The Effect of Uncertainty on Pollution Control Policy

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Abstract

I study a class of differential games of pollution control with profit functions that are polynomial in the global pollution stock. Given an emissions path satisfying mild regularity conditions, a simple polynomial ambient transfer scheme is exhibited that induces it in Markov-perfect equilibrium (MPE). Proposed transfers are a polynomial function of the difference between actual and desired pollution levels; moreover, they are designed so that in MPE no tax or subsidy is ever levied. Their applicability under stochastic pollution dynamics is studied for a symmetric game of polluting oligopolists with linear demand. I discuss a quadratic scheme that induces agents to adopt Markovian emissions strategies that are stationary and linear-decreasing in total pollution. Total expected ambient transfers are always non-positive and increase linearly in volatility and the absolute value of the slope of the inverse demand function. However, if the regulator is interested in inducing a constant emissions strategy then, in expectation, transfers vanish.

Keywords: differential games, stochastic dynamics, nonpoint source pollution, policy design

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1 Introduction

When individual pollution discharges are not observable, a regulator may wish to impose corrective policy measures that are based on observed total (ambient) pollution levels. As a result, there is an extensive literature on ambient transfers as a means of nonpoint-source pollution control going back to the work of Segerson [15], whose analysis builds on earlier theoretical work of Holmstrom [9]. Xepapadeas [18] extends Segerson's contribution to a dynamic setting under both deterministic and random specifications on pollutant accumulation. Since then a significant and growing literature has developed, shedding light into the theoretical design and practical implementation of ambient transfer schemes.

A common criticism of ambient transfers rests on their dependence on total pollution levels and, in particular, the fact that they may result in excessive and inequitable penalties. In an environment with no uncertainty Karp [10], drawing on earlier work of Karp and Livernois [11], investigates these concerns by comparing the tax burdens of (a plausible type of) Pigouvian and ambient taxes. In his model, both tax schemes are linear and evolve over time in an intuitive fashion; moreover, they are designed to induce a common steady state level of pollution. Karp finds that, in many plausible cases, the steady-state tax burden of ambient policy is lower, mitigating some of the concerns regarding its potential inequity.

At the same time, it is possible to design ambient transfers so that, in steady-state equilibrium, no tax or subsidy is ever imposed. In particular, one can make the tax scheme a function of the observed difference between actual and desired pollution levels, ensuring that when that difference is zero transfers accordingly vanish (Xepapadeas [18]). Indeed, Karp and Livernois' [11] ambient scheme (which is revisited in Karp [10]) can be modified in this way as well.[1]

One important point that the literature has largely left unaddressed is *how* desirable steady states are reached.[2] In particular, issues of potential inter-temporal welfare loss (in relation to the social optimum) en route to the equilibrium are not explored. By focusing on entire paths of emissions instead of just steady-state levels, this paper accommodates such concerns.

I initially focus on a class of deterministic infinite horizon differential games of pollution control in which agents' payoffs are polynomial in the total stock of pollution.[3] Moreover, I

[1] It should be noted, however, that the equilibrium analysis in [11] and [10] deals with necessary conditions for a MPE. Moreover, Xepapadeas [18] examines non-degenerate Markovian Nash equilibria, which may or may not be Markov perfect. This paper employs the MPE criterion described in Definition 4.4 of Dockner et al. [4], for which sufficient conditions are given in Theorem 4.4 of the same reference.

[2] An exception is the paper of Benchenkroun and Long [1]. But these authors do not consider ambient transfers.

[3] Specific instances of this model can be found in many previous contributions including [15, 17, 18, 5, 11, 6, 1,

allow for potential irreversibility or hysteresis effects in the pollution accumulation process. Such phenomena are typically observed in many ecological processes, notably so in shallow lake systems (see, for e.g., [13, 12]), and carry profound implications for pollution control policy. Given a particular emissions path satisfying a mild regularity condition, I exhibit a simple ambient transfer scheme that induces it in MPE. The scheme is a polynomial function of the observed difference between actual and desired total pollution and is designed so that, in MPE, no tax or subsidy is levied *at every point in time*, not just at the steady-state. Since induced emissions paths only depend on calendar time and are subgame-perfect, it is less likely that agents will find themselves off equilibrium. Thus, actual pollution levels will, at least in theory, plausibly match desired ones so that no transfers ever occur.[4]

An additional implication of the deterministic analysis is that, with moderate monitoring, first-best outcomes can be achieved in settings in which, without the use of policy, this is structurally impossible.[5] Or, more abstractly, that differential games with "bad" equilibrium properties can be, via the manipulation of the state-dependent component of agents' objective functions, transformed into ones possessing at least one MPE that is obvious and, where applicable, socially desirable.

This neat result breaks down when uncertainty is introduced into the pollutant accumulation process. From a purely technical point of view, the differential game becomes stochastic and its analysis is substantially complicated. Determining the temporal distribution of pollution as a result of agents' emissions rests on solving a stochastic differential equation, an exercise of considerable mathematical difficulty. Moreover, even when such an equation allows for analytical insight, the resulting process will typically fail to have a stationary distribution unless certain modeling assumptions are imposed. Such assumptions, while standard in the literature on stochastic models of economic growth (see Merton [14]), are not natural in a pollution control context.

To keep the analysis tractable, I concentrate on a symmetric model of polluting oligopoly introduced by Benchenkroun and Long [1]. Assuming linear demand, I focus on schemes that induce emissions strategies that are symmetric, stationary, and linear-decreasing in total pollution.

10, 7].

[4]Of course, deviations from the equilibrium can happen for a variety of reasons and are observed in experimental studies. A striking example can be found in Cochard et al. [2] where ambient transfers perform quite poorly. At the same time, and in contrast to [2], Spraggon [16] finds ambient transfers to be effective in inducing socially optimal behavior. These occasionally dramatic discrepancies between experimental studies are not thoroughly understood, though collusion seems to play a prominent role in the inefficiency observed in [2].

[5]Consider the linear quadratic game studied in Dockner and Long [5], which draws on foundational work by Tsutsui and Mino [17]. The best one can hope for in this setting (assuming the discount rate is low enough) is a MPE in nonlinear strategies that leads to socially optimal *steady state* pollution levels.

This class of target strategies is appealing for its simplicity; for this reason, its elements may be thought of as the result of a political process aiming to curb emissions.

Under this specification on target strategies, the stochastic process of total pollution accumulation is a special case of the well-studied Cox-Ingersoll-Ross process [3], which is extensively used in finance and whose probabilistic and asymptotic properties are completely characterized. The underlying stochastic control problem is tractable and it is possible to gauge the effect of ambient transfers. In particular, given a target strategy, I exhibit a simple quadratic ambient transfer scheme that induces it in MPE and provide closed-form expressions for expected transfers at any point in time. These (expected) transfers are always non-positive and their magnitude increases linearly with volatility and the absolute value of the slope of the inverse demand function. Moreover, I show that expected transfers vanish when the regulator wishes to induce a constant emissions strategy. To the best of my knowledge, this is the first paper that provides as precise a probabilistic analysis of dynamic nonpoint-source pollution control policy.

The paper is organized as follows. Section 2 discusses the deterministic model and its policy implications. Section 3 introduces the symmetric stochastic oligopoly game and derives the expected ambient tax burden for a stationary and linear target emissions strategy. Section 4 provides concluding remarks. Technical proofs are collected in the Appendix.

2 The Deterministic Model

Suppose there are n agents who are involved in a pollution-generating economic activity. Agent i's emissions at time t are denoted by $e_i(t)$ and the global stock of pollution by $x(t) \in \Re_+$. Agent i's profit at time t is denoted by

$$\pi_i(\mathbf{e}(t), x(t)),$$

where the function $\pi_i(\cdot) : \Re^{n+1} \mapsto \Re$ is strictly concave in e_i and *polynomial* decreasing in x.[6]

Agents' emissions are constrained by technology and labor, so that for all $i \in \{1, 2, .., n\}$ there exists $e_i^{max} \in \Re_+$ such that $e_i(t) \in \left[0, e_i^{max}\right]$. The time evolution of pollution is governed by the following differential equation

$$\dot{x}(t) = \sum_{i=1}^{n} e_i(t) - g(x(t)), \tag{1}$$

[6] In the nonpoint source pollution literature π_i is typically only a function of e_i, with the damages from pollution entering only in the social welfare function. I relax this assumption to allow for agents potentially incurring some of the costs of total pollution accumulation.

where $g(x)$ is a *polynomial* function that denotes the physical rate of natural purification, which satisfies

$$\lim_{x\to\infty} g(x) = \infty.$$

The function $g(\cdot)$ may have convex-concave nonlinearities in order to capture potential irreversibility or hysteresis effects in the pollution accumulation process.[7]

These assumptions imply that the global stock of pollution will be bounded so that, given an initial pollution level of x_0, $x(t) \in \left[0, x^{max}\right]$, where

$$x^{max} = \max\left\{x_0,\ \ \max\left\{x \in \Re_+ : g(x) = \sum_{i=1}^{n} e_i^{max}\right\}\right\}.$$

Suppose that the regulator imposes an ambient transfer scheme ϕ that is a function of total pollution and calendar time. Focusing on agent i, and denoting other agents' emissions by $\mathbf{e}^*_{-\mathbf{i}}$, ϕ gives rise to the following differential game:

$$\begin{aligned} \max_{e_i(\cdot)} \quad & \int_0^\infty e^{-\delta t}\left[\pi_i(e_i(t), \mathbf{e}^*_{-\mathbf{i}}(t), x(t)) + \phi_i(x(t), t)\right] dt \\ \text{subject to:} \quad & \dot{x}(t) = e_i(t) + \sum_{j\neq i}^{n} e_j^*(t) - g(x(t)) \\ & e_i(t) \in \left[0, e_i^{max}\right],\ \ x(0) = x_0, \end{aligned} \tag{2}$$

where δ is the discount rate. Moreover, the regulator wishes to induce emissions path $\hat{\mathbf{e}}$ where

$$\hat{\mathbf{e}} = \left\{\hat{e}_i(t) :\ \ t \geq 0,\ \ i \in \{1, 2, .., n\}\right\}.$$

Given an initial condition $\hat{x}_0$ on total pollution, $\hat{\mathbf{e}}$ has an associated pollution path $\hat{x}$, where

$$\hat{x} = \left\{\hat{x}(t),\ \ t \geq 0\right\}.$$

Theorem 1 shows that the regulator can induce $\hat{\mathbf{e}}$ in MPE with the use of a relatively simple ambient transfer scheme. First, I introduce some notation. Given a continuously differentiable emission path $\hat{\mathbf{e}}$, define the functions $f_i^{\hat{\mathbf{e}}} : \Re \times [0, \infty) \mapsto \Re$, where

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x, t) \quad = \quad & -\delta \int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x) dx + \frac{\partial}{\partial t}\left[\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x) dx\right] \\ & + \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\left[\sum_{j=1}^{n} e_j(t) - g(x)\right] - \pi_i(\hat{\mathbf{e}}(t), x), \end{aligned} \tag{3}$$

[7]For (a non-polynomial) example applicable to shallow lake dynamics, see Maler et al. [13] and Kossioris et al. [12].

for $i \in \{1, 2, .., n\}$. The model assumptions imply that the functions $f_i^{\hat{\mathbf{e}}}$ are well-defined and polynomial in x. Let $\hat{n}_i \geq 1$ denote the polynomial degree of $f_i^{\hat{\mathbf{e}}}$. The following theorem summarizes the paper's first result.

Theorem 1 *Consider a feasible and continuously differentiable emission path* $\hat{\mathbf{e}}$ *and the functions* $f_i^{\hat{\mathbf{e}}}$ *given by Eq. (3). Suppose that the functions* $V^i(x,t) : \Re \times [0, \infty) \mapsto \Re$*, where*

$$V^i(x,t) = -\int \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x) dx - \int_t^{\infty} f_i^{\hat{\mathbf{e}}}(\hat{x}(s), t) e^{-\delta(s-t)} ds,$$

are bounded from below and satisfy $\limsup_{t \to \infty} e^{-\delta t} V^i(\hat{x}(t), t) \leq 0$*, for all initial conditions* x_0 *and* $i \in \{1, 2, ..., n\}$*. The mechanism*

$$\hat{\phi}_i(x,t) = \sum_{k=1}^{\hat{n}_i} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t), t) \frac{[x - \hat{x}(t)]^k}{k!}, \quad i \in \{1, 2, ..., n\} \tag{4}$$

induces $\hat{\mathbf{e}}$ *in Markov perfect equilibrium.*

Proof. Consider the Hamilton-Jacobi-Bellman (HJB) equation for agent i, assuming that other agents choose the control paths $\hat{\mathbf{e}}_{-\mathbf{i}}$,

$$\delta V^i(x,t) - V_t^i(x,t) = \max_{e_i \in \left[0, e_i^{max}\right]} \left\{ \pi_i(e_i, \hat{\mathbf{e}}_{-\mathbf{i}}(t), x) + \hat{\phi}_i(x,t) + V_x^i(x,t) \left[e_i + \sum_{j \neq i} \hat{e}_j(t) - g(x) \right] \right\}. \tag{5}$$

To ensure that agent i's best response is given by $\hat{e}_i(t)$, the right-hand-side of Eq. (5) must be maximized at that level of emissions. As the function π_i is strictly concave in e_i, it is sufficient to impose that the value function $V^i(x,t)$ satisfy

$$V_x^i(x,t) = -\frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x). \tag{6}$$

Since $\hat{\mathbf{e}}(t)$ is by definition feasible no constraints are violated. Eq. (6) in turn implies

$$V^i(x,t) = -\int \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x) dx + \hat{A}_i(t), \tag{7}$$

where $\hat{A}_i(t)$ is a function that is, for the moment, unspecified. Substituting the value function given by (7) into the HJB conditions obtains the following equation

$$\delta \left[-\int \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x) dx + \hat{A}_i(t) \right] - \frac{\partial}{\partial t} \left[-\int \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x) dx + \hat{A}_i(t) \right] =$$
$$= \pi_i(\hat{\mathbf{e}}(t), x) + \hat{\phi}_i(x,t) - \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(t), x) \left[\sum_{j=1}^{n} \hat{e}_j(t) - g(x) \right]. \tag{8}$$

Recalling Eq. (3) and rearranging terms, Eq. (8) may be written in the following way

$$
\begin{aligned}
& \hat{\phi}_i(x,t) - \delta \hat{A}_i(t) + \frac{d}{dt}\hat{A}_i(t) = f_i^{\hat{\mathbf{e}}}(x,t) \\
\Rightarrow \quad & \sum_{k=1}^{\hat{n}} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!} - \delta \hat{A}_i(t) + \frac{d}{dt}\hat{A}_i(t) = f_i^{\hat{\mathbf{e}}}(x,t) \\
\Rightarrow \quad & \sum_{k=1}^{\hat{n}} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!} - f_i^{\hat{\mathbf{e}}}(x,t) - \delta \hat{A}_i(t) + \frac{d}{dt}\hat{A}_i(t) = 0. \qquad (9)
\end{aligned}
$$

Considering the Taylor expansion of $f_i^{\hat{\mathbf{e}}}(x,t)$ (recall that $f_i^{\hat{\mathbf{e}}}$ is polynomial in x) about $(\hat{x}(t),t)$, Eq. (9) obtains the following differential equation

$$
\frac{d}{dt}\hat{A}_i(t) - \delta \hat{A}_i(t) - f_i^{\hat{\mathbf{e}}}(\hat{x}(t),t) = 0. \qquad (10)
$$

Solving differential equation (10) yields

$$
\hat{A}_i(t) = e^{\delta t}\left[A_i(0) + \int_0^t f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)e^{-\delta s}ds\right].
$$

Setting

$$
\hat{A}_i(0) = -\int_0^{\infty} f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)e^{-\delta s}ds
$$

implies the particular solution

$$
\hat{A}_i(t) = -\int_t^{\infty} f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)e^{\delta(t-s)}ds.
$$

The theorem's assumptions imply that $V^i(x,t)$ satisfies sufficient conditions for optimality given by Theorem 4.4 in Dockner et al. [4]. ■

Remarks. Theorem 1 gives rise to two immediate corollaries.

Corollary 1 *All feasible and continuously differentiable target paths* $\hat{\mathbf{e}}$ *for which* $\frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t),x)$ *and* $\frac{\partial^2}{\partial t \partial e_i}\pi_i(\hat{\mathbf{e}}(t),x)$ *are bounded for all* $i \in \{1,2,..,n\}$ *satisfy the assumptions of Theorem 1.*

Proof. Recall that under our assumptions both $\mathbf{e}$ and x are bounded. The result follows. ■

As an example, Corollary 1 is satisfied for profit functions π_i that are separable in $\mathbf{e}$ and x, provided the target path $\hat{\mathbf{e}}$ is bounded from below by a strictly positive number (an interior assumption that is typically true of first-best solutions) and does not change too rapidly. These modeling assumptions are present in many well-studied dynamic games of pollution control.

Corollary 2 *Eq. (4) implies that* $\hat{\phi}(\hat{x}(t), t) \equiv \mathbf{0}$. *Thus, in MPE, the mechanism prescribed by Theorem 1 ensures that no transfers are ever made.*

Remarks. The practical relevance of Corollaries 1 and 2 hinges on the equilibrium concept employed in the analysis. On this score, Theorem 1 implies that a desirable emissions trajectory may be induced in Markov-perfect *open-loop* equilibrium. This finding suggests that agents are less likely to deviate from the equilibrium path as they do not condition their emissions on anything else but calendar time, knowing that their actions will constitute a best response regardless of perceived pollution levels. Indeed, the predictive capacity of this equilibrium concept is, at least in theory, quite robust. As a result, provided there is no uncertainty in the evolution of the global pollution stock, Corollary 2 indicates that it is unlikely for any tax or subsidy to ever be levied. In this sense, the deterministic problem is relatively easy to address.

3 A Model of Polluting Oligopolists with Stochastic Dynamics

In a physical environment in which pollutant accumulation evolves stochastically, state-dependent policy needs to be designed with caution. This is because Corollary 2 no longer holds and actual transfers will have to be made between agents and the regulating authority. Appropriate policy tools should arguably result in transfers that are moderate, or at the very least predictable (in a probabilistic sense). Xepapadeas [18] incorporates stochastic dynamics in his model but focuses on long-run asymptotics and does not discuss the dynamic effect of policy implementation. In this section, I attempt to address some of these issues in a systematic fashion.

Adopting the model of polluting oligopolists by Benchenkroun and Long [1], suppose there are n identical agents producing a homogeneous good. I take the output of each agent to equal his emissions and assume that each agent has a constant unit cost $c \geq 0$. Furthermore, assume that the underlying demand for the produced good is linear so that the inverse demand function $P(\cdot)$ is given by

$$P(\mathbf{e}) = A - b \sum_{j=1}^{n} e_j.$$

Hence, agent i's profit at time t is given by

$$\pi_i(\mathbf{e}(t), x(t)) = \left[A - b \sum_{j=1}^{n} e_j(t) \right] e_i(t) - c e_i(t).$$

Departing from a deterministic physical environment I follow Xepapadeas [18] and assume that the evolution of the pollution stock is governed by the following stochastic differential equation

$$dx(t) = \left[\sum_{j=1}^{n} e_j(t) - \beta x(t) \right] dt + \sigma \sqrt{x(t)} dW_t, \ \ x(0) = \hat{x}_0 \tag{11}$$

where $\beta > 0$, W_t is a Wiener process and $\sigma > 0$. Thus, pollutant accumulation is a diffusion process with instantaneous drift of $\sum_{j=1}^{n} e_i(t) - \beta x(t)$ and variance $\sigma \sqrt{x(t)}$.

Suppose that the regulator is interested in inducing a symmetric, stationary and linear-decreasing Markovian emissions *strategy*[8] $\hat{\mathbf{e}}$ so that

$$\hat{e}_i(x) = \frac{E}{n} - \frac{\gamma}{n} \cdot x, \ \ i \in \{1, 2, .., n\} \tag{12}$$

where $\gamma \geq 0$ and $E > 0$.

The attentive reader will notice that the class of target strategies given by Eq. (12), in combination with the fact that (in a stochastic framework) the state space is now unbounded, implies that emission rates may take negative values.[9] For reasons of analytical tractability that will become apparent, I relax the condition on feasible emissions so that, in contrast to the deterministic case, $e_i \in \Re$. This kind of simplifying assumption is commonly made even in the deterministic literature on games of pollution control (e.g., Dockner and Long [5], Benchenkroun and Long [1], Karp [10], Kossioris et al. [12]). At the same time, it should be noted that in my framework target strategies may be chosen so that negative emission rates occur with arbitrarily low probability (see Proposition 1). From a practical standpoint, negative emission rates may correspond to abatement measures agents undertake when total pollution exceeds critically high levels.

With this target strategy specification, the pollutant dynamics (11) can be rewritten in the following way:

$$dx(t) = (\beta + \gamma) \left[\frac{E}{\beta + \gamma} - x(t) \right] dt + \sigma \sqrt{x(t)} dW_t, \ \ x(0) = \hat{x}_0. \tag{13}$$

Eq. (13) is an instance of the celebrated Cox-Ingersoll-Ross [3] process, which is extensively used in finance. Fortunately, its evolution and steady-state properties are completely characterized. The following proposition summarizes.

[8]In contrast to the deterministic case, and since the problem at hand is stochastic, it is reasonable to condition the regulator's goal on current pollutant accumulation and not on calendar time.

[9]Note, however, that this same class of strategies ensures that total pollution is always non-negative (see Proposition 1) so that stochastic differential equation (11) is well-defined.

Proposition 1 *Stochastic differential equation (13) has a unique solution given by the diffusion process* $\{\hat{x}(t) : t \geq 0\}$ *where*

(a) $\hat{x}(t)$ *has a noncentral chi-square distribution with expectation*

$$\mathbf{E}[\hat{x}(t)] = \hat{x}_0 e^{-(\beta+\gamma)t} + \frac{E}{\beta+\gamma}\left[1 - e^{-(\beta+\gamma)t}\right]$$

and variance

$$\mathbf{Var}[\hat{x}(t)] = \hat{x}_0 \frac{\sigma^2}{\beta+\gamma}\left[e^{-(\beta+\gamma)t} - e^{-2(\beta+\gamma)t}\right] + \frac{E\sigma^2}{2(\beta+\gamma)^2}\left[1 - e^{-(\beta+\gamma)t}\right]^2.$$

(b) If $2E > \sigma^2$ *then* $\{\hat{x}(t) : t \geq 0\}$ *has a stationary distribution that is Gamma*$\left(\frac{2E}{\sigma^2}, \frac{\sigma^2}{2(\beta+\gamma)}\right)$.

Due to its relevance for financial applications, much numerical analysis has been undertaken to describe the precise nature of this noncentral chi-square distribution.[10] For our purposes, knowledge of its mean and variance will suffice.

In view of Proposition 1, the class of target strategies introduced in Eq. (12) holds considerable appeal. This is because, if somehow induced, its elements lead to an equilibrium pollutant accumulation process that can be described in precise probabilistic terms. Moreover, this same class of target strategies also lends itself to simple policy prescriptions. In particular, a quadratic ambient transfer scheme is presented that induces, in MPE, a linear stationary Markovian strategy, which satisfies the stability condition $2E > \sigma^2$.

Let $\hat{V}(x) : \Re_+ \mapsto \Re$, denote a function such that

$$\hat{V}(x) = -\gamma b \frac{n+1}{2n} x^2 - \left[A - c - \frac{b(n+1)}{n} E\right] x. \tag{14}$$

Slightly modifying the logic of Eq. (3) to fit our particular target strategy in a stochastic framework leads to the following expression

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x) &= \delta \hat{V}(x) - \left[A - b(E - \gamma x)\right] \frac{E - \gamma x}{n} + \frac{c}{n}\left[E - \gamma x\right] \\ &\quad - \hat{V}_x \left[E - (\beta+\gamma)x\right] - \frac{\sigma^2}{2} \hat{V}_{xx} x, \end{aligned} \tag{15}$$

for $i \in \{1, 2, .., n\}$. Similar to the proof of Theorem 1, this function will appear in the HJB equation of the stochastic control problem faced by the agents. The second result of the paper is summarized in the following Theorem.

[10] See Dyrting [8] for a discussion of efficient numerical methods to determine its probability distribution function.

Theorem 2 *Let* $\hat{\mathbf{e}}$ *denote a target Markovian strategy given by Eq. (12) such that* $2E > \sigma^2$*. The ambient transfer scheme* $\hat{\phi}$ *such that*

$$\hat{\phi}_i(x,t) = \frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2}\Big(\mathbf{E}[\hat{x}(t)]\Big)\frac{\big[x - \mathbf{E}[\hat{x}(t)]\big]^2}{2} + \frac{\partial f_i^{\hat{\mathbf{e}}}}{\partial x}\Big(\mathbf{E}[\hat{x}(t)]\Big)\big[x - \mathbf{E}[\hat{x}(t)]\big],\ \ i \in \{1,2,...,n\},$$

where all relevant quantities are defined in Proposition 1 and Eqs. (14) and (15), induces $\hat{\mathbf{e}}$ *in Markov-perfect equilibrium.*

Proof. See Appendix. ■

Note that the function $\frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2}$ is a constant such that

$$\frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2} = -\frac{b\gamma\big[\delta(n+1) + 2n(\beta+\gamma) + 2\beta\big]}{n}.$$

Thus, at any point in time an agent incurs an expected ambient transfer that is equal to

$$-\frac{b\gamma\big[\delta(n+1) + 2n(\beta+\gamma) + 2\beta\big]}{n}\frac{\mathbf{Var}[\hat{x}(t)]}{2}, \tag{16}$$

where $\mathbf{Var}[\hat{x}(t)]$ is given by Proposition 1. This transfer is non-positive and can be clearly seen to be zero in the deterministic $\sigma = 0$ case. Interestingly, it is also zero when γ vanishes (i.e., when the regulator wishes to induce a constant emissions strategy), or when the slope of the inverse demand function is zero. The next proposition gives a precise description of the total discounted cost of policy implementation.

Proposition 2 *The expected total discounted ambient transfer for an agent* i *is equal to*

$$-\sigma^2 b\gamma\left[\frac{\delta(n+1)}{n} + 2(\beta+\gamma) + \frac{2\beta}{n}\right]\frac{E + \delta\hat{x}_0}{\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))}.$$

Proof. I proceed to calculate

$$\mathbf{E}\left[\int_0^\infty e^{-\delta t}\hat{\phi}_i(\hat{x}(t),t)dt\right] = \mathbf{E}\left[\int_0^\infty e^{-\delta t}\left(\frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2}\frac{\big[\hat{x}(t) - \mathbf{E}[\hat{x}(t)]\big]^2}{2} + \frac{\partial f_i^{\hat{\mathbf{e}}}}{\partial x}\Big(\mathbf{E}[\hat{x}(t)]\Big)\big[\hat{x}(t) - \mathbf{E}[\hat{x}(t)]\big]\right)dt\right].$$

By Proposition 1 and Fubini's Theorem, the expectation and integral operators can be interchanged so that

$$\begin{aligned}
\mathbf{E}\Big[\int_0^{\infty} e^{-\delta t}\hat{\phi}_i(\hat{x}(t),t)dt\Big] &= \int_0^{\infty} e^{-\delta t}\frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2}\frac{\mathbf{Var}[\hat{x}(t)]}{2}dt = \frac{\partial^2 f_i^{\hat{\mathbf{e}}}}{\partial x^2}\frac{\sigma^2(E+\delta\hat{x}_0)}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} \\
&= -\frac{b\gamma\big[\delta(n+1)+2n(\beta+\gamma)+2\beta\big]}{n}\frac{\sigma^2(E+\delta\hat{x}_0)}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} \\
&= -\sigma^2 b\gamma\left[\frac{\delta(n+1)}{n}+2(\beta+\gamma)+\frac{2\beta}{n}\right]\frac{E+\delta\hat{x}_0}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))}
\end{aligned} \tag{17}$$

■

Proposition 3 *Expected total ambient transfers are non-positive. They are equal to zero when either b or γ are equal to zero. Their absolute value is decreasing in n, β and δ, increasing in E and γ, and linearly increasing in b, and σ^2.*

Proof. The monotonicity results regarding β and γ can be established by taking the appropriate derivatives of Eq. (17) and observing their signs. All other statements are obvious by inspection. ■

Remarks. Propositions 2 and 3 provide a precise account of the expected tax burden agents will bear. Whether ambient taxation is a viable policy option will, in large part, depend on whether this tax is overly excessive. Since the particulars of the target emissions strategy are up to the regulator's discretion, physical parameters such as volatility and the rate of natural purification will ultimately determine whether ambient transfers should be implemented. Between the two, volatility is arguably more important. This is because Eq. (17) shows that the tax burden diverges with volatility.

An important implication of Theorem 2, is that it is possible to induce strategies that reconcile many different considerations. One may wish, for instance, to induce a linear strategy that maximizes steady-state payoffs while ensuring that the mean and variance of steady-state pollution levels be below certain exogenously determined levels. In view of Proposition 1, determining such a target strategy (i.e., solving for the relevant E and γ) would amount to solving a two-variable nonlinear optimization problem with quadratic constraints.

4 Conclusion

This paper sheds light on the ability of ambient transfers to influence MPE behavior for a large class of differential games of pollution control. The analysis suggests that, under deterministic pollution accumulation, these policy tools are able to induce a wide set of emissions paths. Moreover, proposed schemes are designed so that, in equilibrium, no tax or subsidy is ever levied. The equilibrium concept that is used (Markov-perfect open-loop equilibrium) ensures that deviations from the equilibrium path are, at least in theory, relatively unlikely.

The robustness of these results is tested under a stochastic framework for pollutant accumulation. When physical dynamics are uncertain, it is no longer possible to guarantee zero transfers in equilibrium and it becomes important to gauge the scale of potential taxes or subsidies. This exercise is undertaken for a simple linear oligopoly model and a regulating authority that is interested in inducing symmetric, stationary, and linear-decreasing emissions as a function of total pollution. I derive closed-form expressions for expected ambient transfers at any point in time and find that they are always non-positive, with their magnitude increasing linearly with volatility and the slope of the inverse demand function. In addition, these expected transfers vanish if the regulating authority wishes to induce a constant emission strategy. The simplicity of the results implies that one may solve for the target strategy that maximizes profits subject to the constraint that the mean and variance of steady-state levels of pollution be below certain exogenously determined levels. A careful numerical study of this issue is left for future research.

Appendix

Proof of Theorem 2 Consider the Hamilton-Jacobi-Bellman equation for agent i,

$$\begin{aligned}\delta V^i(x,t) - V_t^i(x,t) \quad &= \quad \max_{e_i}\Big\{\big[A - b\sum_{j\neq i} e_j(x,t) - be_i\big]e_i - ce_i + \hat{\phi}_i(x,t) \\ &\quad + V_x^i(x,t)\Big[e_i + \sum_{j\neq i} e_j(x,t) - \beta x\Big] + V_{xx}^i(x,t)\frac{\sigma^2 x}{2}\Big\}. \qquad (18)\end{aligned}$$

Assuming that other agents choose the stationary Markovian strategies $\hat{e}_j(x) = \frac{E-\gamma x}{n}$ and dropping superscripts, Eq. (18) obtains

$$\begin{aligned}\delta V(x,t) - V_t(x,t) \quad &= \quad \max_{e_i}\Big\{\Big[A - \frac{b(n-1)}{n}(E-\gamma x) - be_i\Big]e_i - ce_i + \hat{\phi}(x,t) \\ &\quad + V_x(x,t)\Big[e_i + \frac{(n-1)}{n}(E-\gamma x) - \beta x\Big] + V_{xx}(x,t)\frac{\sigma^2 x}{2}\Big\}. \qquad (19)\end{aligned}$$

To ensure that agent i's best response is given by $\hat{e}_i(x) = \frac{E-\gamma x}{n}$, the right-hand-side of Eq. (5) must be maximized at that level of emissions. Thus, the value function $V(x,t)$ must satisfy

$$V_x(x,t) = -\Big[A - c - \frac{b(n+1)}{n}(E-\gamma x)\Big]. \qquad (20)$$

Following identical reasoning as in the proof of Theorem 1 the specification of $\hat{\phi}$ ensures that the value function

$$V(x,t) = \underbrace{-\gamma\frac{b(n+1)}{2n}x^2 - \left[A - c - \frac{b(n+1)}{n}E\right]x}_{\hat{V}(x)} - \int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)e^{-\delta(s-t)}ds. \tag{21}$$

solves the HJB equation (19) for the desired maximizing control $\hat{e}_i(x) = \frac{E-\gamma x}{n}$. That is, $V(x,t)$ satisfies the partial differential equation

$$\begin{aligned} \delta V(x,t) - V_t(x,t) &= \left[A - b(E-\gamma x)\right]\frac{E-\gamma x}{n} + \frac{c}{n}\big[E-\gamma x\big] + \hat{\phi}(x,t) \\ &+ \hat{V}_x(x)\big[E-(\beta+\gamma)x\big] + \hat{V}_{xx}(x)\frac{\sigma^2 x}{2}. \end{aligned} \tag{22}$$

But, while this choice of $V(x,t)$ solves the HJB equation, it is not possible to invoke standard sufficiency theorems to establish optimality. This is because the state space is no longer bounded; hence, the candidate value function will not be bounded or even bounded from below. For this reason, it is necessary to use an alternative sufficiency theorem given by Theorem 3.4 in Dockner et al. [4] that relies on finite horizon approximations of the value function. To this end, consider a finite-horizon version of our problem over $t \in [0,T]$ with no salvage function and postulate that a value function of the form

$$V(x,t;T) = A^1(t;T)x^2 + A^2(t;T)x + A^3(t;T) \tag{23}$$

solves the Hamilton-Jacobi-Bellman equation (19) for a maximizing control of $e_i(x) = \big[E-\gamma x\big]/n$, with the added terminal time constraint $V(x,T;T) = 0$. In particular,

$$\begin{aligned} &\left[\delta A^1(t;T) - \frac{d}{dt}A^1(t;T)\right]x^2 + \left[\delta A^2(t;T) - \frac{d}{dt}A^2(t;T)\right]x + \delta A^3(t;T) - \frac{d}{dt}A^3(t;T) \\ = \quad &\left[A - b(E-\gamma x)\right]\frac{E-\gamma x}{n} + \frac{c}{n}\big[E-\gamma x\big] + \hat{\phi}(x,t) \\ &+\big[2A^1(t;T)x + A^2(t;T)\big]\big[E-(\beta+\gamma)x\big] + A^1(t;T)\sigma^2 x. \end{aligned} \tag{24}$$

and

$$A^1(T;T) = A^2(T;T) = A^3(T;T) = 0.$$

Using Eq. (22), it is possible to cancel out $\hat{\phi}(x,t)$ and to rewrite Eq. (24) in the following way

$$\begin{aligned}
&\left[\delta A^1(t;T) - \frac{d}{dt}A^1(t;T) + \frac{\delta\gamma b(n+1)}{2n}\right]x^2 + \left[\delta A^2(t;T) - \frac{d}{dt}A^2(t;T) + \delta\left(A - c - \frac{Eb(n+1)}{n}\right)\right]x \\
&+\delta A^3(t;T) - \frac{d}{dt}A^3(t;T) + \underbrace{\delta\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)e^{-\delta(s-t)}ds - \frac{d}{dt}\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)e^{-\delta(s-t)}ds}_{\text{equivalently to Eq. (10), this expression equals } -f_i^{\hat{\mathbf{e}}}\left(\mathbf{E}[\hat{x}(t)]\right)}. \\
= \quad &\left[2A^1(t;T)x + A^2(t;T) + A - c - \frac{b(n+1)}{n}(E-\gamma x)\right]\big[E - (\beta+\gamma)x\big] + \left[A^1(t;T) + \frac{\gamma b(n+1)}{2n}\right]\sigma^2.
\end{aligned} \tag{25}$$

Collecting the terms involving x^2, $A^1(t;T)$ must satisfy the following differential equation

$$-\frac{d}{dt}A^1(t;T) + \big[\delta + 2(\beta+\gamma)\big]A^1(t;T) = -\gamma\frac{b(n+1)}{2n}(\delta + 2(\beta+\gamma)). \tag{26}$$

The solution of (26) satisfying $A^1(T;T) = 0$ is given by

$$A^1(t;T) = -\gamma\frac{b(n+1)}{2n}e^{(\delta+2(\beta+\gamma))t}\int_t^T(\delta+2(\beta+\gamma))e^{-(\delta+2(\beta+\gamma))s}ds = -\gamma\frac{b(n+1)}{2n}\left[1-e^{-(\delta+2(\beta+\gamma))(T-t)}\right],$$

so that

$$\lim_{T\to\infty}A^1(t;T) = -\gamma b\frac{n+1}{2n}. \tag{27}$$

Similarly, collecting the terms involving x, $A^2(t;T)$ must satisfy

$$\begin{aligned}
&-\frac{d}{dt}A^2(t;T) + \big[\delta+\beta+\gamma\big]A^2(t;T) = -\big[\delta+\beta+\gamma\big]\left[A - c - \frac{b(n+1)}{n}E\right], \\
&+\Bigg[\underbrace{A^1(t;T) + \frac{\gamma b(n+1)}{2n}}_{\frac{\gamma b(n+1)}{2n}e^{-(\delta+2(\beta+\gamma))(T-t)}}\Bigg]\big[2E+\sigma^2\big].
\end{aligned} \tag{28}$$

The solution of (28) satisfying $A^2(T;T) = 0$ is given by

$$A^2(t;T) = -\left[A-c-\frac{b(n+1)}{n}E\right]\left[1-e^{-(\delta+\beta+\gamma)(T-t)}\right]+Ke^{(\delta+\beta+\gamma)t}\int_t^T e^{-(\delta+2(\beta+\gamma))(T-s)}e^{-(\delta+\beta+\gamma)s}ds,$$

where $K = \big[2E+\sigma^2\big]\frac{\gamma b(n+1)}{2n}$. It is easy to see that $A^2(t;T)$ will satisfy

$$\lim_{T\to\infty}A^2(t;T) = -\left[A - c - \frac{b(n+1)}{n}E\right] \tag{29}$$

Finally, $A^3(t;T)$ will need to satisfy

$$\delta A^3(t;T) - \frac{d}{dt}A^3(t;T) = f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(t)]\big) + \left[A_2(t;T) + A - c - \frac{b(n+1)}{n}E\right]E. \tag{30}$$

Using identical reasoning as before it is easy to show that

$$\lim_{T\to\infty} A^3(t;T) = -\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)e^{-\delta(s-t)}ds, \tag{31}$$

so that collecting Eqs. (27), (29), and (31) obtains

$$\lim_{T\to\infty} V(x,t;T) = V(x,t). \tag{32}$$

Finally, it is necessary to examine the limiting properties of $\mathbf{E}\big[V(\hat{x}(t),t)\big]$:

$$\begin{aligned}
\limsup_{t\to\infty} e^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big] &= \limsup_{t\to\infty} e^{-\delta t}\Bigg[-\gamma\frac{b(n+1)}{2n}\mathbf{E}[\hat{x}(t)^2] - \Bigg[A - c - \frac{b(n+1)}{n}E\Bigg]\mathbf{E}[\hat{x}(t)] \\
&\quad - \int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)e^{-\delta(s-t)}ds\Bigg] \\
&= \limsup_{t\to\infty} e^{-\delta t}\Bigg[-\gamma\frac{b(n+1)}{2n}\Big[\mathbf{Var}[\hat{x}(t)] + \big[\mathbf{E}[\hat{x}(t)]\big]^2\Big] - \Bigg[A - c - \frac{b(n+1)}{n}E\Bigg]\mathbf{E}[\hat{x}(t)]\Bigg].
\end{aligned} \tag{33}$$

Given Proposition 1, it is easy to see that the process $\{\hat{x}(t) : t \geq 0\}$ converges to the relevant Gamma distribution in L_2 so that

$$\limsup_{t\to\infty} e^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big] = \lim_{t\to\infty} e^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big] = 0. \tag{34}$$

Given Eqs. (32) and (34), applying the stochastic equivalent of Theorem 3.4 in Dockner et al. [4] completes the proof. ■

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