). This approach shares with the previous one the conditional independence assumption, i.e. conditional on the latent variable Y, the defaults of the obligors are independent random variables. Given this assumption and the individual loss given default $(1-\delta_i) = l_i$ for every name in the pool, the portfolio loss can be written as a sum of independent random variable: $Z = \sum_{i=1}^{n} l_i$. The characteristic function, conditional on the common factor Y, is then given by:

$$\begin{aligned} \mathbb{E}\left\{e^{iuZ|Y}\right\} &= \mathbb{E}\left\{e^{iu\sum_{i=1}^{n}l_i}|Y\right\} \\ &= \mathbb{E}\left\{e^{iu(l_i)}|Y\right\}\mathbb{E}\left\{e^{iul_2}|Y\right\}\ldots\mathbb{E}\left\{e^{iul_n}|Y\right\}. \end{aligned} \tag{2.28}$$

The characteristic function for the $i-th$ obligor can be written as:

$$\begin{aligned} \mathbb{E}\left\{e^{iul_i|Y}\right\} &= e^{iul_i}p_i(t|y) + (1 - p_i(t|y)) \\ &= 1 + [e^{iul_i} - 1]p_i(t|y). \end{aligned} \tag{2.29}$$

Inserting the characteristic function 2.29 into the 2.28 we can express the portfolio characteristic function (conditional on Y) as follows:

$$\mathbb{E}\left\{e^{iuZ|Y}\right\} = \prod_{i=1}^{n}\left(1 + [e^{iul_i} - 1]p_i(t|y)\right), \qquad (2.30)$$

and integrating out the common factor we get the unconditional characteristic function:

$$\hat{h}(u) = \mathbb{E}\left\{e^{iuZ}\right\} = \int_{-\infty}^{\infty}\prod_{i=1}^{n}\left(1 + [e^{iul_i} - 1]p_i(t|y)\right)dF_Y(y). \quad (2.31)$$

The integral in 2.31 can be solved numerically by, e.g., quadrature technique or Monte Carlo simulation. Once we have found the characteristic function of the portfolio loss $\hat{h}(u)$ the density function can be found using the fast Fourier transform[17] which is a computationally efficient.

---

[17]We recall that given the Fourier transform $\hat{h}(u)$, the density function $h(t)$ is given by $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{h}(u)e^{-iut}du$, see [48] for a detailed description of the inversion via FFT.

Chapter 3

Correlation skew

In this chapter we briefly present some of the main concepts used to imply correlation data from the market quotes. We describe implied correlation and different tranches' sensitivity to correlation, the correlation smile obtained from market data, and we present a standard market practice used by practitioners in correlation modelling: base correlation.

3.1   What is the correlation skew?

Although the single factor copula model, particularly in its Gaussian version presented in §2, is in general not able to match the market quotes directly, it has become a market standard for index tranches pricing. Amongst the limitations[1] of this model that can explain its partial failure, there is certainly the fact that Gaussian copula has light tails and this has an important impact on the model's ability to fit market quotes, defaults being rare events. To overcome this restriction it is enough to modify the correlation parameter into the single factor model: other things being equal, increasing the correlation parameter leads to an increase of the probability in the tails, thus leading to either very few or a very large number of defaults. The new loss distributions, generated by varying the correlation parameter, are able to fit the market quotes obtaining the so called "implied correlation" or "compound correlation". Implied correlations obtained with this practice can be interpreted as a measure of the markets view on default correlation.

Implied correlation has been widely used to communicate "tranche correlation" between traders and investors using the Gaussian copula as a common

---

[1]Another consideration is due to the fact that intensity based models generally take into account only idiosyncratic jumps, but also global jumps can be considered, see e.g. Baxter [8].

framework[2]. The possibility to observe market correlation represents a very useful resource for dealers who can check the validity of the assumptions used to price bespoken CDOs, non standardised tranches or other exotic structured credit products, with the market quotes for the relevant standardised index. This practice reveals the existence of a "correlation smile" or "correlation skew" which generally takes, as shown in the figure 3.1, the following shape: the mezzanine tranches typically show a lower compound correlation than equity or senior tranches, and for senior and super senior a higher correlation is necessary than for equity tranche.
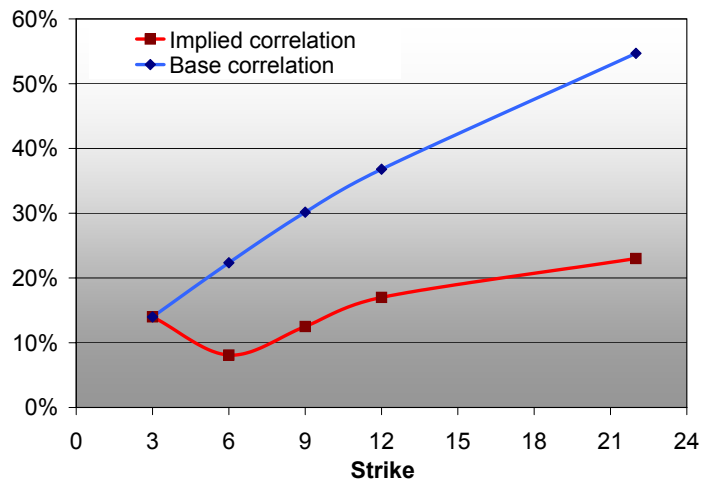

Figure 3.1: Implied correlation and base correlation for DJ iTraxx on 13 April 2006. On the X axis we report the detachment point (or strike) for each tranche and on the Y axis the correlation level for both implied correlation or base correlation.

---

[2]The probability distribution of portfolio losses obtained with a Gaussian copula is very different from the one obtained with a e.g. student t-copula or a normal inverse gaussian copula, as we analyse in §4.1. It is therefore very important to use the same model with implied correlations.

3.2 Analogy with the options market and explanation of the skew

The existence of a correlation skew is far from being obvious: since the correlation parameter associated with the model does not depend on the specific tranche priced, we would expect an almost flat correlation. The correlation smile can be described through an analogy with the more celebrated volatility smile (or skew): i.e. implied volatility in the Black-Scholes model changes for different strikes of the option. The existence of volatility skew can be explained by two main lines of argument: the first approach is to look at the difference between the asset's return distribution assumed in the model and the one implicit in the market quotes[3], while the second is to consider the general market supply and demand conditions, e.g. in a market that goes down traders hedge their positions by buying puts.

The correlation smile can be considered in a similar way: we can find explanations based either on the difference between the distribution assumed by the market and the specific model, or based on general market supply and demand factors. Within this analysis, we report three[4] explanations for the correlation smile which, we believe, can explain the phenomena.

- Although the liquidity in the market has grown consistently over the past few years, *demand and supply circumstances* strongly influence prices. Implied correlations reflect the strong interest by "yield searching" investors in selling protection on some tranches. As reported by Bank of England [10]:"the strong demand for mezzanine and senior tranches from continental European and Asian financial institutions may have compressed spreads relative to those of equity tranches to levels unrepresentative of the underlying risks". To confirm this argument we can add that banks need to sell unrated equity pieces in order to free up regulatory capital under the new Basel II capital requirement rules.

- *segmentation among investors across tranches* as reported by Amato and Gyntelberg [2]:"different investor groups hold different views about correlations. For instance, the views of sellers of protection on equity tranches (e.g. hedge funds) may differ from sellers of protection on mezzanine tranches (e.g. banks and securities firms). However, there is no

---

[3]The Black-Scholes model assumes that equity returns are lognormally distributed while the implicit distribution, see e.g Hull [29], has fatter lower tail and lighter upper tail.

[4]See e.g. [2], [7], [40] and [51] for further details.

compelling reason why different investor groups would systematically hold different views about correlations."

- the *standard single-factor gaussian copula is a faulty model* for the full capital structure of CDOs index tranches. The reasons may be found in the main assumptions of the model: i.e. the implicit default distribution has fatter tails than the Gaussian copula, recovery rates and default are not independent[5], and recovery rates are not fixed but could be stochastic.

3.3 Limitations of implied correlation and base correlation

Implied correlation has been a very useful parameter to evince information from the market: e.g. if there is a widening of the underlying index spread and the implied correlation on the equity piece has dropped, the fair spread of an equity tranche would most likely rise. This would be a consequence of the fact that the market is implying an increased idiosyncratic risk on few names which would affect the junior pieces. This would probably not affect the rest of the capital structure.

The compound correlation is a useful tool, although it has a few problems that limit its utility: as shown in figure 3.2, mezzanine tranches, not being monotonic in correlation, may not have an implied correlation or they may have multiple implied correlation[6] (typically two); there are difficulties in pricing bespoke tranches consistently with standard ones because of the smile; and there is an instability issue with implied correlation, i.e. a small change in the spread for mezzanine tranches would result in a big movement of the implied correlation, as we can see in figure 3.2.

Because of these limitations, market participants have developed the so called "base correlation". The concept was introduced by McGinty *et al.* [40] and can be defined as the correlation implied by the market on (virtual) equity pieces $0 - K_D\%$. In its original formulation it was extracted using the large homogeneous pool assumption. A popular[7] recursion technique can be applied to DJ iTraxx as follows:

- the first base correlation 0-3% coincides with the first equity implied correlation;

---

[5]As reported by Altman *et al.* [1]: "recovery rates tend to go down just when the number of defaults goes up in economic downturns".

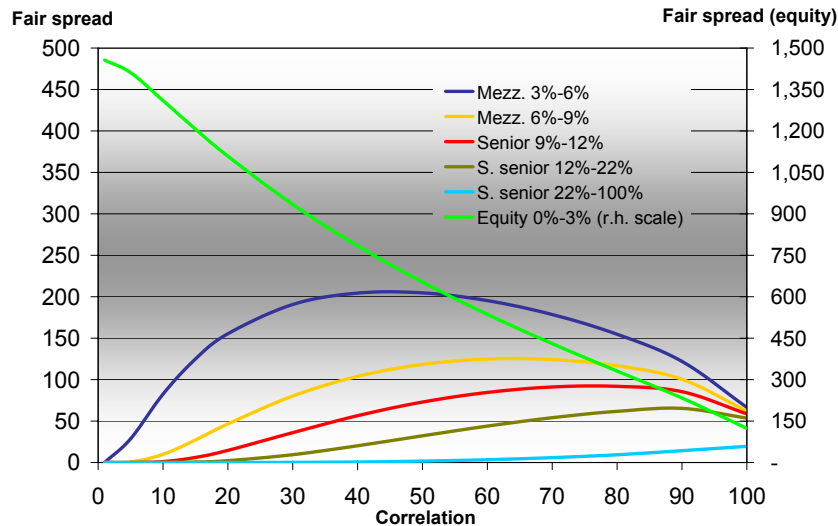[6]Multiple implied correlation are problematic for hedging.)

[7]See e.g. [9].

Figure 3.2: Correlation sensitivity for DJ iTraxx on 13 April 2006. On the X axis we report the correlation parameter and on the Y axis the fair spread value (expressed in basis points) for each tranche. For the equity tranche a different scale for the fair spread is used, which is reported on the right hand side. From the chart we can see that mezzanine tranches are not monotonic in correlation.

- price the 0-6% tranche combining the 0-3% and the 3-6% tranches with premium equal to the 3-6% tranche's fair spread and correlation equal to the 0-3% base correlation;

- The 0-6% price is simply the price of the 0-3% tranche with the 3-6% premium, being a 3-6% tranche with the 3-6% premium equal to zero, and can be used to imply the base correlation for the 0-6% tranche, using the standard pricing model;

- using the 0-6% tranche base correlation recursively, we can then find the price of the 0-6% tranche with the 6-9% fair spread;

- the base correlation of the 0-9% tranche can be implied from the pre-

23

vious price;

- the procedure has to be iterated for all the tranches.

Base correlation represents a valid solution to the shortcomings of implied correlations obtained using the Gaussian copula: especially the problem relating to existence is generally solved (excluding unquoted super senior tranche) and the hedging obtained using base correlation offers better performance than the one obtained using implied correlations[8]. However, this technique is not solving all the problems related to the correlation skew. Amongst the main limitations of base correlation we recall that the valuation of off-market index tranches is not straightforward and depends on the specific interpolative technique of the base correlation.

---

[8]See [9] for a detailed description of the issue.

Chapter 4

Gaussian copula extensions: correlation skew models

The insufficiency of the single factor Gaussian model to match market quotes has already been emphasised by many practitioners and academics and can be summarised as follows:

- relatively low prices for equity and senior tranches;

- high prices for mezzanine tranches.

For a good understanding of the problem it is central to look at the cumulative loss distribution resulting from the Gaussian copula and the one implied by the market, as in figure 4.1: a consistently higher probability is allocated by the market to high default scenarios than the probability mass associated with a Gaussian copula model (which has no particularly fat upper tail). Furthermore, the market assigns a low probability of zero (or few) defaults, while in the Gaussian copula it is certainly higher. In this chapter we look at some of the proposed extensions to the base model and, mixing two of these different approaches, we propose a new variation of correlation skew model.

4.1   Factor models using alternative distributions

A factor model can be set up using different distributions for the common and specific factors. There is a very rich literature[1] on copulas that can be

---

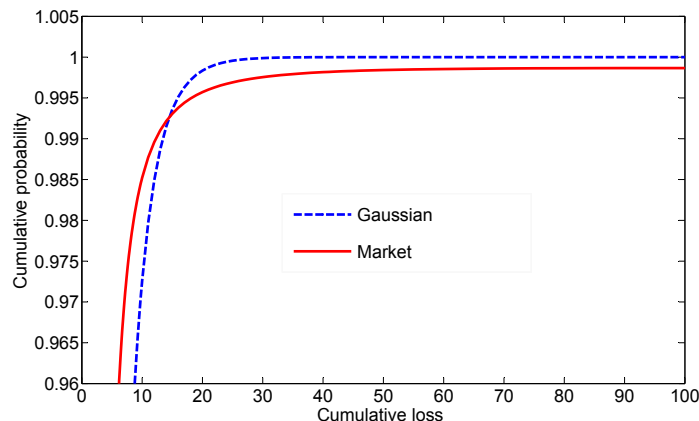[1]We refer to Burtschell *et al.* [14] for a recent comparison of different copula models.

Figure 4.1: In this chart we compare the cumulative portfolio loss using a Gaussian copula and the loss implied by the market. The Y axis reports the cumulative probability, while the X axis reports the cumulative portfolio loss.

used, amongst the most famous we recall the Student t, Archimedean and Marshall-Olkin copulas.

In the sequel two distributions are analysed:

- normal inverse Gaussian;

- α-stable.

The choice of these two distributions is due to their particular versatility and ability to cope with heavy-tailed processes: appropriately chosen values for their parameters can provide an extensive range of shapes of the distribution.

In figure 4.2 these distributions are compared with the Gaussian pdf: the upper and lower tails are higher for both α-stable and NIG, showing that a higher probability is associated with "rare events" such as senior tranche and/or super senior tranche losses[2]. In particular, a fatter lower tail (i.e. associated with low values for the common factor Y) is very important for a correct pricing of senior tranches, thus overcoming the main limitation
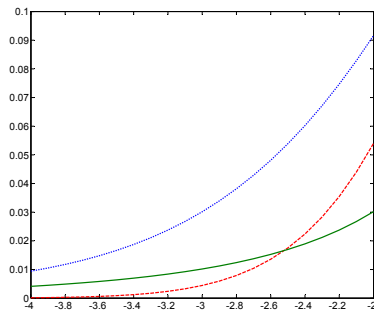
---

[2]An easy calculation shows that, considering the iTraxx index and the standard assumption for constant recovery rate at 40%, for a senior tranche to start losing 19 names defaulting is enough, while for a first super senior tranche 26 names are required, and for a second super senior tranche more than 45 distressed names are needed.
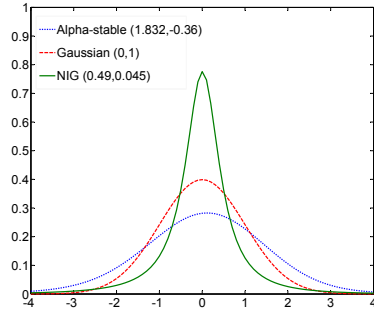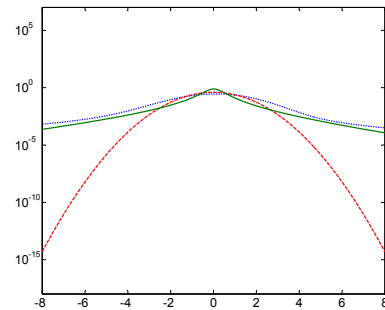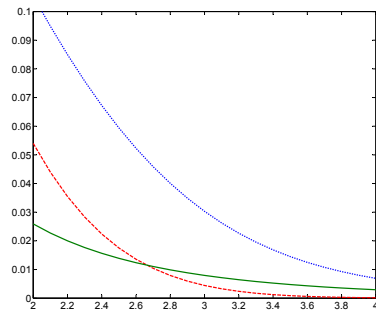
Figure 4.2: The charts compare the pdf for the α-stable, NIG and Gaussian distribution. In the lower charts we focus on the tail dependence analysing upper and lower tails.

associated with Gaussian copula. In the sequel the main characteristics of these statistical distributions are summarised and it is also described how, if appropriately calibrated, they can provide a good fit for CDO tranche quotes.

4.1.1 Normal inverse Gaussian distributions (NIG)

The NIG is a flexible distribution recently introduced in financial applications by Barndorff-Nielsen [6]. A random variable X follows a NIG distribution with parameters α, β, μ and δ if[3] given that $Y \sim IG(\delta\eta, \eta^2)$ with $\eta := \sqrt{(\alpha^2 - \beta^2)}$, then $X|Y = y \sim \Phi(\mu + \beta y, y)$ with $0 \leq |\beta| < \alpha$ and $\delta > 0$. Its

---

[3]For the inverse Gaussian (IG) density function see e.g. [16].

moment generating function is given by:

$$M(t) = \mathbb{E}\left[e^{xt}\right] = e^{\mu t} \frac{e^{\delta \sqrt{\alpha^2 - \beta^2}}}{e^{\delta \sqrt{\alpha^2 - (\beta + t)^2}}} \tag{4.1}$$

and it has two important properties: scaling property and stability under convolution [4].

The NIG can be profitably applied to credit derivatives pricing thanks to its properties, as in Kalemanova *et al.* [33]. The simple model consists of replacing the Gaussian processes of the factor model with two NIG:

$$V_i = \sqrt{\rho} Y + \sqrt{1 - \rho} \epsilon_i \leq k_i \tag{4.2}$$

where V_i is a individual risk process and $Y, \epsilon_i, i = 1, ..., n$ follow independent NIG distributions:

$$Y \sim NIG\left(\alpha, \beta, \frac{-\beta \eta^2}{\alpha^2}, \frac{\eta^3}{\alpha^2}\right),$$

and

$$\epsilon_i \sim NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{-\beta \eta^2}{\alpha^2}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\eta^3}{\alpha^2}\right),$$

where $\eta = \sqrt{(\alpha^2 - \beta^2)}$.

Note that conditioning on the systemic factor Y, the V_i are independent and, using the properties of the NIG distribution, each process V_i follows a standard [5] NIG with parameters $V_i \sim NIG\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta \eta^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\eta^3}{\alpha^2}\right)$. This notation can be simplified and we can write $V_i \sim NIG\left(\frac{1}{\sqrt{\rho}}\right)$.

Under this model, following the general approach described in the previous sections, we can calculate the default barrier k_i, the individual probability of default $p(t|y)$ and, using the LHP approximation, the $F_\infty(x) = \mathbb{P}\{X \leq x\}$.

Given the i-th marginal default probability for the time horizon t, $F_i(t) = \mathbb{P}\{V_i \leq k_i\}$, the default barrier K_i follows:

$$K_i = F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}(F_i(t)). \tag{4.3}$$

---

[4] see e.g. [5],[33].

[5] It can be easily verified that, given $Z \sim NIG\left(\alpha, \beta, \mu, \sigma\right)$ and using that $\mathbb{E}[Z] = \mu + \sigma\frac{\beta}{\eta}$ and $Var[X] = \sigma\frac{\alpha^2}{\eta^3}$, V_i has zero mean and unit variance.

Using the scaling property of the NIG, the individual probability of default is:

$$p(t|y) = F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)} \left(\frac{k - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right). \tag{4.4}$$

Using 4.4 and the large homogeneous portfolio approximation, given any $x \in [0,1]$, the loss distribution as $n \to \infty$ can be written as:

$$
\begin{aligned}
G(x) &= \mathbb{P}\{X \le x\} \\
&= \mathbb{P}\left\{F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)} \left(\frac{k - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \le x\right\} \\
&= 1 - F_{NIG(1)}\left(\frac{k - \sqrt{1-\rho}F^{-1}_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}(x)}{\sqrt{\rho}}\right), \tag{4.5}
\end{aligned}
$$

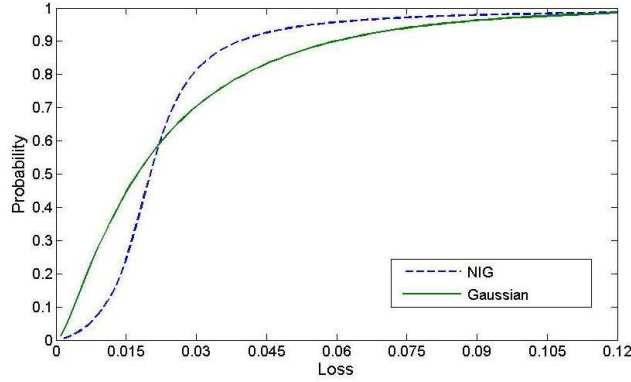which cannot be simplified any further, the NIG distribution not always being symmetric[6].

Figure 4.3: In the chart the cumulative loss probability distributions obtained with the NIG and Gaussian copula have been compared in the lower tail.

---

[6]The NIG is symmetric only in the special case when $\beta = 0$.

4.1.2 α-stable distributions

The α-stable is a flexible family of distributions described by Paul Lévi in the 1920's and deeply studied by, e.g., Nolan [42, 43]. This distribution can be used efficiently for problems where heavy tails are involved and, as introduced by Prange and Scherer in [44], can be profitably used to price CDOs. Following Nolan[7] we can define this family of distributions as follows:

Definition 3 *A random variable X is stable if and only if $X := aZ + b$, where $0 < \alpha \leq 2$, $-1 < \beta \leq 1$, $a > 0$, $b \in \mathbb{R}$ and Z is a random variable with characteristic function:*

$$\mathbb{E}\left[e^{iuZ}\right] = \begin{cases} e^{-|u|^\alpha\left[1-i\beta\tan\left(\frac{\pi\alpha}{2}\right)(u)\right]}, & \text{if} \quad \alpha \neq 1 \\ e^{-|u|\left[1+i\beta tan\left(\frac{2}{\pi}\right)(u)\ln|u|\right]}, & \text{if} \quad \alpha = 1 \end{cases} \tag{4.6}$$

Definition 4 *A random variable X is α-stable $S_\alpha(\alpha, \beta, \gamma, \delta, 1)$ if:*

$$X := \begin{cases} \gamma Z + \delta, & \text{if} \quad \alpha \neq 1 \\ \gamma Z + (\delta + \beta\frac{2}{\pi}\gamma\ln\gamma), & \text{if} \quad \alpha = 1 \end{cases} \tag{4.7}$$

where Z comes from the former definition.

The parameters γ and δ represent the scale and the location. In the sequel the simplified notation for a standard distribution $S_\alpha(\alpha, \beta, 1, 0, 1) := S_\alpha(\alpha, \beta, 1)$ will be used to define the random variables. The family of stable distributions[8] includes widely used distributions, such as Gaussian, Cauchy or Lévi distributions (i.e. a $S_\alpha(2, 0, 1) \sim \Phi(0, 2)$ as shown in figure 4.4).

From figure 4.4 it is possible to observe how decreasing the value for α away from 2 moves probability to the tails and a negative β skews the distribution to the right, producing another distribution with a particularly fat lower tail.

We can apply this distribution to the simple factor model, where the common and the specific risk factors Y and ϵ_i follow two independent α-stable

---

[7]We would like to thank John Nolan for the α-stable distribution code provided. For further details visit http://www.RobustAnalysis.com.

[8]Another definition of stable distribution can be the following:

Definition 5 *A random variable X is stable if for two independent copies X_1, X_2 and two positive constants a and b:*

$$aX_1 + bX_2 \sim cX + d \tag{4.8}$$

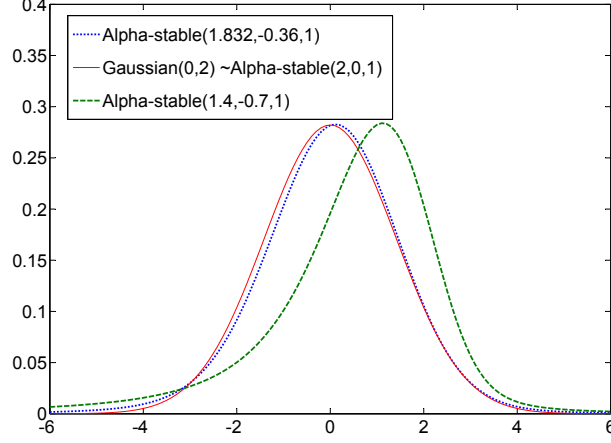*for some positive constant c and some $d \in \mathbb{R}$,*

see e.g. [43].

Figure 4.4: Examples of α-stable and Gaussian distributions.

distributions:

$$Y \sim S_\alpha(\alpha, \beta, 1),$$

and

$$\epsilon_i \sim S_\alpha(\alpha, \beta, 1),$$

this implies, using the properties of the stable distribution, that the firm risk factor follows the same distribution, i.e. $V_i \sim S_\alpha(\alpha, \beta, 1)$.

Given the i-th marginal default probability $F_i(t)$, the default barrier K_i is obtained inverting the relation $\mathbb{P}\{V_i \leq k_i\}$:

$$K_i = F_\alpha^{-1}(F_i(t)). \tag{4.9}$$

The individual probability of default is:

$$p(t|y) = F_\alpha \left(\frac{k - \sqrt{\rho}Y}{\sqrt{1-\rho}} \right). \tag{4.10}$$

Using 4.10 and the large pool approximation, given any $x \in [0,1]$, the loss distribution as $n \to \infty$ can be written as:

$$G(x) = \mathbb{P}\{X \leq x\} = 1 - F_\alpha \left(k - \frac{\sqrt{1-\rho}F_\alpha^{-1}(x)}{\sqrt{\rho}} \right). \tag{4.11}$$

.

4.2   Stochastic correlation

The general model can be expressed as:

$$V_i = \sqrt{\tilde{\rho}_i}Y + \sqrt{1 - \tilde{\rho}_i}\epsilon_i \leq k_i, \tag{4.12}$$

where V_i is a individual risk process $Y, \epsilon_i, i = 1, ..., n$ are independent $\Phi(0,1)$ and $\tilde{\rho}_i$ is a random variable that takes values in $[0,1]$ and it is independent from Y, ϵ_i. The latter independence assumption is particularly important as conditioning upon $\tilde{\rho}_i$ the processes $V_i, i = 1, ..., n$ remain independent $\Phi(0,1)$ and it is possible to calculate:

$$
\begin{aligned}
F_t(t) &= \mathbb{P}\{V_i \leq k_i\} \\
&= \mathbb{P}\{\sqrt{\tilde{\rho}_i}Y + \sqrt{1 - \tilde{\rho}_i}\epsilon_i \leq k_i\} \\
&= \mathbb{E}[\mathbb{P}\{\sqrt{\rho}Y + \sqrt{1 - \rho}\epsilon_i \leq k_i|\tilde{\rho}_i = \rho\}] \\
&= \int_0^1 \mathbb{P}\{\sqrt{\rho}Y + \sqrt{1 - \rho}\epsilon_i \leq k_i|\tilde{\rho}_i = \rho\}dF_{\tilde{\rho}_i}(\rho).
\end{aligned}
$$

$$\tag{4.14}$$

We can then write $F_i(t) = \Phi(k_i)$ and easily find the default threshold.

Under this model the individual probability of default can be calculated for $\tilde{\rho}_i \in [0,1]$:

$$
\begin{aligned}
p_i(t|Y) &= \mathbb{P}\{V_i \leq k_i|Y\} \\
&= \mathbb{P}\{\tau_i \leq t|Y\} \\
&= \int_0^1 \Phi\left[\frac{k_i - \sqrt{\rho}Y}{\sqrt{1 - \rho}}\right] dF_{\tilde{\rho}_i}(\rho).
\end{aligned}
$$

$$\tag{4.16}$$

Following closely Burtschell *et al.* [14, 15], within this framework two different specifications for $\tilde{\rho}_i$ are considered. In one case the correlation random variable follows a binary distribution: $\tilde{\rho}_i = (1 - B_i)\sqrt{\rho} + B_i\sqrt{\gamma}$, where $B_i, i = 1, ..., n$ are independent Bernoulli random variables and $\rho, \gamma \in [0,1]$ are two constants. The general model in 4.12 can be written as:

$$
\begin{aligned}
V_i &= \sqrt{((1-B_i)\sqrt{\rho}+B_i\sqrt{\gamma})}Y + \sqrt{1-((1-B_i)\sqrt{\rho}+B_i\sqrt{\gamma})}\epsilon_i \\
&= B_i\left(\sqrt{\gamma}Y + \sqrt{1-\gamma}\epsilon_i\right) + (1-B_i)\left(\sqrt{\rho}Y + \sqrt{1-\rho}\epsilon_i\right). \quad (4.17)
\end{aligned}
$$

In this simple case the correlation $\tilde{\rho}_i$ can be either equal to ρ or γ conditioning on the value of the random variable B_i. The Bernoulli's parameters are denoted by $p_\gamma = \mathbb{P}\{B_i = 1\}, p_\rho = \mathbb{P}\{B_i = 0\}$.

The expression for the individual default probability, conditioning on Y and B_i and using tower property to integrate out B_i, leads to:

$$
\begin{aligned}
p_i(t|Y) &= \mathbb{P}\{\tau_i \le t|Y\} = \sum_{j=0}^{1}\mathbb{P}\{\tau_i \le t|Y, B_i = j\}\mathbb{P}\{B_i = j\} \\
&= p_\rho\Phi\left[\frac{k_i - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right] + p_\gamma\Phi\left[\frac{k_i - \sqrt{\gamma}Y}{\sqrt{1-\gamma}}\right] \quad (4.18)
\end{aligned}
$$

From 4.18 we can price CDO tranches calculating the loss distribution as in 2.26 for a homogeneous large portfolio:[9]

$$
\begin{aligned}
G(x) &= \mathbb{P}\left\{p_i(t|Y, B_i) \le x\right\} \\
&= \mathbb{P}\left\{\Phi\left[\frac{k(t) - \sqrt{\tilde{\rho}_i}Y}{\sqrt{1-\tilde{\rho}_i}}\right] \le x\right\} \\
&= \mathbb{E}\left[\mathbb{P}\left\{\Phi\left[\frac{k(t) - \sqrt{\tilde{\rho}_i}Y}{\sqrt{1-\tilde{\rho}_i}}\right] \le x\middle| Y, B_i\right\}\right] \\
&= p_\rho\Phi\left[\frac{\sqrt{1-\rho}\Phi^{-1}(x) - k(t)}{\sqrt{\rho}}\right] + p_\gamma\Phi\left[\frac{\sqrt{1-\gamma}\Phi^{-1}(x) - k(t)}{\sqrt{\gamma}}\right].
\end{aligned}
$$
$$
(4.19)
$$

This simple model can be very useful to replicate market prices through the use of two or three possible specifications for the correlation random variable. The ability to fit the market quotes much better comes from the fact that the joint default distribution is not a Gaussian copula anymore as described in section 2.2.1, but it is now a factor copula (see [15]). Particularly

---

[9]A general case can be considered computing the distribution function $p_i(t|Y)$ for each name in the portfolio, then, combining it with fast Fourier transform (see Schönbucher [48] or other algorithms (see e.g. Hull and White [30]), it is possible to price CDOs without using further assumptions.

under the LHP assumption it is quite effective to set up the correlation as follows:

$$\tilde{\rho} = \begin{cases} \rho & \text{with,} & p_\rho \\ \gamma = 0 & \text{with,} & p_\gamma \\ \xi = 1 & \text{with,} & p_\xi. \end{cases} \quad (4.20)$$

In this way we can combine an independence state when $\gamma = 0$, that allows us to highlight the effect of the individual risk, with a perfect correlation state $\xi = 1$ for the latent risk. The idiosyncratic risk can be studied more effectively under independence assumption, since a default occurring does not result in any contagious effects. Therefore this kind of models can have a better performance during crisis periods when compared to a standard model[10]. The common risk can be controlled by increasing the probability of $\xi = 1$ and decreasing the probability associated with the other possible states of the random variable $\tilde{\rho}$, thus moving probability mass on the right tail of the loss distribution and then rising the price for the senior tranches, which are more sensible to rare events.

A second approach to model stochastic correlation, proposed by Burtschell *et al.*[15] admits a more sophisticated way to take into account the systemic risk. Instead of having a high correlation parameter ξ like the previous model, we can set: $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$, where $B_s, B_i, i = 1, ..., n$ are independent Bernoulli random variables and $\rho \in [0, 1]$ is a constant. The Bernoulli's parameters are denoted by $p = \mathbb{P}\{B_i = 1\}, p_s = \mathbb{P}\{B_s = 1\}$. This model has a so called comonotonic state, or perfect correlation state occurring when $B_s = 1$. The general model in 4.12 can be written as:

$$\begin{aligned} V_i & = & ((1 - B_s)(1 - B_i)\rho + B_s)Y + (1 - B_s)(\sqrt{1 - \rho^2}(1 - B_i) + B_i)\epsilon_i \\ & = & \tilde{\rho}_i Y + \sqrt{1 - \tilde{\rho}_i^2}\epsilon_i. \end{aligned} \quad (4.21)$$

Under this specification of the model[11], in analogy with the idea underlying model 4.20, the distribution of the correlation random variable $\tilde{\rho}_i^2$, has

---

[10]As reported by Burtschell *et al.*[15], following the downgrades of Ford and GMAC in May 2005, the higher idiosyncratic risk perceived by the market can be measured incorporating the idiosyncratic risk on each name and in particular on the more risky ones, i.e. the names with wide spreads.

[11]Given $\tilde{\rho}_i = \rho$ we can easily verify that $\text{Corr}[V_i, V_j] = \mathbb{E}[V_i, V_j] = \mathbb{E}[\rho Y + \sqrt{(1 - \rho^2)}\epsilon_i, \rho Y + \sqrt{1 - \rho^2}\epsilon_j] = \rho^2 \mathbb{E}[Y^2] = \rho^2$.

the following distribution: $\tilde{\rho_i}^2 = 0$ with probability $p(1 - p_s)$, $\tilde{\rho_i}^2 = \rho$ with probability $(1 - p)(1 - p_s)$ and $\tilde{\rho_i}^2 = 1$ with probability p_s, which corresponds to the comonotonic state.

The individual default probability can be found conditioning on Y, B_i and B_s:

$$
\begin{aligned}
p_i(t|Y) &= \sum_{l,m=0}^{1} \mathbb{P}\{\tau_i \leq t|Y, B_i = m, B_s = l\}\mathbb{P}\{B_i = m\}\mathbb{P}\{B_s = l\} \\
&= p(1 - p_s)\Phi[k_i(t)] + (1 - p)(1 - p_s)\Phi\left[\frac{k_i(t) - \rho Y}{\sqrt{1 - \rho^2}}\right] + p_s\mathbf{1}_{\{k_i(t)\geq Y\}} \\
&= p(1 - p_s)\Phi[\Phi^{-1}[F_i(t)]] + (1 - p)(1 - p_s)\Phi\left[\frac{k_i(t) - \rho Y}{\sqrt{1 - \rho^2}}\right] + p_s\mathbf{1}_{\{k_i(t)\geq Y\}} \\
&= (1 - p_s)\left(pF_i(t) + (1 - p)\Phi\left[\frac{k_i(t) - \rho Y}{\sqrt{1 - \rho^2}}\right]\right) + p_s\mathbf{1}_{\{k_i(t)\geq Y\}}. \qquad (4.22)
\end{aligned}
$$

This result can be used to find a semianalytical solution for the price of any CDO tranches (see [15]), or alternatively, conditioning on Y and B_S, can be used to approximate the loss distribution under the usual LHP assumption[12]. For the purpose of this work the analysis will be focused on the second approach. Given $x \in [0, 1]$, the loss distribution is then expressed by:

$$
\begin{aligned}
G(x) &= (1 - p_s)\left(\Phi\left[\frac{1}{\rho}\left(\sqrt{1 - \rho^2}\Phi^{-1}\left[\frac{x - pF(t)}{1 - p}\right] - k(t)\right)\right]\mathbf{1}_{\{x\in A\}} + \mathbf{1}_{\{x\in B\}}\right) \\
&\quad + p_s\Phi[-k(t)], \qquad (4.23)
\end{aligned}
$$

where the notation has been simplified defining:

$$A = \{x \in [0, 1]|pF(t) < x < (1 - p) + pF(t)\},$$

and

$$B = \{x \in [0, 1]|x > (1 - p) + pF(t)\}.$$

Proof:

---

[12]Under the LHP approximation, the name dependence in the superscripts can be removed, the default probability law being homogeneous for every single name in the portfolio.

Using the tower property we have:

$$
\begin{aligned}
G(x) &= \mathbb{P}\left\{p(t|Y, B_s) \le x\right\} \\
&= \mathbb{E}\left[\mathbb{P}\left\{(1 - B_s)\left(pF(t) + (1 - p)\Phi\left[\frac{k(t) - \rho Y}{\sqrt{1 - \rho^2}}\right]\right) + B_s \mathbf{1}_{\{k(t) \ge Y\}} \le x \,\middle|\, Y, B_s\right\}\right].
\end{aligned}
$$
(4.24)

Summing over the possible outcomes for B_s we have:

$$
\begin{aligned}
G(x) &= (1 - p_s)\mathbb{E}\left[\mathbb{P}\left\{pF(t) + (1 - p)\Phi\left[\frac{k(t) - \rho Y}{\sqrt{1 - \rho^2}}\right] \le x \,\middle|\, Y\right\}\right] \\
&\quad + p_s\mathbb{E}\left[\mathbb{P}\left\{\mathbf{1}_{\{k(t) \ge Y\}} \le x \,\middle|\, Y\right\}\right] \\
&= (1 - p_s)\mathbb{E}\left[\mathbb{P}\left\{\frac{k(t) - \rho Y}{\sqrt{1 - \rho^2}} \le \Phi^{-1}\left[\frac{x - pF(t)}{1 - p}\right] \,\middle|\, Y\right\}\right] + p_s\mathbb{P}\left\{Y > k(t)\right\} \\
&= (1 - p_s)\left(\Phi\left[\frac{1}{\rho}\left(\sqrt{1 - \rho^2}\Phi^{-1}\left[\frac{x - pF(t)}{1 - p}\right] - k(t)\right)\right]\mathbf{1}_{\{x \in A\}} + \mathbf{1}_{\{x \in B\}}\right) \\
&\quad + p_s\Phi\left[-k(t)\right].
\end{aligned}
$$
(4.25)

In the formula above we recall that:

$$
A = \{x \in [0, 1]|pF(t) < x < (1 - p) + pF(t)\},
$$

and

$$
B = \{x \in [0, 1]|x > (1 - p) + pF(t)\}.
$$

In step three we have used the fact that for $0 \le x < 1$, $\mathbf{1}_{\{k(t) \ge Y\}}$ if and only if $Y > k(t)$ and in step four that, assuming $p < 1$ and $\rho > 0$,

$$
pF(t) + (1 - p)\Phi\left[\frac{k(t) - \rho Y}{\sqrt{1 - \rho^2}}\right]
$$

can take values between

$$
pF(t) \text{ and } (1 - p) + pF(t).
$$

It is therefore possible to write:

(a) if $F(t) < x < (1-p)+pF(t)$, then $-Y \leq \frac{1}{\rho} \left(\sqrt{1-\rho^2}\Phi^{-1}\left[\frac{x-pF(t)}{1-p}\right] - k(t) \right)$

$\Rightarrow \mathrm{P}\left\{ pF(t) + (1-p)\Phi\left[\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}}\right] \leq x \Big| Y \right\} = 1;$

(b) $x \leq F(t)$,

$\Rightarrow \mathrm{P}\left\{ pF(t) + (1-p)\Phi\left[\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}}\right] \leq x \Big| Y \right\} = 0;$

and

(c) $x \geq (1-p) + pF(t)$,

$\Rightarrow \mathrm{P}\left\{ pF(t) + (1-p)\Phi\left[\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}}\right] \leq x \Big| Y \right\} = 1.$

This model can be used to adjust the probability in the upper part of the cumulative loss distribution, i.e. increasing p_s raises the probability of having credit events for all the names in the portfolio affecting the prices of senior tranches. Analogously increasing the idiosyncratic probability q pushes probability towards the left part of the loss distribution, resulting in an increased risk for the junior holder and a lower risk for the senior investors. In the case of the mezzanine tranches the dependence is not always constant, generally not being monotone in correlation ρ.

4.3   Local correlation

The term "local correlation" refers to the idea underlying a model where the correlation factor ρ can be made a function of the common factor Y. This family of models belongs to the stochastic correlation class because, being Y a random variable, the correlation factor $\rho(Y)$ is itself stochastic. This approach was introduced by Andersen and Sidenius [4] with the "random factor loadings" (RFL) model and by Turc *et al.* [51]. The base assumption of these models is very interesting since it attempts to explain correlation through the intuitive relation with the economic cycle: equity correlation tends to be higher during a slump than during a growing economy period.

4.3.1   Random factor loadings

Following closely the line of argument represented by Andersen and Sidenius [4], the general model can be expressed as follows:

$$V_i = a_i(Y)Y + v\epsilon_i + m \leq k_i, \tag{4.26}$$

where V_i is a individual risk process Y, ϵ_i, $i = 1, ..., n$ are independent $N(0, 1)$, v and m are two factors fixed to have zero mean and variance equal to 1 for the variable V_i.

The factor $a_i(Y)$ is a $\mathbb{R} \to \mathbb{R}$ function to which, following the original model [4], can be given a simple two point specification like the following:

$$a_i(Y) = \begin{cases} \alpha & \text{if, } Y \leq \theta \\ \beta & \text{if, } Y > \theta. \end{cases}$$

One can already observe the ability of this model to produce a correlation skew depending on the coefficient α and β: if $\alpha > \beta$ the factor $a(Y)$ falls as Y increases (i.e. good economic cycle) lowering the correlation amongst the names, while the opposite is true when Y falls below θ (i.e. bad economic cycle). In the special case $\alpha = \beta$ the model coincides with the Gaussian copula, but in general both the individual risk process V_i Gaussian and the joint default times do not follow a Gaussian copula.

The coefficients v and m can be easily found solving for

$$\mathbb{E}[V_i] = \mathbb{E}[a_i(Y)Y] + m = 0, \Leftrightarrow \mathbb{E}[a_i(Y)Y] = -m$$
$$\text{Var}[V_i] = \text{Var}[a_i(Y)Y] + v^2 = 1), \Leftrightarrow \text{Var}[a_i(Y)Y] = 1 - v^2$$

then we can calculate the values for m and v:

$$\begin{aligned} \mathbb{E}[a_i(Y)Y] &= \mathbb{E}[\alpha Y \mathbf{1}_{\{Y \leq \theta\}} + \beta Y \mathbf{1}_{\{Y > \theta\}} Y] \\ &= \varphi(\theta)(-\alpha + \beta), \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \mathbb{E}[a_i(Y)^2 Y^2] &= \mathbb{E}[\alpha^2 Y^2 \mathbf{1}_{\{Y \leq \theta\}} + \beta^2 Y^2 \mathbf{1}_{\{Y > \theta\}} Y] \\ &= \alpha^2(\phi(\theta) - \theta\varphi(\theta)) + \beta^2(\theta\varphi(\theta) + 1 - \phi(\theta)), \end{aligned} \tag{4.28}$$

from which[13] the solutions are:

---

[13]In the equations 4.27 and 4.28 was used that $\mathbb{E}[\mathbf{1}_{\{a < x \leq b\}} x] = \mathbf{1}_{\{a \leq b\}}(\varphi(a) - \varphi(b))$ $\mathbb{E}[\mathbf{1}_{\{a < x \leq b\}} x^2] = \mathbf{1}_{\{\alpha \leq b\}}(\Phi(b) - \Phi(a)) + \mathbf{1}_{\{a \leq b\}}(a\varphi(a) - b\varphi(b))$, see [4] Lemma 5 for a proof.

$$m = \varphi(\theta)(\alpha - \beta),$$

and

$$v = \sqrt{1 - \text{Var}[a_i(Y)Y]}.$$

As already recalled, V_i are in general not Gaussian in this model, thus the calculation to find the individual default probability and the default threshold changes. The conditional default probability can be calculated as follows:

$$
\begin{aligned}
p_i(t|Y) &= \mathbb{P}\{V_i \leq k_i|Y\} = \mathbb{P}\{\tau_i \leq t|Y\} \\
&= \mathbb{P}\left\{\alpha Y \mathbf{1}_{\{Y \leq \theta\}} + \beta Y \mathbf{1}_{\{Y > \theta\}}Y + \epsilon_i v + m \leq k_i|Y\right\} \\
&= \mathbb{P}\left\{\epsilon_i \leq \frac{k_i - \left(\alpha Y \mathbf{1}_{\{Y \leq \theta\}} + \beta Y \mathbf{1}_{\{Y > \theta\}}Y\right) - m}{v}\Bigg|Y\right\}. \quad (4.29)
\end{aligned}
$$

Integrating out Y, and using that ϵ_i is a standard Gaussian under this simple specification of the model, the unconditional default probability for the i-th obligor is:

$$
\begin{aligned}
p_i(t) &= \mathbb{E}\left[\mathbb{P}\left\{\epsilon_i \leq \frac{k_i - \left(\alpha Y \mathbf{1}_{\{Y \leq \theta\}} + \beta Y \mathbf{1}_{\{Y > \theta\}}Y\right) - m}{v}\Bigg|Y\right\}\right] \\
&= \int_{-\infty}^{\theta} \Phi\left[\frac{k_i - \alpha Y - m}{v}\right] dF_Y(Y) \\
&\quad + \int_{\theta}^{\infty} \Phi\left[\frac{k_i - \beta Y - m}{v}\right] dF_Y(Y).
\end{aligned}
$$

$$(4.30)$$

From this integral it is straightforward to calculate the default threshold k_i numerically.

Alternatively, using that Y follows a Gaussian distribution, and using some Gaussian integrals[14] a solution of the 4.30 can be found using a bivariate

---

[14]For a proof of the following Gaussian integrals see [4] lemma 1:

$$\int_{-\infty}^{\infty} \Phi\left[ax + b\right]\varphi(x)dx = \Phi\left[\frac{b}{\sqrt{1 + a^2}}\right]$$

$$\int_{-\infty}^{c} \Phi\left[ax + b\right]\varphi(x)dx = \Phi_2\left[\frac{b}{\sqrt{1 + a^2}}, c; \frac{-a}{\sqrt{1 + a^2}}\right]$$

Gaussian cdf[15]:

$$
\begin{aligned}
p_i(t) &= \mathbb{P}\{V_i \le k_i\} \\
&= \Phi_2\left[\frac{\theta - m}{\sqrt{v^2 + \alpha^2}}, \theta; \frac{\alpha}{\sqrt{v^2 + \alpha^2}}\right] + \Phi\left[\frac{\theta - m}{\sqrt{v^2 + \beta^2}}\right] \\
&\quad - \Phi_2\left[\frac{\theta - m}{\sqrt{v^2 + \beta^2}}, \theta; \frac{\beta}{\sqrt{v^2 + \beta^2}}\right].
\end{aligned}
$$

(4.32)

Assuming a large and homogeneous portfolio, given any $x \in [0,1]$ it is possible to find a common distribution function $G(x)$ for $p_i(t|Y)$ as $n \to \infty$ using, as before, the result stated in Vasicek [52]:

$$
\begin{aligned}
G(x) &= 1 - \mathbb{P}(X > x) \\
&= 1 - \mathbb{P}\left\{a(Y)Y \le k_i - v\Phi^{-1}[x] - m\right\} \\
&= 1 - \left(\mathbb{P}\left\{\alpha Y \le \Omega(x), Y \le \theta\right\} + \mathbb{P}\left\{\beta Y \le \Omega(x), Y > \theta\right\}\right) \\
&= 1 - \left(\Phi\left[min\left(\frac{\Omega(x)}{\alpha}, \theta\right)\right] + \mathbf{1}_{\left\{\theta < \frac{\Omega(x)}{\beta}\right\}}\left(\Phi\left[\frac{\Omega(x)}{\alpha}\right] - \Phi[\theta]\right)\right)
\end{aligned}
$$

(4.33)

where $\Omega(x) := k_i - v\Phi^{-1}[x] - m$.

4.4 Stochastic correlation using normal inverse Gaussian

This model represents an attempt to shape the correlation skew and then the market prices combining two of the solutions presented in the previous subsections. The stochastic correlation model, as presented in 4.12, can be efficiently used with NIG distributions for the systemic and the idiosyncratic risks factors $Y, \epsilon_i, i = 1, ..., n,$[16] instead of normal random variables. This

---

[15]Depending on the quadrature technique used 4.30 can be solved directly more efficiently than 4.31, which involves a bivariate Gaussian.

[16]A similar approach was used by Prange and Scherer [44], using α-stable distribution and Lüscher [38], where local correlation (i.e., random factor loading model) was combined with a double NIG.

model, as described in chapter 5, can produce very good results thanks to its ability to model the upper tail of the distribution more accurately with the conjunct use of a fat-tails distribution like NIG and a stochastic correlation model with a comonotonic state.

The model used is the one corresponding to the formula 4.21, and each risk process follows an independent NIG with the following characteristics:

$$Y \sim NIG\left(\alpha, \beta, \frac{-\beta\eta^2}{\alpha^2}, \frac{\eta^3}{\alpha^2}\right),$$

and

$$\epsilon_i \sim NIG\left(\frac{\sqrt{1-\tilde{\rho_i}^2}}{\tilde{\rho_i}}\alpha, \frac{\sqrt{1-\tilde{\rho_i}^2}}{\tilde{\rho_i}}\beta, \frac{\sqrt{1-\tilde{\rho_i}^2}}{\tilde{\rho_i}}\frac{-\beta\eta^2}{\alpha^2}, \frac{\sqrt{1-\tilde{\rho_i}^2}}{\tilde{\rho_i}}\frac{\eta^3}{\alpha^2}\right).$$

Using the independence between the random variables and conditioning on the systemic factor Y and $\tilde{\rho}_i$, the V_i remains independent NIG with parameters $V_i \sim NIG\left(\frac{1}{\tilde{\rho}_i}\right)$, using the simplified notation as in section 4.1.1.

The individual default probability can then be written as follows:

$$
\begin{aligned}
F_i(t) &= \mathbb{P}\{V_i \leq k_i\} \\
&= \mathbb{P}\{\tilde{\rho}_i Y + \sqrt{1-\tilde{\rho}_i{}^2}\epsilon_i \leq k_i\} \\
&= \mathbb{E}[\mathbb{P}\{\rho Y + \sqrt{1-\rho^2}\epsilon_i \leq k_i | \tilde{\rho}_i = \rho\}] \\
&= \sum_{l=1}^{3} F_{NIG(\frac{1}{\tilde{\rho}_i})}(k_i(t))\mathbb{P}\{\tilde{\rho}_i = \rho_l\}, \quad (4.34)
\end{aligned}
$$

from 4.34 the default threshold $k_i(t)$ can be calculated numerically[17].

Once we have found the default threshold, we can proceed similarly to the original model to determine the conditional default probability:

$$
\begin{aligned}
p_i(t|Y, B_s) &= (1 - B_s)\left(pF_{NIG(1)}(k_i(t)) + (1-p)F_{NIG\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}\left[\frac{k_i(t) - \rho Y}{\sqrt{1-\rho^2}}\right]\right) \\
&\quad + B_s \mathbf{1}_{\{k_i(t) \geq Y\}}. \quad (4.35)
\end{aligned}
$$

The loss distribution under the usual LHP approximation is then given by:

---

[17]Note that it is no longer possible to calculate $k_i(t)$ through direct inversion as it has been done before in the original model, where $F(t) = \Phi(k(t))$.

$$
\begin{aligned}
G(x) &= (1-p_s)\left(F_{NIG1}\left[\frac{1}{\rho}\left(\sqrt{1-\rho^2}F_{NIG2}^{-1}\left[\frac{x-pF_{NIG1}(k(t))}{1-p}\right]-k(t)\right)\right]\mathbf{1}_{\{x\in A\}}+\mathbf{1}_{\{x\in B\}}\right)\\
&\quad + p_s\left(1-F_{NIG1}\left[k(t)\right]\right),
\end{aligned}
\tag{4.36}
$$

where the notation has been simplified by defining:

$$
A = \{x\in[0,1] | pF_{NIG1}(k(t)) < x < (1-p)+pF_{NIG1}(k(t))\},
$$

$$
B = \{x\in[0,1] | x > (1-p)+pF_{NIG1}(k(t))\},
$$

$$
F_{NIG2} = F_{NIG\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)},
$$

and

$$
F_{NIG1} = F_{NIG(1)}.
$$

Proof:
Using the tower property we have:

$$
\begin{aligned}
G(x) &= \mathbb{P}\left\{p(t|Y,B_s) \le x\right\}\\
&= \mathbb{E}\left[\mathbb{P}\left\{(1-B_s)\left(pF_{NIG1}(k(t))+(1-p)F_{NIG2}\left[\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}}\right]\right)+B_s\mathbf{1}_{\{k(t)\ge Y\}} \le x \middle| Y,B_s\right\}\right].
\end{aligned}
$$

Summing over the possible outcomes of the variable B_s we have:

$$
\begin{aligned}
G(x) &= (1-p_s)\mathbb{E}\left[\mathbb{P}\left\{pF_{NIG1}(k(t))+(1-p)F_{NIG2}\left[\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}}\right] \le x \middle| Y\right\}\right] + \tag{4.37}\\
&\quad + p_s\mathbb{E}\left[\mathbb{P}\left\{\mathbf{1}_{\{k(t)\ge Y\}} \le x \middle| Y\right\}\right]\\
&= (1-p_s)\mathbb{E}\left[\mathbb{P}\left\{\frac{k(t)-\rho Y}{\sqrt{1-\rho^2}} \le F_{NIG2}^{-1}\left[\frac{x-pF_{NIG1}(k(t))}{1-p}\right] \middle| Y\right\}\right] + p_s\mathbb{P}\left\{Y > k(t)\right\}\\
&= (1-p_s)\left(F_{NIG1}\left[\frac{1}{\rho}\left(\sqrt{1-\rho^2}F_{NIG2}^{-1}\left[\frac{x-pF_{NIG1}(k(t))}{1-p}\right]-k(t)\right)\right]\mathbf{1}_{\{x\in A\}}+\mathbf{1}_{\{x\in B\}}\right)\\
&\quad + p_s\left(1-F_{NIG1}\left[k(t)\right]\right).
\end{aligned}
$$

$$
\tag{4.38}
$$

In the formula above we recall that:

$$A = \{x \in [0,1] | pF_{NIG1}(k(t)) < x < (1-p) + pF_{NIG1}(k(t))\},$$

and

$$B = \{x \in [0,1] | x > (1-p) + pF_{NIG1}(k(t))\}.$$

In step three was used that for $0 \le x < 1$, $\mathbf{1}_{\{k(t) \ge Y\}}$ if and only if $Y > k(t)$ and in step four, assuming $p < 1$ and $\rho > 0$, we have that

$$pF_{NIG1}(k(t)) + (1-p)F_{NIG2}\left[\frac{k(t) - \rho Y}{\sqrt{1-\rho^2}}\right]$$

can take values between

$$pF_{NIG1}(k(t)),$$

and

$$(1-p) + pF_{NIG1}(k(t)).$$

It is therefore possible to write:

(a) if $F_{NIG1}(k(t)) < x < (1-p) + pF_{NIG1}(k(t))$,

then $-Y \le \frac{1}{\rho}\left(\sqrt{1-\rho^2}F_{NIG2}^{-1}\left[\frac{x - pF_{NIG1}(k(t))}{1-p}\right] - k(t)\right)$

$\Rightarrow P\left\{pF_{NIG1}(k(t)) + (1-p)\Phi\left[\frac{k(t) - \rho Y}{\sqrt{1-\rho^2}}\right] \le x \middle| Y\right\} = 1;$

(b) $x \le F_{NIG1}(k(t))$,

$\Rightarrow P\left\{pF_{NIG1}(k(t)) + (1-p)F_{NIG2}\left[\frac{k(t) - \rho Y}{\sqrt{1-\rho^2}}\right] \le x \middle| Y\right\} = 0;$

and

(c) $x \ge (1-p) + pF_{NIG1}(k(t))$,

$\Rightarrow P\left\{pF_{NIG1}(k(t)) + (1-p)F_{NIG2}\left[\frac{k(t) - \rho Y}{\sqrt{1-\rho^2}}\right] \le x \middle| Y\right\} = 1.$

Chapter 5

Numerical results

In this chapter we review the numerical results for the general Gaussian copula and the other models presented in chapter 4 under the LHP assumption. We apply the pricing tools to the Dow Jones iTraxx 5 Years index series 5 with maturity date 20 June 2011[1] at 13 April 2006. In the sequel the different models are compared in terms of their ability to fit the market quotes, thus fitting the correlation skew.

5.1   Pricing iTraxx with different models

In table 5.1 the prices obtained with the six models under the LHP assumption are compared with the market quotes. Further assumptions are a flat risk free interest rate at 5% and a standard flat recovery rate at 40%. For completeness, the parameters used for the models are summarised in the table.

The calibration of the models has been carried out taking into account two criteria: the least square error (l.s.e.) and the minimum total error in bp. However the calibration was not developed through a full optimisation algorithm, hence a closer fit may be possible.

As observed in chapter 3.1 and 4, correlation skew strongly depends on the loss distribution function associated with the Gaussian copula model, which in particular affects the ability to fit the market quotes for the most senior and junior tranches at the same time. It is possible to see this feature in figures 5.1 and 5.2 where, for the models here considered, the relevant loss distribution functions are drawn. The good and bad results of the models studied in 5.1 can be explained observing the different shapes of the distributions in the lower and upper tails.

---

[1] Data available online at http://www.itraxx.com.

Model	Tranche 0-3%(†)	3-6%	6-9%	9-12%	12-22%	Errors (*) l.s.e.	bp
Market(§)	1226	63	18	9	4		
Gaussian	1,226.7	117.5	23.4	4.7	0.4	2.87	72.3
NIG	1,236.9	66.3	25.0	13.0	5.4	0.47	26.6
α-stable	1,236.7	82.0	24.7	12.4	6.5	0.76	42.3
Stoc. corr.	1223.1	63.9	23.1	7.7	0.6	0.82	13.6
RFL	1276.2	63.2	17.3	15.5	3.2	0.56	58.4
Stoc. corr. NIG	1234.1	62.1	20.6	9.5	3.4	**0.047**	**12.7**

	Parameters ρ	α	β	p	p_s	θ
Gaussian	0.14					
NIG	0.125	0.6	0.1			
α-stable	0.155	1.91	-0.6			
Stoc. corr.	0.407(‡)			0.755	0.035	
RFL		0.45	0.32			-2.39
Stoc. corr. NIG	0.1296(‡)	0.83	-0.015	0.10	0.05	

Table 5.1: Prices in bp and parameter calibration for the DJ iTraxx at 13 April 2006, the portfolio average spread was equal to 31.5 bp.
Notes:
(∗) The "error l.s.e." is calculate summing over the relative errors squared for each traded tranche while the "error bp" is the sum of absolute errors in basis point.
(†) This running premium corresponds to a market standard 24% upfront premium plus 500 bp running.
(§) Source: CDO-CDS Update: Nomura Fixed Income Research, 4/24/2006.
(‡) For stochastic correlation models the parameter in the table corresponds to ρ^2 according to the specification of the model.

Some common characteristics can then be summarised in this example when compared with the Gaussian copula:

- For the equity tranche the default probability is redistributed approximately from the 0-2% to the 2-3% area, being overall the same probability given by the different models in this part of the capital structure.

- For mezzanine tranches there is higher risk associated with the Gaussian copula which is reflected in the prices and in the distribution function; the other models provide in general a good matching on the 3-6%

tranche but most of them tend to overestimate the 6-9% tranche. However, the α-stable distribution could not be calibrated sufficiently well for the first mezzanine part of the capital structure.

- For the senior tranches there is in general a more wide-ranging situation. While Gaussian copula performs poorly due to its thin tails, thus underestimating the risk allocated by the market, the other models tend to fit the market quotes better. In particular fat-tails copulas like NIG and α-stable overestimate the 9-12% and 12-22% tranches, while RFL overestimates the senior but performs well for super senior tranches and stochastic correlation models perform very well especially when used in conjunction with a NIG. In particular, super senior tranches are very difficult to price and are closely related to the shape of the distribution in the upper tail. Both the RFL and stochastic correlation NIG models are able to fit the market quote, but while the former has a very thin upper tail in the 30-60% area, the latter displays a fatter tail also for higher strike values leading to more coherent prices for unquoted super senior tranches.
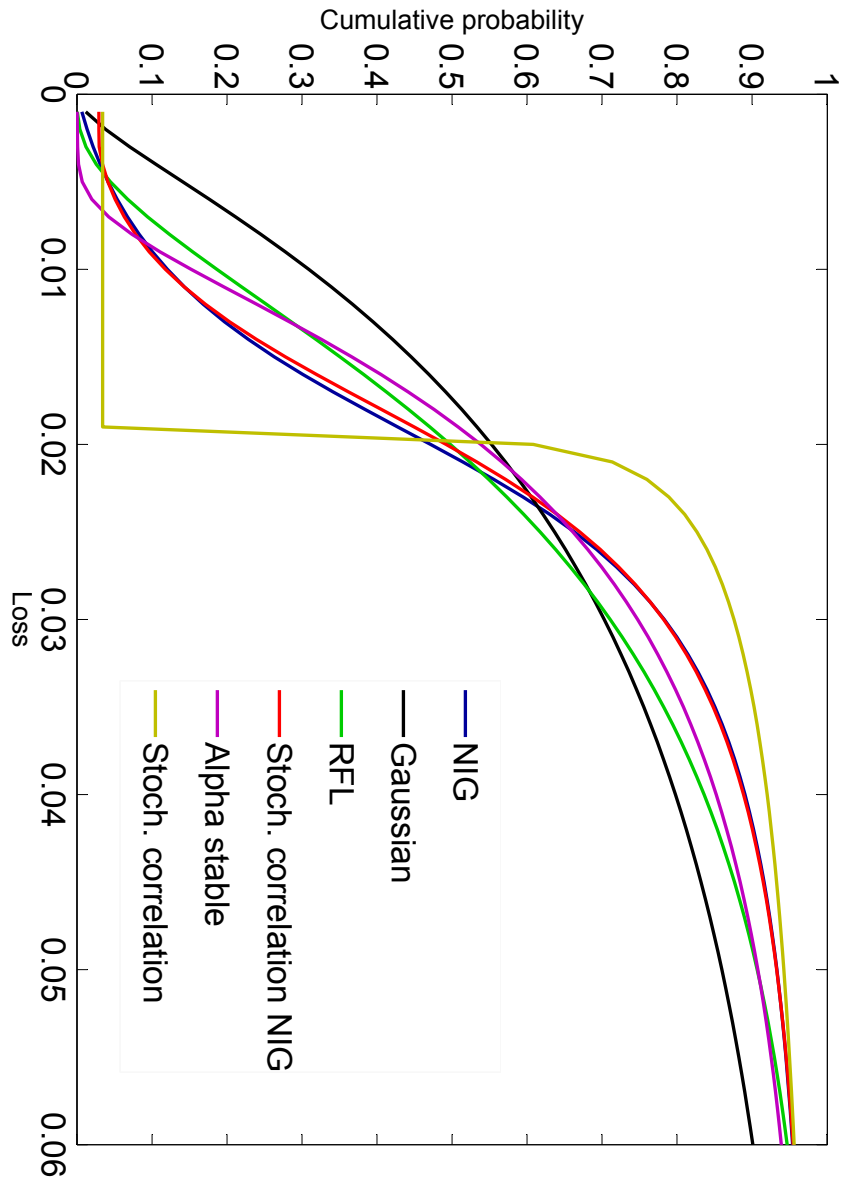
Figure 5.1: In the chart above we compare the loss distribution lower tail for the six models presented in the table 5.1.
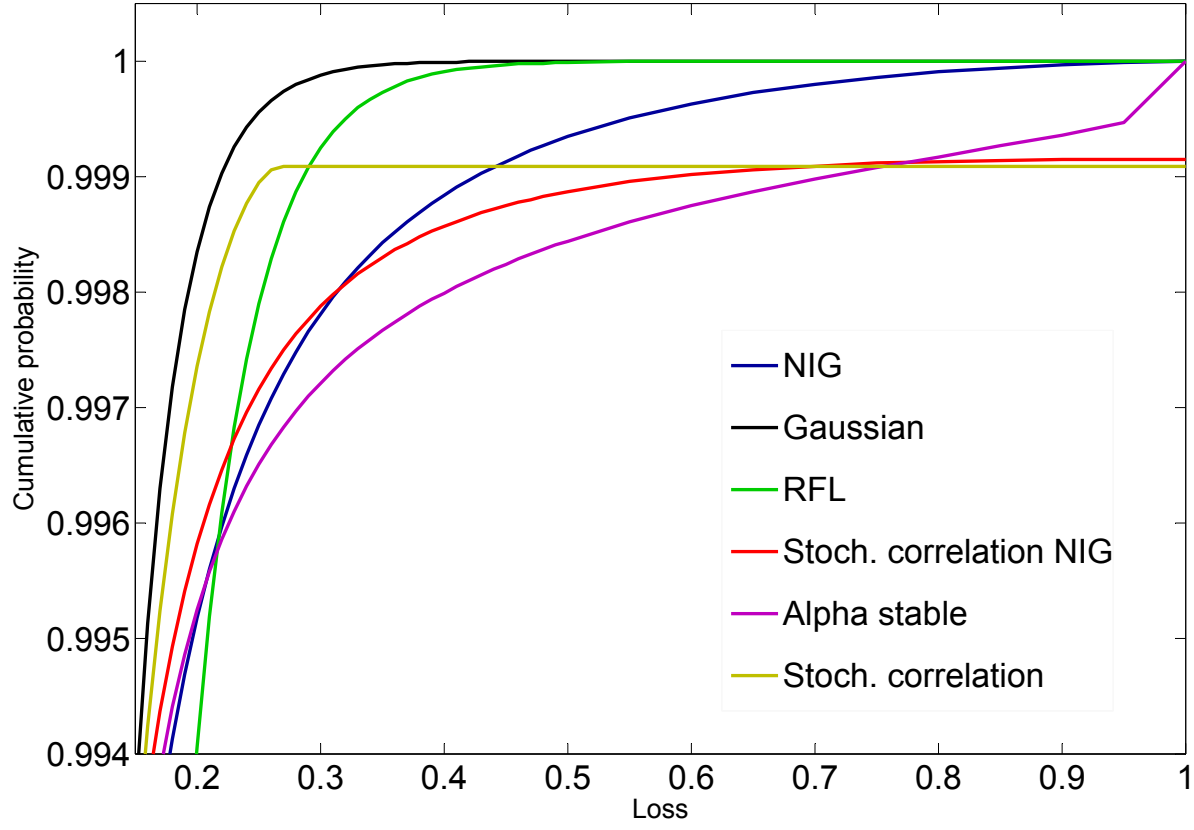
Figure 5.2: In the chart above we compare the loss distribution upper tail for the six models presented in the table 5.1.

5.2 Comparison between the different methods

We believe that it is not trivial to choose a universally accepted indicator/method to compare CDO pricing models, hence in this work we have chosen to use two different indicators: the l.s.e. error and the error in basis points. While the former error indicator tends to penalise the faults in the senior part of the capital structure more, the latter penalises the errors in the junior and mezzanine tranches. As emphasised in table 5.1, amongst the models here considered, the most accurate price match can be obtained using the stochastic correlation NIG model, here presented with the formula 4.36. In particular, this model performs better considering both the error indicators used for the analysis. The good performance of the model in terms of market fit needs to be analysed along with the computational speed: this model is just slightly more complicated then the stochastic correlation, but it is sensibly slower, i.e. a couple of minutes may be required for computation. The NIG distribution recorded a satisfactory performance overall but shows some difficulties in the pricing of senior and super senior tranches. The same consideration applies to α-stable distribution with the addition of a considerable error for the mezzanine 3-6%. The stochastic correlation model has registered a good performance, with the exclusion of the super senior 12-22% tranche which is underestimated by the model. The RFL can fit the market quotes very well but for the senior 9-12% a high bias has been recorded[2].

The fact that the best fit is obtained using together a fat tail distribution and a stochastic specification of the correlation leads as to the conclusion that neither a distribution or a stochastic correlation are able to explain the market prices. Is the necessary to mix the two models in order to achieve a satisfactory fit.

We recall here that the empirical analysis is limited to the LHP assumption, therefore the results above may vary under different assumptions.

---

[2]Note that here we have used a simple two point specification of the Andersen and Sidenius model. The factor $a_i(Y)$ can be modelled using e.g. a three point specification, thus obtaining better results at the cost of a slightly more complicated model(i.e. two parameters would be added in this case).

Acknowledgements

I would like to thank my supervisor Professor Lane P. Hughston and the Financial Mathematics Group at Kings College London. I would also like to thank my Professor Simonetta Rabino from the University of Rome "La Sapienza" for the support given during these years of studies.

Bibliography

[1] E.I. Altman, A. Resti, and A. Sironi (2003) Default Recovery Rates in Credit Risk Modeling: A Review of the Literature and Empirical Evidence, Stern School of Business New York University, Bergamo University and Bocconi University, working paper, online at http://pages.stern.nyu.edu/ ealtman/Review1.pdf.

[2] J. D. Amato and J. Gyntelberg (2005) CDS index tranches and the pricing of credit risk correlations *Bank for International Settlements Quarterly Review*, **March**, online at http://www.bis.org.

[3] L. Andersen, J. Sidenius and S. Basu (2003) All your hedges in one basket, *Risk*, **November**, 6772.

[4] L. Andersen and J. Sidenius (2005) Extension to the gaussian copula: Random recovery and random factor loadings, *Journal of credit risk*, 1(1), 29-70.

[5] O. E. Barndorff-Nielsen (1998) Processes of normal inverse Gaussian type, *Finance and Stochastics* **2**, 41-68.

[6] O. E. Barndorff-Nielsen (1997) Normal inverse Gaussian distributions and stochastic volatility modelling, *Scand. J. Statist.* **24**, 113.

[7] Various authors (2005) *Financial Stability Review, Bank of England*, **June**, pg. 55-58.

[8] M. Baxter (2006) Levy Process Dynamic Modelling of Single-Name Credits and CDO Tranches, *Nomura Fixed Income Quant Group*, working paper, online at www.nomura.com/resources/europe/pdfs/cdomodelling.pdf.

[9] Various authors (2004) Credit derivatives: Valuing and Hedging Synthetic CDO Tranches Using Base Correlations, *Bear Stearns*.

[10] T. Belsham N. Vause (2005) Credit Correlation: Interpretation and Risks, *Financial Stability Review, Bank of England* **December**.

[11] F. Black and J. C. Cox (1976) Valuing corporate securities: some effects of bond indenture provisions, *J. Finance* **31**, 351-367.

[12] C. Bluhm, L. Overbeck, and C. Wagner (2002) *An Introduction to Credit Risk Modeling*, Chapman Hall/CRC Financial Mathematics Series, Boca Raton.

[13] D.C. Brody, L.P. Hughston and A. Macrina (2005) Beyond hazard rates: a new framework for credit-risk modelling, King's College London and Imperial College London, Working paper, online at http://www.defaultrisk.com.

[14] X. Burtschell, J. Gregory, and J-P. Laurent (2005) A comparative analysis of cdo pricing models, ISFA Actuarial School and BNP Parisbas, working paper, online at http://www.defaultrisk.com.

[15] X. Burtschell, J. Gregory, and J-P. Laurent (2005) Beyond the gaussian copula: Stochastic and local correlation, ISFA Actuarial School and BNP Parisbas, working paper, online at http://www.defaultrisk.com.

[16] G. Casella and R. L. Berger (2001) *Statistical Inference*, Duxbury Press, 2nd ed.

[17] G. Chaplin (2005) *Credit Derivatives: Risk Managment, Trading and Investing*, Wiley.

[18] U. Cherubini, E. Luciano and V. Vecchiato (2004) *Copula methods in finance*, Wiley Finance.

[19] D. Duffie (1999) Credit Swap Valuation, *Financial Analyst journal*, **55(1)**, 73-87.

[20] B. Dupire (1994) Pricing with a smile, *Risk* **7**, **January**, 1820.

[21] A. Elizalde (2005) Credit Risk Models IV:Understanding and pricing CDOs, CEMFI and UPNA, working paper, online at www.cemfi.es/ elizalde.

[22] C.C. Finger (2004) Issues in the pricing of synthetic CDOs, *The Journal of Credit Risk*, **1(1)**, 113-124.

[23] A. Friend and E. Rogge (2005) Correlation at first sight, *Economic Notes* **34**, No. 2, 155-183.

[24] S.S. Galiani (2003) Copula functions and their application in pricing and risk managing multiname credit derivative products, Master Thesis Kings College London, working paper, online at http://www.defaultrisk.com.

[25] S.S. Galiani, M. Shchetkovskiy and A. Kakodkar (2004). Factor Models for Synthetic CDO Valuation, *Merrill Lynch Credit Derivatives*.

[26] J. Gregory and J-P. Laurent(2005) I Will Survive, *Risk*, **June**, 103-107.

[27] J. Gregory and J-P. Laurent (2003) Basket Default Swaps, CDOs and Factor Copulas, ISFA Actuarial School and BNP Parisbas, working paper, online at http://www.defaultrisk.com.

[28] L.P. Hughston and S. Turnbull (2000) Credit derivatives made simple. *Risk* **13**, 36-43.

[29] J. Hull (2003) *Options Futures and Other Derivatives* Fifth edition, Prentice Hall.

[30] J. Hull and A. White (2004) Valuation of a CDO and an n-th default CDS without Monte Carlo Simulation. *Journals of Derivatives 12***2**, 8-23.

[31] J. Hull and A. White (2005) The perfect copula, working paper, online at http://www.defaultrisk.com.

[32] R.A. Jarrow and S.M. Turnbull (1995) Pricing derivatives on Fnancial securities subject to credit risk, *J. Finance* **50**, 53-85.

[33] A. Kalemanova, B. Schmidt and R. Werner (2005) The normal inverse gaussian distribution for synthetic cdo pricing, working paper, online at http://www.defaultrisk.com.

[34] D. Lando (1998) On Cox processes and credit risky securities, Department of Operational Research, University of Copenhagen, working paper.

[35] H. E. Leland and K. B. Toft (1996) Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads, *J. Finance* **50**, 789-819.

[36] D. Li. (2000) On default correlation: A copula function approach, *Journal of Fixed Income* **9**,March, 43-54.

[37] D. Li. (1998) Constructing a credit curve, *Credit Risk, a Risk Special report*, **November**, 40-44.

[38] A. Lüscher (2005) The normal Synthetic CDO pricing using the double inverse Gaussian copula with stochastic factor loadings, Diploma Thesis ETH Zurich, working paper, online at www.msfinance.ch/pdfs/AnnelisLuescher.pdf.

[39] R. Mashal and A. Zeevi (2002) Beyond Correlation: Extreme Comovements Between Financial Assets, Columbia University, working paper.

[40] L. McGinty, E. Beinstein, R. Ahluwalia, and M. Watts (2004) Introducing base correlations, *Credit Derivatives Strategy JP Morgan*.

[41] R.C. Merton (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance*, **29**, MIT, 449-470.

[42] J. P. Nolan (1997) Numerical calculation of stable densities and distribution functions, *Commun. Statist. -Stochastic Models* **13**, 759774.

[43] J. P. Nolan (2006) *Stable Distributions - Models for Heavy Tailed Data*, Boston: Birkhäuser, In progress, Chapter 1 online at academic2.american.edu/ jpnolan.

[44] D. Prange and W. Scherer(2006) Correlation Smile Matching with Alpha-Stable Distributions and Fitted Archimedan Copula Models, *Risk Methodology Trading, Model Validation Credit Derivatives, Dresdner Kleinwort Wasserstein*, online at http://www.defaultrisk.com.

[45] L. Schloegl (2006) Stochastic Recovery Rates and No-Arbitrage in the Tranche Markets, Fixed Income Quantitative Research Lehman Brothers, presentation at King's College London Financial Mathematics seminars.

[46] P.J. Schönbucher (2000) Factor Models for Portfolio Credit Risk, Department of Statistics Bonn University, working paper.

[47] P.J. Schönbucher and E. Rogge. (2003) Modeling dynamic portfolio credit risk, working paper, online at www.finasto.uni-bonn.de/ schonbuc/papers/schonbucher rogge dynamiccreditrisk.pdf.

[48] P.J. Schönbucher (2003) Computational Aspects of Portfolio Credit Risk Managment, Risk Lab Workshop Computational Finance Zürich, September, working paper.

[49] P.J. Schönbucher (2003) *Credit derivatives pricing models*, Wiley.

[50] P.A.C. Tavares Nguyen, T.U. Chapovsky and A. Vaysburd (2004). Composite basket model, *Merrill Lynch Credit Derivatives*.

[51] J. Turc, P. Very and D. Benhamou (2005) Pricing CDOs with a smile, *SG Credit Research*.

[52] O. Vasicek (1987) Probability of Loss on Loan Portfolio, KMV Corporation, technical report.

[53] O. Vasicek (1991) Limiting Loan Loss Probability Distribution, KMV Corporation, technical report.

[54] M. van der Voort (2005) Factor copulas: Totally external defaults, ABN AMRO Bank and Erasmus University Rotterdam, working paper.

[55] M. van der Voort (2006) An Implied Loss Model, ABN AMRO Bank and Erasmus University Rotterdam, working paper.