On the Generalized Weitzman's Rule

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Abstract

Weitzman (1976) provides a foundation for net national product of a competitive economy as the annuity equivalent of the present discounted value of maximized consumption. This paper considers how Weitzman's rule should be modified if the competitive equilibrium is affected by the presence of market distortions. The paper first examines the model with external effects of capital in which there are spillovers of knowledge. The paper also studies the model with policy interventions where the policy maker seeks the second best allocation. The central concern of the paper is to elucidate the factors that generate a divergence between net national product and the welfare equivalence of maximized consumption. In discussing each model, the paper presents a typical example that has been widely discussed in the literature.

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1 Introduction

In his seminal contribution, Weitzman (1976) reveals the welfare significance of national income in a dynamic context. He shows that in a competitive economy, the current level of net national product (NNP) is a precise measure of the annuity equivalent of the present discounted value of maximized consumption. This fundamental observation, which we refer to as Weitzman's rule, means that the current level of national income contains all information concerning the welfare of current and future generations. Furthermore, Weitzman's finding plays a key role in considering the relevant issues of dynamic welfare economics such as sustainablility of economic development and intergenerational equity.

Weitzman's (1976) contribution led to a renewal of interest in the welfare implication of national income accounting. Since his fundamental proposition holds for the perfectly competitive economy in which agents are endowed with perfect foresight and have a linear utility function, the subsequent studies have investigated the income-welfare relationship under more general circumstances than those assumed in Weitzman (1976). For example, Asheim (1997) and Pembertin and Ulph (2001) re-examine the concept of NNP when the utility function is concave so that the consumption rate of interest is time dependent, while Arronson and Löfgren (1995) and Weitzman (1998) introduce uncertainty into the base model. In addition, Weitzman (1996) and Asheim (1997) explore the effect of exogenous technical progress. Despite those relevant extensions, there is still a gap between theory and reality if we intend to apply Weitzman's rule to interpret the actual economic data. In reality a market economy does not necessarily satisfy the ideal conditions for the competitive economy. There are many causes that generate market failures. Therefore, once we take into consideration the fact that behavior of a decentralized economy may diverge from that of a command economy, we have to examine alternative market environments in exploring the relationship between nation's wellbeing and the concept of net national product.

The purpose of this paper is to reconsider Weitzman's rule in the presence of market distortions. Among many possible causes for market failure, we focus on Marshallian externalities as well as on policy intervention. The first issue we deal with is the modification of Weitzman's rule for the economy with knowledge spillovers. More specifically, we study a dynamic economy where capital stocks generate knowledge externalities. This kind of external effect of capital was introduced by Arrow (1961) and Romer (1986), who provided one of the theoretical bases for recent development of growth economics. When capital stocks have positive externalities, it is easy to anticipate that NNP underestimates the annuity equivalent of the present discounted value of maximized consumption. We derive a modified version of Weitzman's rule in a general setting. We then apply the modified rule to a simple model of endogenous growth and evaluate the difference between the current level of NNP and the current wealth expressed by a discounted present value of current and future levels of consumption. In the existing literature, Arronson and Löfgren (1996) also discuss the income-welfare relationship in the context of Lucas' (1988) model of growth in which human capital is associated with external effects. Our formulation is more general than their setting. Additionally, in our specific example we conduct numerical experiments to evaluate the accuracy of NNP as a welfare measure.

Our second topic is to investigate how the base result would be modified if the government distorts competitive allocation. In particular, we consider the case where the benevolent government maximizes the private agents' welfare by using distortionary policy tools. When we deal with the optimizing government, we study two alternative strategies of the government: open-loop policy and feedback policy. In each case, NNP cannot be a precise measure of welfare equivalence of maximized consumption. Roughly speaking, the divergence comes from the fact that the marginal value of capital form the private perspective is different from the implicit value of capital form the government perspective. Weitzman's rule still holds if the value of net investment is measured in terms of the implicit prices of capital that support the optimal conditions for the government's plan. However, NNP measured by the market prices fails to evaluate the discounted present value of maximized consumption in a precise manner. After deriving general results, we explore a simple model of dynamic optimal taxation as a typical example.

The remainder of the paper is organized as follows. Section 2 reviews Weitzman's rule. Section 3 treats models with external effects of capital, while Section 4 studies models with policy interventions. Section 5 concludes the paper.

2 The Basic Results

2.1 Weitzman's Rule

Weitzman's rule is established under the following assumptions: (i) preferences and production technology are stationary, (ii) the society's felicity level is expressed as a discounted sum of utilities with a constant discount rate, and (iii) the society attains the intertemporally optimum resources allocation. Given those assumptions, Weitzman's rule may be expressed as

$$Y(t) = u(t) + p(t)\, z(t) = \rho \int_t^{\infty} e^{-\rho(s-t)} u(s)\, ds, \tag{1}$$

where $Y(t)$ is net national product (NNP) in terms of utility at time t, $u(t)$ is the utility level of consumption, $p(t)$ is vector of investment price in terms utility, $z(t)$ is vector of investment and $\rho\,(> 0)$ is a time discount rate. Pointed out by Kemp and Long (1982) and others, (1) is an expression of the Hamilton-Jacobi-Bellman equation for the dynamic optimization problem solved by the command economy that mimics the behavior of the corresponding decentralized economy. More generally, (1) is one of the conservation laws that can be derived by Noether's theorem on invariant transformations in optimizing dynamical systems: see Sato (Chapter 8, 1981) and Sato (1985) for detailed discussions on the income-wealth conservation laws in optimal growth models.[1]

In order to discuss Weitzman's rule in a general setting, we follow the formulation employed by Dasguputa and Mitra (1999). Consider a planning economy in which there are n capital goods and m consumption goods. Let us denote the vectors of capital stocks and consumption goods by $k \in \mathcal{R}_+^n$ and $c \in \mathcal{R}_+^m$, respectively. To avoid the index number problem, we assume that the instantaneous felicity of the society, u, can be expressed as a cardinal utility function

$$u = u(c), \quad u : \mathcal{R}_+^n \to \mathcal{R}_+,$$

which transforms the consumption vector into the level of felicity.[2] The technology set is assumed to be stationary and it is defined as $T \subset \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}_+^n$. The feasibility condition requires that $(u, z, k) \in T$. The technology set is a closed, convex subset of $\mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}_+^n$ and

[1] See also Sato and Ramachandran (Chapter 7, 1998) for expository discussion on the conservation laws..

[2] Dasgupta and Mitra (1999) call u the aggregate consumption. To avoid confusion, we refer to u as utility. It is also to be noted that u can be considered a money metric utility function. Weitzman (Chapter 6, 2002) presents a detailed exposition on the relationship between nominal national income and NNP in terms of utility.

it satisfies the standard regularity conditions.[3] Now define the projection on T in such a way that

$$S = \{(z, k) : (u, z, k) \in T \text{ for some } u\}.$$

Then we may define function $v : S \to R_+$ as follows:

$$v(z, k) = \max\{u : (u, z, k) \in T\}. \tag{2}$$

This function describes the maximum level of utility under given levels of investments and capital stocks. We assume that function $v(z, k)$ is at least twice continuously differentiable and strictly concave on S.

Under given settings, the dynamic optimization problem for the planner is formulated in the following manner:

$$\max \int_0^\infty e^{-\rho t} v(z(t), k(t))\, dt$$

subject to

$$\dot{k}(t) = z(t), \tag{3}$$

$$(z(t), k(t)) \in S, \tag{4}$$

and given initial levels of capital stocks, k_0. Letting $p(t)$ be vector of the capital stock prices (in terms of utility), the maximum principle yields the following conditions for an optimum:

$$p(t) = -v_z(z(t), k(t)), \tag{5}$$

$$\dot{p}(t) = \rho p(t) - v_k(z(t), k(t)), \tag{6}$$

and the transversality condition:

$$\lim_{t \to \infty} e^{-\rho t} p(t) k(t) = 0.$$

The optimal solution satisfies $(3), (4), (5), (6)$, together with (**??**) and the initial condition: $k(0) = k_0$.

Suppose that the above problem has the optimal solution. The value function at time t is then defined as

$$W(k(t)) \equiv \max\left\{\int_t^\infty e^{-(s-t)} v(z(s), k(s))\, ds : \dot{k}(s) = z(s) \text{ and } (z(s), k(s)) \in S\right\}.$$

[3]See Assumption 1 in Dasugupta and Mitra(1999).

When $W(k(t))$ is differentiable with respect to $k(t)$, the Hamilton-Jaobi-Bellman equation is given by

$$\rho W(k(t)) = \max_{z(t)} \{v(z(t), k(t)) + W_k(k(t)) z(t)\}. \tag{7}$$

Since it holds that $W_k(k(t)) = p(t)$, the right hand side of (7) equals the maximized Hamiltonian such that

$$M(k(t), p(t)) \equiv \max_{z(t)} \{v(z(t), k(t)) + p(t) z(t)\}.$$

The maximized Hamiltonian shown above is equal to the sum of consumption and net investment (in terms of utility), so that it represents net national product of a closed economy. Denote

$$Y(t) \equiv M(k(t), p(t)). \tag{8}$$

Then, noting that $v(z(t), k(t))$ expresses the optimum level of utility, (7) establishes Weitzman's rule (1).[4]

It is worth emphasizing that Weitzman's rule may hold in a wide class of dynamic economies. Since the elements in $k(t)$ represent not only stocks of physical capital but also other types of capital that can be devoted to production activities. Therefore, $k(t)$ may include human capital, knowledge capital and non-renewable natural resources. In a similar vein, the technology set would describe learning, education, research and development as well as regular production activities. This means that Weitzman's rule is established in many types of endogenous growth models in which perpetual growth can be sustained in the absence of exogenous technical progress.

2.2 Exogenous Technical Change

As pointed out above, since $k(t)$ may include a wide variety of capital stocks, endogenous technical change can be incorporated into Weitzman's rule, as far as it is fully captured by

[4] It is worth noting that our definition of NNP is the maximized level of consumption plus net investment in terms of (cardinal) utility. As pointed out by Asheim (1997) and Pemberton and Ulph (2001), if there is a single consumption good and the utility function is linear in consumption, the consumption rare of interest rate is constant so that the maximized Hamiltonian expresses the conventional national income in a closed, competitive economy. However, if we assume that the utility function is concave, the consumption rate of interest rate is time dependent and thus the price index problem arises to define a proper concept of national income. In this paper we avoid this issue by using conceptual income and wealth evaluated in terms of utility.

the planner. However, if there is exogenous technical change, the base result (1) should be modified. When exogenous technical changes is anticipated to occur, the production technology set is not stationary. As a result, the maximum utility defined by (2) is expressed as $u^*(t) = v(z(t), k(t), t)$, where $(u(t), z(t), k(t)) \in T(t)$. In this case the maximized Hamiltonian is written as

$$M(k(t), p(t), t) \equiv \max_{z(t)} \{v(z(t), k(t), t) + p(t) z(t)\} \tag{9}$$

and the value function is given by

$$W(k(t), t) \equiv \max \left\{ \int_t^\infty e^{-\rho(s-t)} v(z(s), k(s), s)\, ds : \right.$$
$$\left. \dot{k}(s) = z(s) \text{ and } (z(s), k(s)) \in S(t) \right\}.$$

The Hamilton-Jacobi-Bellman equation (7) is now replaced with

$$\rho W(k(t), t) = M(k(t), p(t), t) + W_t(k(t), t), \tag{10}$$

where $W_t \equiv \partial W / \partial t$. This shows a modified Weitzman's rule with exogenous technical change.[5] By (8) the modified rule (10) is written as

$$\frac{Y(t)}{W(k(t), t)} + \frac{W_t(k(t), t)}{W(k(t), t)} = \rho.$$

Intuitively, this equation is a non-arbitrage condition under which the net rate of return to the current level of wealth, Y/W, plus the anticipated 'capital gain' caused by exogenous technical progress, W_t/W, equals the discount rate.[6]

In order to evaluate $W_t(k(t), t)$, notice that from (6), (8) and (9) the time derivative of

[5] The original value function can be set as

$$\hat{W}(k(t), t) \equiv \max \int_t^\infty e^{-\rho s} v(z(s), k(s), s)\, ds,$$

and hence $e^{-\rho t} W(k(t), t) = \hat{W}(k(t), t)$. The Hamilton-Jacobi-Bellman equation in terms of $\hat{W}(.)$ is

$$-\hat{W}_t(k(t), t) = \max_{z(t)} \{v(z(t), k(t), t) + W_k(k(t), t) z(t)\}.$$

Using $W(.)$, this equation is written as (10).

[6] Hartwick and Long (1999) present a generalized version of (10) for the system in which the discount rate is time dependent.

the net national product (the maximized Hamiltonian) can be written as

$$\begin{aligned}\dot{Y}(t) &= M_k \dot{k}(t) + M_p \dot{p}(t) + M_t \\ &= \rho\left[Y(t) - u(t)\right] + v_t\left(k(t), p(t), t\right). \end{aligned} \tag{11}$$

Provided that $\lim_{\bar{t}\to\infty} \int_t^{\bar{t}} e^{-\rho(s-t)} v_s\left(z(s), k(s), s\right) ds$ exists, (11) yields the following:[7]

$$Y(t) = \rho \int_t^{\infty} e^{-\rho(s-t)} u(s)\, ds - \int_t^{\infty} e^{-\rho(s-t)} v_s\left(z(s), k(s), s\right) ds. \tag{12}$$

Therefore, we obtain

$$W_t\left(k(t), t\right) = \int_t^{\infty} e^{-\rho(s-t)} v_s\left(z(s), k(s), s\right) ds,$$

which shows that $W_t\left(k(t), t\right)$ evaluates the discounted present value of current and future consumption changes due to the exogenous technical progress. The modified rule (12) indicates that the maximum level of utility is given by

$$\rho \int_t^{\infty} e^{-\rho(s-t)} u(s)\, ds = Y(t) + \int_t^{\infty} e^{-\rho(s-t)} v_s\left(z(s), k(s), s\right) ds. \tag{13}$$

In general, technical progress expands the technology set $T(t)$, and hence the maximized instantaneous consumption v increases with time, that is, $v_s(.) > 0$ for all $t \geq 0$. Therefore, in the presence of exogenous technical progress, NNP underestimates the annuity equivalent of sustainable consumption.

Weitzman (1997) derives a simple relationship between the average maximized utility defined by$\bar{u}(t)$ $\left(= \rho \int_t^{\infty} e^{-\rho(s-t)} \bar{u}(t)\, ds\right)$ and the average NNP at t given by $\bar{Y}(t) = \rho \int_t^{\infty} e^{-\rho(s-t)} Y(s)\, ds$. He finds:

$$\bar{u}(t) = \left(1 + \frac{\lambda}{\rho - \gamma}\right) \bar{Y}(t), \tag{14}$$

where $\gamma = \frac{d}{dt}\bar{Y}(t) / \bar{Y}(t)$ and $\lambda = \rho \int_t^{\infty} e^{-(s-t)} \left(\frac{v_s}{v}\right) ds$. Since γ and λ respectively denote the average growth rates of income and technology, the term $\frac{\lambda}{\rho-\gamma}\bar{Y}(t)$ in (14) demonstrates the growth effect of exogenous technical change on the level of maximum utility at time t.[8]

[7] The general solution of (11) is

$$Y(t) = \rho \int_t^{\bar{t}} e^{-\rho(s-t)} c(s)\, ds - \int_t^{\bar{t}} e^{-\rho(s-t)} v_s\left(z(s), k(s), s\right) ds + e^{-\rho(\bar{t}-t)} Y(\bar{t}).$$

Therefore, (12) requires that $\lim_{\bar{t}\to\infty} e^{-\rho(\bar{t}-t)} Y(\bar{t}) = 0$.

[8] Pezzy (2002) modifies (14) by considering environmental issues.

2.3 The Command Optimum and the Market Economy

So far, the fundamental results, (1) and (12) , have been established for the command economy. In applying Weitzman's rule to a market economy, there are at least two issues to be noticed: one is a technical problem and the other is a practical one. First, even though we focus on the perfect-foresight competitive economy in which each market is complete, the necessary conditions for sustaining the competitive equilibrium may diverge from those for the command optimum. Dasugpta and Mitra (1999) find that a competitive equilibrium satisfies (1) if and only if the following 'investment value' transversality condition holds:

$$\lim_{t \to \infty} e^{-\rho t} p(t) \dot{k}(t) = 0, \tag{15}$$

which does not always coincide with the standard 'capital value' transversality condition (**??**) . Dasugpta and Mitra (1999) provide a simple example in which a regular competitive economy with concave utility and production functions does not fulfill (15) so that (1) fails to hold.

Second and more importantly, in the real world the decentralized economies may not attain the command optimum due to the presence of market distortions. There are many causes that make the market economy diverge from the command economy. Among others, in what follows we focus on two types of distortions: Mashallian externalities and policy interventions. As the model with exogenous technical progress suggests, it is easy to anticipate that net national product would diverge from the sustainable consumption in the presence of market distortions. Unlike the case of exogenous technical change, however, the difference between NNP and the annuity equivalent of the discounted present value of maximized consumption depends on the endogenous factors that generate market failures. By deriving modified versions of Weitzman's rule, we consider what factors may yield such a divergence.

3 Externalities

In this section, we treat models of competitive economy in which capital stocks generate external effects. Since the competitive economy cannot internalize externalities, net national product evaluated by the market prices does not equal the welfare equivalence of maximized utility. We focus on the wedge producing the difference between them.

3.1 Weitzman's Rule with External Effects

If the production technology of an individual firm is affected by external effects generated by the aggregate capital in the economy at large, the private technology depends on the social level of capital denoted by $\bar{k}$. Thus the technological feasibility condition is expressed as $(u, z, k) \in T\left(\bar{k}\right)$. This implies that the maximum level of utility can be shown by

$$v = v\left(z, k, \bar{k}\right),$$

where v is defined on $S = \left\{\left(z, k, \bar{k}\right) : (u, z, k) \in T\left(\bar{k}\right) \text{ for some } u\right\}$. Here, we assume that some elements in k represent knowledge capital that satisfy nonexcludabiltity and nonrivalry. According to Kehoe et al. (1992), the equilibrium conditions for a dynamic competitive economy with externalities can be characterized by the optimal solution for a pseudo-planning problem in which the planner is assumed to take the sequence of external effects as given. In our setting, the competitive economy coincides with the optimal solution for the following problem:

$$\max \int_0^{\infty} e^{-\rho t} v\left(z\left(t\right), k\left(t\right), \bar{k}\left(t\right)\right) dt$$

subject to

$$\dot{k}\left(t\right) = z\left(t\right),$$

$$\left(z\left(t\right), k\left(t\right), \bar{k}\left(t\right)\right) \in S,$$

together with the initial level of capital: $k\left(0\right) = k_0$. When solving this problem, the planner takes an anticipated sequence of external effects, $\left\{\bar{k}\left(t\right)\right\}_{t=0}^{\infty}$, as given.

The maximized Hamiltonian for the above problem is given by

$$M\left(k\left(t\right), p\left(t\right), \bar{k}\left(t\right)\right) \equiv \max_{z(t)} \left\{v\left(z\left(t\right), k\left(t\right), \bar{k}\left(t\right)\right) + p\left(t\right) z\left(t\right)\right\}.$$

The optimization conditions under a given sequence of $\left\{\bar{k}\left(t\right)\right\}_{t=0}^{\infty}$ are the following:

$$p\left(t\right) = -v_z\left(z\left(t\right), k\left(t\right), \bar{k}\left(t\right)\right), \tag{16}$$

$$\dot{p}\left(t\right) = \rho p\left(t\right) - v_k\left(z\left(t\right), k\left(t\right), \bar{k}\left(t\right)\right), \tag{17}$$

and the transversality condition: $\lim_{t\to\infty} e^{-\rho t} p\left(t\right) k\left(t\right) = 0$. Since the sequence of $\left\{\bar{k}\left(s\right)\right\}_{s=t}^{\infty}$ is external to the private agents, the value function at time t can be written as a function of $k\left(t\right)$ and t:

$$W\left(k\left(t\right), t\right) \equiv \max \int_t^{\infty} e^{-\rho(s-t)} v\left(z\left(s\right), k\left(s\right), \bar{k}\left(s\right)\right) ds.$$

As well as in the case of exogenous technical change, the Hamilton-Jacobi-Bellman equation is:

$$\rho W\left(k\left(t\right),t\right)=M\left(k\left(t\right),p\left(t\right),\bar{k}\left(t\right)\right)+W_t\left(k\left(t\right),t\right). \tag{18}$$

For analytical simplicity, let us assume that the number of agents is normalized to one. This assumption means that the consistency condition requires that

$$\bar{k}\left(t\right)=k\left(t\right)\text{ for all }t\geq 0. \tag{19}$$

It is now easy to show the following:

Proposition 1 *If consistency condition* (19) *holds, in the presence of external effects of capital, Weitzman's rule* (1) *is modified as*

$$Y\left(t\right)=\rho\int_t^{\infty}e^{-\rho(s-t)}u\left(s\right)ds-\int_t^{\infty}e^{-\rho(s-t)}v_{\bar{k}}\left(z\left(s\right),k\left(s\right),k\left(s\right)\right)z\left(s\right)ds. \tag{20}$$

Proof. Net national product equals the maximized Hamiltonian for the pseudo-planning problem, and therefore from (17) we have

$$\begin{aligned}\dot{Y}\left(t\right)&=M_k\dot{k}\left(t\right)+M_p\dot{p}\left(t\right)+M_t\\&=\rho\left[Y\left(t\right)-u\left(t\right)\right]+v_{\bar{k}}\left(z\left(t\right),k\left(t\right),k\left(t\right)\right)z\left(t\right).\end{aligned}$$

Thus, assuming that $\lim_{\bar{t}\to\infty}\int_t^{\bar{t}}e^{-\rho(s-t)}v_k\left(z\left(s\right),k\left(s\right),\bar{k}\left(t\right)\right)z\left(s\right)ds$ exists, the solution of the above differential equation is (20) ■

Comparing (18) with (20), we find:

$$W_t\left(k\left(t\right),t\right)=\int_t^{\infty}e^{-\rho(s-t)}v_{\bar{k}}\left(z\left(s\right),k\left(s\right),k\left(s\right)\right)z\left(s\right)ds.$$

In other words, the 'capital gain' due to the presence of externalities equals the discounted present value of net investment evaluated by the marginal external effect. Finally, (20) means that

$$\rho\int_t^{\infty}e^{-\rho(s-t)}u\left(s\right)ds=Y\left(t\right)+\int_t^{\infty}e^{-\rho(s-t)}v_{\bar{k}}\left(z\left(s\right),k\left(s\right),k\left(s\right)\right)z\left(s\right)ds. \tag{21}$$

Since NNP of the market economy is defined as $Y\left(t\right)\equiv u\left(t\right)+p\left(t\right)z\left(t\right)$, the right hand side of (21) expresses NNP from the social perspective.

3.2 An Example: Growth with External Increasing Returns

As a typical example of the model with capital externalities, we consider Romer's (1986) model of growth with external increasing returns. For notational simplicity, in this subsection we drop t from the variables unless it is useful to show it explicitly. Suppose that each firm produces a homogenous good by use of a single capital stock, k. The production function of an individual firm is specified as

$$y = f\left(k, \bar{k}\right), \quad f_k > 0, \;\; f_{kk} < 0, \;\; f_{\bar{k}} > 0,$$

where y is output of the firm and $\bar{k}$ denotes the capital stock in the economy at large. The presence of $\bar{k}$ in the production function of each firm represents the spillover effect of knowledge capital. In the following, we normalize the number of firms to one. Thus each variable also expresses the aggregate one and the consistency condition requires that $k = \bar{k}$ holds in every moment.

The market equilibrium condition for the final good is given by

$$y = c + \phi\left(\frac{z}{k}\right)k + \delta k, \tag{22}$$

where $\delta \in (0,1)$ is the depreciation rate of capital. Function $\phi(z/k)$ is assumed to be monotonically increasing and convex in z/k, which shows the presence of investment adjustment costs.[9] The instantaneous utility function of the representative consumer is specified as $u(c) = c^{1-\sigma}/(1-\sigma)$ $(\sigma \in (0,1))$.[10] We normalize the number of households to one as well. Hence, the pseudo-planning problem is to maximize

$$\int_0^\infty e^{-\rho t}\frac{1}{1-\sigma}\left[f\left(k,\bar{k}\right) - \delta k - \phi\left(\frac{z}{k}\right)k\right]^{1-\sigma} dt$$

subject to $\dot{k} = z$, a given sequence of external effects, $\left\{\bar{k}(t)\right\}_t^\infty$, and the initial level of capital, k_0.

[9] Following Romer's (1986) formulation, capital accumulation is determined by

$$z = \dot{k} = g\left(\frac{i}{k}\right)k,$$

where i is net investment spending. Function $g(i/k)$ is assumed to be strictly concave and monotonically increasing in i/k, which represents the presence of adjustment costs of investment. Since i is expressed as $i = g^{-1}(z/k)k$, from the equilibrium condition for the good market we have $\hat{c} + g^{-1}(z/k)k = f\left(k,\bar{k}\right)$. Denoting $g^{-1}(z/k) \equiv \phi(z/k)$, we obtain (22).

[10] Since we have assumed cardinal utility, σ is assumed to be less than 1 to keep $u(\hat{c})$ positive.

The maximized Hamiltonian for this pseudo-planning problem is

$$M\left(k,\bar{k},p\right) \equiv \max_{z(t)} \left\{ \frac{1}{1-\sigma} \left[f\left(k,\bar{k}\right) - \delta k - \phi\left(\frac{z}{k}\right)k \right]^{1-\sigma} + pz \right\}.$$

The first-order condition for optimal investment decision is

$$\left[f\left(k,\bar{k}\right) - \delta k - \phi\left(\frac{z}{k}\right)k \right]^{-\sigma} \phi'\left(\frac{z}{k}\right) = p. \tag{23}$$

By use of (23) , we obtain

$$M_{\bar{k}}\left(k,k\right) = \frac{p f_{\bar{k}}\left(k,k\right)}{\phi'\left(z/k\right)}.$$

Considering the equilibrium condition, $\bar{k} = k$ for all $t \geq 0$, the modified Weitzman's rule (20) is thus given by the following:

$$Y\left(t\right) = \rho \int_t^\infty e^{-\rho(s-t)} c\left(s\right) ds - \int_t^\infty e^{-\rho(s-t)} \frac{p\left(s\right) f_{\bar{k}}\left(k\left(s\right), k\left(s\right)\right) z\left(s\right)}{\phi'\left(z\left(s\right)/k\left(s\right)\right)} ds. \tag{24}$$

Since $p\left(s\right) > 0$, $f_{\bar{k}}\left(k,k\right) > 0$ and $\phi'\left(z/k\right) > 0$, the external effect evaluated by the second term in the right hand side of (24) has a positive value. Therefore, NNP is smaller than the welfare equivalence of maximized consumption.

Now consider a special case where $f\left(k,\bar{k}\right)$ is homogeneous of degree one in k and $\bar{k}$, so that the production function becomes $y = f\left(1,1\right)k$ when $k = \bar{k}$. In this case the production technology has an Ak property. Denoting $z/k = \gamma$, from the optimization condition, p follows

$$\begin{aligned} \dot{p} &= \rho p - v_k\left(z,k,k\right) \\ &= p\left[\rho - \frac{f_k\left(1,1\right) - \delta + \phi\left(\gamma\right) - \phi'\left(\gamma\right)\gamma}{\phi'\left(\gamma\right)}\right]. \end{aligned} \tag{25}$$

Note that (23) is rewritten as $\left[f\left(1,1\right) - \delta - \phi\left(\gamma\right)\right] = pk^{1-\theta}$. In the balanced-growth equilibrium $z/k = \dot{k}/k$ stays constant, so that $pk^{1-\theta}$ is constant over time as well. This means that $-\sigma\dot{k}/k = \dot{p}/p$ and thus from (25) we obtain

$$\gamma = \frac{1}{\sigma}\left[\frac{f_k\left(1,1\right) - \delta + \phi\left(\gamma\right) - \phi'\left(\gamma\right)\gamma}{\phi'\left(\gamma\right)} - \rho\right]. \tag{26}$$

This determines the balanced-growth rate. The right-hand side of the above monotonically decreases with γ, implying that there exists a unique rate of balanced growth. As is well known, the model with an Ak technology with homothetic utility function does not involve transition dynamics and the economy always stays on the balanced-growth path. Noting that

$z(s) = \gamma k(s)$ and $k(s) = k(t)\, e^{\gamma(s-t)}$, we find that the aggregate external effect over an infinite time horizon can be expressed as

$$\begin{aligned} W_t(k(t), t) &= \int_t^{\infty} e^{-\rho(s-t)} f_{\bar{k}}(1,1)\left[f(1,1) - \delta - \phi(\gamma)\right]^{-\sigma} \gamma k(s)^{1-\sigma} ds \\ &= \frac{\left[f(1,1) - \delta - \phi(\gamma)\right]^{-\sigma} f_{\bar{k}}(1,1)\,\gamma}{\rho - (1-\sigma)\gamma} k(t)^{1-\sigma}. \end{aligned}$$

The above show that the external effect on welfare is zero either if there is no externality $(f_{\bar{k}} = 0)$ or if there is no growth $(\gamma = 0)$. On the other hand, using (23), the current level of NNP is given by

$$\begin{aligned} Y(t) &= v(z(t), k(t), k(t)) + p(t)\, z(t) \\ &= \frac{1}{1-\sigma}\left[f(1,1) - \delta - \phi(\gamma)\right]^{1-\sigma} k(t)^{1-\sigma} + \left[f(1,1) - \delta - \phi(\gamma)\right]^{-\sigma} \phi'(\gamma)\,\gamma k(t)^{1-\sigma}. \end{aligned}$$

Therefore, the ratio of the average utility and the external effect, $W_t(k(t), t)/\bar{c}(t)$ is

$$\begin{aligned} \frac{W_t(k(t), t)}{\bar{u}(t)} &= \frac{W_t(k(t), t)}{Y(t) + W_t(k(t), t)} \\ &= \frac{f_{\bar{k}}(1,1)\,\gamma}{(1-\sigma)^{-1}\left[f(1,1) - \delta - \phi(\gamma) + \phi'(\gamma)\,\gamma\right]\left[\rho - (1-\sigma)\gamma\right] + f_{\bar{k}}(1,1)\,\gamma}. \end{aligned}$$

In order to present numerical examples, suppose that the production and adjustment costs functions are respectively specified as

$$y = A k^{\alpha} \bar{k}^{1-\alpha} \;\; (0 < \alpha < 1), \quad \phi(\gamma) = \gamma^{\beta} \;\; (\beta > 1).$$

Then we obtain:

$$f(1,1) = A, \quad f_k = \alpha A, \quad f_{\bar{k}} = (1-\alpha)\, A, \quad \phi(\gamma) - \gamma\phi'(\gamma) = (1-\beta)\,\gamma^{\beta}.$$

Given those specifications, (26) becomes

$$\gamma = \frac{1}{\sigma}\left[\frac{\alpha A - \delta + (1-\beta)\,\gamma^{\beta}}{\beta\gamma^{\beta-1}} - \rho\right].$$

Additionally, $W_{\bar{k}}/\bar{u}(t)$ is written as

$$\frac{W_{\bar{k}}(k(t), t)}{\bar{u}(t)} = \frac{(1-\alpha)\, A\gamma}{(1-\sigma)^{-1}\left[A - \delta + (\beta-1)\,\gamma^{\beta}\right]\left[\rho - (1-\sigma)\,\gamma\right] + (1-\alpha)\, A\gamma}. \tag{27}$$

When there is no adjustment cost for investment, we may set $\beta = 1$. If this is the case, $\gamma = \frac{1}{\sigma}(\alpha A - \delta - \rho)$ so that (**??**) is reduced to

$$\frac{W_{\bar{k}}(k(t), k(t))}{\bar{u}(t)} = \frac{(1-\alpha)\, A\gamma}{(1-\sigma)^{-1}(A - \delta)\left[\rho - (1-\sigma)\,\gamma\right] + (1-\alpha)\, A\gamma}. \tag{28}$$

Let us examine some numerical examples for the case where there are no adjustment cost $(\beta = 1)$.[11] First, consider the following set of parameter values:

$$\alpha = 0.4, \quad \delta = 0.05, \quad \rho = 0.05, \quad A = 0.3, \quad \sigma = 0.8$$

Here the income share of capital, α, is 0.4. The magnitudes of the depreciation rate $\delta = 0.05$ and the discount rate $\rho = 0.05$ are frequently used in the calibrated real business cycle models. The value of σ is assumed to be less than one to make the utility positive, but it is not so small that the elasticity of intertemporal substitution in consumption $(1/\sigma)$ is close to one. The value of A is set to obtain realistic levels of the balanced-growth rate γ. Given those parameter values, the balanced-growth rate is $\gamma = 0.025$ and $W_t/\bar{u}(t) = 0.074$. This means that the current level of NNP expresses almost 93% of the sustainable consumption, so that $Y(t)$ is still a good indicator of welfare measure even in the presence of market failure. However, the result is sensitive to the parameter values, in particular, the long-term growth rate. Now set $\alpha = 0.4$ and $A = 0.35$. Other things being equal, the balanced-growth rate is $\gamma = 0.0625$, so that the economy attains relatively high growth. In this case, $W_t/\bar{u}(t) = 0.1489$, and hence the external effect shares 15% of the sustainable consumption. If $\alpha = 0.4$ and $A = 0.4$, then the balanced-growth rate is $\gamma = 0.075$ and $W_t/\bar{u}(t) = 0.227$.

In the above examples, we set $\alpha = 0.4$. Although this is a plausible magnitude for the income share of capital, the external effect, which equals $1 - \alpha = 0.6$, is too high from the view point of the existing studies on estimation of scale economies: see, for example, Basu and Fernald (1997). Note that if capital k involves human capital, α may take a larger value. To examine the case where k is a composite of physical and human capital, let us assume that $\alpha = 0.8$ and $A = 0.15$. In this case $\gamma = 0.025$ and we have $W_t/\bar{u}(t) = 0.032$. Thus accuracy of NNP as a welfare measure increases with α. If $\alpha = 0.8$ and $A = 0.25$, then $\gamma = 0.075$ and $W_t/\bar{u}(t) = 0.0967$. Consequently, when the income share of capital is high, the welfare effect of capital externality is relatively small even if the economy grows at a rapid rate of 7.25% per year.

Based on his formula (14), Weitzman (1997) claims that the presence of exogenous technical progress may significantly elevate the sustainable consumption so that NNP is much smaller

[11] Unless β has an unusually large value, we obtain the similar results as those shown below even though there are investment adjustment costs.

than the welfare equivalence of maximized consumption. For example, if the rate of technical progress is $\lambda = 0.01$, the growth rate is $\gamma = 0.025$ and $\rho = 0.05$, then $\lambda / (\rho - \gamma) = 0.4$. Therefore, even if the average growth rate is only 2.5% per year, $W_t / \bar{Y}(t) = 0.4$ in the case of exogenous technical progress. In our notation this means that $W_t / \bar{u}(t) = 0.4/1.4 = 0.285$. This relatively large value comes from the fact that Weitzman (1997) assumes that the utility function is linear and capital does not depreciate. In fact, if $\sigma = 0$ and $\delta = 0$ in our model, (28) becomes

$$\frac{W_{\bar{k}}(k(t), k(t))}{\bar{u}(t)} = \frac{(1-\alpha)\gamma}{\rho - \gamma + (1-\alpha)\gamma}$$

When $\alpha = 0.4$, $\gamma = 0.025$ and $\rho = 0.05$, we have $W_{\bar{k}} / \bar{u} = 0.285$. This example demonstrates that preference and capital depreciation would play relevant roles in evaluating the welfare effect of external increasing returns.

4 Policy Intervention

We now turn our attention to the economy where the government distorts the market equilibrium. Since in a decentralized economy private agents take the government's actions as given, the government's intervention plays the similar role as that of external effects discussed above. However, unlike the case of externalities, we should specify government's policy making process in order to evaluate the distorting effects. In what follows, we consider optimization as well as non-optimizing policy makers.

4.1 Weitzman's Rule with Policy Intervention

Suppose that the government's policies affect consumption and production decisions of the private agents. In general, such a situation may be described by assuming that the level of maximum consumption depends on the government's behavior. Let us denote the vector of policy variables by $g(t) \in \Gamma \subset \mathcal{R}^r$, where Γ expresses a set of constraints on the policy-making decisions. Hence, the maximum utility under policy intervention is shown by

$$v(t) = v(z(t), k(t), g(t)). \tag{29}$$

The private agents take the sequence of policy variables, $\{g(t)\}_{t=0}^{\infty}$, as given. As a consequence, the pseudo-planning problem that may characterize the competitive equilibrium is as follows:

$$\max \int_0^{\infty} e^{-\rho(s-t)} v(z(s), k(s), g(s))\, ds$$

subject to

$$\dot{k}(t) = z(t),$$

$$(z(t), k(t), g(t)) \in S,$$

plus an anticipated sequence of policies, $\{g(t)\}_{t=0}^{\infty}$, and the initial value of $k(0)$. The point is that, as well as in models with external effects of capital, the private agents select their optimal plan by taking the government's actions as external effects.

The necessary conditions for an optimum includes

$$p(t) = -v_z(z(t), k(t), g(t)), \tag{30}$$

$$\dot{p}(t) = \rho p(t) - v_k(z(t), k(t), g(t)). \tag{31}$$

From (30) the optimum investment at time t can be written as

$$z(t) = z(k(t), p(t), g(t)). \tag{32}$$

Obviously, if the policy variables are kept constant over time ($g(t) = g$ for all $t \geq 0$), the fundamental formula (1) still holds, so that NNP can fully capture the present value of current and future consumption. Similarly, if the government selects the sequence of $\{g(t)\}_{t=0}^{\infty}$ without considering the private sector's behavior and if private agents perfectly anticipate $\{g(t)\}_{t=0}^{\infty}$, the policy intervention plays essentially the same role as that of exogenous technical change so that we obtain (12). On the other hand, if the policy maker adopts a feedback rule that relates $g(t)$ to the current levels of $z(t)$ and $k(t)$ in such a way that

$$g(t) = \theta(z(t), k(t));\ \theta: \mathcal{R}^n \times \mathcal{R}_+^n \to \mathcal{R}^r,$$

then the effects of policy distortion are similar to the distortion generated by external effects of capital. In particular, if the policy variables depend on the level of capital stocks alone, i.e. $g(t) = \theta(k(t))$, the role of policy distortion is exactly the same as that of capital externalities. In this case, (20) becomes

$$Y(t) = \int_t^{\infty} e^{-\rho(s-t)} c(s)\, ds - \int_t^{\infty} e^{-\rho(s-t)} v_g(k(s), z(s), g(s))\, \theta_k(k(s))\, z(s)\, ds. \tag{33}$$

The second therm in the right-hand-side of (33) shows the total welfare effect of policy intervention evaluated by consumption.[12]

The above argument has assumed that the policy maker's behavior is exogenously specified. Now suppose that the benevolent government chooses its policy variables in order to maximize the welfare of the private sector. If this is the case, the private agents select their investment and consumption plans under a given sequence of policies determined by the policy maker. On the other hand, policy maker decides its optimal policies subject to the optimizing behavior of the private agents as well as to its own constraints such as the budget and resource constraints. Formally speaking, this kind of problem can be formulated as a Stackelberg differential game in which the policy maker plays a leader's role and the private agents are followers. In general, the optimal strategy of the policy maker may be presented either as open-loop policies or as the feedback policies. We will examine those alternative strategies in turn.

First assume that the government adopts an open-loop policy under which each policy variable depends on time alone. By taking (31) and (32) into account, the government maximizes

$$\int_0^\infty e^{-\rho t} v\left(z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right), k\left(t\right), g\left(t\right)\right) dt$$

subject to

$$\dot{k}\left(t\right) = z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right), \tag{34}$$

$$\dot{p}\left(t\right) = \rho p\left(t\right) - v_k\left(z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right), k\left(t\right), g\left(t\right)\right) \tag{35}$$

Namely, the policy maker determines the optimal sequence of $g\left(t\right) \in \Gamma$ in order to maximize a discounted sum of indirect utilities of the representative agent under the constraints of dynamic behaviors of capital stocks and the market prices.[13] The current value Hamiltonian for this problem is given by

$$\begin{aligned}\mathcal{H} = {} & v\left(z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right), k\left(t\right), g\left(t\right)\right) + q\left(t\right) z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right) \\ & + \psi\left(t\right)\left[\rho p\left(t\right) - v_k\left(z\left(k\left(t\right), p\left(t\right), g\left(t\right)\right), k\left(t\right), g\left(t\right)\right)\right],\end{aligned}$$

where $q\left(t\right)$ and $\psi\left(t\right)$ respectively denote the costate variables for $k\left(t\right)$ and $p\left(t\right)$. The open-loop solution for an optimum should satisfy the following conditions:

$$\mathcal{H}_g = v_z z_g + v_g + q\left(t\right) z_g - \psi\left(t\right)\left(v_{kg} z_g + v_{kg}\right) = 0, \tag{36}$$

[12] Note that $\theta\left(k\left(t\right)\right)$ is a vector in $\mathcal{R}^r$ so that $\theta_k\left(k\left(t\right)\right)$ denotes an $r \times n$ matrix. Hence, $v_g \theta_k z$ is a scalor.

[13] The government's optimization is also subject to the transversality condition for the households' plan.

$$\dot{q}(t) = \rho q(t) - v_k - v_z z_k - q(t) z_k + \psi(t)(v_{kz} z_k + v_{kk}), \tag{37}$$

$$\dot{\psi}(t) = \psi(t) v_{kz} z_p - q(t) z_p - v_z z_p, \tag{38}$$

$$\psi(0) = 0, \tag{39}$$

$$\lim_{t\to\infty} e^{-\rho t} q(t) k(t) = 0, \tag{40}$$

plus (34) and (35).[14] Since the initial value of $p(t)$ is not predetermined, the corresponding costate variable, $\psi(t)$, should satisfy the transversality condition at the outset of the planning. Condition (39) presents the transversality condition on $\psi(t)$ at $t = 0$.[15] Examining the conditions displayed above, we find:

Proposition 2 *Suppose that the optimizing government adopts an open-loop policy. Then at the initial period* $(t = 0)$ *Weitzman's rule* (1) *is modified as*

$$Y(0) = \rho \int_0^\infty e^{-\rho t} c(t)\, dt + [p(0) - q(0)] z(0), \tag{41}$$

where $p(0)$ *and* $q(0)$ *respectively denote the private and social prices of capital that satisfy*

$$v_g(z(0), k(0), g(0)) + [q(0) - p(0)] z_g(0) = 0. \tag{42}$$

Proof. Since the government's problem is a standard optimal control problem in which $p(t)$ and $k(t)$ are state variables, the Hamilton-Bellman-Jacobi equation is given by

$$\begin{aligned}\rho W(k(t), p(t)) = \max_{g(t)\in\Gamma} \{ & v(z(k(t), p(t), g(t)), k(t), g(t)) \\ & + W_k(k(t), p(t)) z(k(t), p(t), g(t)) \\ & + W_p(k(t), p(t)) [\rho p(t) - v_k(z(k(t), p(t), g(t)), k(t), g(t))]\},\end{aligned}$$

where $W(.)$ is the value function which satisfies $W_k(.) = q(t)$ and $W_p(.) = \psi(t)$. Thus (39) means that at $t = 0$ we have

$$\begin{aligned}\rho W(k(0), p(0)) = \max_{g(0)\in\Gamma} \{ & v(z(k(0), p(0), g(0)), k(0), g(0)) \\ & + W_k(k(0), p(0)) z(k(0), p(0), g(0)).\end{aligned}$$

[14] In the necessary conditions for an optimum, z_g, v_{gk} and v_{zz} are $r \times n$, $n \times r$ and $n \times n$ matrices, respectively.

[15] See, for example, Bryson and Ho (1975, pp.56-57).

This yields

$$\int_0^\infty e^{-\rho t} c(t)\, dt = c(0) + q(0)\, z(0).$$

Thus $Y(0) \equiv c(0) + p(0)\, z(0)$ and the above equation present (41). Additionally, by use of (30), (36) and (39) we obtain (42). ■

In general, $\psi(t) \neq 0$ for $t > 0$, so that the behavior of $p(t)$ affects the private agents decision. Thus the gap between NNP and the welfare equivalence of maximized consumption equals $[q(t) - p(t)]\, z(t) + \psi(t)\, \dot{p}(t)$ for $t > 0$. The first term in this expression, $[q(t) - p(t)] z(t)$, is the social value of investment that is not captured by the market evaluation. The second term, $\psi(t)\, \dot{p}(t)$, shows the capital gain from the social perspective. Unless the economy is in the steady state where $z(t) = \dot{p}(t) = 0$, NNP generally diverges from the welfare equivalent wealth by reflecting those two terms.

As is well known, the open-loop policy is generally time inconsistent. This is because if the government reoptimizes at period $\hat{t}\,(>0)$, the transeversality condition requires that $\psi\left(\hat{t}\right) = 0$. However, the value of $\psi(t)$ determined by the original program at $t = 0$ does not necessarily satisfy $\psi\left(\hat{t}\right) = 0$. Thus the original plan determined at $t = 0$ is suboptimal if it is re-evaluated at $t = \hat{t} > 0$. This indicates that time consistency requires that the government takes a Markov feedback rule rather than an open-loop strategy. Suppose that optimal investment level selected by the private agents is given by

$$z(t) = z(k(t), g(t)). \tag{43}$$

In words, $z(t)$ does not depends on the market price $p(t)$. If this is the case, the government's optimization behavior is to maximize

$$\int_0^\infty e^{-\rho t} v(z(k(t), g(t)), k(t), g(t))\, dt$$

subject to

$$\dot{k} = z(k(t), g(t)).$$

Since in this case motion of the forward-looking variable, $p(t)$, do not bind the government's planning, the optimization problem for the government is a standard control problem. The Hamilton-Jacobi-Bellman equation for this problem is given by

$$\rho W(k(t)) = \max_{g(t) \in \Gamma} \{v(z(k(t), g(t)), k(t), g(t)) + W_k(k(t))\, z(k(t), g(t))\}. \tag{44}$$

The first-order condition is

$$v_z\left(z\left(k\left(t\right),g\left(t\right)\right),k\left(t\right),g\left(t\right)\right)z_g\left(k\left(t\right),g\left(t\right)\right)+v_g\left(z\left(k\left(g\right),g\left(t\right)\right),k\left(t\right),g\left(t\right)\right)$$
$$+W_k\left(k\left(t\right)\right)z_g\left(k\left(t\right),g\left(t\right)\right)=0,$$

which gives the optimal feedback rule in such a way that

$$g\left(t\right)=\theta\left(k\left(t\right)\right). \tag{45}$$

Observe that, unlike (33), the above rule is derived by solving the optimization problem for the government.

If the policy rule is given by the above, it is easy to see that the modified version of (1) is as follows:

Proposition 3 *If the government adopts a feedback policy rule, the following holds for all* $t \geq 0$*:*

$$Y\left(t\right)=\rho\int_t^{\infty}e^{-\rho(s-t)t}u\left(s\right)ds+\left[q\left(t\right)-p\left(t\right)\right]z\left(t\right), \tag{46}$$

where $p\left(t\right)$ *and* $q\left(t\right)$ *satisfies*

$$v_g\left(z\left(t\right),k\left(t\right),\theta\left(k\left(t\right)\right)\right)+\left[q\left(t\right)-p\left(t\right)\right]z_g\left(t\right)=0. \tag{47}$$

Proof. Since $W_k\left(k\left(t\right)\right)=q\left(t\right)$, equation (44) is written as

$$\rho\int_t^{\infty}e^{-\rho(s-t)}u\left(s\right)ds=u\left(t\right)+q\left(t\right)z\left(t\right).$$

Hence, using $Y\left(t\right)=c\left(t\right)+p\left(t\right)z\left(t\right)$, we obtain (44). In addition, (30), (42) and $W'\left(k\left(t\right)\right)=q\left(t\right)$ gives (47). ■

Consequently, if the policy maker can take a Markov feedback strategy, the modified rule (41) that is satisfied only at $t=0$ under the open-loop policy holds for all $t \geq 0$. As shown by (46), the difference between NNP and the welfare equivalence of maximized consumption reflects the divergence between the market and social values of capital. Since sign of $[q\left(t\right)-p\left(t\right)]z\left(t\right)$ cannot be specified without imposing further conditions, accuracy of NNP as an indicator of welfare measure depends on the policy rule taken by the government. For example, if both $k\left(t\right)$ and $g\left(t\right)$ are scalars, (47) becomes

$$q\left(t\right)-p\left(t\right)=-\frac{v_g\left(z\left(k\left(t\right),g\left(t\right)\right),k\left(t\right),g\left(t\right)\right)}{z_g\left(k\left(t\right),g\left(t\right)\right)}.$$

Therefore, if policy intervention has opposite impacts on consumption and investment, that is, either if $v_g < 0$ and $z_g > 0$ or if $v_g > 0$ and $z_g < 0$), then NNP underestimates (overestimates) the welfare equivalent wealth according to net investment z is positive (negative). Conversely, if v_g and z_g have the same sign, we obtain the opposite outcome.

4.2 An Example: Optimal Income Taxation

As a typical example, let us consider a simple model of dynamic optimal taxation. There is a single good and its production function is given by

$$y = k^{\alpha}, \quad 0 < \alpha < 1, \tag{48}$$

where y and k denote per capita output and capital, repetitively. The government levies a proportional income tax on output to finance its spending. We assume that the government does not rely on debt finance. Letting $g \in (0, 1)$ be the rate of income tax and s be the per capita government's spending, the flow budget constraint for the government is

$$s = gy. \tag{49}$$

The households' felicity depends on consumption, c, and the public expenditure, s. The representative agent maximizes a discounted sum of utilities

$$U = \int_0^{\infty} e^{-\rho t} u(c, s)\, dt,$$

where the instantaneous utility function is specified as

$$u(c, s) = \frac{c^{1-\sigma} - 1}{1 - \sigma} + \frac{s^{1-\xi} - 1}{1 - \xi}; \ 0 < \sigma, \xi < 1. \tag{50}$$

Here, we assume that the public spending has a positive effect on the households' welfare. The flow budget constraint for the household is the following:[16]

$$(1 - g)\, y = c + z + \delta k. \tag{51}$$

Therefore, the aggregate consumption is defined as

$$c = \frac{[(1 - g)\, k^{\alpha} - z - \delta k]^{1-\sigma} - 1}{1 - \sigma} + \frac{(gk^{\alpha})^{1-\xi} - 1}{1 - \xi} \equiv v(z, k, g).$$

[16] Observe that in view of (49), (51) also shows that the market equilibrium condition for the final good.

The household maximizes $\int_0^\infty v\,(z,k,g)\, e^{-\rho t}dt$ subject to $\dot{k} = z$ and an anticipated sequence of $\{g\,(t)\,, s\,(t)\}_{t=0}^{\infty}.$

Denoting the market price of capital by p, the optimization conditions for the household include the following:

$$c^{-\sigma} = [(1-g)\,k^{\alpha} - z - \delta k]^{-\sigma} = p, \tag{52}$$

$$\dot{p} = p\left[\rho + \delta - \alpha\,(1-g)\,k^{\alpha-1}\right], \tag{53}$$

$$\dot{k} = z, \tag{54}$$

$$\lim_{t\to\infty} p\,(t)\, e^{-\rho t} k\,(t) = 0. \tag{55}$$

First, assume that the government adopts an open-loop strategy. By taking its own budget constraint (49) into account, the fiscal authority controls the rate of tax g ($\in [0,1)$) to maximize

$$\int_0^\infty e^{-\rho t}\left[\frac{p^{1-\frac{1}{\sigma}}-1}{1-\sigma} + \frac{(gk^{\alpha})^{1-\xi}-1}{1-\xi}\right]dt$$

subject to $(53)\,,\ (54)\,,$ the initial condition $k\,(0) = k_0$, and the households' transversality condition $(55)\,.$ Set up the Hamiltonian function for the government's problem:

$$\mathcal{H} = \frac{p^{1-\frac{1}{\sigma}}-1}{1-\sigma} + \frac{(gk^{\alpha})^{1-\xi}-1}{1-\xi} + q\left[(1-g)\,k^{\alpha} - p^{-\frac{1}{\sigma}} - \delta k\right] + \psi\left[\rho + \delta - \alpha\,(1-g)\,k^{\alpha-1}\right].$$

The optimum conditions are $(53)\,,\ (54)$ and the following:

$$(gk^{\alpha})^{-\xi} = q - \frac{\alpha\psi}{k}, \tag{56}$$

$$\dot{q} = q[\rho + \delta - \alpha\,(1-g)\,k^{\alpha-1}] - \psi\alpha\,(1-\alpha)\,(1-g)\,k^{\alpha-2} - \alpha g k^{\alpha-1}\,(gk^{\alpha})^{-\xi}, \tag{57}$$

$$\dot{\psi} = \rho\psi - \frac{1}{\sigma}p^{-\frac{1}{\sigma}}, \tag{58}$$

together with the transversality conditions: $\psi\,(0) = 0$ and $\lim_{t\to\infty} e^{-\rho t} q\,(t)\,k\,(t) = 0$. Combining (56) with (57) gives

$$\dot{q} = q\left[\rho + \delta - \alpha k^{\alpha-1}\right] - \psi\alpha\,(1-\tau-\alpha)\,k^{\alpha-2}. \tag{59}$$

On the other hand, (56) yields

$$g = \left[q - \frac{\alpha\psi}{k}\right]^{\frac{1}{\xi}} k^{-\alpha}. \tag{60}$$

Substituting (60) into (53), (54), (57) and (58), we obtain a complete adynamic system with respect to k, p, q and ψ. In this formulation, the Hamilton-Jacobi-Bellman equation yields

$$\rho W(k(t)) = \bar{u}(t) = u(t) + q(t) z(t) + \psi(t) \dot{p}(t) \tag{61}$$

Net national product measured by the market prices is defined as $Y(t) \equiv c(t) + p(t) z(t)$. Thus (61) is rewritten as

$$Y(t) = \rho \int_t^\infty e^{-\rho s} u(s) \, ds - [q(t) - p(t)] z(t) - \psi(t) \dot{p}(t).$$

The difference between NNP and the annuity equivalent of the discounted present value of consumption is expressed as

$$\begin{aligned}
&[q(t) - p(t)] z(t) + \psi(t) \dot{p}(t) \\
&\quad = [q(t) - p(t)] \left[k(t)^\alpha - \left(q(t) - \frac{\alpha \psi(t)}{k(t)} \right)^{\frac{1}{\xi}} - p(t)^{-\frac{1}{\alpha}} - \delta k(t) \right] \\
&\qquad + \psi(t) \left[\rho + \delta - \alpha k(t)^{\alpha - 1} + \left(q(t) - \frac{\alpha \psi(t)}{k(t)} \right)^{\frac{1}{\xi}} k(t) \right].
\end{aligned}$$

Since the initial values of $k(t)$ and $\psi(t)$ are given, if there is a two-dimensional stable saddle path converging to the steady state equilibrium, $q(t)$ and $p(t)$ are written as functions of $k(t)$ and $\psi(t)$. Therefore, on the converging saddle path, the residual term shown above can be evaluated numerically by specifying parameter values involved in the model.

Next, examine the case in which the government can select a feedback rule In order to obtain a closed-form solution, we assume that the inverse of elasticity of intertemporal substitution in consumption, σ, is equal to α.[17] In this special case, from (51), (52) and (54) we obtain the following:

$$\frac{\dot{p}}{p} = \rho + \delta - \alpha (1 - g) k^{\alpha - 1}, \tag{62}$$

$$\frac{\dot{k}}{k} = (1 - g) k^{\alpha - 1} - \frac{1}{k p^{1/\alpha}} - \delta. \tag{63}$$

[17] This is one of the special case where the feedback policy rule can be analytically derived. Another well known example in which we can treat the feedback rule analytical is to assume that $\sigma = 1$ so that $u(c, g) = \log c + \left(g^{\xi - 1} - 1\right) / (1 - \xi)$: see Xie (1997) and Lansing (1999). Mino (2001) shows that the similar results can be obtained in a model with labor-leisure choice.

Hence, letting $x \equiv kp^{1/\alpha}$, (62) and (63) yield:

$$\frac{\dot{x}}{x} = \rho + \left(\frac{1}{\alpha} - 1\right)\delta - \frac{1}{x}.$$

This differential equation has a unique stationary solution and it is totally unstable. Therefore, x exhibits a diverging behavior unless x stays on the stationary point at the initial period. If x diverges from the stationary point, either the transversality condition or the feasibility condition will be ultimately violated. This implies that x always keeps its steady state level so that it holds that $x = \rho + (1/\alpha - 1)\,\delta$ for all $t \geq 0$. As a consequence, the following condition is satisfied:

$$p = (\mu k)^{-\alpha}, \tag{64}$$

where $\mu \equiv \rho + (1/\alpha - 1)\,\delta$.

Consequently, using (64) , the objective of the policy maker is to maximize

$$\int_0^\infty e^{-\rho t}\left[\frac{(\mu k)^{1-\alpha} - 1}{1-\alpha} + \frac{(gk^\alpha)^{1-\xi} - 1}{1-\xi}\right] dt$$

subject to

$$\dot{k} = (1-g)\,k^\alpha - (\mu + \delta)\,k.$$

The necessary conditions for an optimum involve the following conditions:

$$(gk^\alpha)^{-\xi} = q, \tag{65}$$

$$\acute{q} = q[(\rho + \mu + \delta)q - \alpha\,(1-g)\,k^{a-1}] - (gk^\alpha)^{-\xi}\,\alpha g k^{\alpha-1}.$$

From the above equations and $\mu = \rho + (1/\alpha - 1)\,\delta$, we have

$$\dot{q} = q\left[\left(2\rho + \frac{\delta}{\alpha}\right) - \alpha k^{\alpha-1}\right]. \tag{66}$$

Condition (65) yields the optimal tax rate:

$$g\,(t) = W'\,(k\,(t))^{-\frac{1}{\xi}}\,k\,(t)^{-\alpha} \equiv \theta\,(k\,(t))\,. \tag{67}$$

This gives the feedback control rule for $g\,(t)$. Using (67), the value function satisfies the following Hamilton-Jacobi-Bellman equation:

$$\rho W\,(k\,(t)) = \frac{(\mu k\,(t))^{1-\alpha} - 1}{1-\alpha} + \frac{(W'\,(k\,(t)))^{\frac{\xi-1}{\xi}} - 1}{1-\xi} + W'\,(k\,(t))\left[k\,(t)^\alpha - W'\,(k\,(t))^{-\frac{1}{\xi}} - (\rho+\delta)\,k\,(t)\right].$$

For the purpose of obtaining a closer look at the relationship between $p(t)$ and $q(t)$, we further assume that $\theta = \sigma$ so that $\xi = \alpha$. Then we have

$$\frac{\dot{q}}{q} - \frac{\dot{p}}{p} = \rho + \left(\frac{1}{\alpha} - 1\right)\delta = \mu.$$

As a result, it holds that $q(t)/p(t) = e^{\mu t}[q(0)/p(0)]$, which yields:

$$q(t) - p(t) = p(t)\left[e^{\mu t}\frac{q(0)}{p(0)} - 1\right]$$

This means that if $q(0) > p(0)$, it holds that $q(t) > p(t)$ for all ≥ 0. If $q(0) < p(0)$, then $q(t) < p(t)$ for $e^{\mu t} < p(0)/q(0)$ and $q(t) > p(t)$ for $e^{\mu t} > p(0)/q(0)$. Remembering that $p(0) = [\mu k_0]^{-\alpha}$ and $q(0) = W'(k_0)$, the sign of $q(t) - p(t)$ follows

$$\text{sign}\,[q(t) - p(t)] = \text{sign}\,\left\{t - \frac{1}{\mu}\log\left[\frac{W'(k_0)}{(\mu k_0)^{-\alpha}}\right]\right\}.$$

Suppose that $W'(k_0) < (\mu k_0)^{-\alpha}$. Further, assume that the initial level of capital k_0 is less than the steady state level of k and thus the economy converges to the steady state with positive investment Given those conditions, NNP overestimates the annuity equivalent of the present discounted value of maximized consumption for $0 < t < (1/\mu)\log\left[W'(k_0)/(\mu k_0)^{-\alpha}\right]$. After time passes the critical point, NNP becomes smaller than the welfare equivalent wealth. In contrast, if k_0 exceeds the steady-state level of k so that economy converges to the steady state with negative investment, we obtain the opposite results. It is to be pointed out that the critical time length,$(1/\mu)\log\left[W'(k_0)/(\mu k_0)^{-\alpha}\right]$, may be relatively short. For example, if $\alpha = 0.4$, $\rho = \delta = 0.05$, then $\mu = 0.125$. Hence, when $q(0)/p(0) = W'(k_0)/(\mu k_0)^{-\alpha} = 1.2$, the critical time is 1.458. This shows that NNP is over the welfare equivalent wealth for less than two years if k_0 is smaller than the steady state level of k. When $q(0)/p(0) = 2.0$, the critical time is 5.545. If capital involves human capital so that α has a higher value such as 0.8, we have $\mu = 0.0625$. This the switching time point is 11.04 if $q(0)/p(0) = 2.0$, while it is 2.91 if $q(0)/p(0) = 1.2$.

5 Conclusion

This paper has generalized Weitzman's rule by introducing market distortions into the basic framework. If a market economy satisfies the ideal conditions so that it mimics the behavior of the corresponding command economy, Weitzman's fundamental result ensures that net national product serves as an exact welfare measure. However, since market economies generally fail to attain the command optimum because of the presence of various forms of distortions, there may exist a gap between net national product and the annuity equivalent of the discounted present value of maximized consumption. In this paper, we have examined models with market distortions and have explored how the fundamental rule is modified if distortions contaminate the basic relationship between national income and welfare.

It is to be noted that when the market economy does not attain the command optimum, the welfare significance of national income depends upon the magnitude of the contamination effect generated by market distortions. As some of the numerical examples in Section 3.2 show, the divergence between NNP and the discounted present value of current and future consumption can be sufficiently small. In such a case, NNP is still a good measure of welfare. If the divergence has a significantly large value, we should be careful in using NNP as a representation of the welfare equivalence wealth. When evaluating the contamination effect, we should derive explicit forms of modified Weitzman's rules that take the distortionary effects into account. Needless to say, the modified rules presented in this paper are far from general. We have only touched upon the issue by exploring specific examples. The income-welfare relationship in alternative market environments deserves further investigation.

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