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A Sequential Approach to the Characteristic Function and the Core in Games with Externalities*

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Abstract. This paper proposes a formulation of coalitional payoff possibilities in games with externalities, based on the assumption that forming coalitions can exploit a "first mover advantage". We derive a characteristic function and show that when outside players play their best response noncooperatively, the core is nonempty in games with strategic complements. We apply this result to Cournot and Bertrand games and to public goods economies.

Keywords. Core, cooperative games, externalities.

JEL Classification. C7

1 Introduction

This paper proposes a formulation of coalitional payoff possibilities in games with externalities, based on the assumption that forming coalitions can choose their strategies before the outside players. The idea that forming coalition can move first is motivated by the observation that in many economic environment, objections to cooperative agreements are carried out by directly choosing strategy in the ongoing strategic form game. Firms defecting from an industrial cartel can simply set a lower price; countries wishing not to comply with internationally agreed pollution abatements can simply set higher levels of production, and so on. In these cases, forming coalitions seem to exploit a positional advantage, very much as Stackelberg leaders, while outside players optimally react as followers. We accordingly construct a characteristic function assigning to each coalition its equilibrium payoff in an appropriately defined sequential game in which it moves as a Stackelberg leader. We

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study the core of the associated cooperative game and establish sufficient conditions *on the underlying strategic form game* for the existence of core imputations.

The problem of defining a characteristic function in games with externalities is a wellknown problem in cooperative game theory. Since the payoff of each player is affected by all the strategic choices made in the game, coalitional possibilities cannot be defined independently of the behaviour of external players. The relevance of these considerations becomes apparent once we observe that externalities are a common feature of most economic problems in which group formation is a relevant issue: cartel formation in oligopolies, international cooperation on trade, monetary and environmental issues, joint ventures, R&D associations and so on. This problem was in fact already considered by Von Neumann and Morgenstern (1944), who conceived the characteristic function of a coalitional game as the maximal aggregate payoff that a coalition can *guarantee* to its members (see also Aumann (1959)). Their formulation, in which players in the complementary coalition minimize the payoff of the forming coalition, does not address the problem of how coalitions make rational predictions about the reaction of excluded players. This is an open and highly debated issue in the theory of cooperation. Some important contributions have attempted to develope a general analysis by imposing consistency requirements on the overall coalition structure induced by the formation of a coalition (see Ray and Vohra (1997), Ray and Vohra (1999)), or by studying games of coalition formation (Hart and Kurz (1983), Bloch (1996), Bloch (1997), Yi (1997)). Other contributions have addressed specific economic problems by directly introducing assumptions on the predicted behaviour of outside players. In particular, Chander and Tulkens (1997) study the core of an economy with multilateral externalities adopting the following logical construct. A *coalition formation rule*, given *ex ante*, specifies the predicted coalition structure induced by the formation of every coalition. In particular, they propose that all players excluded from a forming coalition simply organize themselves into singletons. This rule is strictly related to the gamma game of coalition formation studied by Hart and Kurtz (1983), in which the decision of a subcoalition of players to separate from an existing coalition induces the remaining players to split up into singletons.[1] Given this coalition formation rule, the strategies induced in the underlying strategic form game by the formation of a coalition S are then naturally determined by letting S and each player outside S simultaneously maximize their own payoff. In this logic, Chander and Tulkens determine the value $v(S)$ as the Nash equilibrium payoff of S in the strategic form game played by S, acting as a single player, and by excluded players, acting as singletons. Because of the simultaneity of strategic choices, we will refer to this approach as *simultaneous conversion* of a strategic form game. To give a simple

[1] This rule should be contrasted with the other rule studied by Hart and Kurz, the delta rule, prescribing that all players announcing the same coalition finally belong to the same coalition (not necessarily the one they announced). Differently from the gamma game, if a coalition of players leaves a bigger coalition, the remaining players form the complementary coalition. For alternative asumptions on how excluded coalitions reorganize after defections, see Carraro and Siniscalco (1993).

example, consider three firms with linear technology competing à la Cournot in a linear demand market. Let a and b be the demand parameters and c be the marginal cost. If all firms merge together, they get the monopoly payoff $v(\{1,2,3\}) = \frac{A}{4}$, where $A = (a-c)^2/b$. If two firms, say firms 1 and 2, jointly leave the merger, a simultaneous duopoly game is played between the joint firm $\{1,2\}$ and firm 3, with equilibrium payoff $v(\{1,2\}) = \frac{A}{9}$. Similarly, if a single firm i leaves the merger, a triopoly game is played, with symmetric payoffs $\frac{A}{16} = v(\{i\})$ (these payoffs are obtained from the general expression $\frac{A}{(n-s+2)^2}$ expressing firms' profits in an n firm oligopoly). It can easily be checked that the equal split imputation $\left(\frac{A}{12}, \frac{A}{12}, \frac{A}{12}\right)$ is in the core. However, other asymmetric imputations belong to the core, such as $\left(\frac{3A}{32} \cdot \frac{3A}{32}, \frac{A}{16}\right)$, giving player 3 his reservation value $v(\{3\})$ and equally splitting the rest between the other two players.

The simultaneous conversion implicitly assumes that coalitional payoffs originate in two stages: a coalition formation stage, in which the coalition structure forms; a strategic form game, in which Nash strategies are played by each coalition. In fact, Nash strategies are a predictable outcome only if all elements of the game (the set of players, *i.e.*, the elements of the newly formed coalition structure, their payoff functions and strategy sets) are commonly known. In other terms, deviations from a generally agreed joint strategy are carried out by first publicly abandoning the negotiation process (as, for instance, a group of countries leaving the international negotiation table) and then playing the Nash equilibrium strategies of the induced simultaneous game. Although appropriate in certain cooperative environments, the simultaneous conversion fails to capture the dynamic nature of coalition formation that we claimed is common to several economic problems. As we argued at the beginning, coalitions can often deviate by directly choosing an alternative strategy in the underlying game, as do firms defecting from an industrial cartel by directly and unexpectedly setting a lower price. In order to explore this idea, we construct a characteristic function formally expressing the assumption that forming coalitions can move first. We stress here that we do not attempt to endogenize the coalition structure induced by a deviation, but we adopt the gamma assumption used in Chander and Tulkens (1997).[2] More precisely, we derive the coalitional value $v_\phi(S)$ as the perfect equilibrium payoff of S in the sequential game in which S chooses a strategy as leader, and the players in the complementary coalition $N \backslash S$ react simultaneously and noncooperatively. We refer to this operation as *sequential conversion* of the gamma game, denoted by (N, v_ϕ).

As an illustration of this approach, consider again the Cournot oligopoly example used above for the simultaneous conversion. As before, the grand coalition obtains the monopolistic profit $v(\{1,2,3\}) = \frac{A}{4}$. Now, if firms 1 and 2 jointly leave the merger, a Stackelberg quantity setting game with the joint firm $\{1,2\}$ as leader

[2] Although the definition of a characteristic function along these lines is compatible with any arbitrary coalition formation rule, we choose to focus on the gamma rule from the beginning to avoid confusion. However, we point out that proposition 1 extends to any coalition formation rule.

and firm 3 as follower originates. The leader's payoffs is given by $v(\{1,2\}) = \frac{A}{8}$. If a single firm, say firm 1, deviates, it becomes the leader in the sequential game in which firms 2 and 3 simultaneously set their quantities at the second stage. Firm 1's profit in equilibrium is given by $\frac{A}{12} = v(\{1\})$ (these numbers are obtained from the general expression $\frac{A}{4(n-s+1)}$ expressing the payoff of a leader with $(n-s)$ followers in a Stackelberg game). We use this example to discuss the main properties of the sequential conversion, formally established in the paper. We first note that every proper subcoalition of players does strictly better under the sequential conversion than under the simultaneous conversion. This directly implies that the sequential core is strictly included in the simultaneous core. As will be shown in theorem 1, this is a generic property for smooth games. Secondly, in the above example the sequential core consists of a unique, symmetric, imputation. Under the assumed linear structure, this remains true for any number of players. Unfortunately, the greater power of deviating coalitions under the sequential conversions yields an empty core as soon as the linear structure of the example is abandoned. In fact, since the above Cournot oligopoly exhibits strategic substitutes, forming coalitions (moving as Stackelberg leaders) enjoy a first mover advantage, and the profitability of deviations rule out the existence of stable imputations. Following the same intuition, stable cooperative outcomes could emerge when the power of deviating coalitions is less strong, that is, when the game has strategic complements. We show in proposition 1 that all smooth, symmetric games with strategic complements have a nonempty core. This nonemptiness result trivially extends to the simultaneous conversion.

The paper is organized as follows: the next section presents the general setup, introducing and comparing the simultaneous and the sequential conversions. Section 3 presents our existence result. Section 4 illustrates, in the framework of some wellknown economic applications, the mechanics underlying the existence result: the core is nonempty when leaders cannot exploit their positional advantage too much. Finally, section 5 concludes the paper.

2 The Model

2.1 Setup

We consider a set of players $N = \{1, .., i, .., n\}$, each endowed with a set $X_i \subset \mathbf{R}$ of feasible actions and a payoff function $u_i : X \to \mathbf{R}$, where $X \equiv \prod_{i \in N} X_i$. For each $S \subseteq N$ we denote by $u_S : X \to \mathbf{R}$ the function defined for all $x \in X$ by $u_S(x) \equiv \sum_{i \in S} u_i(x)$. We assume that utility is transferable, so that $u_S(x)$ is a well defined index of the aggregate utility of S. We will only consider continuous payoff functions. The strategic form game $\Gamma = \left(N, (X_i, u_i)_{i \in N}\right)$ is obtained from the above elements. A Nash equilibrium $\overline{x}$ of the game Γ is defined in the usual way. We will be considering games $\Gamma(S, x_S)$ derived by Γ restricting the set of players to a coalition $N \backslash S$ and fixing the strategies of the players in S to some vector x_S, with payoff functions defined in the obvious way. We will assume throughout the paper that the game Γ and $\Gamma(S, x_S)$ admits a unique Nash Equilibrium for all $S \subset N$ and $x_S \in X_S$. Although this may seem a strong restriction, we anticipate here that for the class of games covered by theorem 1 (supermodular games) our assumption of symmetric externalities (assumption 2) guarantees that Nash equilibria are Pareto rankable. We may therefore argue that, if multiple equilibria should exist, the optimal equilibrium would be chosen through some pre-play communication.

We will associate to the game Γ various cooperative games (N, v) by specifying characteristic functions $v : 2^N \to R_+$, where $v(S)$ expresses the maximal aggregate payoff attainable by coalition S in Γ. An imputation for (N, v) is a vector $z \in \mathbf{R}^n_+$ such that $\sum_{i \in N} z_i \leq v(N)$ and $z_i \geq v(i)$ for all $i \in N$.

Definition 1. *The core of the cooperative game (N, v), denoted $C(N, v)$, is the set of imputations $z \in \mathbf{R}^n_+$ such that $\sum_{i \in S} z_i \geq v(S)$ for all $S \subseteq N$.*

2.2 Simultaneous Conversion

As argued in the introductory section, the simultaneous approach to the derivation of a characteristic function for the game Γ views the value $v(S)$ as resulting from an implicit two stage process. At the first stage players announce coalitions, and a coalition structure including S forms according to some specific coalition formation rule. At the second stage, the formed coalitions play the Nash equilibrium strategy of the induced game. In this paper we will consider the gamma coalition formation rule, predicting that if a coalition S forms and breaks the agreement within the grand coalition, no other coalition forms, and outside players split up into singletons. This rule seems appropriate in some specific institutional settings: in some instances of international environmental agreements, for instance, treaties require the formation of at most one coalition (see, for instance, Murdoch and Sandler (1997) on the regulation of chlorofluorocarbon emissions). Similarly, the assumption of one coalition with fringe outside players is extensively used in the theory of industrial organization for the analysis of horizontal mergers (see Salant et al.(1983), Deneckere and Davidson (1985), Shaffer (1995)).

Formally, we associate with each coalition S the coalition structure $\pi_\gamma(S)$ whose elements are S and all players outside S as singletons. Letting $\Gamma(\pi_\gamma(S))$ denote the strategic form game played by the elements of $\pi_\gamma(S)$, the characteristic function $v_\gamma(S)$ is thus defined as the aggregate payoff of S in the (unique) Nash equilibrium $\bar{x}$ of the game $\Gamma(\pi_\gamma(S))$, *i.e.*,

$$v_\gamma(S) = \sum_{i \in S} u_i(\bar{x}) . \tag{1}$$

2.3 Sequential Conversion

The sequential conversion captures the idea that in some situations, coalitions can deviate from a joint agreement by simply changing their strategies in the underlying normal form game. Outside players, at least for some transitional period, have to react to coalitional deviations by choosing their strategies very much as followers in a Stackelberg game. Again, although the characteristic function can be defined under this approach for any arbitrary coalition formation rule, we will consider the specific case of the gamma rule. Let, as before, $\pi_\gamma(S)$ denote the coalition structure in which only S contains more than one player. Let $\mathbf{\Psi}(\pi_\gamma(S))$ be the sequential game in which S moves first choosing an action $x_S \in X_S$ and, at the second stage, the other elements of $\pi_\gamma(S)$ simultaneously choose an element out of their respective strategy sets. Let the function $f_{N\backslash S} : X_S \to X_{N\backslash S}$ map a joint strategy x_S of coalition S into the Nash equilibrium of the game $\Gamma(S, x_S)$, with f_j denoting its projection on the j-th element:

$$f_j(x_S) \in \underset{x_j \in X_j}{\arg\max}\, u_j(x_S, x_j, f_k(x_S)_{k \in N\backslash S\backslash j}), \ \forall j \in N\backslash S. \tag{2}$$

A perfect equilibrium of $\mathbf{\Psi}(\pi_\gamma(S))$ is a pair $(x_S^*, f_{N\backslash S})$ such that:

$$x_S^* \in \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i(x_S, f_{N\backslash S}(x_S)). \tag{3}$$

We denote by $x^*(S)$ the strategy profile $(x_S^*, f_{N\backslash S}(x_S^*))$. The assumption of continuous payoffs and the closedness property of the Nash correspondence graph (see, for instance, Fudenberg and Tirole (1991)) imply that S faces a continuous maximization problem in (3) so that, by Weiestrass' theorem, a perfect equilibrium of $\mathbf{\Psi}(\pi_\gamma(S))$ always exists. The characteristic function is here defined by assigning to each coalition S its aggregate payoff at the relevant perfect equilibrium:

$$v_\phi(S) = \sum_{i \in S} u_i(x^*(S)). \tag{4}$$

2.4 Sequential versus Simultaneous Conversion

In this section we examine the relation between the core of the cooperative games obtained under the simultaneous and sequential conversions of a given strategic form game Γ (henceforth, simultaneous and sequential cores). We first note that since the joint strategy $\bar{x}_S$ is a feasible choice for S in the maximization problem (3), and since $\bar{x}_{N\backslash S} = f_{N\backslash S}(\bar{x}_S)$, it follows that $v_\phi(S) \geq v_\sigma(S)$ for all $S \subset N$. In turns, this implies that the sequential core is weakly included in the simultaneous core. A more interesting question is whether this inclusion is strict under additional assumptions on the game Γ.

Assumption 1. *The function u_i is twice differentiable, $i = 1, 2, ..., n$.*

Given differentiability of payoffs, we can write the equilibrium change in S's payoff induced by a change in the strategy of its i-th member at a point x in the interior of X as follows:

$$\frac{du_S(x)}{dx_i} = \sum_{j \in N\backslash S} \left[\frac{\partial u_S(x)}{\partial x_i} + \frac{\partial f_j(x_S)}{\partial x_i} \frac{\partial u_S(x)}{\partial x_j} \right].$$

Considering the Nash equilibrium $\bar{x}$ of the game $\Gamma\left(\pi_\gamma\left(S\right)\right)$ (provided this is interior), we can use an envelope argument to express the change in S's payoff induced by an infinitesimal change dx_S in its joint strategy:

$$du_S\left(\bar{x}\right) = \sum_{i\in S}\sum_{j\in N\setminus S}\left[\frac{\partial f_j\left(\bar{x}_S\right)}{\partial x_i}\frac{\partial u_S\left(\bar{x}\right)}{\partial x_j}\right]dx_i.$$

It follows that coalition S can strictly increase its payoff with respect to the Nash payoff at $\bar{x}$ whenever the term $\left(\frac{\partial f_j(\bar{x}_S)}{\partial x_i}\frac{\partial u_S(\bar{x})}{\partial x_j}\right)$ is non-null for some $i \in S$ and $j \in N\setminus S$. Under this condition, $v_\phi\left(S\right) > v_\gamma\left(S\right)$. The term $\frac{\partial f_j(\bar{x}_S)}{\partial x_i}$ can be obtained by implicit differentiation of the first order necessary conditions for a Nash equilibrium of the game $\Gamma(S, \bar{x}_S)$:

$$\frac{\partial f_j\left(\bar{x}_S\right)}{\partial x_i} = -\frac{\frac{\partial^2 u_j(\bar{x})}{\partial x_i \partial x_j} + \sum\limits_{\substack{k\neq j\\ k\in N\setminus S}}\frac{\partial^2 u_j(\bar{x})}{\partial x_k \partial x_j}\frac{\partial f_k(\bar{x}_S)}{\partial x_i}}{\frac{\partial^2 u_j(\bar{x})}{\partial x_j^2}}.$$

This leads to the following result:

Proposition 1. *Let Γ be a strategic form game Γ satisfying assumption 1 and such that, for all $S \subset N$, the Nash equilibrium of the game $\Gamma\left(S, \bar{x}_S\right)$ is unique and interior. For all $S \subset N$, let $\frac{\partial^2 u_j(\bar{x})}{\partial x_i \partial x_j} \neq 0$ for at least one player $i \in S$ and one player j for which $\frac{\partial u_S(\bar{x})}{\partial x_j} \neq 0$. Then, either the cores of the associated games (N, v_ϕ) and (N, v_γ) are both empty or the former is strictly included in the latter.*

Proof. >From the above discussion, under the assumptions of this proposition, for all S we have $v_\phi(S) > v_\gamma(S)$. We also already know that $C\left(N, v_\phi\right) \subseteq C\left(N, v_\sigma\right)$. Then we just need to show that $C\left(N, v_\phi\right)\setminus C\left(N, v_\gamma\right) \neq \emptyset$. Note first that being defined by a series of linear weak inequalities, the set $C\left(N, v_\gamma\right)$ is closed and convex. The boundary of $C\left(N, v_\gamma\right)$ contains all allocations $z \in \mathbf{R}^n$ such that $\sum\limits_{i\in S} z_i = v_\gamma\left(S\right)$ for some $S \subset N$. By closedness of $C\left(N, v_\gamma\right)$, such boundary is included in $C\left(N, v_\gamma\right)$. Since $v_\gamma\left(S\right) < v_\phi\left(S\right)$, we can thus pick an arbitrary allocation w on the boundary relative to S, for which $\sum\limits_{i\in S} w_i = v_\gamma\left(S\right)$. Since $v_\gamma\left(S\right) < v_\phi\left(S\right)$, it follows that S improves upon w and $w \notin C\left(N, v_\phi\right)$.

3 A Class of Games with a Nonempty Core

In this section we identify a class of games allowing for a nonempty core under the sequential conversion. Here, the issue of existence is of particular interest; as suggested by the Cournot example presented in the introduction, the increased power of deviating coalitions with respect to the simultaneous conversion could yield an empty core. In particular, this can happen in games with strategic substitutes. We show in theorem 1 that all symmetric smooth games with strategic complements have a nonempty core. This existence result, when applied to the simultaneous conversion, can itself be regarded as a new contribution (we note here that the only existence result for the core of (N, v_γ) was obtained by construction by Chander and Tulkens (1997) for a multilateral externalities game with quasilinear preferences). An intuitive exposition of the proof is as follows. Games with strategic complements have the property that Stackelberg followers are better off than leaders. This is a wellknown result for symmetric duopolies (see, for instance, Gal-Or (1985)), and is extended to our setting of n players in lemmas 1 and 2. Suppose that a coalition, acting as leader, could improve on the efficient equal split allocation. Since every follower is better off than every leader, the sum of payoffs after the deviation would exceed the sum of the equal split allocation, violating efficiency of the latter.

Given differentiability of payoff functions, strategic complementarity is equivalent to the following condition:

$$\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \geq 0 \; \forall i, j \in N, i \neq j, \forall x \in X. \tag{5}$$

A wellknown theorem by Milgrom and Roberts (1990) directly implies that if payoffs satisfy (5) then the function $f_{N \setminus S}$ is non decreasing in x_S (see Theorem 4.2.2 in Topkis (1998)).[3]

For our main result, stated in theorem 1, we need some additional assumptions.

Assumption 2. (Symmetric players). *Players have identical strategy sets and payoff functions in the following sense: there exists* X *such that* $X_i = X$ *for all* $i \in N$ *and a function* $u : X \times X^{n-1} \to R$ *such that* $u(x,y) = u_i(x_i, x_{N \setminus i})$ *for all* $i \in N$ *and for all* $\left[(x,y), \left(x_i, x_{N\setminus i}\right)\right] \in X^n \times X^n$ *such that* $(x,y) = \left(x_i, x_{N \setminus i}\right)$.

Assumption 3. (Symmetric externality). *Either* $\dfrac{\partial u_i(x)}{\partial x_j} \geq 0$ *for all* $i, j \in N$ *and* $x \in X$ *or* $\dfrac{\partial u_j(x)}{\partial x_i} \leq 0$ *for all* $i, j \in N$ *and* $x \in X$.

Assumption 4. (Strict concavity). $u_i(x)$ *strictly concave in* x_i *for all* $i \in N$.

Assumption 2 requires that all players are identical in the sense that they all have the same strategy set and identical preferences over their own strategies and their competitors' strategies. The second assumption has been shown to play a crucial role in various cooperative game theory results (see, for instance, Milgrom and Roberts (1996), Yi (1999)), and requires that the sign of the effect of each player's action on

[3] In order to apply this result, we here exploit the fact that R is a chain (which, together with condition (8), implies that the game is supermodular) and the assumption of a unique Nash equilibrium (that implies that $f_{N \setminus S}$ is singlevalued, so that the greatest and least elements coincide).

the payoff of the rest of players is the same. We will denote the case of a positive sign as "positive externality" and the case of a negative sign as "negative externality".

To simplify notation, for a given action profile x_S we will denote by $\left(x_{S\backslash i}, y\right)$ the vector $(x_1, ...x_{i-1}, y, x_{i+1}, .., x_s)$.

Lemma 1. *Let $S \subset N$ and consider $x^* (S)$.*
(a) If externalities are positive, then $i \in S$ and $j \in N \backslash S$ imply $x_i^* \geq x_j^*$;
(b) If externalities are negative, then $i \in S$ and $j \in N \backslash S$ imply $x_i^* \leq x_j^*$.

Proof. **(a).** We proceed by contradiction. Suppose $x_i^* < x_j^*$ for some $i \in S$ and $j \in N \backslash S$. The next series of inequalities follows:

$$\frac{\partial u_i \left(x_S^*, x_{N\backslash S}^*\right)}{\partial x_i} > \frac{\partial u_i \left(x_{S\backslash i}^*, x_j^*, x_{N\backslash S}^*\right)}{\partial x_i} \geq$$
$$\frac{\partial u_i \left(x_{S\backslash i}^*, x_j^*, x_{(N\backslash S)\backslash j}^*, x_i^*\right)}{\partial x_i} = \frac{\partial u_j \left(x_S^*, x_{N\backslash S}^*\right)}{\partial x_j} = 0 \tag{6}$$

The first inequality follows by the strict concavity (assumption 4); the second by condition (5); the third by assumption 2, and the fourth by the equilibrium conditions defining the equilibrium reaction function $f_{N\backslash S}$. Note next that every $i \in S$ first order condition of problem (3) can be rewritten as

$$\sum_{h\in S} \left(\frac{\partial u_h \left(x^*\right)}{\partial x_i} + \sum_{j\in N\backslash S} \frac{\partial u_h \left(x^*\right)}{\partial x_j} \frac{\partial f_j \left(x_S^*\right)}{\partial x_i} \right) \equiv 0. \tag{7}$$

Let us examine an arbitrary element h of the summation over S in (7): by the assumption of case (a) of this lemma, the first term is non-negative if $h \neq i$; moreover, by (6), this term is strictly positive for $h = i$. This facts, together with the fact that $f_{N\backslash S}$ is increasing, imply that condition (7) can be satisfied only if $\frac{\partial u_h(x^*)}{\partial x_j} < 0$ for some $h \in S$, which contradicts the assumption of the lemma.

(b). The same contradiction argument used for case (a) can be proved by inverting the inequality signs in (6) in the appropriate manner.

Lemma 2. *Let $S \subset N$ and consider $x^*(S)$. If $j \in N \backslash S$ and $i \in S$ then $u_j(x^*) \geq u_i(x^*)$.*

Proof. The following inequalities hold for all $j \in N \backslash S$ and $i \in S$:

$$u_j \left(x_S^*, x_{N\backslash S}^*\right) \geq u_j \left(x_S^*, x_{(N\backslash S)\backslash j}^*, x_i^*\right) \geq u_j \left(x_{S\backslash i}^*, x_j^*, x_{(N\backslash S)\backslash j}^*, x_i^*\right). \tag{8}$$

The first part is implied by condition (2), while the second follows from lemma 1 and assumption (3). By the assumption of symmetric players, we also have

$$u_j \left(x_{S\backslash i}^*, x_j^*, x_{(N\backslash S)\backslash j}^*, x_i^*\right) = u_i \left(x_S^*, x_{N\backslash S}^*\right). \tag{9}$$

Inequalities (8) and (9) imply

$$u_j\left(x^*\right) \geq u_i\left(x^*\right),$$

which proves the result.

Theorem 1. *Let the game Γ satisfy assumptions 1-4 and exhibit strategic complementarity. Then, the associate cooperative game (N, v_ϕ) has a nonempty core.*

Proof. We prove the theorem showing that the equal split allocation giving $\frac{v_\phi\left(N\right)}{n}$ to each player in N is in the core of the game (N, v_ϕ). Suppose not, so that $v_\phi\left(S\right) > v_\phi\left(N\right)$ for some $S \subset N$. By lemma 2, the maximal payoff of players in S is weakly lower than the minimal payoff of players in $N \backslash S$. This implies that

$$\frac{\sum_{j \in N \backslash S} u_j\left(x^*\right)}{n-s} \geq \frac{\sum_{i \in S} u_i\left(x^*\right)}{s} = \frac{v_\phi\left(S\right)}{s},$$

so that

$$\frac{v_\phi\left(S\right)}{s} > \frac{v_\phi\left(N\right)}{n} \Rightarrow \frac{\sum_{j \in N \backslash S} u_j\left(x^*\right)}{n-s} > \frac{v_\phi\left(N\right)}{n}.$$

This in turns implies that

$$s\frac{\sum_{i \in S} u_i\left(x^*\right)}{s} + \left(n-s\right)\frac{\sum_{j \in N \backslash S} u_j\left(x^*\right)}{n-s} > s\frac{v_\phi\left(N\right)}{n} + \left(n-s\right)\frac{v_\phi\left(N\right)}{n}$$

or

$$\sum_{i \in N} u_i\left(x^*\right) > v_\phi\left(N\right)$$

which contradicts efficiency of $v_\phi(N)$.

The following corollary directly follows from the fact that $v_\phi(S) \geq v_\gamma(S)$ for all S.

Corollary 1. *Let the game Γ satisfy assumptions 1-4and exhibit strategic complementarity. Then, the associate cooperative game (N, v_γ) has a nonempty core..*

4 Discussion and Applications

Theorem 1 establishes sufficient conditions for the non emptiness of the sequential core defined in the present paper. The main condition, strategic complementarity in the sense of Bulow et al. (1985), is a property of the game in strategic form underlying the cooperative game. Crucial to our result is the wellknown property of games with strategic complements to generate nondecreasing best replies; in particular, the supermodularity of payoff functions implies that the Nash responses of players outside a forming coalition are a nondecreasing function of the strategies of coalitional members. In line with some well established results in the theory of industrial organization, this property ensures that excluded players, acting as followers, are better off than players in the leading coalition.[4] Deviations by proper subcoalitions of players are therefore not very profitable, while the grand coalition, not affected by this "deviator's curse", produces a sufficiently big aggregate payoff for stable cooperative outcomes to exist.

In this section we discuss our result and its main requirement in relation to some notable economic applications of game theory: Cournot oligopolies, Bertrand oligopolies and public goods economies. The analysis of Cournot games clearly illustrates the mechanics at work in theorem 1: as long as best replies are such that leading coalitions have not a "too" big positional advantage, stable cooperative outcomes exist. Bertrand games are traditionally games with strategic complements. We discuss them here as an example of the property of the sequential core to act as a refinement of the simultaneous core (see proposition 1). Finally, we work out an example replicating the economy with multilateral externalities studied by Chander and Tulkens (1997). This case is interesting both because it first motivated the use of the gamma core solution concept (see the exhaustive discussion in their paper) and because it generates a game with strategic substitutes for which the sequential core is empty although the simultaneous core is always nonempty.

4.1 Cournot Games

Some recent contributions (Amir (1996), see also Vives (2000)) have shown that the sufficient condition for a Cournot game without costs to be log-supermodular (and so best replies to be increasing) is a log-convex inverse demand function $P(.)$. Following Amir's example, consider the symmetric Cournot oligopoly with inverse demand function $P(X) = (X+1)^{-\alpha}$, where X is aggregate production and $\alpha \geq n$, with zero production costs and no capacity limits. In this case the Cournot game is log-supermodular, best reply functions are increasing and have a unique symmetric intersection. For this case, our theorem 1 implies that the sequential core is nonempty. This can be easily checked by assigning the numerical values $n = 3$ and $\alpha = 3$; in this case, the characteristic function under the sequential conversion is

$$v(S) = \frac{\alpha^{-2\alpha}}{\alpha - 1}\left(\alpha^2 - \alpha + (1-\alpha)(n-s)\right)^{\alpha}.$$

It can be shown that the term $\dfrac{v(S)}{s}$ is monotonically increasing in s; the equal split allocation giving

$$\frac{v(N)}{n} = \frac{(\alpha-1)^{\alpha-1}\alpha^{-\alpha}}{3}$$

to all players is therefore contained in the core. As shown by Amir (1996), introducing linear costs can make the game non log-supermodular and best replies non

[4] More precisely, lemma 2 proves that each outside players is better off than each coalitional member at the relevant sequential equilibrium.

monotonic. As an example, let costs of each firm be defined by the term $\frac{x}{4}$. It is immediately clear that, for $n = 2$ and $\alpha = 1$,

$$v(\{i\}) = .104 > .0261 = \frac{v(N)}{n}$$

implying that the core is empty.

However, the (sufficient) condition used in theorem 1 is far from being minimal: it is easy to construct examples of games with strategic substitutes and for which core allocations exist. A simple Cournot example can be used to illustrate how the same mechanics underlying the result of theorem 1 determine the nonemptiness of the core when strategies are substitutes. In particular, the core is nonempty when best replies are not *too* decreasing or, in other terms, when strategies are not too substitute. This in turn requires that the profit function does not decrease *too much* with other firms' output, a property mainly depending on the level of log-concavity of the inverse demand function $P(.)$. To show this, let $P(X) = (a - X)^{\beta}$, with $a > X$ (for $\alpha = 1$, this becomes the usual linear demand case). Note that $P(.)$ is log-concave (and the game is not log-supermodular) for $\beta > 0$, and best replies are decreasing. When production costs are zero, the Cournot game admits a unique Nash equilibrium $\bar{x}$ with $\overline{x}_i = \frac{a}{\beta+n}$ for every $i \in N$. Also, very simple algebra yields the following characteristic function:

$$v(S) = a^{\beta+1}\beta^{2\beta}(\beta+1)^{-\beta-1}(\beta+n-s)^{-\beta}.$$

By computing the difference between the equal split allocation and what a single player obtains by deviating as leader, we get

$$\frac{v_{\phi}(N)}{n} - v_{\phi}(\{i\}) = \beta^{\beta}(\beta+n-1)^{-\beta} n - 1 < 0 \Leftrightarrow \beta < 1.$$

It follows that when the demand is strictly concave ($\beta < 1$) the core is empty. However, when $\beta = 1$, the core is nonempty with the equal split allocation as a unique element. It is also easy to show that for $\beta > 1$ (convex inverse demand) the equal split allocation still belongs to the core. We conclude that in this case nonemptiness of the core only requires a not too strong log-concavity of $P(.)$. This ensures that the marginal revenue of each firm does not decrease too much with the rivals' output and hence a deviating coalition, by expanding its output as leader (see lemma 1) does not exploit too much its first mover advantage against complementary players. When this is the case, the sequential core of a Cournot game, which is a natural "strategic substitute" game, turns out to be nonempty.

4.2 Bertrand Games

Consider first the traditional symmetric Bertrand game with a homogeneous good, with market demand $D(p) = a - p$, $a > p$ and cost function $C(x) = \frac{x^2}{2}$. At the unique Bertrand equilibrium, the price equals the average cost. It follows that the core of the associated game (N, v_γ) includes all Pareto efficient allocations, corresponding to all possible distributions of the value $v_\gamma(N) = \frac{1}{2}a^2\frac{n}{2n+1}$. The sequential conversion yields the characteristic function $v_\phi(S) = \frac{1}{2}s\frac{a^2}{1+2n+n^2-s^2}$, directly implying that $C(N, v_\phi) \subset C(N, v_\gamma)$ (to see this, note that all imputations giving less than $\frac{1}{2}\frac{a^2}{2n+n^2}$ to at least one player are not in the core of the game (N, v_ϕ)).

The more complex case of differentiated goods can be used to illustrate the effect of complementarity on the relation between the simultaneous and sequential core. Denecker and Davidson (1985) show that when cartels (or mergers) and the fringe of outside firms set their price *à la* Nash, cartels are increasingly profitable in the number of members. Moreover, they also show that merging is more convenient for low degrees of product differentiation. Using the present paper's terminology, their theorem 2 proves that the term $\dfrac{v_\gamma(S)}{s}$ for their game is monotonically increasing in s, implying that the core of the associated game (N, v_γ) always contains at least the equal split allocation.

Turning now to the sequential conversion, note first that all assumption in proposition 1 for strict inclusion of the sequential core are satisfied in the above example. To have a more precise idea of the effect of the degree of good differentiation on the relation between the core of the games (N, v_ϕ) and (N, v_γ), consider the following example of a symmetrically differentiated triopoly with inverse demand function given by $p_i = a - x_i - b\sum_{j\neq i} x_j$. Note that (see Bloch (1995) and Shubik and Leviatan (1980) goods are complements for the range of $b \in [((n-1)/n), 0]$ and substitutes for $b \in [0, 1]$. Taking the direct demand function $x_i = \alpha - \beta p_i + \gamma\sum_{j\neq i} p_j$, with $\alpha = \frac{a}{(n-1)b+1}$, $\beta = \frac{1-(n-2)b}{(1-b)((n-1)b+1)}$ and $\gamma = \frac{b}{(1-b)((n-1)b+1)}$, we obtain the following coalitional values:

$$v_\gamma(\{i\}) = \frac{1}{4}\frac{\alpha^2\beta}{(\beta-\gamma)^2}, \qquad v_\phi(\{i\}) = \frac{1}{4}\frac{\alpha^2(2\beta+\gamma)^2}{(2\beta-\gamma)\left(2\beta^2-\beta\gamma-2\gamma^2\right)},\ i = 1, 2, 3$$

$$v_\gamma(\{i,j\}) = \frac{\alpha\left(2\beta^2-\gamma\beta-\gamma^2\right)}{\left(2\beta^2-2\gamma\beta-\gamma^2\right)}, \quad v_\phi(\{i,j\}) = \frac{1}{2}\frac{\alpha^2\left(2\beta-\beta\gamma-\gamma^2\right)(2\beta+\gamma)}{\left(2\beta^2-\gamma^2-2\beta\gamma\right)^2}$$

$$v_\gamma(\{1,2,3\}) = \frac{3}{4}\frac{\alpha^2}{(\beta-2\gamma)}, \qquad v_\phi(\{1,2,3\}) = \frac{3}{4}\frac{\alpha^2}{(\beta-2\gamma)}.$$

Numerical simulations show that when goods from complements ($b \leq 0$) become substitutes ($0 < b \leq 1$), the worth of every single firm as leader $v_\phi(\{i\})$ increases slightly more than its worth in the simultaneous case $v_\gamma(\{i\})$, while for every intermediate coalition (see figure below), $v_\phi(\{i,j\})$ (thick line) increases substantially more than $v_\gamma(\{i,j\})$ (thin line). As a consequence, for lower degrees of product differentiation, the sequential core becomes increasingly smaller than the simultaneous core. This is in line with intuition: in price competition, when goods become increasingly substitute, there is a higher incentive to reply to a rivals' rise of price with an even higher rise. A higher degree of strategic complementarity implies more upward sloping best replies. There is thus less incentive for deviating coalitions to

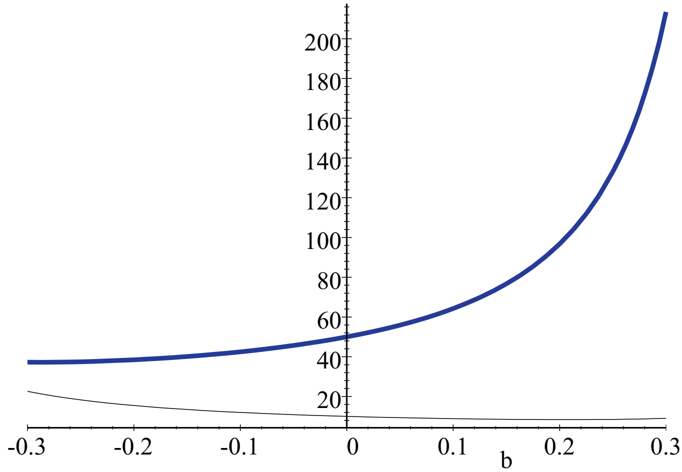

Fig. 1.

leave the grand coalition as Stackelberg leaders and, as a result, the sequential core shrinks.

4.3 Public Goods Economies

Consider an economy with two commodities, a public good q and a private good y. The set of agents is $N = \{1, 2, ., i, ., n\}$. Each agent $i \in N$ is endowed with the amount w_i of the private good. The public good is produced by each agent i out of the private good, with convex cost function

$$C(q) = \frac{q^2}{2}.$$

We let $q = (q_i)_{i \in N}$ and $Q = \sum_N q_i$. Quasilinear preferences are represented by the utility function

$$u_i(q, x_i) = (Q - \alpha Q^2) + y_i.$$

For this economy, the efficient production of public good implied by Samuelson's conditions is

$$Q^* = \frac{n^2}{1 + 2n^2\alpha}.$$

This setup is formally equivalent to the one considered by Chander and Tulkens (1997), and satisfies their assumption 1" for $\alpha \geq \frac{1}{2}$. This implies, by their main theorem, that for this range of values of α the gamma core (or simultaneous core in the present paper's terminology) is nonempty and includes the unique efficient allocation in which production costs are shared according to relative marginal valuations of the public good at the efficient production levels $(q_1^*, ..., q_n^*)$:

$$y_i = w_i - \frac{v_i'(Q^*)}{\sum_{j\in N} v_j'(Q^*)} \sum_{j\in N} \frac{q_j^*}{2} = w_i - \frac{1}{n}\left(\frac{Q^*}{n}\right)^2. \tag{10}$$

We will show that for this economy the sequential core is empty. We proceed by showing that each player, by deviating alone as a Stackelberg leader, improves upon the allocation given by (10). We do not solve the whole maximization problem of a single leader. It is sufficient to show that there exists one strategy for player i which gives him a higher payoff than the proposed allocation, given the equilibrium reaction functions of the other $(n-1)$ agents. Consider then the strategy of producing a zero amount of public good, i.e., $q_i = 0$. The reaction $f_{N\backslash i}(q_i = 0)$ is given by the following first order conditions:

$$1 - 2\alpha q_j (n-1) = q_j, \quad \forall j \neq i,$$

yielding

$$q_j = \frac{1}{1 + 2\alpha(n-1)}, \quad \forall j \neq i$$

and, aggregating, a total produced public good equal to

$$Q = \frac{n-1}{1 + 2\alpha(n-1)}.$$

We are now able to compare agent i's payoff after his deviation, or, in other terms, the value $v_\phi(\{i\})$, with his payoff in the proposed allocation given in (10), that we label u_i^*:

$$u_i^* = \frac{n^2}{1+2n^2\alpha} - \alpha\left[\frac{n^2}{1+2n^2\alpha}\right]^2 + w_i - \frac{1}{2n}\left(\frac{n}{1+2n^2\alpha}\right)^2;$$

$$v_\phi(\{i\}) = \frac{n-1}{1+2\alpha(n-1)} + w_i - \alpha\left[\frac{n-1}{1+2\alpha(n-1)}\right]^2.$$

It can be checked that $v_\phi(\{i\}) > u_i^*$ for $n \geq 2$ and $\alpha \geq 0.5$, implying that the allocation given by (10) is not in the sequential core. The following table reports figures for some values of n and $\alpha = 0.5$:

n	2	10	50	100
$v_\phi(\{i\}) - u_i^*$	0.224	0.8	0.96	0.98

This result directly implies that for this economy the core of the game (N, v_ϕ) is empty. Since $\sum_{i\in N} u_i^* \geq \sum_{i\in N} u_i(q, y_i)$ for all feasible vectors $(q, y_1, ..., y_n)$, it follows that every other feasible imputation z is such $z_j < u_j^*$ for some $j(z) \in N$. Since we have shown that $v_\phi(\{i\}) - u_i^* > 0$ for all $i \in N$, it follows that every feasible imputation z is objected by players $j(z)$ and is therefore not in the core.

5 Concluding Remarks

In this paper we have proposed a sequential approach to the determination of the characteristic function in games with externalities. The appropriateness of this approach depends on the strategic context in which players choose their actions. The sequential conversion seems appropriate in settings in which players' actions are perfectly monitored and players can fully commit to their strategies. In these cases, the derived characteristic function simply formalizes the assumption that a forming coalition anticipates the (optimal) reaction of outside players who face its formation as a *fait accompli.* On the other hand, a simultaneous conversion seems appropriate when the formation of a deviating coalition can be monitored before its strategy choice, and all strategies are chosen only once the new coalition structure has formed. The sequential structure of the game characterizing the payoff of forming coalitions has proved useful to establish sufficient conditions for the nonemptiness of the core. The crucial property for nonemptiness (strategic complementarity) is often encountered (and easily testable) in economic applications. Moreover, the property of the sequential core to act as a *refinement* of the simultaneous core yields interesting results in some economic examples.

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