On Time Consistency in Stackelberg Differential Games

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Abstract

This paper explores a class of Stackelberg differential games in which the open-loop strategies of the leader satisfies time consistency. We show that in this class of games the open-loop equilibrium coincides with the corresponding feedback equilibrium. The analytical framework used in this paper involves the models examined by the several recent contributions to the time consistency issue as special cases.

JEL Classification: C6, C7

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1 Introduction

It is well known that in the open-loop Stackelberg differential games, the optimal strategies of the leader are in general time inconsistent. Namely, the sequence of the strategies of the leader decided at the initial period will be suboptimal when they are re-evaluated at a later period. Recently, Xie (1997), Lansing (1999) and Long and Shimomura (1999) present counterexamples to this proposition. They examine a class of Stackelberg differential games in which the open-loop equilibrium satisfies time consistency. Although those authors presented a variety of models with different economic implications, their models have a common feature: the trajectories of the costate variables of the follower's problem are independent of the leader's strategies. Precisely speaking, in their models the open-loop solution coincides with the corresponding feedback solution. Focusing on this fact, the present paper clarifies the conditions under which the open-loop Stackelberg strategies are time consistent. The analytical framework we use in this paper involves the models studied by the authors mentioned above as special cases.

2 A Stackelberg Differential Game

2.1. *The Analytical Framework*

For expositional simplicity, we assume that the optimization problems of the leader and the follower involve the same state variables and the same dynamic constraints. In addition, both the leader and the follower are assumed to have an infinite time horizon with a common discount rate. These assumptions are not essential for the main results derived below. Assume that the follower maximizes

$$U = \int_0^\infty e^{-\rho t} u\left(x_t, c_t, z_t\right) dt$$

subject to

$$\dot{x}_t = g\left(x_t, c_t, z_t\right), \tag{1}$$

where $x_t \in R^n \times [0, \infty]$ is the vector of state variables, and $c_t \in \Omega_c \subseteq R^m \times [0, \infty]$ and $z_t \in \Omega_z \subseteq R^r \times [0, \infty]$ respectively denote vectors of control variables of the follower and the leader. The initial value of x_t is also given for the follower. In the case of open-loop

Stackelberg game, the leader announces the whole sequence of its control variables, $\{z_t\}_{t=0}^{\infty}$, which are functions of time. The follower takes this sequence as given and selects own optimal strategy, $\{c_t\}_{t=0}^{\infty}$. Then taking the optimal reactions of the follower into account, the leader maximizes its objective functional

$$V = \int_0^{\infty} e^{-\rho t} v\left(x_t, c_t, z_t\right) dt$$

subject to the optimization conditions of the follower, the dynamic constraint (1), and the initial condition on x_t. Note that U and V may be identical, as in the case where the benevolent government (the leader) selects its policies to maximizes the welfare of the public (the follower).

To derive the optimization conditions of the follower, set up the Hamiltonian function such that

$$\mathcal{H}\left(x_t, c_t, z_t, p_t\right) = u\left(x_t, c_t, z_t\right) + p_t g\left(x_t . c_t, z_t\right).$$

Assuming the existence of interior solutions, the necessary conditions for an optimum are:

$$u_c\left(x_t, c_t, z_t\right) + p_t g_c\left(x_t, c_t, z_t\right) = 0, \tag{2}$$

$$\dot{p}_t = p_t\left[\rho - g_x\left(x_t, c_t, z_t\right)\right] - u_x\left(x_t, c_t, z_t\right), \tag{3}$$

together with (1) and the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} p_t x_t = 0. \tag{4}$$

From (2) the follower's control variables may be expressed as

$$c_t = c\left(x_t, z_t, p_t\right). \tag{5}$$

Accordingly, by use of (3) and (5), the leader's optimization problem is given by

$$\max_{\{z_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} v\left(x_t, c\left(x_t, p_t, z_t\right), z_t\right) dt$$

subject to

$$\dot{x}_t = g\left(x_t, c\left(x_t, p_t, z_t\right), z_t\right), \tag{6}$$

$$\dot{p}_t = p_t\left[\rho - g_x\left(x_t, c\left(x_t, p_t, z_t\right)\right), z_t\right] - u_x\left(x_t, c\left(x_t, p_t z_t\right), z_t\right), \tag{7}$$

together with (4) and the initial condition on x_t. The optimal choice of $\{z_t\}_{t=0}^{\infty}$ for this problem gives the open-loop Stackelberg strategies of the leader.

2.2. *The Time-Consistency Condition*

Let us define the value function of the leader's problem as follows:

$$\hat{V}(x_t, p_t) = \max_{\{z_s\}_{s=t}^{\infty}} \left\{ \int_t^{\infty} e^{-\rho(t-s)} v(x_s, c(x_s, p_s, z_s), z_s)\, ds \ \text{ s.t. } (1), (4), (6), (7) \right\}.$$

Since p_t is the vector of costate variable of the follower's optimization problem, its initial value may be selected by the leader. Hence, if we assume that $\hat{V}$ satisfies concavity, the initial value of the follower's costate variables p_t should be chosen to maximize $\hat{V}(x_0, p_0)$ under a given level of x_0. The first-order condition is

$$\mu_0 = \hat{V}_p(x_0, p_0) = 0, \tag{8}$$

where μ_t denote the vector of costate variables for p_t. This gives one of the transverality conditions for the leader's problem.[1] If the leader reoptimizes at period $t > 0$, by the same reason it should hold $\mu_t = \hat{V}_p(x_t, p_t) = 0$. However, the value of μ_t starting from $\mu_0\ (= 0)$ is not generally zero, implying that the leader's planning set at the initial period exhibits time inconsistency.

The well-known result shown above can be restated in the following manner. First, consider the value function of the follower:

$$\hat{U}(x_t, Z_t) = \max_{\{c_s\}_{s=t}^{\infty}} \left\{ \int_t^{\infty} e^{-\rho(s-t)} u(x_s, c_s, z_s)\, ds \ \text{ s.t. } (1) \right\},$$

where Z_t denotes the sequence of leader's control variables after t, i.e. $Z_t = \{z_s\}_{s=t}^{\infty}$. Then assuming that the value function is differentiable with respect to x, we obtain

$$p_t = \hat{U}_x(x_t, Z_t). \tag{9}$$

This demonstrates that the levels of costate variables time t depend on the entire sequence of the leader's control variables after t. Thus the current levels of the follower's control

[1] See Bryson and Ho (1975, pp56-57). Dockner et al. (2000) present a general discussion about the time consistency issue in differential games.

variables also depend on the future values of the leader's control variables in such a way that $c_t = c\left(x_t, \hat{U}_x(x_t, Z_t), z_t\right)$. By use of this relation, the dynamic equation that describes behavior of the state variables is expressed as

$$\dot{x}_t = g\left(x_t, c\left(x_t, \hat{U}_x(x_t, Z_t), z_t\right), z_t\right). \tag{10}$$

As a result, the leader's optimization problem can be rewritten as:

$$\max_{\{z_t\}_{t=o}^{\infty}} \int_0^{\infty} e^{-\rho t} v\left(x_t, c\left(x_t, z_t, \hat{U}_x(x_t.Z_t)\right), z_t\right) dt$$

subject to (10) and the initial level of x_t. It is now obvious that the instantaneous value of $v(x_t, c_t, z_t)$ depends not only on the control variable of the leader at time t but also on the entire sequence of the leader's strategies after t, $Z_t = \{z_x\}_{s=t}^{\infty}$. Therefore, the optimization planning of the leader does not generally have a recursive property, and thus the time consistency fails to hold in the open-loop Stackelberg equilibrium.

As emphasized above, the source of time inconsistency is that the vector of costate variables for the follower's problem, p_t,depends upon the values of the leader's control variables after t. Considering this fact, we immediately obtain the following proposition:

Proposition 1 *The open-loop solution of a Stackelberg differential game satisfies time consistency, if the follower's value function is additively separable between the state variables and the control variables of the leader, that is,* $\hat{U}(x_t, Z_t)$ *is written as*

$$\hat{U}(x_t, Z_t) = \phi(x_t) + \psi(Z_t). \tag{11}$$

Proof. If the value function of the follower is expressed as (11), from (9) we obtain

$$p_t = \phi_x(x_t). \tag{12}$$

In this case, the costate variables in the follower's optimization are independent of the leader's strategies. Hence the initial level of p_t is determined by x_0 alone, so that the transversality condition (8) cannot hold. When (11) is satisfied, the follower's control variables are free from the future values of strategies of the leader and thus they are functions of the current levels of x_t and z_t such as $c_t = c(x_t, \phi(x_t), z_t)$. Hence, the optimization problem the leader solves is to maximize

$$\int_0^{\infty} e^{-\rho t} v(x_t, c(x_t, \phi(x_t), z_t), z_t)\, dt$$

subject to

$$\dot{x}_t = g(x_t, c(x_t, \phi(x_t), z_t), z_t) \tag{13}$$

and the initial condition on x_t. Since in this setting the objective functional of the leader satisfies time additive separability, the optimization problem of the leader has a recursive property. Accordingly, the leader's value function depends on x_t alone and it is given by

$$\hat{V}(x_t) = \max_{\{z_s\}_{s=t}^{\infty}} \left\{ \int_t^{\infty} e^{-\rho(t-s)} u(x_s, c(x_s, \phi(x_s), z_s))\, ds \text{ s.t. } (13) \right\}.$$

This function should satisfy the Hamilton-Jacobi-Bellman equation:

$$\rho \hat{V}(x_t) = \max_{z_s \in \Omega_z} \left\{ u(x_t, c(x_t, \phi(x_t), z_t), z_t) + \hat{V}_x(x_t)\, g(x_t, c(x_t, \phi(x_t), z_t), z_t) \right\}.$$

The first-order condition for maximization yields

$$u_c c_z + u_z + \hat{W}_x [g_c c_z + g_z] = 0.$$

Using the above condition, the optimal strategies of the leader may be expressed as $z_t = z(x_t)$, which is the feedback strategy of leader. Since the leader's planning is a 'game against the nature', the feedback strategies coincide with the open-loop strategies. □

Although this proposition presents a sufficient condition for establishing a time-consistent Stackelberg equilibrium, the separability assumption of the follower's value function is too broad for practical applications. In the next section we examine narrower conditions in the context of a simple example.

3 A Simple Example

In what follows, we assume that x_t, c_t and z_t are scalers. The instantaneous objective function of the follower is assumed to be additively separable between c_t and z_t, and it does not involves the state variable:

$$u(x_t, c_t, z_t) = v(c_r) + w(z_t), \tag{14}$$

In addition, $g(.)$ is specified as

$$\dot{x}_t = g(x_t, c_t, z_t) = \alpha(z_t) f(x_t) + \beta(c_t) h(x_t), \tag{15}$$

where $\alpha(z_t)$ and $\beta(c_t)$ are monotonic and strict concave functions. Then the follower's optimal conditions (2) and (3) respectively become

$$v'(c_r) + p_t \beta'(c_t) h(x_t) = 0, \tag{16}$$

$$\dot{p}_t = p_t \left[\rho - \alpha(z_t) f(x_t) - \beta(c_t) h'(x_t)\right]. \tag{17}$$

Given the above specifications, the condition under which the follower's value function is additively spareable between x_t and $Z_t \, (= \{z_s\}_{s=t}^{\infty})$ may be summarized as follows:

Proposition 2 *Suppose that the instantaneous objective function of the follower and the dynamic constraint are respectively given by (14) and (15). Then the value function of the follower is additively separable between x_t and Z_t, if the following holds:*

$$\beta(c_t) = \frac{\rho}{f'(x_t)} \left[\frac{h'(x_t)}{f'(x_t)} - \frac{h(x_t)}{f(x_t)}\right]^{-1}. \tag{18}$$

Proof. In view of (15) and (17), the trajectory of (x_t, p_t) on the x-p space satisfies the relation below:[2]

$$\frac{dp_t}{dx_t} = \frac{p_t \left[\rho - \alpha(z_t) f'(x_t) - \beta(c_t) h'(x_t)\right]}{\alpha(z_t) f(x_t) + \beta(c_t) h(x_t)}, \tag{19}$$

where c_t fulfills (16). Since (9) holds for any feasible x_t, we obtain $dp_t/dk_t = \hat{U}_{kk}(x_t, Z_t)$ on the optimal trajectory of the follower's problem. This means that if the right hand side of (19) is written as a function of x_t alone, then p_t also depends only on x_t. Rewriting (19) as

$$\frac{dp_t}{dx_t} . \frac{1}{p_t} = \frac{f'(x_t)}{f(x_t)} \left[\frac{\rho}{f'(x_t)} - \alpha(z_t) - \frac{\beta(c_t) h'(x_t)}{f'(x_t)}\right] \left[\alpha(z_t) + \frac{\beta(c_t) h(x_t)}{f(x_t)}\right]^{-1},$$

we find that if (18) holds, the above becomes

$$\frac{dp_t}{dx_t} . \frac{1}{p_t} = -\frac{f'(x_t)}{f(x_t)}. \tag{20}$$

[2] If the model involves a feasigle steady state where $\dot{p} = \dot{x} = 0$, condition (19) is defined out of the steady state.

This shows that p_t is a function of x_t. Denoting $p_t = \phi'(x_t)$, (20) is expressed as

$$\frac{\phi''(x_t)\, x_t}{\phi'(x_t)} = -\varepsilon(x_t), \tag{21}$$

where $\varepsilon(x_t)\,(= f'(x_t)\, x_t / f(x_t))$ denotes the elasticity of $f(x_t)$ function. By solving (21) with respect to $\phi'(x_t)$, we obtain

$$\phi'(x) = \exp\left(-A \int \frac{\varepsilon(x)}{x} dx\right), \tag{22}$$

which means that the value function of the follower may be expressed as

$$\hat{U}(x_t, Z_t) = \int \exp\left(-A \int \frac{\varepsilon(x)}{x} dx\right) dx + \psi(Z_t) + B,$$

where A and B are constants.□

When (18) is fulfilled so that the separability condition holds, (16) becomes

$$v'(c_t) + \phi'(x_t)\, h(x_t)\, \beta'(c_t) = 0.$$

As a result, by use of (18) we obtain

$$\phi'(x_t) = -\frac{v'(\Delta(x_t))}{\beta'(\Delta(x_t))\, h(x_t)}, \tag{23}$$

where

$$\Delta(x_t) = \beta^{-1}\left(\frac{\rho}{f'(x_t)}\left[\frac{h'(x_t)}{f'(x_t)} - \frac{h(x_t)}{f(x_t)}\right]^{-1}\right).$$

In the simplest case where $\beta(c_t) = c_t$ and $h(x_t) = -1$, (18) reduces to

$$c_t = \frac{\rho f(x_t)}{f'(x_t)}. \tag{24}$$

Thus (23) becomes

$$\phi'(x_t) = v'\left(\frac{\rho f(x_t)}{f'(x_t)}\right).$$

This means that by (22) $v(c_t)$ and $\varepsilon(x)$ functions must satisfy:

$$v'\left(\frac{\rho x_t}{\varepsilon(x_t)}\right) = \exp\left(-A \int \frac{\varepsilon(x)}{x} dx\right). \tag{25}$$

In this example, the Hamilton-Jacobi-Bellman equation for the leader's problem is given by

$$\rho \hat{V}(x_t) = \max_{z_t}\left\{v\left(x_t, \frac{\rho f(x_t)}{f'(x_t)}, z_t\right) + \hat{V}'(x_t)\left[\alpha(z_t)\, f(x_t) - \frac{\rho f(x_t)}{f'(x_t)}\right]\right\}.$$

Therefore, the optimal choice of z_t must satisfy

$$v_z\left(x_t, \frac{\rho f(x_t)}{f'(x_t)}, z_t\right) + \hat{V}'(x_t)\,\alpha'(z_t)\, f(x_t) = 0.$$

This gives the stationary feedback solution of $z_t = z(x_t)$.

If the elasticity of $f(x_t)$ is constant ($\phi''(x)\,x/\phi'(x) = -\varepsilon$), then (25) becomes

$$v'\left(\frac{\rho x_t}{\varepsilon}\right) = e^A x_t^{-\varepsilon}.$$

Hence, $v(c_t)$ should take a CES form such that

$$v(c_t) = \frac{Bc^{1-\varepsilon} - 1}{1-\varepsilon}.$$

The is the case corresponds to the finding by Xie (1997)[3].

Finally, it is to be noted that Lansing (1999) and Long and Shimomura (1999) examine the models in which $v(c_t) = \log c_t$ and $f(x_t) = x_t$, and thus their examples are special case of the above with $\varepsilon = 1$.[4] In this case, from (24) the optimal choice of c_t is $c_t = \rho x_t$. Therefore, the behavior of x_t is $\dot{x}_t = [\alpha(z_t) - \rho]\, x_t$, which yields $x_s = x_t \exp\left(\int_t^s (\alpha(z_\xi) - \rho)\, d\xi\right)$. The value function of the follower is thus given by

$$\hat{U}\left(x_t, \{z_s\}_{s=t}^{\infty}\right) = \log x_t + \int_t^{\infty} e^{(t-s)\rho}\left[\int_t^s [\alpha(z_\xi) - \rho]\, d\xi + w(z_s)\right] ds + \log\rho.$$

4 A Concluding Remark

This paper studies the conditions under which the open-loop equilibrium in a Stackelberg differential game coincides with the corresponding feedback equilibrium so that the open-loop solution holds time consistency. The separability condition on the follower's value function

[3] Xie (1997) considers a dynamic optimal taxation problem in a representative agent economy which is close to Chamley's (1986) model. The instantaneous utility function of the household is assumed to be $u(c) + w(z)$, where c is consumption and z is the government spending. Capital accumulation is described by $\dot{x} = (1-\tau) f(x) - c$, where x is capital stock and τ denotes rate of income tax that satisfies $z = \tau f(x)$. The main example analyzed by Xie (1997) is the case where $u(c) = c^{1-\varepsilon}/(1-\varepsilon)$ and $f(x) = x^{\varepsilon}$.

[4] By assuming a logarithmic utility, Lansing (1999) re-examines the optimal income redistribution issue addressed by Judd (1985). Long and Shimomura (1999) applies a model with logarithmic utility to international economics.

shown above is obviously a restrictive one. Even in a simple model with a single state variable, it needs specific forms of functions involved in the model. However, this special class of models would be very helpful to consider what will happen if the time consistency condition is imposed on the Stackelberg dynamic games. The feedback solution of the differential games is generally hard to examine analytically except for the linear-quadratic models. Therefore, our special examples may serve to clarify the difference between the time inconsistent and time consistent policies in a dynamic game setting.[5]

[5] See also Kemp, Long and Shimomura (1993) for further applications.

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