Social choice with approximate interpersonal comparisons of well-being

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1 September 2009

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Abstract

Some social choice models assume that precise interpersonal comparisons of utility (either ordinal or cardinal) are possible, allowing a rich theory of distributive justice. Other models assume that absolutely no interpersonal comparisons are possible, or even meaningful; hence all Pareto-efficient outcomes are equally socially desirable. We compromise between these two extremes, by developing a model of 'approximate' interpersonal comparisons of well-being, in terms of a partial preorder on the space of psychophysical states. We then define and characterize 'approximate' versions of the classical egalitarian and utilitarian social welfare orderings. We show that even very weak assumptions about interpersonal comparability can select relatively small subsets of the Pareto frontier as being socially optimal.

The philosophical and practical problems surrounding interpersonal utility comparisons (IPUC) are well known; see e.g. Sen (1979), Griffin (1986), Davidson (1986), Gibbard (1986), Barrett and Hausman (1990), Fleurbaey and Hammond (2004), Hausmann and McPherson (2006; §7.2), and especially Elster and Roemer (1991). The apparent meaninglessness (or at least, practical impossibility) of IPUC has elicited at least five responses. One response is to restrict welfare economics to questions of Pareto efficiency only (Robbins, 1935, 1938). Economists then can only recommend policies which are clearly Pareto-superior to the status quo. If no policy alternative is Pareto-superior to any other, then the choice between them is a 'political' question, and not the business of economists.

A second, opposite approach is to axiomatize that some specific form of IPUC *is* possible, and investigate which social choice rules arise naturally under this IPUC axiom. This approach was pioneered by Sen (1970b), and has been spectacularly successful; see d'Aspremont and Gevers (2002) for a summary. However, since it explicitly sidesteps the question of *how* IPUC could be possible, the resulting theoretical edifice is in danger of being no more than an academic exercise. If someone rejects the IPUC hypotheses of the theorems, she can dismiss their conclusions. Even if she accepts the IPUC hypotheses in

*I am grateful to Klaus Nehring and Clemens Puppe for reading a draft of this paper and for their valuable comments.

principle, she may be unable to translate an abstract social choice theorem into a concrete policy recommendation, without some way to operationalize the required IPUC between real people.

Thus, a third approach is to 'pseudo-operationalize' IPUC, by using money as a proxy for utility. In some schemes, a policy is recommendable if it can be *made* Pareto-superior to the status quo, once the 'winners' pay the 'losers' adequate financial compensation — either hypothetical (e.g. the Kaldor-Hicks (1939) compensation principle) or actual (e.g. the Thompson (1966) insurance mechanism). In Groves (1973)-Clarke (1971) mechanisms, preference-strength is identified with 'willingness to pay'; people vote for public policy by bidding sums of money. However, such 'money-metric utilitarianism' favours the rich, who tend to assign less marginal utility to each dollar than the poor (*ceteris paribus*), due to risk-aversion and/or satiation. A wealthy minority can literally 'buy' its preferred policy alternative by bidding a sufficiently large sum of money.

A fourth approach explicitly rejects *any* kind of IPUC (monetary or otherwise), and considers social choice mechanisms which select a point on the Pareto frontier using only the profile of individual's (noncomparable) preferences, constrained by 'procedural' criteria such as 'Monotonicity' or 'Independence of Irrelevant Alternatives'. In the theory of bargaining (i.e. social choice over a convex set, with cardinal noncomparable utility), the Nash (1950), 'relative-egalitarian' (Kalai and Smorodinsky, 1975), and 'relative utilitarian' (Segal, 2000) solutions achieve this goal. However, there are other bargaining solutions which *do* require explicit IPUC, and which are uniquely characterized by combinations of axioms which may be quite desirable in some situations; see Kalai (1977) or Myerson (1981). Thus, a rejection of IPUC does not necessarily yield desirable bargaining outcomes. Furthermore, in the theory of voting (i.e. social choice over a discrete set, with ordinal noncomparable preferences) Arrow's Impossibility Theorem essentially says that there is no 'satisfactory' voting rule which eschews IPUC.

A fifth approach is to altogether reject 'welfarism' —the idea that social choices should be determined by utility data —and instead argue that these choices should be based on more tangible or objective measures of quality of life, such as Rawls' (1971) 'primary goods', Sen's (1985, 1988) 'functionings and capabilities', Cohen's (1993) 'midfare', Roemer's (1996) 'advantage', or the 'quality-adjusted life-years' of healthcare economics (Tsuchiya and Miyamoto, 2009).[1] However, this trades the problem of IPUC for another, equally thorny problem. 'Quality of life' is comprised of many factors: mental and physical health, wealth and economic opportunity, political and personal freedom, social prestige, quality of personal relationships, etc. —and each of these factors must be split into several subfactors to be properly quantified. But to have an appropriate object for optimization on the part of the social planner, we must combine all of these variables into a single numerical 'index', which purports to measure 'overall quality of life'. What is the correct way to define this index? Why are we justified in employing the *same* index for two people with wildly different preferences and life-goals? Any attempt to answer these questions rapidly becomes embroiled in philosophical issues which are dangerously close to the questions of IPUC we were trying to escape in the first place.

[1] I thank Clemens Puppe for this observation.

In reality, it *does* seem possible to make at least crude interpersonal comparisons. For example, if Zara and her family and friends are physically comfortable, healthy, and safe, while Juan and his family and friends are suffering in a concentration camp or dying of hemorrhagic fever, it seems fairly uncontroversial to assert that Zara's utility is higher than Juan's. Likewise, if Zara scores much higher than Juan in *every* item on a comprehensive list of measures of health, well-being, and quality of life, then again it seems plausible that Zara's utility is higher than Juan's.

Of course, if Zara, Juan and their families are in roughly equal physical circumstances, and they both have roughly equal scores on all measures of well-being, then it is difficult to say who is happier; such 'high-precision' IPUC might not be possible. However, we will show that even even a crude, 'low-precision' IPUC can be leveraged to define social preference relations which are far more complete than the Pareto ordering. Furthermore, only such low-precision IPUC are required to decide many public policy issues; e.g. whether to transfer wealth from the fabulously rich to the abject poor; whether to spend public resources on emergency medical care or disaster relief; whether to quarantine a few people to protect millions from a deadly plague, etc.

Changing minds. Some kind of IPUC is implicit whenever social policy makes 'redistributive' choices (moving along the Pareto frontier). IPUC is also necessary when the psychologies of the agents are themselves variables which can be modified by policy. Most social choice models assume a fixed population of agents with fixed preferences over the set of possible states of the (physical) world; we then seek the 'optimal' world-state according to some ordering determined by these preferences. Each agent's preferences presumably arise from her 'psychology', which is assumed to be exogenous and immutable. However, in some situations, her psychology is endogenous and mutable. For example, if the agent is mentally ill (e.g. clinically depressed), and we provide her with appropriate therapy (e.g. antidepressants), then she effectively becomes a slightly different person, with different preferences (e.g. she may no longer wish to kill herself). Furthermore, different therapies may lead to slightly *different* post-therapeutic individuals. Thus, a social choice over psychotherapeutic alternatives necessarily involves comparing the preferences of different people.

Another, more long-term example involves the use of propaganda campaigns to mold public preferences. For example, in developing countries with excessive population growth, governments sometimes try to persuade their citizens to prefer smaller family sizes. It is debatable whether such campaigns are successful; but if they *were*, then the post-propaganda population would contain people with *different preferences* than the pre-propaganda population; furthermore, different propaganda campaigns might lead to different preference profiles. Hence any social choice over propaganda campaigns is again a choice over worlds containing slightly different individuals.

An even longer-term example is social choice involving future generations. Different policies will lead to *different* future populations, with different psychologies and different preferences. For example, suppose it were discovered that a certain genetic variation caused mild chronic depression in 20% of the population. Suppose, furthermore, that it was possible to entirely eliminate this genetic variation from future generations through a systematic campaign of gene therapy, thereby presumably saving 20% of all future persons from genetically induced depression. We face a choice of whether or not to launch the gene therapy campaign; this is a choice over two different possible futures, with two psychologically different societies. An especially acute version of the 'future generations' problem confronts attempts to derive constitutions from an 'original position', as in Buchanan and Tullock (1962) and Rawls (1971).

To address these issues, we introduce a space Ψ of 'psychological types' as well as the usual space Φ of personal 'physical states'. An individual's 'psychophysical state' is thus an ordered pair $(\psi, \phi) \in \Psi \times \Phi$. The element ϕ encodes the person's current health, wealth, physical location, consumption bundle, etc. The element ψ encodes the individual's personality, mood, knowledge, beliefs, values, desires, and any other relevant 'psychological' information.[2] Thus, each $\psi \in \Psi$ defines some preference order $(\underset{\psi}{\preceq})$ over Φ.

By definition, Ψ is the space of all possible human psychologies *which could ever exist*; hence the set $\{\underset{\psi}{\preceq}\}_{\psi \in \Psi}$ is the set of all possible preference relations which could ever be part of any profile. A particular 'society' $\boldsymbol{\psi} \in \Psi^{\mathcal{I}}$ is obtained by making some selection from Ψ (here, $\mathcal{I}$ is set indexing the population). Societies change over time, and some of these changes may be socially desirable. Hence, the true space of 'social alternatives' is not $\{\boldsymbol{\psi}\} \times \Phi^{\mathcal{I}}$ for some fixed $\boldsymbol{\psi} \in \Psi^{\mathcal{I}}$. The true space of social alternatives is $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, and it is over *this* space which the social planner must optimize. We refer to an element of $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ as a *world*.

Intertemporal comparisons. Further evidence that people have at least some limited faculty for IPUC is the fact that people remember their pasts and choose their futures. Define a preorder $(\rightsquigarrow)$ on Ψ, where $\psi_1 \rightsquigarrow \psi_2$ means "ψ_2 is a possible future self of ψ_1". Equivalently: ψ_2 *remembers* being ψ_1 at some point during her past, and ψ_1 *anticipates* possibly becoming ψ_2 at some point during her future. Thus, $\mathcal{P}(\psi) := \{\psi' \in \Psi \; ; \; \psi' \rightsquigarrow \psi\}$ is ψ's set of *past selves*, and $\mathcal{F}(\psi) := \{\psi' \in \Psi \; ; \; \psi \rightsquigarrow \psi'\}$ is ψ's set of *possible future selves*. If ψ has accurate memory of her own past, she can correctly make judgements of the form, "I was happier in university than I was in high school", or "I would be happier now to study piano than I would have been as a teenager." This means that she can make interpersonal comparisons between elements of $\mathcal{P}(\psi) \times \Phi$. On the other hand, to be able to make optimal intertemporal choices, she must choose between various possible futures, perhaps involving different future selves; she therefore must make accurate comparisons between elements of $\mathcal{F}(\psi) \times \Phi$. For example, a person choosing whether to get an education, try a new experience, avoid 'temptation', undergo psychotherapy, meditate in search of 'inner

[2] A philosophical 'materialist', who identifies the mind with the brain, will object that this model presupposes an objective distinction between a person's 'mental' state ψ and her ''physical' state ϕ —a distinction which is untenable, because mental states *are* physical states. However, materialism commits one to an even more radical embrace of interpersonal comparisons. For the materialist, it is impossible to separate changes in physical state from changes in psychological state, so *all* personal choices (e.g. what to eat for dinner) potentially involve 'interpersonal' comparisons. However, it is not clear how to develop a theory of social choice if one insists on such an amorphous view of personal identity.

peace', or take a psychoactive drug (especially an addictive one) is clearly choosing amongst possible 'future selves' in $\mathcal{F}(\psi)$. Also, the idea that people can be held partly 'responsible' for their preferences (e.g. for deliberately cultivating 'expensive tastes', for maintaining a more or less 'cheerful' disposition, or for emiserating themselves with unrealistic life-goals) implicitly presupposes some ability to choose over $\mathcal{F}(\psi)$. However, once we recognize that people routinely make interpersonal comparisons across $[\mathcal{P}(\xi)\cup\mathcal{F}(\xi)]\times\Phi$, it seems plausible that they can make interpersonal comparisons involving at least some other elements of $\Psi \times \Phi$.

Contents. Section §1 deals with technical preliminaries. In §2, we introduce a model of 'approximate' IPUC in a purely ordinal framework, in the form of a *weak interpersonal preference ordering* ('wipo'): a (partial) preorder on the space $\Psi \times \Phi$ of psychophysical states. In §3 (still in an ordinal setting), we use wipos to define (partial) preorders over $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, which we call *social preferences over worlds* ('sprows'). We focus on two natural examples: the 'Suppes-Sen' sprow (§3.1) and the 'approximate egalitarian' sprow (§3.2). In §4, we introduce *hedometers*: ordinal utility functions which are compatible with the interpersonal comparisons determined by a wipo. We show that certain wipos are entirely characterized by their associated family of hedometers (Theorem 4.2). In §5 we study *welfarist sprows*, which are obtained by coupling a social welfare ordering on $\mathbb{R}^{\mathcal{I}}$ with a collection of hedometers. Theorem 5.6 shows that the approximate egalitarian sprow is maximal in the class of welfarist sprows which ensure 'minimal equity', while being decisive between all 'fully comparable' pairs of worlds (the smallest class for which one could reasonably require decisiveness).

In §6 we turn to a cardinal utility framework. A *lottery* is a probability distribution over $\Psi \times \Phi$, and a *wipol* is a (partial) preorder over lotteries, which satisfies something like the von Neumann-Morgenstern axioms. A *world-lottery* is a probability measure over the set of all possible worlds. In §6.1, we consider (partial) preorders over world-lotteries (*sprowls*); we show that any sprowl must extend and refine the *approximate utilitarian* sprowl, which ranks world-lotteries according to the 'per capita average lottery' (Theorem 6.5). In §6.2, we show that wipols can often be characterized using a convex cone of cardinal hedometers (Theorem 6.7), and the approximate utilitarian sprowl can be interpreted as maximizing the per capita average expected value of these hedometers on $\Psi \times \Phi$ (Corollary 6.8).

In §7, we consider a rather different model of approximate interpersonal comparisons, obtained by treating the hedometer as a random variable. This leads to a 'profile-independent' version of Harsanyi's Social Aggregation Theorem (Theorem 7.2) and also provides a purely 'welfarist' account of the importance of personal liberty (§7.2). Finally, in §8, we construct three more mathematically complicated models of wipos, based on specific psychological assumptions about how interpersonal comparisons could be made.

To facilitate reading, all but the simplest proofs are relegated to an appendix. It is not necessary to read all these sections in order. The following figure illustrates the lattice of logical dependencies between the sections.

$$\begin{tikzcd}
\S 1 \arrow[rr] & & \S 2 \arrow[rrr] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & & \S 3 \arrow[dr] & \\
& \S 6 & \S 7 & \S 8 & \S 4 \arrow[rr] & & \S 5
\end{tikzcd}$$

Related literature. Some ideas presented here have precursors in the literature. Sen (1970a, 1972, and Chapter 7* of 1970b) was the first to suggest using 'approximate' interpersonal comparisons to define an incomplete social ordering over the space of social alternatives; he developed a model quite similar to the the 'approximate utilitarian' sprowl of §6.1. A similar model was recently explored by Baucells and Shapley (2006, 2008). Theorem 4.2 is closely related to a result of Ok (2002), while Theorem 6.7 generalizes a theorem of Shapley and Baucells (1998) and Dubra et al. (2004).[3] Finally, the wipo construction in §8.3 is inspired by the ideas of Ortuño-Ortin and Roemer (1991).

1 Preliminaries

Let $\mathcal{X}$ be a set. A *preorder* on $\mathcal{X}$ is a binary relation $(\preceq)$ which is transitive and reflexive, but not necessarily complete or antisymmetric. A *complete order* is a preorder $(\preceq)$ such that, for all $x, y \in \mathcal{X}$, either $x \preceq y$ or $y \preceq x$. (For example, a social welfare order (SWO) is a complete order on $\mathbb{R}^{\mathcal{I}}$.) A *linear order* is a complete order which is also antisymmetric (for all $x, y \in \mathcal{X}$, if $x \preceq y \preceq x$, then $x = y$.) We will assume each individual's preferences can be described by a complete order (not necessarily linear), but that interpersonal comparisons can only be described by an (incomplete) preorder.

The *symmetric factor* of $(\preceq)$ is the relation $(\approx)$ defined by $(x \approx x') \Leftrightarrow (x \preceq x'$ and $x' \preceq x)$. The *asymmetric factor* of $(\preceq)$ is the relation $(\prec)$ defined by $(x \prec x') \Leftrightarrow (x \preceq x'$ and $x' \not\preceq x)$. If neither $x \preceq x'$ nor $x' \preceq x$ holds, then x and x' are *incomparable*; we then write $x \not\prec\!\!\!\succ x'$. If $(\underset{1}{\preceq})$ and $(\underset{2}{\preceq})$ are two partial orders on $\mathcal{X}$, then $(\underset{2}{\preceq})$ *extends* $(\underset{1}{\preceq})$ if, for all $x, x' \in \mathcal{X}$: $(x \underset{1}{\preceq} x') \implies (x \underset{2}{\preceq} x')$. It follows that $(x \underset{1}{\approx} x') \implies (x \underset{2}{\approx} x')$, while $(x \underset{2}{\prec} x') \implies (x \underset{1}{\prec} x'$ or $x \underset{1}{\not\prec\!\!\!\succ} x')$. (In particular, *every* preorder is extended by the 'trivial' preorder where $x \approx x'$ for all $x, x' \in \mathcal{X}$). We say $(\underset{2}{\preceq})$ *refines* $(\underset{1}{\preceq})$ if, for all $x, x' \in \mathcal{X}$:

$$\left(x \underset{1}{\prec} x'\right) \implies \left(x \underset{2}{\prec} x'\right) \quad \text{and} \quad \left(x \underset{1}{\approx} x'\right) \implies \left(x \underset{2}{\preceq} x' \text{ or } x \underset{2}{\succeq} x'\right).$$

That is: every pair of elements which is comparable under $(\underset{1}{\preceq})$ remains comparable under $(\underset{2}{\preceq})$, and the asymmetric part of $(\underset{2}{\preceq})$ extends the asymmetric part of $(\underset{1}{\preceq})$. (Thus, if $x \underset{2}{\approx} x'$,

[3] Shapley and Baucells (1998) were explicitly concerned with group preferences assuming partial interpersonal utility comparisons, but Ok (2002) and Dubra et al. (2004) only briefly allude to this problem; they are mainly concerned with modelling *individual* choice under incomplete preferences.

then either $x \underset{1}{\approx} x'$ or $x \underset{1}{\not\prec} x'$.) For example, the 'lexmin' SWO refines the 'maxmin' SWO (see Example 5.1 below).

If $(\underset{2}{\preceq})$ extends *and* refines $(\underset{1}{\preceq})$, then for all $x, x' \in \mathcal{X}$, we have

$$\left(x \underset{1}{\preceq} x'\right) \Longrightarrow \left(x \underset{2}{\preceq} x'\right) \quad \text{and} \quad \left(x \underset{1}{\prec} x'\right) \Longrightarrow \left(x \underset{2}{\prec} x'\right). \tag{1}$$

Let $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$ be a collection of preorders on $\mathcal{X}$ (where Λ is some indexing set). The ***meet*** of $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$ is the preorder $(\underset{M}{\preceq})$ defined by $(x \underset{M}{\preceq} x') \Leftrightarrow (x \underset{\lambda}{\preceq} x', \ \forall\, \lambda \in \Lambda)$. To clarify the meanings of these concepts, and for later reference, we state the following facts.

Lemma 1.1 *Let $\mathcal{X}$ be a set and let $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$ be a collection of preorders on $\mathcal{X}$.*

(a) *Let $(\underset{M}{\preceq})$ be the meet of $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$. Then $(\underset{M}{\preceq})$ is also a preorder on $\mathcal{X}$. For every $\lambda \in \Lambda$, the preorder $(\underset{\lambda}{\preceq})$ extends $(\underset{M}{\preceq})$* (but doesn't necessarily refine it).

(b) *Let $(\preceq)$ be a preorder on $\mathcal{X}$, and suppose that, for every $\lambda \in \Lambda$, the preorder $(\underset{\lambda}{\preceq})$ extends and refines $(\preceq)$. Then $(\underset{M}{\preceq})$ also extends and refines $(\preceq)$.*

(c) *Let $(\underset{1}{\preceq})$ be a complete order on $\mathcal{X}$, and let $(\underset{2}{\preceq})$ be another preorder.*

$$\left((\underset{2}{\preceq}) \textit{ either extends or refines } (\underset{1}{\preceq})\right) \Longrightarrow \left((\underset{2}{\preceq}) \textit{ is also a complete order on } \mathcal{X}\right).$$
$$\left((\underset{2}{\preceq}) \textit{ extends and refines } (\underset{1}{\preceq})\right) \Longrightarrow \left((\underset{2}{\preceq}) \textit{ is identical with } (\underset{1}{\preceq})\right).$$

(d) *Let $(\underset{1}{\preceq})$ and $(\underset{2}{\preceq})$ be complete orders on $\mathcal{X}$. Then*

$$\left((\underset{2}{\preceq}) \textit{ extends } (\underset{1}{\preceq})\right) \Longleftrightarrow \left((\underset{1}{\preceq}) \textit{ refines } (\underset{2}{\preceq})\right).$$

(e) *Let $(\underset{1}{\preceq})$ and $(\underset{2}{\preceq})$ be strict preorders on $\mathcal{X}$.* (i.e. the relations $(\underset{1}{\approx})$ and $(\underset{2}{\approx})$ are trivial). *Then*

$$\left((\underset{1}{\preceq}) \textit{ extends } (\underset{2}{\preceq})\right) \Longleftrightarrow \left((\underset{1}{\preceq}) \textit{ refines } (\underset{2}{\preceq})\right).$$

(f) *Let $(\underset{1}{\preceq})$ and $(\underset{2}{\preceq})$ be linear orders on $\mathcal{X}$. Then*

$$\left((\underset{1}{\preceq}) \textit{ extends } (\underset{2}{\preceq})\right) \Longleftrightarrow \left((\underset{1}{\preceq}) \textit{ refines } (\underset{2}{\preceq})\right) \Longleftrightarrow \left((\underset{1}{\preceq}) \textit{ is identical with } (\underset{2}{\preceq})\right).$$

2 Weak interpersonal preference orderings

Let Ψ be the space of psychological states, and let Φ be the space of personal physical states. For any $\psi \in \Psi$, let $(\underset{\psi}{\preceq})$ be a complete order on Φ, describing the preferences of a ψ-type personality. We can also regard $(\underset{\psi}{\preceq})$ as a (very incomplete) preorder on $\Psi \times \Phi$, such that, for any distinct $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$, we have $(\psi_1, \phi_1) \underset{\psi}{\preceq} (\psi_2, \phi_2)$ if and only if $\psi_1 = \psi = \psi_2$ and $\phi_1 \underset{\psi}{\preceq} \phi_2$. A *weak interpersonal preference ordering* (or *wipo*) is a preorder $(\preceq)$ on $\Psi \times \Phi$ which satisfies two axioms:

(W1) (*Nonpaternalism*) For any $\psi \in \Psi$, the preorder $(\preceq)$ extends and refines $(\underset{\psi}{\preceq})$. That is: for all $\phi_1, \phi_2 \in \Phi$,

$$\Big((\psi, \phi_1) \preceq (\psi, \phi_2)\Big) \iff \Big(\phi_1 \underset{\psi}{\preceq} \phi_2\Big) \quad \text{and} \quad \Big((\psi, \phi_1) \prec (\psi, \phi_2)\Big) \iff \Big(\phi_1 \underset{\psi}{\prec} \phi_2\Big).$$

(W2) (*Minimal interpersonal comparability*) For all $\psi_1, \psi_2 \in \Psi$, and all $\phi_1 \in \Phi$, there exists some $\phi_2 \in \Phi$ such that $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$, and there exists some $\phi_2' \in \Phi$ such that $(\psi_2, \phi_2') \preceq (\psi_1, \phi_1)$.

Axiom (W2) just says there exists at least one physical state (possibly very extreme) which is clearly better for ψ_2 than the physical state ϕ_1 is for ψ_1, and one physical state for ψ_2 which is clearly worse for ψ_2 than the ϕ_1 is for ψ_1. If $(\preceq)$ was a complete ordering on $\Psi \times \Phi$, we would have a complete system of interpersonal utility level comparisons —but we will presume that $(\preceq)$ is normally quite incomplete.

The incompleteness of $(\preceq)$ can be interpreted either 'epistemologically' or 'metaphysically'. In the *epistemological* interpretation, we suppose there is, in reality, an underlying complete order $(\underset{*}{\preceq})$ on $\Psi \times \Phi$, which extends and refines $(\preceq)$, and which describes the 'true' interpersonal comparison of well-being between different psychophysical states. However, $(\underset{*}{\preceq})$ is unknown to us (and perhaps, unknowable). The partial preorder $(\preceq)$ reflects our incomplete knowledge of $(\underset{*}{\preceq})$.

In the *metaphysical* interpretation, there is *no* underlying true, complete ordering of $\Psi \times \Phi$; if $\psi_1 \neq \psi_2$, then it is only meaningful to compare (ψ_1, ϕ_1) and (ψ_2, ϕ_2) when they yield unambiguously different levels of well-being (e.g. because ϕ_1 is a state of great suffering and ϕ_2 is a state of great happiness). The partial preorder $(\preceq)$ encodes all the interpersonal comparisons which can be meaningfully made between different psychological types. If $(\psi_1, \phi_1) \not\succeq\!\!\not\preceq (\psi_2, \phi_2)$, then it is simply *meaningless* to inquire which of (ψ_1, ϕ_1) or (ψ_2, ϕ_2) experiences a greater level of well-being.

A physics analogy may clarify this distinction. Suppose Ψ represents spatial position, and Φ represents some time measurement, so that an ordered pair (ψ, ϕ) represents an event which occurred at position ψ at time ϕ. Suppose the relation "$(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$" means: "The event (ψ_1, ϕ_1) happened before the event (ψ_2, ϕ_2)". In the epistemological interpretation, the comparison between ϕ_1 and ϕ_2 is subject to some 'measurement error', which may depend on the distance from ψ_1 to ψ_2 (say, because it is difficult to determine the exact time of occurrence of far away events). This measurement error might make it impossible for us to determine whether $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$ or $(\psi_2, \phi_2) \preceq (\psi_1, \phi_1)$ —but in the setting of classical physics, *one* of these two statements is definitely true. However, in the setting of special relativity, if (ψ_2, ϕ_2) occurs outside of the 'light cone' of (ψ_1, ϕ_1), then *neither* statement is true; event (ψ_2, ϕ_2) occurred neither before nor after (ψ_1, ϕ_1). Indeed, the words 'before' and 'after' only have meaning for events which occur inside one another's light cones.

We will generally remain agnostic about whether to adopt the epistemological or metaphysical interpretation. However, some of our analysis (e.g. the concept of 'hedometers') clearly tends towards the epistemological interpretation.

2.1 Weak interpersonal comparisons of utility

Suppose that $\Phi = \mathbb{R}$; that is, each person's physical state can be entirely described by a single real number (measuring her 'well-being' or 'utility'). For all $\psi \in \Psi$, we suppose that $(\underset{\psi}{\preceq})$ is the standard linear ordering on $\mathbb{R}$; however, different individuals potentially have different 'utility scales', so given $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, it is not necessarily possible to compare (ψ_1, r_1) and (ψ_2, r_2) if $\psi_1 \neq \psi_2$. A wipo on $\Psi \times \mathbb{R}$ is thus a *weak interpersonal comparison of utility* (or *wicu*).

Example 2.1: Let d be a metric on Ψ (measuring the 'psychological distance' between individuals).

(a) Suppose all individuals have cardinal utility functions with the same scale (so for any $\psi, \psi' \in \Psi$ and $r_1 < r_2 \in \mathbb{R}$, the change from (ψ, r_1) to (ψ, r_2) represents the same 'increase in happiness' for ψ as the change from (ψ', r_1) to (ψ', r_2) represents for ψ'). However, suppose the 'zeros' of different people's utility functions are set at different locations (so $(\psi, 0)$ is not necessarily equivalent to $(\psi', 0)$). The precise deviation between utility zeros of two individuals is unknown, but it is bounded by psychological distance between them. Formally, let $c > 0$ and $\gamma \in (0, 1)$ be constants. For any $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, stipulate that $(\psi_1, r_1) \prec (\psi_2, r_2)$ iff $r_1 + c \cdot d(\psi_1, \psi_2)^\gamma < r_2$, while $(\psi_1, r_1) \approx (\psi_2, r_2)$ iff $(\psi_1, r_1) = (\psi_2, r_2)$. See Figure 1(a,b).

(b) Suppose all individuals have cardinal utility functions with the same zero point (so for all ψ, ψ', the point $(\psi, 0)$ is equivalent to $(\psi', 0)$ —perhaps being the utility of some 'neutral' state, like nonexistence or eternal unconsciousness). However, different utility functions have different scales. The precise deviation between utility scales of two individuals is unknown, but it is bounded by psychological distance between them. Formally, let $c > 1$ be a constant. For any $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, stipulate that $(\psi_1, r_1) \prec (\psi_2, r_2)$ if either $r_1 \geq 0$ and $c^{d(\psi_1, \psi_2)} \cdot r_1 < r_2$; or $r_1 < 0$ and $c^{-d(\psi_1, \psi_2)} \cdot r_1 < r_2$. Meanwhile, $(\psi_1, r_1) \approx (\psi_2, r_2)$ iff either $(\psi_1, r_1) = (\psi_2, r_2)$ or $r_1 = 0 = r_2$. See Figure 1(c). $\diamondsuit$

Now let Φ be any space of physical states, and for each $\psi \in \Psi$, let $u_\psi : \Phi \longrightarrow \mathbb{R}$ be a utility function representing the preference order $(\underset{\psi}{\preceq})$ on Φ. Let $(\underset{*}{\preceq})$ be a wicu on $\Psi \times \mathbb{R}$;

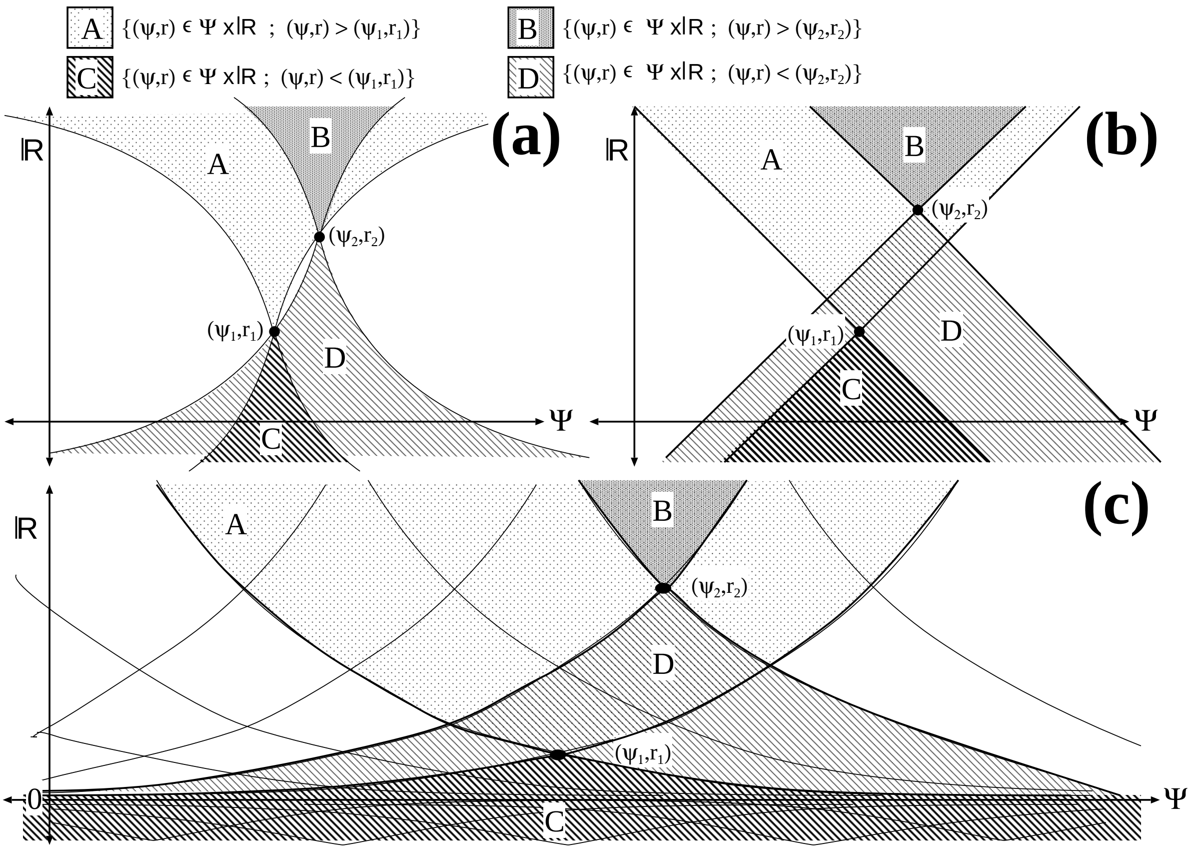

Figure 1: Upper and lower contour sets for the wicus on $\Psi \times \mathbb{R}$ from Example 2.1. Here, for visualization purposes, we suppose that $\Psi \subseteq \mathbb{R}$, with the Euclidean metric. (a) The wicu from Example 2.1(a), with $\gamma = 1/2$. The contour sets are bounded by curves of the form $y = \pm\sqrt{|x|}$. (b) The wicu from Example 2.1(a), with $\gamma = 1$. The contour sets are bounded by lines. Note that we must have $\gamma \leq 1$ so that, if $(\psi_1, r_1) \preceq (\psi_2, r_2)$, then the upper contour set of (ψ_2, r_2) is contained in the upper contour set of (ψ_1, r_1) (as required by transitivity). (c) The wicu from Example 2.1(b). The contour sets are bounded by exponential curves of the form $y = c^{\pm x}$.

then we can define a *wicu-mediated* wipo $(\preceq)$ on $\Psi \times \Phi$ by:

$$\Big((\psi_1, \phi_1) \preceq (\psi_2, \phi_2)\Big) \iff \Big((\phi_1, u_{\psi_1}(\phi_1)) \underset{*}{\preceq} (\phi_2, u_{\psi_2}(\phi_2))\Big). \tag{2}$$

2.2 Hedometers

Suppose there was a scientific instrument which, when applied to any person, could objectively measure her current happiness or well-being in some standard units. Call this hypothetical instrument a *hedometer*, and represent it as a function $h : \Psi \times \Phi \longrightarrow \mathbb{R}$. Thus, if $h(\psi, \phi) < h(\psi, \phi')$, then psychology ψ is happier in physical state ϕ' than in state ϕ. Thus, the hedometer yields an ordinal utility function representing the preference ordering $(\underset{\psi}{\preceq})$ of any fixed psychological type ψ. However, since h objectively measures utility in standard units, it can also be used to make interpersonal comparisons: if $h(\psi, \phi) < h(\psi', \phi')$, then, objectively, psychology ψ' is happier in physical state ϕ' than psychology ψ is in state ϕ.

Unfortunately, no such instrument exists, and even we had a putative hedometer in front of us, there would be no way of verifying its accuracy. However, suppose we have a collection of *possible hedometers*; that is, a set $\mathcal{H}$ of functions $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that:

- For all $\psi \in \Psi$, the function $h(\psi, \bullet) : \Phi \longrightarrow \mathbb{R}$ is an ordinal utility function for the preference ordering $(\underset{\psi}{\preceq})$.
- For any $(\psi, \phi), (\psi', \phi') \in \Psi \times \Phi$, if (ψ', ϕ') is *much* happier than (ψ, ϕ), then $h(\psi', \phi') > h(\psi, \phi)$ (but not conversely).

One of the elements of $\mathcal{H}$ is the 'true' hedometer, but we don't know which one. Thus, we could define a wipo $(\underset{\mathcal{H}}{\preceq})$ on $\Psi \times \Phi$ as follows: for all $(\psi, \phi), (\psi', \phi') \in \Psi \times \Phi$,

$$\Big((\psi, \phi) \underset{\mathcal{H}}{\preceq} (\psi', \phi')\Big) \quad \Longleftrightarrow \quad \Big(h(\psi, \phi) \;\leq\; h(\psi', \phi'), \text{ for all } h \in \mathcal{H}\Big). \tag{3}$$

We will see later that many wipos can be represented in this way (Theorem 4.2).

Example 2.2: (Wipo by jury) Let $\mathcal{J}$ be some jury of individuals, and assume each $j \in \mathcal{J}$ possesses a *complete* wipo $(\underset{j}{\preceq})$ on $\Psi \times \Phi$, which expresses j's own (subjective) interpersonal comparisons of well-being. The orders $\{\underset{j}{\preceq}\}_{j \in \mathcal{J}}$ may disagree with one another (although all of them must satisfy axiom (W1)). Let $(\underset{\mathcal{J}}{\preceq})$ be the meet of the collection $\{\underset{j}{\preceq}\}_{j \in \mathcal{J}}$; then Lemma 1.1(a,b) implies that $(\underset{\mathcal{J}}{\preceq})$ is a wipo.[4]

Suppose each of the complete orders $(\underset{j}{\preceq})$ can be represented by a function $h_j : \Psi \times \Phi \longrightarrow \mathbb{R}$. Then the jury's wipo $(\underset{\mathcal{J}}{\preceq})$ is defined by eqn.(3). $\diamondsuit$

In §8, we will develop more technically complicated examples of wipos, based on more detailed and plausible psychological models of interpersonal comparability. First, however, in §3-§7, we will apply wipos to make social welfare judgements.

3 Social preferences over worlds

Let $\mathcal{I}$ be a finite set (representing a population). A ***society*** is an element of $\boldsymbol{\psi} \in \Psi^{\mathcal{I}}$, which assigns a 'psychology' ψ_i to each member i of the population $\mathcal{I}$. A ***situation*** is an element $\boldsymbol{\phi} \in \Phi^{\mathcal{I}}$ which assigns a physical state ϕ_i to each $i \in \mathcal{I}$. A ***world*** is an ordered

[4] Note that we must require *unanimous* consensus in the definition of $(\underset{\mathcal{J}}{\preceq})$; if we merely required majoritarian or supermajoritarian support [e.g. we say $(\psi_1, \phi_1) \underset{\mathcal{J}}{\preceq} (\psi_2, \phi_2)$ if at least 66% of all $j \in \mathcal{J}$ think $(\psi_1, \phi_1) \underset{j}{\preceq} (\psi_2, \phi_2)$], then the relation $(\underset{\mathcal{J}}{\preceq})$ could have cycles.

pair $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ —that is, a society together with a situation. If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is a permutation, and $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, then define

$$\sigma(\boldsymbol{\psi},\boldsymbol{\phi}) \quad := \quad (\boldsymbol{\psi}',\boldsymbol{\phi}'), \qquad \text{where } \psi'_i := \psi_{\sigma(i)} \text{ and } \phi'_i := \phi_{\sigma(i)} \text{ for all } i \in \mathcal{I}. \qquad (4)$$

Let $(\preceq)$ be a wipo on $\Psi \times \Phi$. A $(\preceq)$-*social preference over worlds* (or *sprow*) is a preorder $(\trianglelefteq)$ on $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ which satisfies two properties:

(Par$^{\trianglelefteq}$) For any $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1),(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$,

$$\begin{aligned} &\Big((\psi_i^1,\phi_i^1)\preceq(\psi_i^2,\phi_i^2),\ \forall i \in \mathcal{I}\Big) \implies \Big((\boldsymbol{\psi}^1,\boldsymbol{\phi}^1) \trianglelefteq (\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)\Big), \\ \text{and } &\Big((\psi_i^1,\phi_i^1)\prec(\psi_i^2,\phi_i^2),\ \forall i \in \mathcal{I}\Big) \implies \Big((\boldsymbol{\psi}^1,\boldsymbol{\phi}^1) \triangleleft (\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)\Big). \end{aligned}$$

(Anon$^{\trianglelefteq}$) For all $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, if $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, then $(\boldsymbol{\psi},\boldsymbol{\phi}) \overset{\vartriangle}{\approx} \sigma(\boldsymbol{\psi},\boldsymbol{\phi})$. (Here, $(\overset{\vartriangle}{\approx})$ is the symmetric factor of $(\trianglelefteq)$).

Axiom (Anon$^{\trianglelefteq}$) makes sense because the elements of $\mathcal{I}$ are merely 'placeholders', with no psychological content. All information about the 'psychological identity' of individual i is encoded in the 'psychological state variable' ψ_i. Thus, if $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1),(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$ are two worlds, and $\psi_i^1 \neq \psi_i^2$, then it may not make any sense to compare the welfare of (ψ_i^1,ϕ_i^1) with (ψ_i^2,ϕ_i^2) (unless such a comparison is allowed by $(\preceq)$), because ψ_i^1 and ψ_i^2 represent *different people* (even though they have the same index). On the other hand, if $\psi_i^1 = \psi_j^2$, then it makes perfect sense to compare (ψ_i^1,ϕ_i^1) with (ψ_j^2,ϕ_j^2), even if $i \neq j$, because ψ_i^1 and ψ_j^2 are in every sense the *same* person (even though this person has different indices in the two worlds).

Axiom (Par$^{\trianglelefteq}$) is sometimes called 'Weak Pareto'. We might also consider sprows which also satisfy the following 'Strong Pareto' property:

(SPar$^{\trianglelefteq}$) For any $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1),(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, if $(\psi_i^1,\phi_i^1)\preceq(\psi_i^2,\phi_i^2)$ for all $i \in \mathcal{I}$, and $(\psi_i^1,\phi_i^1)\prec(\psi_i^2,\phi_i^2)$ for some $i \in \mathcal{I}$, then $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1) \trianglelefteq (\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$.

3.1 The Suppes-Sen sprow

The *Suppes-Sen* sprow[5] $(\underset{s}{\trianglelefteq})$ is defined as follows: for any $(\boldsymbol{\psi},\boldsymbol{\phi}),(\boldsymbol{\psi}',\boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, $(\boldsymbol{\psi},\boldsymbol{\phi})\underset{s}{\trianglelefteq}(\boldsymbol{\psi}',\boldsymbol{\phi}')$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, $(\psi_i,\phi_i) \preceq (\psi'_{\sigma(i)},\phi'_{\sigma(i)})$. We will see shortly that $(\underset{s}{\trianglelefteq})$ is the 'minimal' $(\preceq)$-sprow, which is extended (and often refined) by every other $(\preceq)$-sprow (see Proposition 3.4(b)).

[5]This sprow is based on the *grading principle*, a partial social welfare order defined by Suppes (1966) on $\mathbb{R}^2$, and extended to $\mathbb{R}^n$ by Sen (1970b, §9*1-§9*3, pp.150-156). It was later named the 'Suppes-Sen' ordering by Saposnik (1983), who showed that, on $\mathbb{R}^n$, it is equivalent to the rank-dominance ordering.

Example 3.1: (*Cost-benefit analysis*)

Given two worlds $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, let $\mathcal{I}_\downarrow := \{i \in \mathcal{I};\ (\psi_i^1, \phi_i^1) \succ (\psi_i^2, \phi_i^2)\}$ be the set of 'losers' under the change from world $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ to world $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$, and let $\mathcal{I}_\uparrow := \{i \in \mathcal{I};\ (\psi_i^1, \phi_i^1) \prec (\psi_i^2, \phi_i^2)\}$ be the set of 'winners'. Let $\mathcal{I}_0 := \mathcal{I} \setminus (\mathcal{I}_\downarrow \sqcup \mathcal{I}_\uparrow)$ be everyone else. Suppose that:

- There is a bijection $g : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ such that, for every $i \in \mathcal{I}_0$, $(\psi^1_{g(i)}, \phi^1_{g(i)}) \approx (\psi_i^2, \phi_i^2)$;
- There is an injection $h : \mathcal{I}_\downarrow \longrightarrow \mathcal{I}_\uparrow$ such that, for all $i \in \mathcal{I}_\downarrow$,

$$(\psi^1_{h(i)}, \phi^1_{h(i)}) \quad \preceq \quad (\psi_i^2, \phi_i^2) \quad \prec \quad (\psi_i^1, \phi_i^1) \quad \preceq \quad (\psi^2_{h(i)}, \phi^2_{h(i)}). \tag{5}$$

Thus, we can pair up every 'loser' i in $\mathcal{I}_\downarrow$ with some 'winner' $h(i)$ in $\mathcal{I}_\uparrow$ such that the gains for $h(i)$ clearly outweigh the losses for i in the change from $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ to $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$.

Claim 3.1*: $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$.

Proof. Define $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ as follows: $\sigma(i) := g(i)$ for all $i \in \mathcal{I}_0$; $\sigma(i) := h(i)$ for all $i \in \mathcal{I}_\downarrow$; $\sigma(i) := h^{-1}(i)$ for all $i \in h(\mathcal{I}_\downarrow) \subseteq \mathcal{I}_\uparrow$; and $\sigma(i) := i$ for all other $i \in \mathcal{I}_\uparrow \setminus h(\mathcal{I}_\downarrow)$.

It remains to show that $(\psi_i^1, \phi_i^1) \preceq (\psi^2_{\sigma(i)}, \phi^2_{\sigma(i)})$ for all $i \in \mathcal{I}$. There are three cases: (1) $i \in \mathcal{I}_0$; (2) $i \in \mathcal{I}_\downarrow$ or $i \in h(\mathcal{I}_\downarrow)$; and (3) $i \in \mathcal{I}_\uparrow \setminus h(\mathcal{I}_\downarrow)$.

(1): If $i \in \mathcal{I}_0$, then $(\psi_i^1, \phi_i^1) \approx (\psi^2_{g(i)}, \phi^2_{g(i)}) = (\psi^2_{\sigma(i)}, \phi^2_{\sigma(i)})$ by definition of g.

(2): If $i \in \mathcal{I}_\downarrow$ and $j = h(i) \in \mathcal{I}_\uparrow$, then $(\psi_j^1, \phi_j^1) \preceq (\psi_i^2, \phi_i^2) \prec (\psi_i^1, \phi_i^1) \preceq (\psi_j^2, \phi_j^2)$. However, $\sigma(i) = j$ and $\sigma(j) = i$; hence $(\psi_i^1, \phi_i^1) \preceq (\psi^2_{\sigma(i)}, \phi^2_{\sigma(i)})$ and $(\psi_j^1, \phi_j^1) \preceq (\psi^2_{\sigma(j)}, \phi^2_{\sigma(j)})$.

(3): If $i \in \mathcal{I}_\uparrow \setminus f(\mathcal{I}_\downarrow)$, then $\sigma(i) = i$ and $(\psi_i^1, \phi_i^1) \prec (\psi_i^2, \phi_i^2)$; so $(\psi_i^1, \phi_i^1) \prec (\psi^2_{\sigma(i)}, \phi^2_{\sigma(i)})$.

◇ Claim 3.1*

For example, suppose $\mathcal{I} = \{i, j\}$, fix $\psi_i, \psi_j \in \Psi$, and let $\boldsymbol{\phi}^1, \boldsymbol{\phi}^2 \in \Phi^{\mathcal{I}}$ be two situations such that $\phi_i^1 \underset{\psi_i}{\preceq} \phi_i^2$ while $\phi_j^2 \underset{\psi_j}{\preceq} \phi_j^1$. Thus, a change from situation $\boldsymbol{\phi}^1$ to $\boldsymbol{\phi}^2$ would help Isolde (i) and hurt Jack (j) —thus, neither situation is Pareto-preferred to the other. Borrowing Harsanyi's well-known example, suppose I have an extra ticket to a Chopin concert which I can't use, and let $\boldsymbol{\phi}^1$ be the situation where I give the ticket to Jack, while $\boldsymbol{\phi}^2$ is the situation where I give the ticket to Isolde. Both Isolde and Jack want the ticket. However Isolde is a classical pianist and Chopin fanatic who has been complaining bitterly for months that she couldn't get a ticket to this sold-out concert, whereas Jack doesn't even like classical music; he only wants the ticket because going to any concert is slightly preferable to spending a boring evening at home. Assume that, other than the concert issue, Jack and Isolde have roughly similar levels of well-being. Then we might reasonably suppose that $(\psi_i, \phi_i^1) \preceq (\psi_j, \phi_j^2) \preceq (\psi_j, \phi_j^1) \preceq (\psi_i, \phi_i^2)$. Thus, the change from $\boldsymbol{\phi}^1$ to $\boldsymbol{\phi}^2$ helps Isolde *more* than it hurts Jack, so $\boldsymbol{\phi}^2$ is socially preferable to $\boldsymbol{\phi}^1$; hence $(\boldsymbol{\psi}, \boldsymbol{\phi}^1) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}, \boldsymbol{\phi}^2)$. (To see this, set $\mathcal{I}_\downarrow := \{j\}$, $\mathcal{I}_\uparrow := \{i\}$, and $h(j) := i$ in eqn.(5).) ◇

Note that we can perform the interpersonal 'cost-benefit analysis' in Example 3.1 without even a utility function, much less a complete system of IPUC. However, even if the wipo $(\preceq)$ is a complete ordering on $\Psi \times \Phi$, the sprow $(\underset{s}{\trianglelefteq})$ is still a very partial ordering of $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. In Example 3.1, the number of 'big winners' in $\mathcal{I}_{\uparrow}$ must exceed the number of losers (even small losers) in $\mathcal{I}_{\downarrow}$, so that every loser can be matched up with some 'big winner' whose gains outweigh her losses. Thus, $(\underset{s}{\trianglelefteq})$ might not recognize the social value of a change $\boldsymbol{\phi}^1 \rightsquigarrow \boldsymbol{\phi}^2$ where a wealthy 51% majority $\mathcal{I}_{\downarrow}$ sacrifices a pittance so that destitute 49% minority $\mathcal{I}_{\uparrow}$ can gain a fortune —something which classic utilitarianism *would* recognize. In particular, it is necessary, but *not* sufficient, for a clear majority to support the change $\boldsymbol{\phi}^1 \rightsquigarrow \boldsymbol{\phi}^2$; thus, $(\underset{s}{\trianglelefteq})$ is actually much less decisive than simple majority vote.

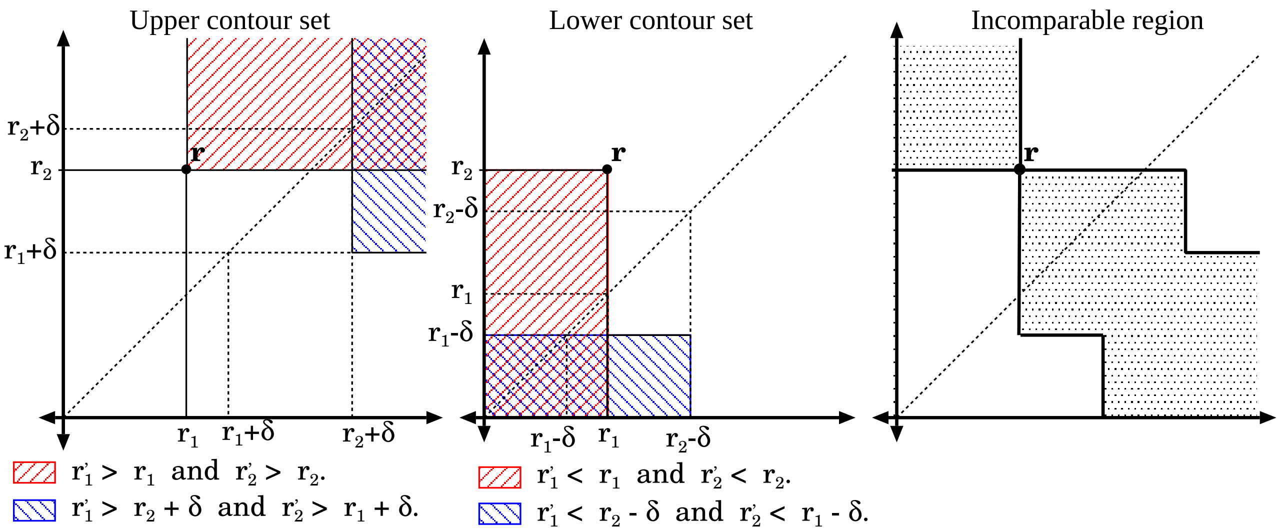

Figure 2: Upper and lower contour sets of the relation $(\underset{s,\boldsymbol{\psi}}{\blacktriangleleft})$ on $\mathbb{R}^2$ induced by the Suppes-Sen sprow $(\underset{s}{\trianglelefteq})$ in Example 3.2(a). Each contour set contains two overlapping regions, corresponding to the two possible conditions implying the relation $\mathbf{r}' \underset{s,\boldsymbol{\psi}}{\blacktriangleright} \mathbf{r}$ (or vice versa).
The sprow of Example 3.2(b) generates similar pictures: simply replace '$r_j - \delta$' with 'r_j/C' and '$r_j + \delta$' with '$C\, r_j$' everywhere. The difference between Examples 3.2(a) and (b) is in scaling. Using the sprow of Example 3.2(b), if we multiply $\mathbf{r}$ by a scalar, we see exactly the same pictures. However, using the sprow of 3.2(a), if we multiply $\mathbf{r}$ by, say, 2, then the 'incomparable' region (right) will be only half as wide.

Example 3.2: Suppose that $\Phi = \mathbb{R}$, as in §2.1. Then for any $(\boldsymbol{\psi}^1, \mathbf{r}^1), (\boldsymbol{\psi}^2, \mathbf{r}^2) \in \Psi^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}}$, $(\boldsymbol{\psi}^1, \mathbf{r}^1) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}^2, \mathbf{r}^2)$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, $(\psi_i^1, r_i^1) \preceq (\psi^2_{\sigma(i)}, r^2_{\sigma(i)})$. Let $\mathcal{I} = \{1, 2\}$ and fix $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \Psi^{\mathcal{I}}$; then $(\underset{s}{\trianglelefteq})$ induces a preorder $(\underset{s,\boldsymbol{\psi}}{\blacktriangleleft})$ on $\mathbb{R}^2$, where, for all $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^2$, we have $\mathbf{r}' \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}$ iff $(\boldsymbol{\psi}, \mathbf{r}') \underset{s}{\trianglelefteq} (\boldsymbol{\psi}, \mathbf{r})$.

(a) Let $(\preceq)$ be the wipo on $\Psi \times \mathbb{R}$ from Example 2.1(a), and let $\delta := c \cdot d(\psi_1, \psi_2)^\gamma$. Fix $\mathbf{r} \in \mathbb{R}^2$. For any $\mathbf{r}' \in \mathbb{R}^2$, $\mathbf{r}' \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}$ iff either $r_1' \leq r_1$ and $r_2' \leq r_2$, or $r_2' \leq r_1 - \delta$ and $r_1' \leq r_2 - \delta$. See Figure 2.

(b) Let $(\preceq)$ be the wipo on $\Psi \times \mathbb{R}$ from Example 2.1(b), and let $C := c^{d(\psi_1, \psi_2)}$. Then for any $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^2$, $\mathbf{r}' \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}$ iff either $r_1' \leq r_1$ and $r_2' \leq r_2$, or $r_2' \leq r_1/C$ and $r_1' \leq r_2/C$. $\diamondsuit$

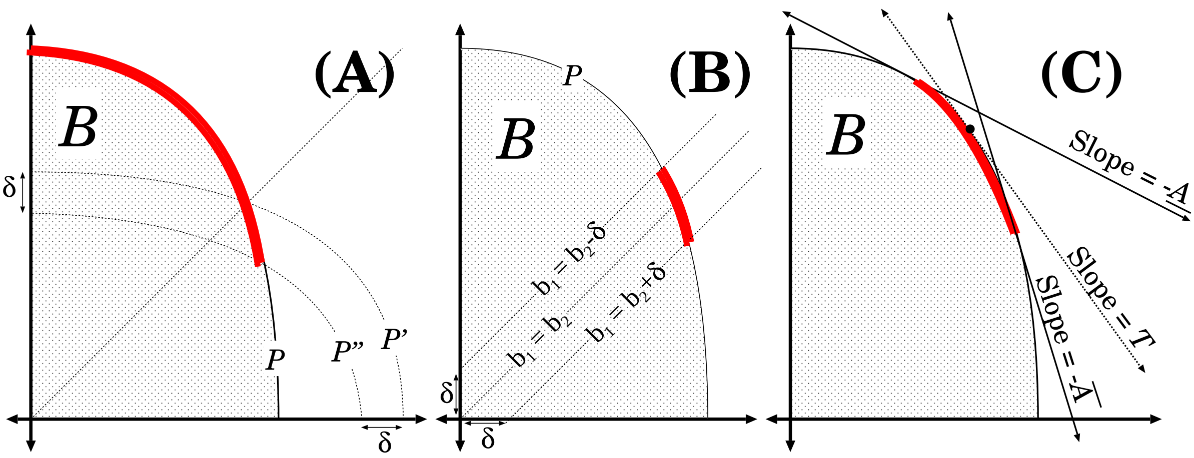

Figure 3: Solving bilateral bargaining problems with sprows. **(A)** The Suppes-Sen bargaining solution set of Example 3.3. **(B)** The approximate egalitarian bargaining solution of Example 3.6. **(C)** The approximate utilitarian bargaining solution set of Example 6.3.

Example 3.3: (Bargaining problems) Let $\mathcal{B} \subset \mathbb{R}^2$ be some compact, convex set —for example, the set of feasible utility profiles in a bilateral bargaining problem. Let $\mathcal{P}$ be the Pareto frontier of $\mathcal{B}$. Classic bargaining solutions prescribe a single point on $\mathcal{P}$ (usually by maximizing some SWO over $\mathcal{B}$).

Fix $\boldsymbol{\psi} \in \Psi^2$, and let $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$ be the preorder on $\mathbb{R}^2$ from Example 3.2. An incomplete preorder like $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$ generally will not identify a unique 'optimal' solution on $\mathcal{P}$, but it can isolate some subset of points in $\mathcal{B}$ which are $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$-undominated in $\mathcal{B}$. A point $\mathbf{b} \in \mathcal{B}$ is $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$-*undominated* in $\mathcal{B}$ if there is no $\mathbf{b}' \in \mathcal{B}$ such that $\mathbf{b} \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{b}'$. This means: (1) There is no $\mathbf{b}' \in \mathcal{B}$ which Pareto-dominates $\mathbf{b}$; and (2) There is no $\mathbf{b}' \in \mathcal{B}$ such that $b_1 < b'_2 - \delta$ and $b_2 < b'_1 - \delta$.

Let $\mathcal{P}'$ be the reflection of $\mathcal{P}$ across the diagonal. Let $\mathcal{P}'' := \mathcal{P} - (\delta, \delta)$; then $\mathbf{b}$ is $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$-undominated if (1) $\mathbf{b} \in \mathcal{P}$ and (2) There is no $\mathbf{b}' \in \mathcal{P}''$ which Pareto-dominates $\mathbf{b}$. The set of $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}})$-undominated solutions is shown in Figure 3(A). ◇

Proposition 3.4 *Let* $(\preceq)$ *be a wipo.*

(a) $(\underset{s}{\trianglelefteq})$ *is a* $(\preceq)$*-sprow.*

(b) *If* $(\trianglelefteq)$ *is any* $(\preceq)$*-sprow, then* $(\trianglelefteq)$ *extends* $(\underset{s}{\trianglelefteq})$.

If $(\trianglelefteq)$ *also satisfies* (SPar$^{\trianglelefteq}$)*, then* $(\trianglelefteq)$ *also refines* $(\underset{s}{\trianglelefteq})$.

(c) (Pareto Indifference) *Let* $(\trianglelefteq)$ *be any* $(\preceq)$*-sprow, and let* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. *If* $(\psi_i^1, \phi_i^1) \approx (\psi_i^2, \phi_i^2)$, *for all* $i \in \mathcal{I}$, *then* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \overset{\triangle}{\approx} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$.

(d) *If* $\{\underset{\lambda}{\trianglelefteq}\}_{\lambda\in\Lambda}$ *is a collection of* $(\preceq)$*-sprows* (where Λ is some indexing set)*, and* $(\trianglelefteq)$ *is their meet, then* $(\trianglelefteq)$ *is also a* $(\preceq)$*-sprow.*

3.2 Approximate egalitarianism

Given a wipo $(\preceq)$ on $\Psi\times\Phi$, the $(\preceq)$-*approximate egalitarian* sprow $(\underset{æ}{\trianglelefteq})$ on $\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}}$ is defined as follows: For any $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1),(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)\in\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}}$,

$$\Big((\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)\underset{æ}{\trianglelefteq}(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)\Big) \iff \begin{pmatrix}\text{There is a function } f:\mathcal{I}\longrightarrow\mathcal{I} \text{ (possibly not injective)}\\ \text{such that, for all } i\in\mathcal{I},\ \ (\psi^1_{f(i)},\phi^1_{f(i)})\preceq(\psi^2_i,\phi^2_i)\end{pmatrix}.$$

In other words, for every person i in the world $(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$, no matter how badly off, we can find some person $f(i)$ in the world $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)$ who is even worse off. In particular, this means that even the 'worst off' people in $(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$ (i.e. elements of $\mathcal{I}$ which are 'minimal' with respect to $(\preceq)$) are still better off than someone in $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)$. If $(\preceq)$ is a complete ordering on $\Psi\times\Phi$, then all people in world $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)$ are comparable with all people in $(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$, and $(\underset{æ}{\trianglelefteq})$ is equivalent to the classical 'maximin' egalitarian social welfare ordering.

Example 3.5: Suppose that $\Phi=\mathbb{R}$, as in §2.1. Then for any $(\boldsymbol{\psi}^1,\mathbf{r}^1),(\boldsymbol{\psi}^2,\mathbf{r}^2)\in\Psi^{\mathcal{I}}\times\mathbb{R}^{\mathcal{I}}$, $(\boldsymbol{\psi}^1,\mathbf{r}^1)\underset{æ}{\trianglelefteq}(\boldsymbol{\psi}^2,\mathbf{r}^2)$ iff there is a function $f:\mathcal{I}\longrightarrow\mathcal{I}$ (possibly not injective) such that, for all $i\in\mathcal{I}$, $(\psi^1_{f(i)},r^1_{f(i)})\preceq(\psi^2_i,r^2_i)$.

Let $\mathcal{I}=\{1,2\}$ and fix $\boldsymbol{\psi}=(\psi_1,\psi_2)\in\Psi^{\mathcal{I}}$; then $(\underset{æ}{\trianglelefteq})$ induces a preorder $(\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft})$ on $\mathbb{R}^2$, where $\mathbf{r}'\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft}\mathbf{r}$ iff $(\boldsymbol{\psi},\mathbf{r}')\underset{æ}{\trianglelefteq}(\boldsymbol{\psi},\mathbf{r})$. In particular, let $(\preceq)$ be the wipo of Example 2.1(a), and let $\delta:=c\cdot d(\psi_1,\psi_2)^\gamma$. For any $\mathbf{r},\mathbf{r}'\in\mathbb{R}^2$, $\mathbf{r}'\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft}\mathbf{r}$ iff either (1) $r_1\leq r_1'$ and $r_2\leq r_2'$; or (2) $r_1\leq r_1'$ and $r_1\leq r_2'-\delta$; or (3) $r_2\leq r_2'$ and $r_2\leq r_1'-\delta$. See Figure 4. ♢

Example 3.6: (Bargaining problems) Let $\mathcal{B}\subset\mathbb{R}^2$ be some compact, convex set (e.g. a bargaining set), as in Example 3.3. Let $(\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft})$ be the preorder on $\mathbb{R}^2$ from Example 3.5, and let $\mathcal{P}$ be the Pareto frontier of $\mathcal{B}$. We have:

Claim 3.6*: *For any* $\mathbf{b}\in\mathcal{B}$, $\Big(\mathbf{b}$ *is* $(\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft})$*-undominated*$\Big)\iff\Big(\mathbf{b}\in\mathcal{P}$ *and* $|b_1-b_2|\leq\delta\Big)$.

[See Figure 3(B)]. ♢

Given $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1),(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)\in\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}}$, let $\mathcal{I}_\downarrow$, $\mathcal{I}_\uparrow$, and $\mathcal{I}_0$ be as in Example 3.1. We say $(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$ is a *Hammond equity improvement* over $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)$ if

- There is a bijection $g:\mathcal{I}_0\longrightarrow\mathcal{I}_0$ such that, for every $i\in\mathcal{I}_0$, $(\psi^1_{g(i)},\phi^1_{g(i)})\approx(\psi^2_i,\phi^2_i)$;
- There is a injection $h:\mathcal{I}_\downarrow\longrightarrow\mathcal{I}_\uparrow$ such that, for all $i\in\mathcal{I}_\downarrow$,

$$(\psi^1_{h(i)},\phi^1_{h(i)})\quad\preceq\quad(\psi^2_{h(i)},\phi^2_{h(i)})\quad\preceq\quad(\psi^2_i,\phi^2_i)\quad\preceq\quad(\psi^1_i,\phi^1_i). \tag{6}$$

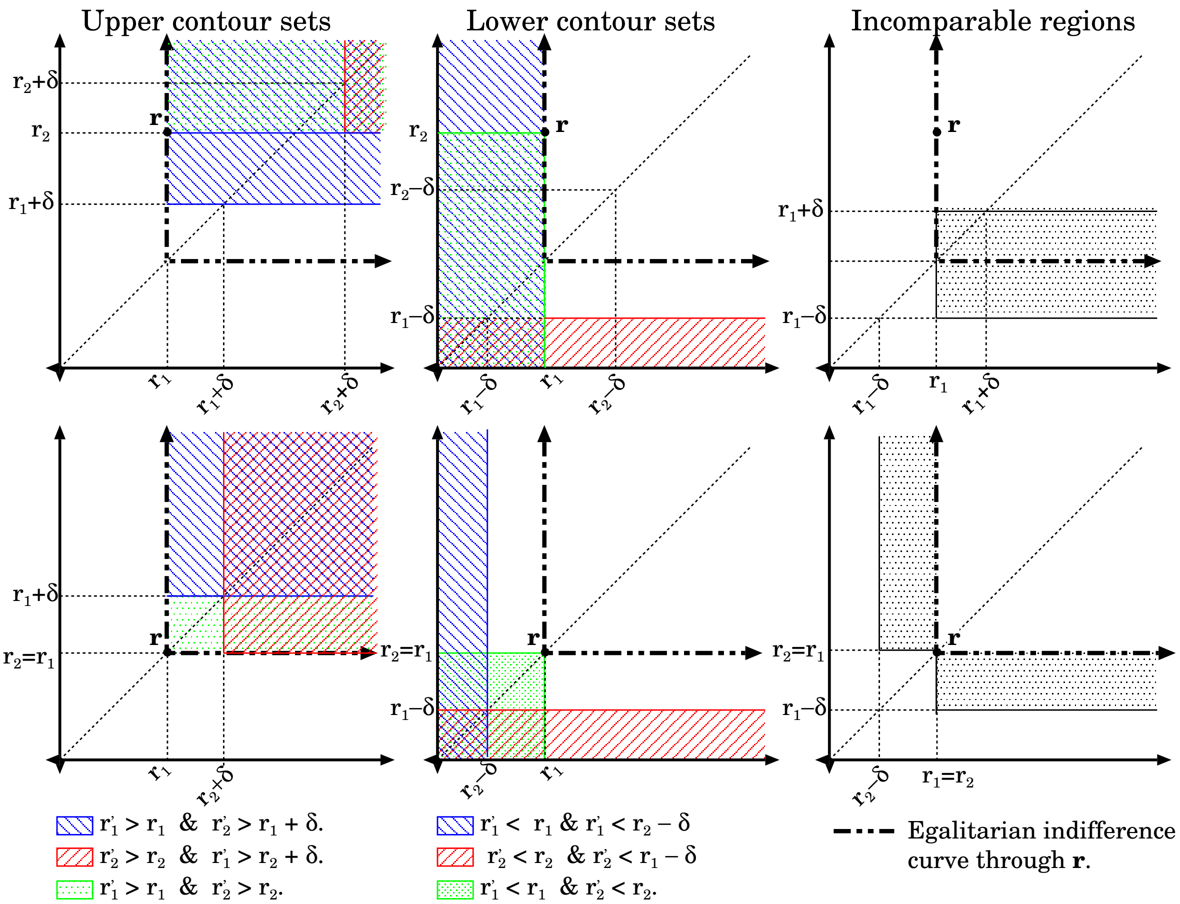

Figure 4: Contour sets for the relation ($\underset{\text{æ},\psi}{\blacktriangleleft}$) induced on $\mathbb{R}^2$ by approximate egalitarian sprow ($\underset{\text{æ}}{\trianglelefteq}$) in Example 3.5. *Left:* the upper contour sets for two choices of $\mathbf{r} \in \mathbb{R}^2$. *Middle:* the lower contour sets. Each contour set contains three overlapping regions, corresponding to the three possible conditions implying the relation $\mathbf{r}' \underset{\text{æ},\psi}{\blacktriangleright} \mathbf{r}$ (or vice versa). *Right:* The incomparable regions $\{\mathbf{r}' \in \mathbb{R}^2\ ;\ \mathbf{r}' \bowtie \mathbf{r}\}$. For reference, we also show the indifference curve of the classical egalitarian (i.e. maximin) SWO.

In other words, we can pair up every 'loser' i in $\mathcal{I}_\downarrow$ with some 'winner' $h(i)$ in $\mathcal{I}_\uparrow$ such that Hammond's (1976) equity condition is satisfied: both before and after the change, i is better off than $h(i)$, but the change narrows the gap between them.

For example, recall the 'concert ticket' story from Example 3.1, but now with a different scenario. Suppose Isolde and Jack have roughly equally strong desires to attend the concert. However, Isolde is a miserable, depressed person, whereas Jack is a happy, contented person. Isolde will be less happy than Jack no matter who gets the ticket; thus, we have $(\psi_i, \phi_i^1) \preceq (\psi_i, \phi_i^2) \preceq (\psi_j, \phi_j^2) \preceq (\psi_j, \phi_j^1)$. Thus, the change from $\boldsymbol{\phi}^1$ to $\boldsymbol{\phi}^2$ reduces inequality, so it is a Hammond equity improvement. (To see this, set $\mathcal{I}_\downarrow := \{j\}$, $\mathcal{I}_\uparrow := \{i\}$, and $h(j) := i$ in eqn.(6).)

The next result says that the approximate egalitarian sprow ($\underset{\text{æ}}{\trianglelefteq}$) is 'Hammond equity promoting'.

Proposition 3.7 *For any* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$*, if* $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ *is a Hammond equity improvement over* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$*, then* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{æ}}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$*.*

Proof. Let $g : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ and $h : \mathcal{I}_\downarrow \longrightarrow \mathcal{I}_\uparrow$ be as in eqn.(6). Define $f : \mathcal{I} \longrightarrow \mathcal{I}$ as follows: For all $i \in \mathcal{I}_0$, let $f(i) := g(i)$. For all $i \in \mathcal{I}_\downarrow$, let $f(i) := h(i)$. For all $i \in \mathcal{I}_\uparrow$, let $f(i) = i$. Then clearly, for all $i \in \mathcal{I}$, we have $(\psi^1_{f(i)}, \phi^1_{f(i)}) \preceq (\psi^2_i, \phi^2_i)$; hence $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{æ}}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$, as desired. □

4 Hedometers

In §2.2, we showed how to define a wipo from a collection of 'possible hedometers'. We now reverse this construction. Given any wipo $(\preceq)$ on $\Psi \times \Phi$, a ***hedometer*** for $(\preceq)$ is any function $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that, for all $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$,

$$\begin{aligned} & \Big((\psi_1, \phi_1) \preceq (\psi_2, \phi_2)\Big) \implies \Big(h(\psi_1, \phi_1) \le h(\psi_2, \phi_2)\Big) \\ \text{and} \quad & \Big((\psi_1, \phi_1) \prec (\psi_2, \phi_2)\Big) \implies \Big(h(\psi_1, \phi_1) < h(\psi_2, \phi_2)\Big). \end{aligned} \tag{7}$$

Let $\mathcal{H}_{\mathcal{ED}}(\preceq)$ be the set of hedometers for $(\preceq)$; this set is nonempty under fairly general conditions.[6] In general, the converses of the implications (7) are false (otherwise $(\preceq)$ would in fact be a complete preorder on $\Psi \times \Phi$). However, if $\psi_1 = \psi = \psi_2$ for some $\psi \in \Psi$, then (7) implies

$$\Big((\psi, \phi) \preceq (\psi', \phi')\Big) \iff \Big(h(\psi, \phi) \le h(\psi', \phi')\Big), \quad \text{for all } \phi, \phi' \in \Phi.$$

In other words, the function $\Phi \ni \phi \mapsto h(\psi, \phi) \in \mathbb{R}$ is an ordinal utility function for the preference order $(\underset{\psi}{\preceq})$. Thus, $\mathcal{H}_{\mathcal{ED}}(\preceq)$ is nonempty only if the individual preference orders in $\{\underset{\psi}{\preceq}\}_{\psi \in \Psi}$ can each be represented by a utility function. The wipo $(\preceq)$ is called ***hedometric*** if $\mathcal{H}_{\mathcal{ED}}(\preceq) \neq \emptyset$ and, for any $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$, we have

$$\Big((\psi_1, \phi_1) \preceq (\psi_2, \phi_2)\Big) \iff \Big(h(\psi_1, \phi_1) \le h(\psi_2, \phi_2), \text{ for all } h \in \mathcal{H}_{\mathcal{ED}}(\preceq)\Big). \tag{8}$$

[6] If $\mathcal{X}$ is a set and $(\preceq)$ is a (partial) preorder on $\mathcal{X}$, then a *utility function* for $(\preceq)$ is any function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that, for all $x, y \in \mathcal{X}$, $(x \preceq y) \Rightarrow (u(x) \le u(y))$ and $(x \prec y) \Rightarrow (u(x) < u(y))$. Thus, a 'hedometer' is just a utility function for a wipo on $\Psi \times \Phi$; we use the term 'hedometer' to avoid confusion with the utility functions representing the (complete) preferences of the individuals.

Even incomplete preorders admit utility functions under fairly general conditions. For example, if $\mathcal{X}$ is a space of lotteries and $(\preceq)$ satisfies versions of the vNM axioms of 'Linearity' and 'Continuity', then Aumann (1962) showed that $(\mathcal{X}, \preceq)$ admits a linear utility function, thereby extending the classic result of von Neumann and Morgenstern (1947). If $\mathcal{X}$ is a topological space and $(\prec)$ is irreflexive, continuous, separable, and 'spacious', then Peleg (1970) showed that $(\mathcal{X}, \prec)$ admits a continuous utility function —a result analogous to Debreu's (1954) representation theorem. With weaker topological hypotheses, Jaffray (1975) and Sondermann (1980) give existence theorem for semicontinuous utility functions (analogous to Rader's (1963) theorem). Finally, Richter (1966) showed that any choice function satisfying a weak consistency property ('congruence') could be represented as maximizing some utility function.

Recently, Ok (2002) has established necessary conditions for a preorder to be representable by a collection of utility functions. In our terminology, Ok's Theorem 3 (2002) says that a wipo $(\preceq)$ is hedometric if it is *upper-separable*, meaning there is a countable subset $\Xi \subseteq \Psi \times \Phi$ which is both *dense* [i.e. for all $(\psi_1, \phi_1) \prec (\psi_2, \phi_2) \in \Psi \times \Phi$, there exists some $(\psi, \phi) \in \Xi$ such that $(\psi_1, \phi_1) \prec (\psi, \phi) \prec (\psi_2, \phi_2)$] and *upper-dense* [i.e. for all $(\psi_1, \phi_1) \not\succ\!\!\!\prec (\psi_2, \phi_2) \in \Psi \times \Phi$, there exists some $(\psi, \phi) \in \Xi$ such that $(\psi_1, \phi_1) \preceq (\psi, \phi) \not\succ\!\!\!\prec (\psi_2, \phi_2)$].[7,8] We will show that a wipo is hedometric under hypotheses which are similar, but more natural in the setting of weak interpersonal comparisons. We say the wipo $(\preceq)$ is ***regular*** if, for every $\psi_0 \in \Psi$, the following holds:

(R1) The ordering $(\underset{\psi_0}{\preceq})$ can be represented by an ordinal utility function $u_{\psi_0} : \Phi \longrightarrow \mathbb{R}$.

(R2) For all $\phi_0 \in \Phi$, and all $(\psi_1, \phi_1) \in \Psi \times \Phi$:

[i] If $(\psi_0, \phi_0) \prec (\psi_1, \phi_1)$, then $\exists\ \phi_0' \in \Phi$ such that $(\psi_0, \phi_0) \prec (\psi_0, \phi_0') \prec (\psi_1, \phi_1)$.

[ii] If $(\psi_0, \phi_0) \succ (\psi_1, \phi_1)$, then $\exists\ \phi_0' \in \Phi$ such that $(\psi_0, \phi_0) \succ (\psi_0, \phi_0') \succ (\psi_1, \phi_1)$.

[iii] If $(\psi_0, \phi_0) \not\succ\!\!\!\prec (\psi_1, \phi_1)$, then $\exists\ \phi_0', \phi_0'' \in \Phi$ such that $(\psi_1, \phi_1) \not\succ\!\!\!\prec (\psi_0, \phi_0') \prec (\psi_0, \phi_0) \prec (\psi_0, \phi_0'') \not\succ\!\!\!\prec (\psi_1, \phi_1)$.

(R3) For any $\phi_0 \underset{\psi_0}{\prec} \phi_0' \in \Phi$, and any $\psi_1 \in \Psi$:

[i] There exists $\phi_1 \in \Phi$ such that $(\psi_0, \phi_0) \preceq (\psi_1, \phi_1)$ but $(\psi_0, \phi_0') \not\preceq (\psi_1, \phi_1)$.

[ii] There exists $\phi_1' \in \Phi$ such that $(\psi_1, \phi_1') \preceq (\psi_0, \phi_0')$ but $(\psi_1, \phi_1') \not\preceq (\psi_0, \phi_0)$.

Example 4.1: (a) Let $\Phi = \mathbb{R}$, let d be a metric on Ψ, and let $(\preceq)$ be one of the distance-based wicus on $\Psi \times \mathbb{R}$ from Example 2.1. Then $(\preceq)$ is regular.

(b) Let Φ be any space of physical states. For each $\psi \in \Psi$, let $u_\psi : \Phi \longrightarrow \mathbb{R}$ be a utility function representing the preference order $(\underset{\psi}{\preceq})$ on Φ, such that the image $u_\psi(\Phi) \subseteq \mathbb{R}$ is dense. Let $(\underset{*}{\preceq})$ be one of the distance-based wicus on $\Psi \times \mathbb{R}$ from Example 2.1, and let $(\preceq)$ be the wicu-mediated wipo on $\Psi \times \Phi$ defined by eqn.(2). Then $(\preceq)$ is regular. $\diamondsuit$

We now come to the main result of this section.

Theorem 4.2 *Any regular wipo is hedometric.*

[7] Ok (2002) also establishes a similar representation theorem using *semicontinuous* utility functions on partially ordered *topological* spaces, as well as sufficient conditions for a partial order to be 'representable' by a *finite* collection of utility functions; see also Yılmaz (2008). Much earlier, Dushnik and Miller (1941) showed that any irreflexive partial order was the intersection of all its linear extensions; this result was extended to preorders by Donaldson and Weymark (1998), and to a very broad class of binary relations by Duggan (1999). However, the linear extensions involved in these intersections cannot generally be represented by utility functions.

[8] I am grateful to Klaus Nehring for his comments on an earlier version of this section, which indirectly led me to discover Ok (2002) and related work.

Note that (R1) is really a statement about the individuals' preference orders, and has nothing to do with interpersonal comparisons *per se*. Furthermore, we do not require the utility functions in (R1) to be (semi)continuous or otherwise 'well-behaved', so (R1) is satisfied under very general conditions. Axiom (R2) is an 'interposition' axiom, similar to Ok's (2002) definitions of 'dense' and 'upper-dense'; however we require the interposed points to belong in the fibre $\{\psi_0\} \times \Phi$, rather than to some countable dense subset of $\Psi \times \Phi$. (R2) is best understood as a very mild 'continuity' condition. Intuitively, (R2)[i] says: if the wipo ($\preceq$) judges that Zara, in her current physical state, is strictly less happy than Juan in his current physical state, then we can make a very slight (but nonzero) improvement to her state (say, give her 0.01 cents) which still leaves her strictly less happy than Juan. Axiom (R2)[ii] says the same thing, but with 'happier' instead of 'less happy'. Axiom (R2)[iii] says: if Zara, in her current physical state, is ($\preceq$)-incomparable to Juan in his current physical state, then we can very slightly improve or harm her state and she will still be incomparable to Juan.

Finally, axiom (R3) says that the wipo ($\preceq$) is sensitive enough that any improvement in Zara's well-being (i.e. from ϕ_0 to ϕ_0'), can be roughly 'matched' with some improvement in Juan's well-being, in the following sense. [i] The worst state for Juan which is ($\preceq$)-better than ϕ_0 (which must be at least as bad as ϕ_1) is strictly worse than the worst state for Juan which is ($\preceq$)-better than ϕ_0' (which must be strictly better than ϕ_1). [ii] The best state for Juan which is ($\preceq$)-worse than ϕ_0' (which must be at least as good as ϕ_1') is strictly better than the best state for Juan which is ($\preceq$)-worse than ϕ_0 (which must be strictly worse than ϕ_1').

5 Welfarist sprows

A *social welfare order* (SWO) is a complete preorder ($\underline{\blacktriangleleft}$) on $\mathbb{R}^{\mathcal{I}}$ satisfying two axioms:

(Par$\underline{\blacktriangleleft}$) For any $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$, if $r_i \leq r_i'$ for all $i \in \mathcal{I}$, then $\mathbf{r} \underline{\blacktriangleleft} \mathbf{r}'$. If $r_i < r_i'$ for all $i \in \mathcal{I}$, then $\mathbf{r} \blacktriangleleft \mathbf{r}'$.

(Anon$\underline{\blacktriangleleft}$) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, and $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, then $\mathbf{r} \overset{\blacktriangle}{\approx} \sigma(\mathbf{r})$.

Example 5.1: (a) The *egalitarian* SWO ($\underset{e}{\underline{\blacktriangleleft}}$) is defined as follows. For all $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^{\mathcal{I}}$, $\mathbf{r}^1 \underset{e}{\underline{\blacktriangleleft}} \mathbf{r}^2$ if and only if $\min_{i \in \mathcal{I}} (r_i^1) \leq \min_{i \in \mathcal{I}} (r_i^2)$. (Thus, $\mathbf{r}^1 \underset{e}{\overset{\blacktriangle}{\approx}} \mathbf{r}^2$ whenever $\min_{i \in \mathcal{I}} (r_i^1) = \min_{i \in \mathcal{I}} (r_i^2)$.)

(b) Suppose $\mathcal{I} := [1...I]$. Let $\overset{\nearrow}{\mathbb{R}}^{\mathcal{I}} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{I}} \, ; \, r_1 \leq r_2 \leq \cdots \leq r_I\}$. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, let $\overset{\nearrow}{\mathbf{r}} \in \overset{\nearrow}{\mathbb{R}}^{\mathcal{I}}$ be the element obtained by arranging the entries of $\mathbf{r}$ in ascending order —e.g. $\overset{\nearrow}{r}_1 := \min_{i \in \mathcal{I}} r_i$ and $\overset{\nearrow}{r}_I := \max_{i \in \mathcal{I}} r_i$. For any $k \in [1...I]$, the *rank k dictatorship* SWO ($\underset{k}{\underline{\blacktriangleleft}}$) is defined on $\mathbb{R}^{\mathcal{I}}$ by $\mathbf{r} \underset{k}{\underline{\blacktriangleleft}} \mathbf{r}'$ iff $\overset{\nearrow}{r}_k \leq \overset{\nearrow}{r}_k'$ (thus, ($\underset{e}{\underline{\blacktriangleleft}}$) is the rank 1 dictatorship).

(c) The *lexmin* SWO ($\underset{\text{lex}}{\underline{\blacktriangleleft}}$) is defined as follows: $\mathbf{r}^1 \underset{\text{lex}}{\blacktriangleleft} \mathbf{r}^2$ iff there exists some $j \in [1...I]$ such that $\overset{\nearrow}{r}_k = \overset{\nearrow}{r}_k'$ for all $k \in [1...j)$, while $\overset{\nearrow}{r}_j < \overset{\nearrow}{r}_j'$. Meanwhile, $\mathbf{r}^1 \underset{\text{lex}}{\overset{\blacktriangle}{\approx}} \mathbf{r}^2$ iff $\mathbf{r}^1 = \mathbf{r}^2$. $\diamondsuit$

Let $(\preceq)$ be a wipo, and let $\mathcal{H}_{\mathcal{ED}}(\preceq)$ be the set of 'possible hedometers' for $(\preceq)$ (see §4). If $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$ and $(\boldsymbol{\psi}, \boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, then define $\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi}) := (h(\psi_i, \phi_i))_{i \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$.

Proposition 5.2 *Let* $(\preceq)$ *be a wipo on* $\Psi \times \Phi$*. Let* $(\blacktriangleleft)$ *be a complete preorder on* $\mathbb{R}^{\mathcal{I}}$ *satisfying axiom* (Par$^{\blacktriangleleft}$)*, and let* $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ *be some function. Define the preorder* $(\underset{h}{\trianglelefteq})$ *on* $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ *by* $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{h}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}')$ *iff* $\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi}) \underline{\blacktriangleleft} \mathbf{h}(\boldsymbol{\psi}', \boldsymbol{\phi}')$*. Then*

$$\left(\underset{h}{\trianglelefteq} \textit{ is a } (\preceq)\textit{-sprow} \right) \iff \left(h \in \mathcal{H}_{\mathcal{ED}}(\preceq) \textit{ and } (\underline{\blacktriangleleft}) \textit{ is a SWO} \right).$$

Corollary 5.3 *Let* $(\preceq)$ *be a wipo on* $\Psi \times \Phi$*, and let* $\mathcal{H} \subseteq \mathcal{H}_{\mathcal{ED}}(\preceq)$ *be some collection of hedometers. Let* $(\underline{\blacktriangleleft})$ *be a SWO on* $\mathbb{R}^{\mathcal{I}}$*, and define the preorder* $(\underset{\mathcal{H}}{\trianglelefteq})$ *on* $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ *by*

$$\left((\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{\mathcal{H}}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}') \right) \quad \iff \quad \left(\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi}) \ \underline{\blacktriangleleft}\ \mathbf{h}(\boldsymbol{\psi}', \boldsymbol{\phi}'), \textit{ for all } h \in \mathcal{H} \right).$$

Then $(\underset{\mathcal{H}}{\trianglelefteq})$ *is a* $(\preceq)$*-sprow.*

Proof. Combine Proposition 5.2 with Lemma 3.4(d). □

The set $\mathcal{H}_{\mathcal{ED}}(\preceq)$ generally contains many possible hedometers, which could yield different, contradictory sprows in Proposition 5.2. Corollary 5.3 mitigates this problem by requiring 'unanimity' over some 'representative sample' $\mathcal{H}$ of hedometers. What constitutes a representative sample? The most conservative choice would be to set $\mathcal{H} = \mathcal{H}_{\mathcal{ED}}(\preceq)$. Thus, for any SWO $(\underline{\blacktriangleleft})$ on $\mathbb{R}^{\mathcal{I}}$, the $(\preceq, \underline{\blacktriangleleft})$-*welfarist*[9] *sprow* $(\trianglelefteq)$ is defined as follows: for all $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$,

$$\left((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \trianglelefteq (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \right) \iff \left(\text{For all } h \in \mathcal{H}_{\mathcal{ED}}(\preceq), \ \ \mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underline{\blacktriangleleft} \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \right). \tag{9}$$

The welfarist sprow (9) seems most plausible when $(\preceq)$ is hedometric (as in §4). But it is well-defined whenever $\mathcal{H}_{\mathcal{ED}}(\preceq) \neq \emptyset$.

Proposition 5.4 *Let* $(\preceq)$ *be a regular wipo* (see §4)*. Let* $(\underset{e}{\underline{\blacktriangleleft}})$ *be the egalitarian SWO in* Example 5.1(a)*. The* $(\preceq, \underset{e}{\underline{\blacktriangleleft}})$*-welfarist sprow is the approximate egalitarian sprow* $(\underset{\text{æ}}{\trianglelefteq})$ *from* §3.2*. In other words, for any* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$,

$$\left((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{æ}}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \right) \iff \left(\min_{i \in \mathcal{I}} h(\psi_i^1, \phi_i^1) \leq \min_{i \in \mathcal{I}} h(\psi_i^2, \phi_i^2), \ \ \forall\ h \in \mathcal{H}_{\mathcal{ED}}(\preceq) \right).$$

[9] A weaker form of 'welfarism' simply requires the social ordering of two worlds to be entirely determined by the pattern of individual preferences between those worlds; this follows from 'Pareto Indifference', which holds for any sprow, by Proposition 3.4(c). We here use 'welfarism' in the stronger sense employed by Sen (1970b) and d'Aspremont and Gevers (2002): the social ordering is determined by comparing the values of the individual's *utility functions* (or in this case, a hedometer) on the the two worlds. For a detailed discussion of this distinction, see (d'Aspremont and Gevers, 2002, §3.3.1, p.489-494)

In general, a $(\preceq)$-sprow will be a very incomplete preorder on $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, because $(\preceq)$ itself is an incomplete preorder of $\Psi \times \Phi$. Say that two worlds $(\boldsymbol{\psi}, \boldsymbol{\phi}), (\boldsymbol{\psi}', \boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ are *fully* $(\preceq)$*-comparable* if the set $\{(\psi_i, \phi_i)\}_{i\in\mathcal{I}} \cup \{(\psi'_i, \phi'_i)\}_{i\in\mathcal{I}}$ is totally ordered by $(\preceq)$. (For example, fix $\psi \in \Psi$, and suppose $(\boldsymbol{\psi}, \boldsymbol{\phi})$ and $(\boldsymbol{\psi}', \boldsymbol{\phi}')$ are 'ψ-clone worlds' where $\psi_i = \psi'_i = \psi$ for all $i \in \mathcal{I}$; then $(\boldsymbol{\psi}, \boldsymbol{\phi})$ and $(\boldsymbol{\psi}', \boldsymbol{\phi}')$ are fully $(\preceq)$-comparable). In this case, a $(\preceq)$-sprow really has no excuse for failing to order $(\boldsymbol{\psi}, \boldsymbol{\phi})$ relative to $(\boldsymbol{\psi}', \boldsymbol{\phi}')$, since every element of $\{(\psi_i, \phi_i)\}_{i\in\mathcal{I}}$ is $(\preceq)$-comparable to every element of $\{(\psi'_i, \phi'_i)\}_{i\in\mathcal{I}}$. A $(\preceq)$-sprow $(\trianglelefteq)$ is *minimally decisive* if $(\boldsymbol{\psi}, \boldsymbol{\phi})$ and $(\boldsymbol{\psi}', \boldsymbol{\phi}')$ are $(\trianglelefteq)$-comparable whenever they are fully $(\preceq)$-comparable.

Example 5.5: The approximate egalitarian sprow $(\underset{\text{æ}}{\trianglelefteq})$ (see §3.2) is minimally decisive. To see this, suppose $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ and $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ are fully $(\preceq)$-comparable. Then there exists some $m \in \{1, 2\}$ and some $j \in \mathcal{I}$ such that $(\psi^m_j, \phi^m_j) \preceq (\psi^n_i, \phi^n_i)$ for all $(n, i) \in \{1, 2\} \times \mathcal{I}$. Suppose $m = 1$, and define $f : \mathcal{I} \longrightarrow \mathcal{I}$ by $f(i) = j$ for all $i \in \mathcal{I}$; then we have $(\psi^1_{f(i)}, \phi^1_{f(i)}) = (\psi^1_j, \phi^1_j) \preceq (\psi^2_i, \phi^2_i)$ for all $i \in \mathcal{I}$; hence $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{æ}}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. $\diamondsuit$

We will now show that very few welfarist sprows are minimally decisive, and among these, only the approximate egalitarian sprow has a desirable 'equity' property. To explain this, suppose $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ are fully $(\preceq)$-comparable. The *rank structure* of the pair $((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2))$ is the complete order $(\underline{\lessdot})$ on $\{1, 2\} \times \mathcal{I}$ defined as follows: for all $n, m \in \{1, 2\}$ and $i, j \in \mathcal{I}$, $(n, i) \underline{\lessdot} (m, j)$ if and only if $(\psi^n_i, \phi^n_i) \preceq (\psi^m_j, \phi^m_j)$. We will require the following axiom of 'minimal richness' for $(\preceq)$:

(MR) For any complete order $(\underline{\lessdot})$ on $\{1, 2\} \times \mathcal{I}$, there exist fully $(\preceq)$-comparable $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ and $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ in $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ whose rank structure is $(\underline{\lessdot})$.

This is a very mild condition, which is satisfied by almost any collection of preferences. For example, suppose there exists some subset $\Phi' \subseteq \Phi$ with $|\Phi'| \geq 2 \times |\mathcal{I}|$, and some $\psi \in \Psi$ such that $(\underset{\psi}{\preceq})$ is a strict ordering of Φ'; then $(\preceq)$ satisfies (MR). (Let $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ be 'ψ-clone societies' with $\psi_i = \psi'_i = \psi$ for all $i \in \mathcal{I}$; then pick $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi'_i\}_{i\in\mathcal{I}}$ from Φ' to obtain any desired rank structure).

We will use the following 'minimal' version of Hammond's equity condition:

(MinEq$^{\trianglelefteq}$) There exist $(\boldsymbol{\psi}, \boldsymbol{\phi}), (\boldsymbol{\psi}', \boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ and $i, j \in \mathcal{I}$ such that:

- **(q1$^{\trianglelefteq}$)** $(\psi_i, \phi_i) \prec (\psi'_i, \phi'_i) \preceq (\psi'_j, \phi'_j) \prec (\psi_j, \phi_j)$.
- **(q2$^{\trianglelefteq}$)** $(\psi_i, \phi_i) \preceq (\psi_k, \phi_k) \approx (\psi'_k, \phi'_k)$ for all $k \in \mathcal{I} \setminus \{i, j\}$; and
- **(q3$^{\trianglelefteq}$)** $(\boldsymbol{\psi}, \boldsymbol{\phi}) \trianglelefteq (\boldsymbol{\psi}', \boldsymbol{\phi}')$.

We now come to the main result of this section.

Theorem 5.6 *Let* $(\preceq)$ *be a wipo on* $\Psi \times \Phi$ *which satisfies* (MR)*, with* $\mathcal{H}_{\mathcal{ED}}(\preceq) \neq \emptyset$*. Let* $(\underline{\blacktriangleleft})$ *be a SWO on* $\mathbb{R}^{\mathcal{I}}$*, and let* $(\trianglelefteq)$ *be the* $(\preceq, \underline{\blacktriangleleft})$*-welfarist sprow on* $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$*.*

(a) $(\trianglelefteq)$ *is minimally decisive if and only if* $(\blacktriangleleft)$ *refines a rank dictatorship SWO* [Example 5.1(b)].

(b) *If* $(\trianglelefteq)$ *is minimally decisive and satisfies* (MinEq)*, then* $(\trianglelefteq)$ *is extended by the approximate egalitarian sprow* $(\underset{æ}{\trianglelefteq})$.

(c) *If* $(\trianglelefteq)$ *refines* $(\underset{æ}{\trianglelefteq})$*, then* $(\trianglelefteq)$ *is minimally decisive and satisfies* (MinEq).

(d) $(\trianglelefteq)$ *extends* $(\underset{æ}{\trianglelefteq})$ *if and only if* $(\trianglelefteq)$ *is* $(\underset{æ}{\trianglelefteq})$.

Let $\mathcal{W}(\preceq)$ be the set of all welfarist sprows for the wipo $(\preceq)$, and consider the partial order relation "$\subseteq$" on $\mathcal{W}(\preceq)$ (i.e. $(\underset{1}{\trianglelefteq}) \subseteq (\underset{2}{\trianglelefteq})$ iff $(\underset{2}{\trianglelefteq})$ extends $(\underset{1}{\trianglelefteq})$). If $(\preceq)$ is regular, then Proposition 5.4 says that $(\underset{æ}{\trianglelefteq}) \in \mathcal{W}(\preceq)$. In this case, Theorem 5.6(d) says that $(\underset{æ}{\trianglelefteq})$ is a local $(\subseteq)$-maximum in $\mathcal{W}(\preceq)$, while Theorem 5.6(b) says that $(\underset{æ}{\trianglelefteq})$ is the *global* $(\subseteq)$-maximum for the set of minimally decisive and minimally equitable elements of $\mathcal{W}(\preceq)$. However, Theorem 5.6 applies even when $(\preceq)$ is not regular, or even hedometric (so $(\underset{æ}{\trianglelefteq})$ itself might not be in $\mathcal{W}(\preceq)$).

Example 5.7: Let $(\underset{\text{lex}}{\blacktriangleleft})$ be the lexmin SWO [Example 5.1(c)], and let $(\underset{\text{lex}}{\trianglelefteq})$ be the $(\preceq, \underset{\text{lex}}{\blacktriangleleft})$-welfarist sprow. Then $(\underset{\text{lex}}{\trianglelefteq})$ is minimally decisive (by Lemma 5.8 in the Appendix) and satisfies (MinEq). If $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{lex}}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$, then $\mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{lex}}{\blacktriangleleft} \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ for all $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$; hence $\mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{æ}{\blacktriangleleft} \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ for all $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$, so $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{æ}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Thus, $(\underset{æ}{\trianglelefteq})$ extends $(\underset{\text{lex}}{\trianglelefteq})$. ♢

6 Weak interpersonal comparisons of lotteries

The theory of wipos and sprows developed in sections 2-5 cannot model decision-making under uncertainty. We now remedy this. Let $\mathbb{P}(\Phi)$ be the space of probability distributions over Φ (with respect to some sigma algebra on Φ). For all $\psi \in \Psi$, let $(\underset{\psi}{\preceq})$ be a complete preorder on $\mathbb{P}(\Phi)$ which satisfies the von Neumann-Morgenstern ('vNM') axioms. (Thus, $(\underset{\psi}{\preceq})$ could be represented as maximizing the expected value of a cardinal utility function). Let $\mathbb{P}(\Psi)$ be the space of probability distributions over Ψ (with respect to some sigma algebra on Ψ), and let $\mathbb{P}(\Psi \times \Phi)$ be the space of *lotteries* —that is, probability measures on the product sigma algebra on $\Psi \times \Phi$. For any $\delta \in \mathbb{P}(\Psi)$ and $\rho \in \mathbb{P}(\Phi)$, let $\delta \otimes \rho \in \mathbb{P}(\Psi \times \Phi)$ denote the unique lottery over $\Psi \times \Phi$ such that $(\delta \otimes \rho)(\Psi' \times \Phi') = \delta(\Psi') \cdot \rho(\Phi')$ for all measurable subsets $\Psi' \subseteq \Psi$ and $\Phi' \subseteq \Phi$.

Let $\mathfrak{P} \subseteq \mathbb{P}(\Psi \times \Phi)$ be a convex set of lotteries. A *weak interpersonal preference order over lotteries* (or *wipol*) is a preorder $(\preceq)$ on $\mathfrak{P}$ which satisfies the following axioms:

(Nonpat$^{\preceq}$) (*Nonpaternalism*) For all $\mu \in \mathbb{P}(\Psi)$ and $\rho_1, \rho_2 \in \mathbb{P}(\Phi)$, if $\mu \otimes \rho_1$ and $\mu \otimes \rho_2$ are in $\mathfrak{P}$, and if $\mu\Big\{\psi \in \Psi \ ; \ \rho_1 \underset{\psi}{\preceq} \rho_2\Big\} = 1$, then $\mu \otimes \rho_1 \preceq \mu \otimes \rho_2$.

(Lin$^{\preceq}$) (*Linearity*) For all $\rho, \rho_1', \rho_2' \in \mathfrak{P}$ and $s, s' \in (0,1)$ with $s + s' = 1$, $\Big(\rho_1' \preceq \rho_2'\Big) \implies \Big((s\rho + s'\rho_1') \preceq (s\rho + s'\rho_2')\Big)$.

Axiom (Lin$^{\preceq}$) is a version of the standard vNM linearity axiom. To illustrate (Nonpat$^{\preceq}$), fix $\psi \in \Psi$, and let $\delta_\psi \in \mathbb{P}(\Psi)$ be the 'sure thing' distribution such that $\delta_\psi\{\psi\} = 1$. If $\delta_\psi \otimes \rho_1$ and $\delta_\psi \otimes \rho_2$ are in $\mathfrak{P}$, then (Nonpat$^{\preceq}$) implies a more familiar 'Nonpaternalism' condition similar to (W1):

$$\Big(\delta_\psi \otimes \rho_1 \ \preceq\ \delta_\psi \otimes \rho_2\Big) \iff \Big(\rho_1 \underset{\psi}{\preceq} \rho_2\Big).$$

The intuitive arguments for the existence of a wipol on $\mathfrak{P}$ parallel the arguments made in the introduction for the existence of a wipo on $\Psi \times \Phi$: we have some (limited) ability to compare the welfare of people in different psychophysical states —especially when these are our own potential future psychophysical states —and this ability should extend to some ability to compare the welfare of people confronting *lotteries* over psychophysical states.

Example 6.1: For any $\rho \in \mathbb{P}(\Psi \times \Phi)$ and measurable function $h : \Psi \times \Phi \longrightarrow \mathbb{R}$, we define

$$h^*(\rho) \quad := \quad \int_{\Psi \times \Phi} h(\psi, \phi)\ d\rho(\psi, \phi). \tag{10}$$

Let $\mathcal{H}$ be a collection of functions $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that, for every $\psi \in \Psi$ and every $h \in \mathcal{H}$, the function $h(\psi, \bullet) : \Phi \longrightarrow \mathbb{R}$ is a vNM cardinal utility function representing $(\underset{\psi}{\preceq})$. Define the ordering $(\preceq)$ on $\mathfrak{P}$ as follows: for any $\rho, \rho' \in \mathfrak{P}$,

$$\Big(\rho \preceq \rho'\Big) \iff \Big(h^*(\rho) \leq h^*(\rho') \text{ for all } h \in \mathcal{H}\Big). \tag{11}$$

Then $(\preceq)$ is a wipol on $\mathfrak{P}$. (Proposition 6.7 below shows that many reasonable wipols can be represented in this fashion.) $\diamondsuit$

6.1 Social preferences over world lotteries

Any policy chosen by the social planner will result in a ***world-lottery***: a probability distribution $\boldsymbol{\rho}$ over $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. To decide the 'best' policy, the social planner must formulate a preference relation $(\trianglelefteq)$ over $\mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$. For any world-lottery $\boldsymbol{\rho} \in \mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$, and any $i \in \mathcal{I}$, let $\rho_i \in \mathbb{P}(\Psi \times \Phi)$ be the lottery on the ith coordinate induced by $\boldsymbol{\rho}$. That is, for any measurable subset $\mathcal{U} \subset \Psi \times \Phi$,

$$\rho_i[\mathcal{U}] \quad := \quad \boldsymbol{\rho}\big\{(\boldsymbol{\psi}, \boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}\ ;\ (\psi_i, \phi_i) \in \mathcal{U}\big\}. \tag{12}$$

For any convex subset $\mathfrak{P} \subseteq \mathbb{P}(\Psi \times \Phi)$, let $\mathfrak{P}^{\otimes \mathcal{I}} := \big\{\boldsymbol{\rho} \in \mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})\ ;\ \rho_i \in \mathfrak{P},\ \forall\ i \in \mathcal{I}\big\}$; this is a convex subset of $\mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$.

If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, define $\sigma : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ as in eqn.(4). For any $\boldsymbol{\rho} \in \mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$, we define $\sigma(\boldsymbol{\rho}) = \boldsymbol{\rho}'$ as follows:

$$\text{For any measurable subset } \mathcal{U} \subseteq \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}, \qquad \boldsymbol{\rho}'[\mathcal{U}] \quad := \quad \boldsymbol{\rho}\left[\sigma^{-1}(\mathcal{U})\right]. \tag{13}$$

It is easy to check that $\sigma[\mathfrak{P}^{\otimes\mathcal{I}}] = \mathfrak{P}^{\otimes\mathcal{I}}$. If $(\preceq)$ is a wipol on $\mathfrak{P}$, then a $(\preceq)$-*social preference order over world-lotteries* (or $(\preceq)$-*sprowl*) is a preorder $(\trianglelefteq)$ on $\mathfrak{P}^{\otimes\mathcal{I}}$ with the following properties:

(Par$^{\trianglelefteq}$) For all $\boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathfrak{P}^{\otimes\mathcal{I}}$, if $\rho_i \preceq \rho_i'$ for all $i \in \mathcal{I}$, then $\boldsymbol{\rho} \trianglelefteq \boldsymbol{\rho}'$. Also, if $\rho_i \prec \rho_i'$ for all $i \in \mathcal{I}$, then $\boldsymbol{\rho} \triangleleft \boldsymbol{\rho}'$.

(Anon$^{\trianglelefteq}$) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, then for all $\boldsymbol{\rho} \in \mathfrak{P}^{\otimes\mathcal{I}}$, $\boldsymbol{\rho} \overset{\triangle}{\approx} \sigma(\boldsymbol{\rho})$.

(Lin$^{\trianglelefteq}$) For all $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2' \in \mathfrak{P}^{\otimes\mathcal{I}}$, and $s, s' \in [0, 1]$ with $s + s' = 1$, if $\boldsymbol{\rho}_1 \trianglelefteq \boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_1' \trianglelefteq \boldsymbol{\rho}_2'$, then $(s\boldsymbol{\rho}_1 + s'\boldsymbol{\rho}_1') \trianglelefteq (s\boldsymbol{\rho}_2 + s'\boldsymbol{\rho}_2')$.

Axioms (Par$^{\trianglelefteq}$) and (Anon$^{\trianglelefteq}$) are the world-lottery versions of the eponymous axioms in §3. Axiom (Lin$^{\trianglelefteq}$) is just the von Neumann-Morgenstern linearity axiom.

Fix a world-lottery $\boldsymbol{\rho} \in \mathfrak{P}^{\otimes\mathcal{I}}$. For all $i \in \mathcal{I}$, let $\rho_i \in \mathfrak{P}$ be as in eqn.(12). Define the *per capita average lottery*

$$\overline{\rho} \quad := \quad \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \rho_i \ \in \ \mathfrak{P}. \tag{14}$$

The *approximate utilitarian* $(\preceq)$-sprow $(\underset{u}{\trianglelefteq})$ is then defined:

$$\forall \boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathfrak{P}^{\otimes\mathcal{I}}, \qquad \left(\boldsymbol{\rho} \underset{u}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left(\overline{\rho} \preceq \overline{\rho}'\right). \tag{15}$$

For example, suppose $(\preceq)$ is defined in terms of a family $\mathcal{H}$ of 'utility functions' $h : \Psi \times \Phi \longrightarrow \mathbb{R}$, as in Example 6.1. For any $h \in \mathcal{H}$, clearly, $h^*(\overline{\rho}) = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} h^*(\rho_i)$ is the per capita average expected value of h in the world-lottery described by $\boldsymbol{\rho}$. Thus, combining statements (11) and (15) yields:

$$\left(\boldsymbol{\rho} \underset{u}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left(\sum_{i \in \mathcal{I}} h^*(\rho_i) \leq \sum_{i \in \mathcal{I}} h^*(\rho_i'), \text{ for all } h \in \mathcal{H}\right). \tag{16}$$

Corollary 6.8 (below) shows that, for many reasonable wipols, the approximate utilitarian sprowl can be represented as in (16).

Example 6.2: Fix $\boldsymbol{\psi} \in \Psi^{\mathcal{I}}$. Then $(\underset{u}{\trianglelefteq})$ induces a preference order $(\underset{u,\boldsymbol{\psi}}{\trianglelefteq})$ on $\mathbb{P}(\Phi^{\mathcal{I}})$, where, for all $\boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathbb{P}(\Phi^{\mathcal{I}})$

$$\left(\boldsymbol{\rho} \underset{u,\boldsymbol{\psi}}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left((\delta_{\boldsymbol{\psi}} \otimes \boldsymbol{\rho}) \underset{u}{\trianglelefteq} (\delta_{\boldsymbol{\psi}} \otimes \boldsymbol{\rho}')\right). \tag{17}$$

(Here $\delta_{\boldsymbol{\psi}} \in \mathbb{P}(\Psi^{\mathcal{I}})$ is the 'sure thing' probability measure with $\delta_{\boldsymbol{\psi}}\{\boldsymbol{\psi}\} = 1$).

If the wipol $(\preceq)$ is defined in terms of a family $\mathcal{H}$ of utility functions, as in Example 6.1, then statements (16) and (17) together become:

$$\left(\boldsymbol{\rho} \underset{u,\boldsymbol{\psi}}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left(\sum_{i \in \mathcal{I}} h(\psi_i, \rho_i) \leq \sum_{i \in \mathcal{I}} h(\psi_i, \rho_i'), \text{ for all } h \in \mathcal{H}\right). \tag{18}$$

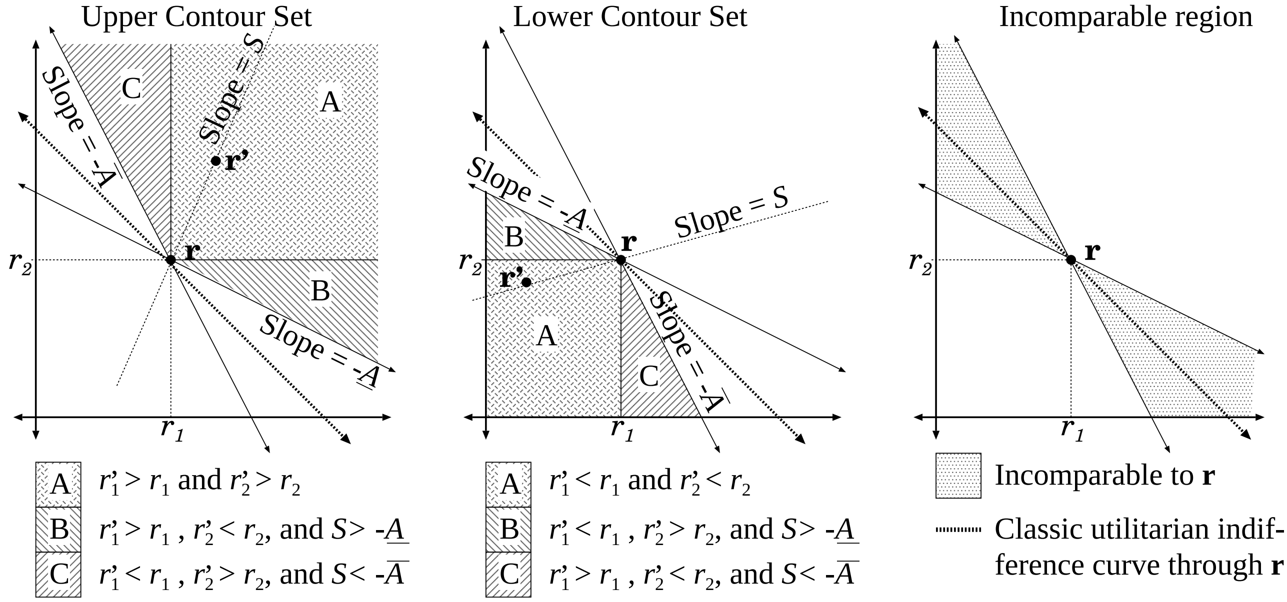

Figure 5: Contour sets for the relation $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$ induced on $\mathbb{R}^2$ by approximate utilitarian sprowl $(\underset{u}{\trianglelefteq})$ in Example 6.2 *Left:* the upper contour set of $\mathbf{r}$. *Middle:* the lower contour set of $\mathbf{r}$. Each contour set contains three disjoint regions, corresponding to the three possible conditions implying the relation $\mathbf{r}' \underset{u,\boldsymbol{\psi}}{\blacktriangleright} \mathbf{r}$ (or vice versa). *Right:* The incomparable regions $\{\mathbf{r}' \in \mathbb{R}^2 \; ; \; \mathbf{r}' \not\bowtie \mathbf{r}\}$. For reference, we also show the indifference curve of the classical utilitarian SWO.

[Here, we define $h(\psi, \rho) := h^*(\delta_\psi \otimes \rho)$ for all $\psi \in \Psi$ and $\rho \in \mathbb{P}(\Phi)$, where h^* is as in eqn.(10).]

Fix $h_0 \in \mathcal{H}$. For all $h \in \mathcal{H}$ and $i \in \mathcal{I}$, there exist constants $w_i = w_i(h) \in \mathbb{R}_+$ and $b_i = b_i(h) \in \mathbb{R}$ such that, for all $\phi \in \Phi$, we have $h(\psi_i, \phi) \; = \; w_i \cdot h_0(\psi_i, \phi) + b_i$ (because both $h(\psi_i, \bullet)$ and $h_0(\psi_i, \bullet)$ are vNM cardinal utility functions for $(\underset{\psi_i}{\preceq})$). For any $h \in \mathcal{H}$, define the 'weight vector' $\mathbf{w}(h) := (w_i(h))_{i \in \mathcal{I}} \in \mathbb{R}_+^{\mathcal{I}}$. Next, define $\mathcal{W} := \{\mathbf{w}(h) \; ; \; h \in \mathcal{H}\} \subseteq \mathbb{R}_+^{\mathcal{I}}$. Then statement (18) becomes:

$$\left(\boldsymbol{\rho} \underset{u,\boldsymbol{\psi}}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left(\sum_{i \in \mathcal{I}} w_i \cdot h_0(\psi_i, \rho_i) \leq \sum_{i \in \mathcal{I}} w_i \cdot h_0(\psi_i, \rho'_i), \text{ for all } \mathbf{w} \in \mathcal{W}\right). \qquad (19)$$

(The constants $\{b_i(h)\}_{i \in \mathcal{I}}$ are irrelevant because they cancel from both sides of the right-hand inequality in (18), for any fixed $h \in \mathcal{H}$.)

In particular, suppose $\mathcal{I} = \{1, 2\}$; then $\mathcal{W} \subseteq \mathbb{R}_+^2$. Let $\underline{A} := \inf\{w_1/w_2 \; ; \; \mathbf{w} \in \mathcal{W}\}$ and $\overline{A} := \sup\{w_1/w_2 \; ; \; \mathbf{w} \in \mathcal{W}\}$. As shown in Figure 5, we define a preorder $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$ on $\mathbb{R}^2$ by:

$$\left(\mathbf{r} \underset{u,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}'\right) \iff \begin{pmatrix} \text{either (A)} \;\; r'_1 \geq r_1 \text{ and } r'_2 \geq r_2; \\ \text{or} \quad \text{(B)} \;\; r'_1 \geq r_1, \;\; r'_2 \leq r_2 \text{ and } S \geq -\underline{A}; \\ \text{or} \quad \text{(C)} \;\; r'_1 \leq r_1, \;\; r'_2 \geq r_2, \text{ and } S \leq -\overline{A} \end{pmatrix}, \text{ where } S \; := \; \frac{r'_2 - r_2}{r'_1 - r_1}. \quad (20)$$

(That is, S is the slope of the line through $\mathbf{r}$ and $\mathbf{r}'$.) We have:

Claim 6.2*: *Let $\boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathbb{P}(\Phi^{\mathcal{I}})$, and for $i \in \{1,2\}$, let $r_i := h_0(\psi_i, \rho_i)$ and $r'_i := h_0(\psi_i, \rho'_i)$, to obtain vectors $\mathbf{r}$ and $\mathbf{r}'$ in $\mathbb{R}^2$. Then $\left(\boldsymbol{\rho} \underset{u,\boldsymbol{\psi}}{\trianglelefteq} \boldsymbol{\rho}'\right) \iff \left(\mathbf{r} \underset{u,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}'\right)$.* ♢

Example 6.3: (Bargaining problems) Let $\mathcal{B} \subset \mathbb{R}^2$ be some compact, convex set (e.g. a bargaining set), as in Example 3.3. Let $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$ be the preorder on $\mathbb{R}^2$ from Example 6.2. If $\mathbf{b} \in \mathcal{B}$, then $\mathbf{b}$ is $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$-undominated iff the wedge $\left\{\mathbf{r}' \in \mathbb{R}^2 \ ; \ \mathbf{r}' \underset{u,\boldsymbol{\psi}}{\blacktriangleright} \mathbf{r}\right\}$ intersects $\mathcal{B}$ only at $\mathbf{b}$. Thus, if $\mathcal{P}$ is the Pareto frontier of $\mathcal{B}$, then any $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$-undominated point of $\mathcal{B}$ must be in $\mathcal{P}$. Furthermore, if $\mathbf{b} \in \mathcal{P}$, and T is the slope of the tangent line to $\mathcal{P}$ at $\mathbf{b}$, then $\mathbf{b}$ is $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$-undominated iff $-\overline{A} \leq T \leq -\underline{A}$.[10] The set of $(\underset{u,\boldsymbol{\psi}}{\blacktriangleleft})$-undominated points in $\mathcal{B}$ is shown in Figure 3(C). ♢

Remark 6.4. 'Approximate utilitarian' social orderings defined like formula (16) have appeared at least twice before in the literature. Sen (1970a, 1972, and Chapt. 7* of 1970b) defined partial social orderings over a space $\mathcal{X}$ of social alternatives (not necessarily lotteries) by computing weighted utilitarian sums for all weight vectors in some convex cone $\mathcal{W} \subseteq \mathbb{R}^{\mathcal{I}}_{+}$, as in (19). Under certain axioms, he showed that one could define a one-parameter family $\{\underset{\alpha}{\preceq}\}_{\alpha\in[0,1]}$ of such social orderings, where $(\underset{0}{\preceq})$ is the Pareto ordering (no comparability), and $(\underset{1}{\preceq})$ is the classic utilitarian SWO (full comparability). (Indeed, Sen explicitly motivates his approach as an attempt to compromise between these extremes.) If $0 \leq \alpha < \beta \leq 1$, then $\mathcal{W}_\beta \subset \mathcal{W}_\beta$, so that $(\underset{\beta}{\preceq})$ extends and refines $(\underset{\alpha}{\preceq})$, and represents a greater degree of interpersonal comparability (e.g. in Figure 5(C), the grey noncomparability wedges for $(\underset{\beta}{\preceq})$ are thinner than those for $(\underset{\alpha}{\preceq})$). Unfortunately, aside from Fine (1975), there seems to have been little followup on Sen's idea.

More recently, Baucells and Shapley (2006, 2008) have developed a theory which assigns, to every subcoalition $\mathcal{J} \subseteq \mathcal{I}$, a (partial) preorder $(\underset{\mathcal{J}}{\blacktriangleleft})$ on a space $\mathbb{P}(\mathcal{X})$ of lotteries over some set $\mathcal{X}$. They impose the *Extended Pareto Rule* (EPR): For any $\rho, \rho' \in \mathbb{P}(\mathcal{X})$, and any two disjoint coalitions $\mathcal{J}, \mathcal{K} \subset \mathcal{I}$, if $\rho \underset{\mathcal{J}}{\blacktriangleleft} \rho'$ and $\rho \underset{\mathcal{K}}{\blacktriangleleft} \rho'$, then $\rho \underset{\mathcal{J}\sqcup\mathcal{K}}{\blacktriangleleft} \rho'$. Using a version of Proposition 6.9 (below), they argue that all coalitions must have preference orderings defined by cones of utility functions, as in (16). In particular, two-person coalitions must have preference orderings as in formula (20): for any $i, j \in \mathcal{I}$, there exists constants $\overline{A}_{ij} \geq \underline{A}_{ij} > 0$ providing upper and lower bounds for the 'conversion rates' from i's vNM utility function to j's vNM utility function. Baucells and Shapley (2008, §2.5) then show that, for any distinct $i, j, k \in \mathcal{I}$, EPR imposes an interesting 'no arbitrage' condition on the conversion rates for the pairs $\{i,j\}$, $\{j,k\}$ and $\{i,k\}$ (similar to Lemma 4.3(f) in the Appendix). Furthermore, EPR forces $(\underset{\{i,j,k\}}{\blacktriangleleft})$ to lie in the intersection of regions determined by $(\underset{\{i,j\}}{\blacktriangleleft})$, $(\underset{\{j,k\}}{\blacktriangleleft})$ and $(\underset{\{i,k\}}{\blacktriangleleft})$, so that $(\underset{\{i,j,k\}}{\blacktriangleleft})$ will often come closer to a complete ordering than any of the three pairwise orderings. In general, EPR forces larger coalitions to have more complete orderings (Baucells and Shapley, 2006, Prop.3). In particular, if there exists a spanning

[10]If $\mathbf{b}$ is a corner point of $\mathcal{P}$, then this inequality must hold for *all* tangent lines at $\mathbf{b}$.

tree in $\mathcal{I}$ such that all links in this tree are two-person coalitions with complete preferences (i.e. $\overline{A}_{ij} = \underline{A}_{ij}$), then EPR forces $(\underset{\mathcal{I}}{\blacktriangleleft})$ (the preference order of the grand coalition $\mathcal{I}$) to be a complete, vNM preference relation on $\mathbb{P}(\mathcal{X})$, generated by a weighted average of the utility functions of all $i \in \mathcal{I}$ (Baucells and Shapley, 2008, Thm.4); this can be seen as a generalization of Harsanyi's (1955) Social Aggregation Theorem. Baucells and Shapley (2006, Thm.6) obtain a similar result when $\mathcal{I}$ can be covered by system of overlapping subcoalitions, each having a complete vNM preference. Baucells and Sarin (2003) have applied this theory to multicriteria decision-making. $\bigtriangledown$

Recall that Proposition 3.4(b) says the Suppes-Sen sprow $(\underset{s}{\trianglelefteq})$ is the 'minimal' $(\preceq)$-sprow, which is extended by every other sprow. Similarly, $(\underset{u}{\trianglelefteq})$ is the 'minimal' $(\preceq)$-sprowl.

Theorem 6.5 *Let $(\preceq)$ be a wipol on $\mathfrak{P}$. Every $(\preceq)$-sprowl on $\mathfrak{P}^{\otimes\mathcal{I}}$ extends and refines the approximate utilitarian $(\preceq)$-sprowl $(\underset{u}{\trianglelefteq})$.*

6.2 Wipol hedometers

Let $\mathfrak{P} \subseteq \mathbb{P}(\Psi\times\Phi)$ be a convex subset, and let $(\preceq)$ be a wipol on $\mathfrak{P}$. A *lottery hedometer* is a measurable function $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that, for all $\rho_1, \rho_2 \in \mathfrak{P}$,

$$\Big(\rho_1 \preceq \rho_2\Big) \Longrightarrow \Big(h^*(\rho_1) \leq h^*(\rho_2)\Big),$$

where h^* is defined as in eqn.(10). (Thus, for any $\psi \in \Psi$, the function $h(\psi, \bullet) : \Phi \longrightarrow \mathbb{R}$ is a vNM cardinal utility function representing the preference order $(\underset{\psi}{\preceq})$.) Let $\mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)$ be the set of lottery hedometers for $(\preceq)$. In §4 we saw that certain wipos were entirely characterized by their family of hedometers. The same is true for wipols which are continuous with respect to a suitable topology on $\mathfrak{P}$. To explain this, we must briefly ascend to a higher level of abstraction.

Let $(\mathcal{X}, \mathfrak{S})$ be *measurable space* —i.e. $\mathcal{X}$ is a set, and $\mathfrak{S}$ is a sigma-algebra on $\mathcal{X}$. Let $\mathfrak{M}$ be a vector space of finite signed measures on $(\mathcal{X}, \mathfrak{S})$. For any measurable function $f : \mathcal{X} \longrightarrow \mathbb{R}$, define the linear functional $f^* : \mathfrak{M} \longrightarrow \mathbb{R}$ by $f^*(\mu) := \int_{\mathcal{X}} f \, d\mu$ for all $\mu \in \mathfrak{M}$. Let

$$\mathfrak{P} \quad := \quad \{\rho \in \mathfrak{M}\ ;\ \rho[\mathcal{S}] \geq 0 \text{ for all } \mathcal{S} \in \mathfrak{S}, \text{ and } \rho[\mathcal{X}] = 1\} \qquad (21)$$

be the (convex) set of probability measures in $\mathfrak{M}$. We say that a topology on $\mathfrak{M}$ is *convex-separable* if (1) $\mathfrak{P}$ is a closed subset of $\mathfrak{M}$; and (2) for any closed, convex $\mathcal{B} \subset \mathfrak{M}$, and any $\mu \in \mathfrak{M} \setminus \mathcal{B}$, there exists some measurable function $f : \mathcal{X} \longrightarrow \mathbb{R}$ such that $f^*(\mu) < \inf[f^*(\mathcal{B})]$ (where $f^*(\mathcal{B}) := \{f^*(\beta)\ ;\ \beta \in \mathcal{B}\}$).

Example 6.6: (a) Let $\mathcal{X}$ be a finite set, let $\mathfrak{S}$ be its power set, and let $\mathfrak{M}$ be the set of all finite measures on $\mathfrak{S}$; then $\mathfrak{M}$ can be identified with $\mathbb{R}^{\mathcal{X}}$ in an obvious way. The Euclidean topology on $\mathbb{R}^{\mathcal{X}}$ is convex-separable by the Separating Hyperplane Theorem.

(b) Let $\mathfrak{M}^*$ be the vector space of continuous linear functions from $\mathfrak{M}$ into $\mathbb{R}$. Suppose the topology on $\mathfrak{M}$ is locally convex, and suppose that for every $F \in \mathfrak{M}^*$, there is some measurable function $f : \mathcal{X} \longrightarrow \mathbb{R}$ such that $F = f^*$. Then $\mathfrak{M}$ is convex-separable (Conway, 1990, Corol.IV.3.10, p.111).

(c) In particular, let $\mathcal{X}$ be any locally compact topological space, let $\mathfrak{S}$ be its Borel sigma algebra, and let $\mathfrak{M}$ be the space of all finite signed measures on $(\mathcal{X}, \mathfrak{S})$. Let $\mathcal{C}(\mathcal{X})$ be the Banach space of continuous real-valued functions on $\mathcal{X}$ with the uniform norm; then $\mathfrak{M} = \mathcal{C}(\mathcal{X})^*$ by the Riesz Representation Theorem (Conway, 1990, Thm.III.5.7, p.75). Endow $\mathfrak{M}$ with resulting weak* topology; then $\mathfrak{M}$ is locally convex (Conway, 1990, Exm.IV.1.8, p.101), and $\mathfrak{M}^* = \mathcal{C}(\mathcal{X})$ (Conway, 1990, Thm.V.1.3, p.125). Thus, $\mathfrak{M}$ is convex-separable in the weak* topology, by example (b).

(d) Alternately, let $(\mathcal{X}, \mathfrak{S})$ be an abstract measurable space, fix some finite measure λ on $\mathcal{X}$, and let $\mathfrak{M}$ be the set of all measures on $(\mathcal{X}, \mathfrak{S})$ which are absolutely continuous relative to λ; then $\mathfrak{M}$ can be identified with the Banach space $\mathbf{L}^1(\mathcal{X}, \mathfrak{S}, \lambda)$ via the Radon-Nikodym theorem (Conway, 1990, Thm.C.7, p.380). In the resulting topology, the space $\mathfrak{M}$ is convex-separable by example (b), because for any $F \in \mathbf{L}^1(\mathcal{X}, \mathfrak{S}, \lambda)^*$ there is some $f \in \mathbf{L}^\infty(\mathcal{X}, \mathfrak{S}, \lambda)$ such that $F = f^*$ (Conway, 1990, Thm.III.5.6, p.75). $\diamondsuit$

A preorder $(\preceq)$ on $\mathfrak{P}$ is *continuous* if it satisfies the following axiom:

(Cont$^{\preceq}$) For all $\rho, \rho' \in \mathfrak{P}$, and sequences $\{\rho_n\}_{n=1}^\infty$ and $\{\rho'_n\}_{n=1}^\infty$, if $\rho_n \xrightarrow[n\to\infty]{} \rho$ and $\rho'_n \xrightarrow[n\to\infty]{} \rho'$, and $\rho_n \preceq \rho'_n$ for all $n \in \mathbb{N}$, then $\rho \preceq \rho'$.

A wipol satisfying this continuity hypothesis is 'hedometric' in the sense of §4:

Theorem 6.7 *Suppose $\mathfrak{P}$ has a convex-separable topology, and let $(\preceq)$ be a wipol on $\mathfrak{P}$ satisfying axiom* (Cont$^{\preceq}$). *For any $\rho, \rho' \in \mathfrak{P}$, we have*

$$\Big(\rho \preceq \rho'\Big) \iff \Big(h^*(\rho) \le h^*(\rho') \text{ for all } h \in \mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)\Big).$$

Let $(\underset{u}{\blacktriangleleft})$ be the classic utilitarian SWO on $\mathbb{R}^{\mathcal{I}}$. In the terminology of §5, $(\underset{u}{\trianglelefteq})$ is then the '$(\preceq, \underset{u}{\blacktriangleleft})$-welfarist' sprowl:

Corollary 6.8 *Let $\mathfrak{P}$ be convex-separable, let $(\preceq)$ be a continuous wipol on $\mathfrak{P}$ and let $(\underset{u}{\trianglelefteq})$ be the approximate utilitarian $(\preceq)$-sprowl. For any $\boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathfrak{P}^{\otimes \mathcal{I}}$, we have*

$$\Big(\boldsymbol{\rho} \underset{u}{\trianglelefteq} \boldsymbol{\rho}'\Big) \iff \left(\sum_{i\in\mathcal{I}} h^*(\rho_i) \le \sum_{i\in\mathcal{I}} h^*(\rho'_i), \text{ for all } h \in \mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)\right).$$

Theorem 6.7 is actually a special case of a broader result.

Theorem 6.9 *Let $(\mathcal{X}, \mathfrak{S})$ be any measurable space, and let $\mathfrak{M}$ be a convex-separable topological vector space of measures on $\mathcal{X}$. Let $\mathfrak{P} \subset \mathfrak{M}$ be the set* (21) *of probability measures, and let $(\preceq)$ be any preorder on $\mathfrak{P}$ satisfying axioms* (Lin$^{\preceq}$) *and* (Cont$^{\preceq}$). *Then there is a set $\mathcal{U}$ of measurable functions from $\mathcal{X}$ to $\mathbb{R}$ such that, for all $\rho, \rho' \in \mathfrak{P}$, $\Big(\rho \preceq \rho'\Big) \iff \Big(u^*(\rho) \le u^*(\rho') \text{ for all } u \in \mathcal{U}\Big)$.*

Shapley and Baucells (1998; Theorem 1.8, p.12) proved the special case of Theorem 6.9 where $\mathcal{X}$ is a finite set, and $\mathfrak{M} = \mathbb{R}^{\mathcal{X}}$ with the Euclidean topology (as in Example 6.6(a)).[11] Likewise, Dubra et al. (2004) proved the special case of Theorem 6.9 where $\mathcal{X}$ is a compact metric space and $\mathfrak{M}$ is the set of signed Borel measures on $\mathcal{X}$ with the weak* topology (a subcase of Example 6.6(c)).[12] Our proof (in the appendix) uses the same basic strategy as these earlier results: first, represent $(\preceq)$ using a closed, convex cone in $\mathfrak{M}$, and then show that any point *not* in this cone can be 'separated' from it by a linear functional on $\mathfrak{M}$, which in turn can be represented by a measurable function on $\mathcal{X}$. However, Theorem 6.9 is more general than the previous results, so we must work at a higher level of abstraction. Ironically, the resulting argument is actually more elementary: our proof uses only basic linear algebra and topology, whereas Dubra et al. (2004) use a fair amount of functional analysis. (Of course, some functional analysis will be required to verify that a particular $\mathfrak{P}$ is convex-separable in the first place, as can be seen in Examples 6.6(b,c,d) above).

7 Stochastic utilitarianism

We now turn to a very different model of approximate interpersonal utility comparisons. Suppose there exists a complete wipo $(\preceq)$ on $\Psi \times \Phi$ which, in principle, would allow us to make precise interpersonal comparisons of well-being. The wipo $(\preceq)$ is described by a 'true hedometer' $h : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that, for all $(\psi, \phi), (\psi', \phi') \in \Psi \times \Phi$, $(\psi, \phi) \preceq (\psi', \phi')$ if and only if $h(\psi, \phi) \leq h(\psi', \phi')$. However, the exact structure of $(\preceq)$ is unknown. We model this by representing h as a random variable. That is, we introduce a probability space Ω, and represent h by a measurable function $H : \Psi \times \Phi \times \Omega \longrightarrow \mathbb{R}$.

This model has at at least three interpretations. In the first interpretation, we suppose that, for all $\psi \in \Psi$, we have perfect knowledge of the individual preference ordering $(\underset{\psi}{\preceq})$, which can be described by a cardinal utility function $u_\psi : \Phi \longrightarrow \mathbb{R}$. However, the different utility functions $\{u_\psi\}_{\psi \in \Psi}$ are expressed on different 'scales', and the correct interpersonal calibration is unknown to us. For all $\psi \in \Psi$, there are (unknown) constants $a_\psi > 0$ and $b_\psi \in \mathbb{R}$, such that that, for all $\phi \in \Phi$, $h(\psi, \phi) = a_\psi \, u_\psi(\phi) + b_\psi$. We don't know the vectors $\mathbf{a} := (a_\psi)_{\psi \in \Psi} \in \mathbb{R}_+^\Psi$ and $\mathbf{b} := (b_\psi) \in \mathbb{R}^\Psi$ so we model them as a random variables. Thus, in this model, $\Omega := \mathbb{R}_+^\Psi \times \mathbb{R}^\Psi$ (with some probability measure), and $H : \Psi \times \Phi \times \mathbb{R}_+^\Psi \times \mathbb{R}^\Psi \longrightarrow \mathbb{R}$ is defined by $H(\psi, \phi, \mathbf{a}, \mathbf{b}) := a_\psi \circ u_\psi(\phi) + b_\psi$, for all $(\psi, \phi, \mathbf{a}, \mathbf{b}) \in \Psi \times \Phi \times \mathbb{R}_+^\Psi \times \mathbb{R}^\Psi$.

In the second interpretation, we suppose we *know* the true hedometer h, so in principle we could make precise interpersonal comparisons. However, we have incomplete knowledge of the psychological 'type' of each person (as in a Bayesian game). We suppose there is some space Ξ of 'true' psychological types (which are hidden), and interpret Ψ as a space of 'publicly visible' personality types. The true hedometer is a known function $h : \Xi \times \Phi \longrightarrow \mathbb{R}$, such that $h(\xi, \bullet) : \Phi \longrightarrow \mathbb{R}$ is the (correctly calibrated) cardinal utility function of a person whose true type is ξ. If a person's visible personality is $\psi \in \Psi$, then her true psychological

[11] Remark 6.4 discusses some applications of this theorem by Baucells and Shapley (2006, 2008).

[12] I thank Klaus Nehring for referring me to Dubra et al. (2004) (and through them, to Shapley and Baucells (1998)) after reading an early draft of this section.

type $\xi(\psi) \in \Xi$ is unknown to us, and thus modelled as a random variable. Formally, we introduce a probability space Ω and define a measurable function $\xi : \Psi \times \Omega \longrightarrow \Xi$. We then define $H : \Psi \times \Phi \times \Omega \longrightarrow \mathbb{R}$ by $H(\psi, \phi, \omega) := h[\xi(\psi, \omega),\ \phi]$.

The third interpretation combines both forms of ambiguity. That is, we assume that we have incomplete knowledge of the psychological types of the individuals, and we also have incomplete knowledge of the correct calibration we need to compare utility functions between individuals.

7.1 A stochastic social aggregation theorem

Let $\mathcal{X}$ be a set of social alternatives, and let $\mathbb{P}(\mathcal{X})$ be the set of lotteries over these alternatives. Let $\mathcal{I}$ be a set of individuals. Harsanyi (1955, 1976) presented the following result as a strong argument for utilitarianism.

Social Aggregation Theorem. *For each $i \in \mathcal{I}$, let $(\underset{i}{\preceq})$ be a vNM preference relation on $\mathbb{P}(\mathcal{X})$, represented by vNM utility function $u_i : \mathcal{X} \longrightarrow \mathbb{R}$. Let $(\trianglelefteq)$ be the social planner's vNM preference relation over $\mathbb{P}(\mathcal{X})$, and suppose $(\trianglelefteq)$ satisfies:*

(Par) *For any $\rho, \rho' \in \mathbb{P}(\mathcal{X})$, if $\rho \underset{i}{\preceq} \rho'$ for all $i \in \mathcal{I}$, then $\rho \trianglelefteq \rho'$.*

Then there exist nonnegative constants $\{c_i\}_{i \in \mathcal{I}} \subset \mathbb{R}_+$ such that $(\trianglelefteq)$ is represented by the vNM utility function $U : \mathcal{X} \longrightarrow \mathbb{R}$ defined by $U(x) := \sum_{i \in \mathcal{I}} c_i u_i(x)$ for all $x \in \mathcal{X}$. □

Unfortunately, because of its 'single-profile' framework, the SAT is *not* an argument for utilitarianism. It does *not* prescribe a particular weighted utilitarian social choice function which the social planner must employ, independent of the profile of individual vNM preferences. Instead, the SAT says that, *given* a profile $\{\underset{i}{\preceq}\}_{i \in \mathcal{I}}$ of individual vNM preferences, and given a collective vNM preference $(\trianglelefteq)$ (generated through whatever means), if $(\trianglelefteq)$ satisfies (Par) for the profile $\{\underset{i}{\preceq}\}_{i \in \mathcal{I}}$, then $(\trianglelefteq)$ can always be 'rationalized' as utilitarianism *ex post facto*, by a suitable choice of constants $\{c_i\}_{i \in \mathcal{I}}$. The constants $\{c_i\}_{i \in \mathcal{I}}$ might depend on the particular profile $\{\underset{i}{\preceq}\}_{i \in \mathcal{I}}$. A proper characterization of utilitarianism must specify some constants $\{c_i\}_{i \in \mathcal{I}}$ independent of the particular profile $\{\underset{i}{\preceq}\}_{i \in \mathcal{I}}$, and must apply to all conceivable profiles. See Weymark (1991) or Mongin (1994) for further discussion.

Let $\mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$ be the space of probability distributions over $\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. For any $\boldsymbol{\rho} \in \mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$ and $i \in \mathcal{I}$, let $\rho_i \in \mathbb{P}(\Psi \times \Phi)$ be the projection of $\boldsymbol{\rho}$ onto the ith coordinate, as defined by eqn.(12) in §6.1. Fix $\omega \in \Omega$, and let $H^*(\rho_i, \omega)$ be the ρ_i-expected value of H, given ω. That is:

$$H^*(\rho_i, \omega) \quad := \quad \int_{\Psi \times \Phi} H(\psi, \phi, \omega)\ d\rho_i(\psi, \phi).$$

Given ω, assume that individual i has a preference relation $(\underset{\omega,i}{\preceq})$ over $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$ defined by $(\boldsymbol{\rho}\underset{\omega,i}{\preceq}\boldsymbol{\rho}') \iff (H^*(\rho_i,\omega) \leq H^*(\rho'_i,\omega))$. This is a vNM preference relation on $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$, with vNM utility function $h_i^\omega : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \mathbb{R}$ defined by $h_i^\omega(\boldsymbol{\psi},\boldsymbol{\phi}) := H(\psi_i,\phi_i,\omega)$.

As in §6.1, the social planner must formulate preference relation $(\trianglelefteq)$ over $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$. If $(\trianglelefteq)$ satisfies the vNM axioms, then it can be represented by a vNM utility function $U : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \mathbb{R}$. The problem is that the correct choice of U may depend on the true value of ω, which is unknown to the planner. For any measurable subset $\mathcal{S} \subseteq \Omega$, if the planner 'observes' $\mathcal{S}$ (i.e. if she acquires enough information to know that $\omega \in \mathcal{S}$), then we suppose she formulates a vNM preference relation $(\underset{\mathcal{S}}{\trianglelefteq})$ on $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$, described by a vNM utility function $U_{\mathcal{S}} : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \mathbb{R}$. Let $\mathfrak{S}$ be the sigma-algebra on Ω and let $\pi : \mathfrak{S} \longrightarrow [0,1]$ be the probability measure. We suppose that the family $\{U_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ of utility functions satisfies the following 'Bayesian consistency' condition:

(Bayes) For any $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ and any countable collection $\{\mathcal{S}_n\}_{n=1}^\infty \subset \mathfrak{S}$ of disjoint measurable sets, if $\mathcal{S} = \bigsqcup_{n=1}^\infty \mathcal{S}_n$, then $U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi}) \ = \ \frac{1}{\pi(\mathcal{S})} \sum_{n=1}^\infty \pi(\mathcal{S}_n)\, U_{\mathcal{S}_n}(\boldsymbol{\psi},\boldsymbol{\phi})$.

Intuitively, this says that the family $\{U_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ behaves as if $U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi})$ is the expected value of the unknown 'true' social utility of the world $(\boldsymbol{\psi},\boldsymbol{\phi})$, conditioned on the observation $\mathcal{S}$. Indeed, we have the following:

Lemma 7.1 *Suppose the family $\{U_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ satisfies* (Bayes). *Then there exists a measurable function $U : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \times \Omega \longrightarrow \mathbb{R}$ such that, for any $\mathcal{S} \in \mathfrak{S}$ and $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$,*

$$U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi}) \quad = \quad \frac{1}{\pi(\mathcal{S})} \int_{\mathcal{S}} U_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \ d\pi[\omega]. \tag{22}$$

Intuitively, $U_\omega : \Psi \times \Phi \longrightarrow \mathbb{R}$ is the vNM utility function which the planner would employ if she knew that the true value was ω. Let $(\underset{\omega}{\trianglelefteq})$ be the vNM preference relation on $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$ represented by U_ω. For all $\omega \in \Omega$, we assume $(\underset{\omega}{\trianglelefteq})$ satisfies the following axioms:

(Par) For all $\boldsymbol{\rho},\boldsymbol{\rho}' \in \mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$, if $\boldsymbol{\rho}\underset{\omega,i}{\preceq}\boldsymbol{\rho}'$ for all $i \in \mathcal{I}$, then $\boldsymbol{\rho}\underset{\omega}{\trianglelefteq}\boldsymbol{\rho}'$.

(Anon) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, then for all $\boldsymbol{\rho} \in \mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$, $\boldsymbol{\rho} \underset{\omega}{\overset{\triangle}{\approx}} \sigma(\boldsymbol{\rho})$ [where $\sigma(\boldsymbol{\rho})$ is defined by eqn.(13)].

(Nonindiff) The ordering $(\underset{\omega}{\trianglelefteq})$ is not totally indifferent over $\mathbb{P}(\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}})$.

(Welf) There exists a function $F : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$ such that, for any $\omega \in \Omega$ and $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$, if $r_i := H(\psi_i,\phi_i,\omega)$ for all $i \in \mathcal{I}$, then $U_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \ = \ F(\mathbf{r})$.

The meanings of axioms (Par), (Anon), and (Nonindiff) are clear. Axiom (Welf) says that the function U is welfarist: $U_\omega(\boldsymbol{\psi},\boldsymbol{\phi})$ is entirely determined by the values of $H(\psi_i,\phi_i,\omega)$

(for all $i \in \mathcal{I}$), independent of ω (see footnote #9 in §5). Loosely speaking, this ensures that U cannot assign more 'weight' to some values of ω than others.

For any $\mathcal{S} \in \mathfrak{S}$, define $\overline{h}_{\mathcal{S}} : \Psi \times \Phi \longrightarrow \mathbb{R}$ by

$$\overline{h}_{\mathcal{S}}(\psi, \phi) \quad := \quad \frac{1}{\pi(\mathcal{S})} \int_{\mathcal{S}} H(\psi, \phi, \omega) \; d\pi[\omega], \qquad \text{for all } (\psi, \phi) \in \Psi \times \Phi. \tag{23}$$

In words: $\overline{h}_{\mathcal{S}}(\psi, \phi)$ is the *expected value* of the random hedometer H for the personal psychophysical state (ψ, ϕ), conditional on observing the event $\mathcal{S}$. We now have:

Theorem 7.2 *Let* $\{(\underset{\mathcal{S}}{\trianglelefteq})\}_{\mathcal{S}\in\mathfrak{S}}$ *be a collection of vNM preference relations on* $\mathbb{P}(\Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}})$ *satisfying axioms* (Bayes)*,* (Par)*,* (Anon)*,* (Nonindiff)*, and* (Welf)*. Then for any* $\mathcal{S} \in \mathfrak{S}$*, the relation* $(\underset{\mathcal{S}}{\trianglelefteq})$ *seeks to maximize the expected value of the utilitarian social welfare function* $\overline{U}_{\mathcal{S}} : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \mathbb{R}$ *defined by*

$$\overline{U}_{\mathcal{S}}(\boldsymbol{\psi}, \boldsymbol{\phi}) \quad := \quad \sum_{i\in\mathcal{I}} \overline{h}_{\mathcal{S}}(\psi_i, \phi_i), \qquad \textit{for all } (\boldsymbol{\psi}, \boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}. \tag{24}$$

Note that this model entirely obviates the 'single-profile' criticism of Harsanyi's original SAT. By definition, $\Psi \times \Omega$ encodes the space of all possible human psychologies *which could ever exist*; hence the hedometer H encodes all possible vNM preference relations which could ever manifest in *any* profile. Thus, Theorem 7.2 does not presuppose any particular profile; it prescribes $\overline{U}_{\mathcal{S}}$ as the social welfare function which the social planner must employ when she observes $\mathcal{S}$, independent of the profile of individual vNM preferences which actually obtains.

Also, for practical purposes, this model does not require the social planner to have precise information about people's true preferences. The hidden variable ω could contain a lot of information; indeed, the model is even applicable when Ψ is trivial, so that *all* information about people's true preferences is hidden from the social planner. However, the model *does* require the social planner to have a correct model of the probability distribution of preferences, even if she doesn't know which preferences actually obtain (i.e. the planner must know the true hedometer $H : \Psi \times \Phi \times \Omega \longrightarrow \mathbb{R}$, even if she doesn't know the true value of ω). Also, in keeping with the rest of this paper, the model assumes that interpersonal comparisons of utility are possible in principle, even if they are ambiguous in practice.

7.2 On liberty

Welfarist social choice theory has been criticized for not recognizing the value of personal liberty.[13] Suppose $\mathcal{X}$ is a feasible set, and individual i has utility function $u_i : \mathcal{X} \longrightarrow \mathbb{R}$. Let x^* be the u_i-maximal element of $\mathcal{X}$. Intuitively, we feel that a social policy which allows i to choose x^* herself is more desirable than a social policy which forces x^* upon her —even though both policies yield the same utility for i. Formally, we can imagine a

[13] See Dowding and van Hees (2009) for a summary of this debate.

policy which allows i to choose any element from some subset $\mathcal{F} \subseteq \mathcal{X}$; the larger $\mathcal{F}$ is, the more 'freedom' it offers i, and hence, the more desirable the policy.

However, this account is puzzling, because by definition, elements of the set $\mathcal{X}$ are supposed to encode *all* information relevant to i's happiness or well-being, as measured by u_i. Furthermore, any 'freedom' offered by $\mathcal{F}$ is clearly a function of the 'quality' of the elements of $\mathcal{F}$ as well as their quantity. For example, if $\mathcal{F}'$ is obtained by adding an extremely undesirable option (e.g. 'execution at dawn') to $\mathcal{F}$, then we would not feel that $\mathcal{F}'$ offers i 'more freedom' than $\mathcal{F}$. This is because when i 'freely chooses' an element from $\mathcal{F}$, we suppose what she really does is solve an optimization problem; adding options which are obviously grossly suboptimal does not enhance her optimization opportunities. However, if this 'optimization' view of free choice is correct, then once $\mathcal{F}$ contains the global optimum x^*, it seems futile to add any other options, because *any* other element of $\mathcal{X}$ is suboptimal, relative to x^*. Hence any measure of 'freedom' which accounts for the 'quality' of elements in $\mathcal{F}$ leads us back to welfarism.

However, this objection assumes that we *know* the u_i-optimal element of $\mathcal{X}$, because we know u_i. In reality, our knowledge of u_i is imperfect (cf. the second ('Bayesian game') interpretation of the random hedometer model). Even in a purely welfarist framework, liberty then acquires instrumental value: by offering i a larger feasible set $\mathcal{F}$ to freely choose from, we increase the probability that $\mathcal{F}$ contains her true optimum x^* (which is unknown to us); more generally, we increase the *expected value* of $\max_{x\in\mathcal{F}} u_i(x)$.

As before, let Ω be a probability space, and $H : \Psi \times \Phi \times \Omega \longrightarrow \mathbb{R}$ represent a 'random hedometer'. Fix a society $\boldsymbol{\psi} \in \Psi^{\mathcal{I}}$. Suppose that social policy does not determine a single point $\boldsymbol{\phi} \in \Phi^{\mathcal{I}}$; instead, a social policy determines, for each $i \in \mathcal{I}$, some subset $\mathcal{F}_i \subseteq \Phi$, leaving i the freedom to choose any element of $\mathcal{F}_i$. Presumably i chooses $\arg\max_{\phi\in\mathcal{F}_i} H(\psi_i, \phi_i, \omega)$. Let us refer to the collection $(\mathcal{F}_i)_{i\in\mathcal{I}}$ as a ***freedom allocation***.[14] Given a choice between two freedom allocations $\mathbf{F} := (\mathcal{F}_i)_{i\in\mathcal{I}}$ and $\mathbf{F}' := (\mathcal{F}'_i)_{i\in\mathcal{I}}$, a utilitarian social planner will choose $\mathbf{F}$ over $\mathbf{F}'$ if it offers a higher expected utility sum, conditional on individual optimization; that is, if

$$\int_\Omega \sum_{i\in\mathcal{I}} \max_{\phi_i\in\mathcal{F}_i} H(\psi_i, \phi_i, \omega)\ d\omega \quad > \quad \int_\Omega \sum_{i\in\mathcal{I}} \max_{\phi'_i\in\mathcal{F}'_i} H(\psi_i, \phi'_i, \omega)\ d\omega.$$

Thus, a stochastic utilitarian may deem it socially optimal to grant considerable liberty to citizens.

8 Models of wipos

This section describes three models of how a weak interpersonal preference ordering might be constructed. These are not practical operationalizations, but are aimed instead at prov-

[14]Presumably the social planner is constrained in the sort of freedom allocations she can offer, but we will refrain from formally modelling these constraints. Also, we are unrealistically assuming that each $i \in \mathcal{I}$ can choose a point in $\mathcal{F}_i$ independent of the choices made by other $j \in \mathcal{I}$. In reality, the agents might interact (e.g. trade) and their choices will be interdependent, resulting in an I-player game.

ing a philosophical point: under plausible assumptions, weak interpersonal comparisons *can* be well-defined in principle.

Most of the following constructions are obtained by 'stitching together' a collection of preorders. Let $\mathcal{X}$ be a set and let $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$ be a collection of preorders on $\mathcal{X}$ (where Λ is some indexing set). The *join* of $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$ is the transitive closure $(\underset{J}{\preceq})$ of the union of the relations $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$. That is, $x\underset{J}{\preceq}x'$ if there exists a sequence $x = x_0\underset{\lambda_1}{\preceq}x_1\underset{\lambda_2}{\preceq}\cdots\underset{\lambda_N}{\preceq}x_N = x'$, where $\lambda_n \in \Lambda$ for all $n \in [1...N]$. It is easy to see that $(\underset{J}{\preceq})$ is itself a preorder (it is transitive and reflexive), which extends $(\underset{\lambda_0}{\preceq})$ for every $\lambda_0 \in \Lambda$. However, the asymmetric factor $(\underset{J}{\prec})$ does not necessarily extend the asymmetric factor $(\underset{\lambda_0}{\prec})$, because elements which are strictly ordered by $(\underset{\lambda_0}{\preceq})$ may be rendered equivalent through some chain of relations in the union of $\{\underset{\lambda}{\preceq}\}_{\lambda\in\Lambda}$. Many of the technical complications which follow arise in response to this problem.

8.1 Nonexample: Wipos based on multiple desiderata

We begin with a *nonexample*, which shows that one obvious strategy for defining a wipo fails. Let $\mathbf{q} : \Phi\longrightarrow\mathbb{R}^K$ be some function, such that, for all $k \in [1...K]$, the component $q_k : \Phi\longrightarrow\mathbb{R}$ is some quantitative measure of 'quality of life'. For example, some of the coordinates of $\mathbf{q}$ might be the consumption levels of various physical goods; others might be various measures of physical health, or welfare indicators such as education level or opportunities for participation in the social, cultural and political life of the community; others might try to measure more intangible desiderata such as autonomy, security, dignity, liberty, or self-actualization. Some coordinates of $\mathbf{q}$ could measure Rawls' (1971) 'primary goods' or Sen's (1985, 1988) 'functionings and capabilities'. Define preorder $(\underset{\mathbf{q}}{\preceq})$ on $\Psi \times \Phi$ by

$$\Big((\psi,\phi)\underset{\mathbf{q}}{\preceq}(\psi',\phi')\Big) \iff \Big(q_k(\phi) \leq q_k(\phi') \text{ for every } k \in [1...K]\Big).$$

Suppose the collection $\{q_1, \ldots, q_K\}$ is comprehensive enough that, for any $\psi \in \Psi$, and any $\phi, \phi' \in \Phi$, if $(\psi,\phi)\underset{\mathbf{q}}{\prec}(\psi,\phi')$, then $\phi\underset{\psi}{\preceq}\phi'$ (but not conversely). Thus, if we define $(\underset{*}{\preceq})$ to be the join of $(\underset{\mathbf{q}}{\preceq})$ and $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$, then we would expect $(\underset{*}{\preceq})$ to be a wipo. However, Pattanaik and Xu (2007; §3, Proposition 1) have shown that this is false, as long as different individuals have even slightly different preferences over Φ (a principle they call 'minimal relativism'). The problem is that the definition of $(\underset{\mathbf{q}}{\preceq})$ clearly forces $(\psi,\phi)\underset{\mathbf{q}}{\approx}(\psi',\phi')$ (and hence, $(\psi,\phi)\underset{*}{\approx}(\psi',\phi')$), whenever $\mathbf{q}(\phi) = \mathbf{q}(\phi')$. This is in fact a very strong assumption of interpersonal preference comparison, and leaves individuals with essentially no room to differ in their preference orderings.

To illustrate the problem, suppose $K = 2$, let $\phi_1, \phi_2 \in \Phi$, and suppose $\mathbf{q}(\phi_1) = (1,2)$, while $\mathbf{q}(\phi_2) = (2,1)$; thus, neither $\phi_1 \underset{\mathbf{q}}{\prec} \phi_2$ nor $\phi_2 \underset{\mathbf{q}}{\prec} \phi_1$. Let $\psi, \psi' \in \Psi$, and suppose $\phi_1 \underset{\psi}{\prec} \phi_2$ while $\phi_2 \underset{\psi'}{\prec} \phi_1$. Suppose we can find some ϕ_1' very 'close' to ϕ_1 such that $\mathbf{q}(\phi_1')$ is close to $(1,2)$ but dominates it; say $\mathbf{q}(\phi_1') = (1.01,\ 2.01)$. Thus, $\phi_1 \underset{\mathbf{q}}{\prec} \phi_1'$, but assuming ψ has continuous preferences, we have $\phi_1' \underset{\psi}{\prec} \phi_2$. Next, find some ϕ_2' very 'close' to ϕ_2, such that $\mathbf{q}(\phi_2')$ is close to $(2,1)$ but dominates it; say $\mathbf{q}(\phi_2') = (2.01,\ 1.01)$. Thus, $\phi_2 \underset{\mathbf{q}}{\prec} \phi_2'$, but assuming ψ' has continuous preferences, we have $\phi_2' \underset{\psi'}{\prec} \phi_1$. Putting it all together, we get

$$\phi_1 \underset{\mathbf{q}}{\prec} \phi_1' \underset{\psi}{\prec} \phi_2 \underset{\mathbf{q}}{\prec} \phi_2' \underset{\psi'}{\prec} \phi_1.$$

Thus, if $(\underset{*}{\preceq})$ is the join of $(\underset{\mathbf{q}}{\preceq})$, $(\underset{\psi}{\preceq})$ and $(\underset{\psi'}{\preceq})$, then we get $\phi_1 \underset{*}{\preceq} \phi_1' \underset{*}{\preceq} \phi_2 \underset{*}{\preceq} \phi_2' \underset{*}{\preceq} \phi_1$, so that $\phi_1 \underset{*}{\approx} \phi_2$, contradicting the fact that $\phi_1 \underset{\psi}{\prec} \phi_2$.

8.2 Wipos based on envy and pity

Suppose that each individual can attempt interpersonal comparisons between herself and other people, but not between two other people. Formally, for each $\psi_1, \psi_2 \in \Psi$, let $(\underset{\psi_1,\psi_2}{\preceq})$ be a wipo on $\{\psi_1, \psi_2\} \times \Phi$ which agrees with $(\underset{\psi_1}{\preceq})$ on $\{\psi_1\} \times \Phi$ and agrees with $(\underset{\psi_2}{\preceq})$ on $\{\psi_2\} \times \Phi$. The order $(\underset{\psi_1,\psi_2}{\preceq})$ is a ψ_1-type person's comparison between herself and a ψ_2-type person; if $(\psi_1, \phi_1) \underset{\psi_1,\psi_2}{\prec} (\psi_2, \phi_2)$, then we might say that ψ_1 'envies' ψ_2; whereas if $(\psi_1, \phi_1) \underset{\psi_1,\psi_2}{\succ} (\psi_2, \phi_2)$, then we might say that ψ_1 'pities' ψ_2. 'Self-knowledge' requires $(\underset{\psi_1,\psi_2}{\preceq})$ to agree with $(\underset{\psi_1}{\preceq})$, while 'nonpaternalism' requires $(\underset{\psi_1,\psi_2}{\preceq})$ to agree with $(\underset{\psi_2}{\preceq})$.

These interpersonal comparisons might not be correct; for example, ψ_1 might envy ψ_2, while ψ_2 simultaneously envies ψ_1 (i.e. we might have $(\psi_1, \phi_1) \underset{\psi_1,\psi_2}{\prec} (\psi_2, \phi_2)$ while $(\psi_2, \phi_2) \underset{\psi_2,\psi_1}{\prec} (\psi_1, \phi_1)$). However, if both ψ_1 *and* ψ_2 agree that ψ_1 is happier, we might take this to mean that ψ_1 objectively *is* happier than ψ_2. In other words, we could define a relation $(\underset{\&}{\preceq})$ on $\Psi \times \Phi$ by

$$\Big((\psi_1, \phi_1) \underset{\&}{\preceq} (\psi_2, \phi_2)\Big) \iff \Big((\psi_1, \phi_1) \underset{\psi_1,\psi_2}{\preceq} (\psi_2, \phi_2) \text{ and } (\psi_1, \phi_1) \underset{\psi_2,\psi_1}{\preceq} (\psi_2, \phi_2)\Big). \tag{25}$$

Unfortunately, the relation $(\underset{\&}{\preceq})$ defined by (25) might not be a wipo, because it might violate condition (W1): there may exist $\psi \in \Psi$ and $\phi, \phi' \in \Phi$ such that $\phi' \underset{\psi}{\succ} \phi$, but $(\psi, \phi') \underset{\&}{\preceq} (\psi, \phi)$.

Example 8.1: $\Psi = \{0, 1, 2\}$, let $\Phi = \mathbb{Z}$, and suppose $(\underset{\psi}{\preceq})$ is the standard ordering on $\mathbb{Z}$ for all $\psi \in \Psi$. Suppose that each $\psi \in \Psi$ believes that $(\psi - 1, \phi + 1) \underset{\psi,\psi-1}{\prec} (\psi, \phi) \underset{\psi,\psi+1}{\prec} (\psi + 1, \phi - 1)$, for all $\phi \in \mathbb{Z}$ (here, we perform addition in Ψ mod 3, so that $2 + 1 \equiv 0$ mod 3, etc.). Thus, for all $\psi \in \Psi$, if $\psi' = \psi + 1$ mod 3, then the orderings $(\underset{\psi,\psi'}{\preceq})$ and $(\underset{\psi',\psi}{\preceq})$ agree on $\{\psi, \psi'\} \times \Phi$, so definition (25) is in force. But $(0, 9) \underset{\&}{\preceq} (1, 8) \underset{\&}{\preceq} (2, 7) \underset{\&}{\preceq} (0, 6)$. Taking the transitive closure, we get $(0, 9) \underset{\&}{\preceq} (0, 6)$, which contradicts the fact that $9 \underset{0}{\succ} 6$. $\diamondsuit$

The system of envy/pity relations $\{\underset{\psi_1,\psi_2}{\preceq}\}_{\psi_1,\psi_2\in\Psi}$ is ***consistent*** if the following holds: for any $(\psi_1,\phi_1),(\psi_2,\phi_2)\in\Psi\times\Phi$ with $(\psi_1,\phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2,\phi_2)$ and $(\psi_1,\phi_1)\underset{\psi_2,\psi_1}{\preceq}(\psi_2,\phi_2)$, and any $(\psi',\phi')\in\Psi\times\Phi$:

- if $(\psi',\phi')\underset{\psi',\psi_1}{\precneq}(\psi_1,\phi_1)$, then also $(\psi',\phi')\underset{\psi',\psi_2}{\precneq}(\psi_2,\phi_2)$;
- if $(\psi',\phi')\underset{\psi',\psi_2}{\succneq}(\psi_2,\phi_2)$, then also $(\psi',\phi')\underset{\psi',\psi_1}{\succneq}(\psi_1,\phi_1)$.

This weak transitivity condition requires ψ' to respect any $\{\psi_1,\psi_2\}$-interpersonal comparisons on which both ψ_1 and ψ_2 agree. For example, if both ψ_1 and ψ_2 think that ψ_2 is happier than ψ_1, and ψ' envies ψ_1 then she must also envy ψ_2. (However, if ψ_1 and ψ_2 disagree about their relative happiness levels, then ψ' is not obliged to be consistent with either of them).

Proposition 8.2 *Suppose that, for any $\psi_1,\psi_2\in\Psi$, the relation $(\underset{\psi_1,\psi_2}{\preceq})$ is a wipo on $\{\psi_1,\psi_2\}\times\Phi$. Also suppose the system $\{\underset{\psi_1,\psi_2}{\preceq}\}_{\psi_1,\psi_2\in\Psi}$ is consistent. Then $(\underset{\&}{\preceq})$ is a wipo.*

8.3 Wipos from local expertise

Ortuño-Ortin and Roemer (1991) propose a model of interpersonal comparisons based on 'local expertise'. For each $\psi\in\Psi$, let $\mathcal{N}_\psi\subset\Psi$ be a 'neighbourhood' of the point ψ, and assume that a ψ-type individual is capable of constructing a 'local' wipo $(\underset{\psi}{\preceq})$ over $\mathcal{N}_\psi\times\Phi$. We can justify ψ's ability to make interpersonal comparisons of well-being over $\mathcal{N}_\psi\times\Phi$ in at least two ways:

- Each psychology $\nu\in\mathcal{N}_\psi$ is so 'psychologically similar' to ψ that a ψ-person can completely empathize with a ν-person, and accurately compare of their levels of well-being.
- $\mathcal{N}=\mathcal{P}(\psi)\cup\mathcal{F}(\psi)$, where $\mathcal{P}(\psi)$ and $\mathcal{F}(\psi)$ are the past and possible future psychologies of type ψ. As argued under the heading *Intertemporal Comparisons* in the introduction, ψ must be able to make interpersonal comparisons over $\mathcal{P}(\psi)$ and $\mathcal{F}(\psi)$, because she remembers her past and can make choices about her future.

We will require the system $\{\mathcal{N}_\psi,\underset{\psi}{\preceq}\}_{\psi\in\Psi}$ to satisfy the following consistency condition:

(RO) If $\mathcal{N}_{\psi_1}\cap\mathcal{N}_{\psi_2}\neq\emptyset$, then the local wipos $(\underset{\psi_1}{\preceq})$ and $(\underset{\psi_2}{\preceq})$ agree on $(\mathcal{N}_{\psi_1}\cap\mathcal{N}_{\psi_2})\times\Phi$.

(This condition is quite natural if we suppose that $(\underset{\psi_1}{\preceq})$ and $(\underset{\psi_2}{\preceq})$ are both fragments of some some underlying 'objectively true' interpersonal comparison structure.) We then define a global relation $(\underset{RO}{\preceq})$ as the join of $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$.

The relation $(\underset{RO}{\preceq})$ is not necessarily a wipo, because it might violate condition (W1). When stitching together the local relations $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$, we may introduce a *preference cycle*

$$(\psi_1,\phi_1)\underset{\psi_1}{\preceq}(\psi_2,\phi_2)\underset{\psi_2}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N,\phi_N)\underset{\psi_N}{\preceq}(\psi_1,\phi_1')\underset{\psi_1}{\prec}(\psi_1,\phi_1).$$

Taking the transitive closure, we get $(\psi_1,\phi_1)\underset{RO}{\preceq}(\psi_1,\phi_1')$, contradicting the fact that $\phi_1'\underset{\psi_1}{\succ}\phi_1'$.

Example 8.3: Suppose $\Psi=\{0,1,2,3\}$, and let $\mathcal{N}_\psi:=\{j-1,j,j+1\}$ for all $\psi\in\Psi$ (where we perform addition mod 4, so that $3+1\equiv 0$ mod 4, etc.). Let $\Phi=\mathbb{Z}$, and suppose $(\underset{\psi}{\preceq})$ is the standard ordering on $\mathbb{Z}$ for all $\psi\in\Psi$. Suppose that each $\psi\in\Psi$ believes that $(\psi-1,\phi+1)\underset{\psi}{\prec}(\psi,\phi)\underset{\psi}{\prec}(\psi+1,\phi-1)$, for all $\phi\in\mathbb{Z}$. . Then each pair of local wipos agrees on their overlap, but $(0,9)\underset{RO}{\preceq}(1,8)\underset{RO}{\preceq}(2,7)\underset{RO}{\preceq}(3,6)\underset{RO}{\preceq}(0,5)$. Taking the transitive closure, we get $(0,9)\underset{RO}{\preceq}(0,5)$, which contradicts the fact that $5\underset{0}{\prec}9$. ◇

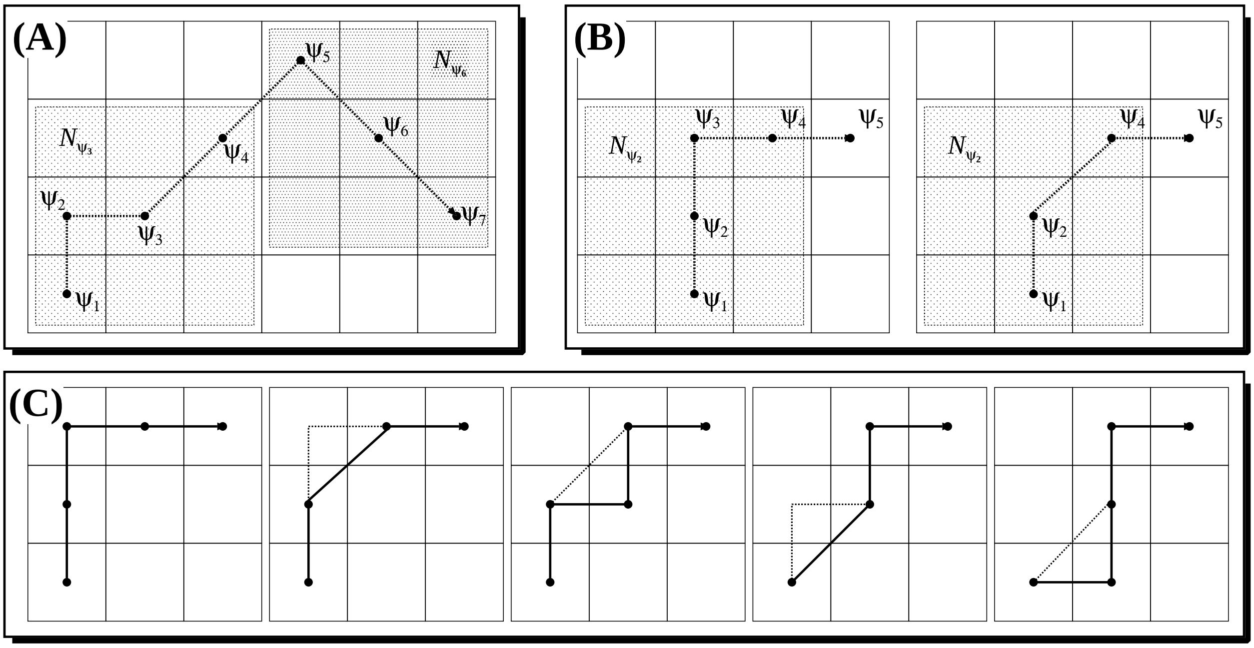

Figure 6: **(A)** A chain. Here, $\Psi=\mathbb{Z}^2$, and for all $\psi\in\Psi$, $\mathcal{N}_\psi=\{\psi'\in\Psi;\ |\psi_1-\psi_1'|\leq 1$ and $|\psi_2-\psi_2'|\leq 1\}$. We have shaded $\mathcal{N}_{\psi_3}$ and $\mathcal{N}_{\psi_6}$ to illustrate. **(B)** An elementary homotopy, obtained by deleting the element ψ_3 from the chain. **(C)** A sequence of elementary homotopies yields a homotopy from the far-left chain to the far-right chain.

To prevent preference cycles, we need additional conditions. Given a neighbourhood system $\mathfrak{N}:=\{\mathcal{N}_\psi\}_{\psi\in\Psi}$ and two points $\psi,\psi'\in\Psi$, an $\mathfrak{N}$-*chain* from ψ to ψ' is a sequence $\psi=\psi_0,\psi_1,\psi_2,\ldots,\psi_N=\psi'$ such that, for all $n\in[1...N]$, $\psi_n\in\mathcal{N}_{\psi_{n-1}}$ (see Figure 6(A)). We say that $\mathfrak{N}$ *chain-connects* Ψ if any two points in Ψ can be connected with an $\mathfrak{N}$-chain. If $\boldsymbol{\psi}:=(\psi_0,\psi_1,\ldots,\psi_{n-1},\psi_n,\psi_{n+1},\cdots,\psi_N)$ is an $\mathfrak{N}$-chain and $\psi_{n+1}\in\mathcal{N}_{\psi_{n-1}}$, then

$\boldsymbol{\psi}' := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_{n+1}, \cdots, \psi_N)$ is also an $\mathfrak{N}$-chain; we say that $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}$ are related by ***elementary homotopy***, and write $\boldsymbol{\psi} \underset{\epsilon}{\simeq} \boldsymbol{\psi}'$ (see Figure 6(B)). Note that $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ have the same endpoints. Two $\mathfrak{N}$-chains $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ are ***homotopic*** if $\boldsymbol{\psi}$ can be converted into $\boldsymbol{\psi}'$ through a sequence of elementary homotopies —that is, there is a sequence of $\mathfrak{N}$-chains $\boldsymbol{\psi} = \boldsymbol{\psi}_1 \underset{\epsilon}{\simeq} \boldsymbol{\psi}_2 \underset{\epsilon}{\simeq} \cdots \underset{\epsilon}{\simeq} \boldsymbol{\psi}_N = \boldsymbol{\psi}'$ (see Figure 6(C)). It follows that $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ must have the same endpoints.

The $\mathfrak{N}$-chain $\boldsymbol{\psi}$ is ***closed*** if $\psi_N = \psi_0$. We say $\boldsymbol{\psi}$ is ***trivial*** if $\psi_0 = \psi_1 = \cdots = \psi_N$. We say $\boldsymbol{\psi}$ is ***nullhomotopic*** if $\boldsymbol{\psi}$ is chain-homotopic to a trivial chain. The neighbourhood system $\mathfrak{N} := \{\mathcal{N}_\psi\}_{\psi\in\Psi}$ is ***simply connected*** if it chain-connects Ψ, and any closed chain is nullhomotopic.

Example 8.4: (a) Suppose Ψ is a simply connected topological space (e.g. $\Psi = \mathbb{R}^N$), and for each $\psi \in \Psi$, let $\mathcal{N}_\psi$ be a simply connected open neighbourhood of ψ (e.g. a ball). Then the system $\mathfrak{N}$ is simply connected.

(b) Suppose $\Psi = \mathbb{Z}^N$, and for all $\psi \in \Psi$, let $\mathcal{N}_\psi$ be the unit box around ψ —that is, $\mathcal{N}_\psi := \left\{\psi' \in \mathbb{Z}^N \; ; \; |\psi'_n - \psi_n| \leq 1, \; \forall n \in [1...N]\right\}$. Then $\mathfrak{N}$ is simply connected.

(c) The system in Example 8.3 is *not* simply connected. For example, the sequence $(0, 1, 2, 3, 0)$ is a closed chain, but it is not nullhomotopic. ♢

The local wipo $(\underset{\psi}{\preceq})$ on $\mathcal{N}_\psi$ is ***indifference-connected*** if, for any $\phi \in \Phi$ and $\nu \in \mathcal{N}_\psi$, there is some ϕ' such that $(\psi, \phi)\underset{\psi}{\approx}(\nu, \phi')$ (Roemer and Ortuño-Ortin call this property 'continuity'). Indifference-connectedness is a very strong property: in particular it implies that each wipo $(\underset{\psi}{\preceq})$ is a complete ordering of $\mathcal{N}_\psi \times \Phi$. The system of local wipos $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$ is ***consistent*** if each wipo $(\underset{\psi}{\preceq})$ can be extended to an indifference-connected wipo $(\underset{\psi}{\hat{\preceq}})$, such that the system $\{\mathcal{N}_\psi, \underset{\psi}{\hat{\preceq}}\}_{\psi\in\Psi}$ still has property (RO). Intuitively, this means that the wipo $(\underset{\psi}{\preceq})$ represents ψ's incomplete perception of some underlying, objectively true system of complete interpersonal comparisons, encoded by $\{\underset{\psi}{\hat{\preceq}}\}_{\psi\in\Psi}$. It is not necessary for us to have explicit knowledge of $\{\underset{\psi}{\hat{\preceq}}\}_{\psi\in\Psi}$ —only to know that it exists.

Proposition 8.5 *Suppose $\mathfrak{N}$ is simply connected, and the system $\{\mathcal{N}_\psi, \underset{\psi}{\preceq}\}_{\psi\in\Psi}$ satisfies* (RO).

(a) *If the system $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$ is consistent, then the global relation $(\underset{RO}{\preceq})$ is a wipo.*

(b) *Suppose, for all $\psi \in \Psi$, that $(\underset{\psi}{\preceq})$ is indifference-connected. Then the global relation $(\underset{RO}{\preceq})$ is wipo and a complete order on $\Psi \times \Phi$.*

Ortuño-Ortin and Roemer (1991) prove two special cases of Proposition 8.5(b): the case $\Psi = \mathbb{Z}^N$ described in Example 8.4(b), and the case when $\Psi = \mathbb{R}^N$, where, for each

$\psi \in \mathbb{R}^N$, the neighbourhood $\mathcal{N}_\psi$ is arc-connected and has radius at least ϵ around ψ, for some fixed $\epsilon > 0$. However, Proposition 8.5(b) requires indifference-connectedness, which may be an unreasonably strong assumption even for 'local' interpersonal comparisons.

8.4 Wipos from infinitesimal expertise

One might object that even the 'local' wipos posited in §8.3 assume an unrealistic level of interpersonal comparability. In response to this objection, we now consider a model which posits only 'infinitesimal' interpersonal comparisons. This will require some elementary differential geometry; see Warner (1983) for background.

Suppose Ψ and Φ are differentiable manifolds, and let $\Psi \times \Phi$ have the product manifold structure. For any $\psi \in \Psi$, let $\mathsf{T}_\psi \Psi$ be the tangent space of Ψ at ψ; for any $\phi \in \Phi$, we similarly define the tangent spaces $\mathsf{T}_\phi \Phi$ and $\mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi) \cong \mathsf{T}_\psi \Psi \times \mathsf{T}_\phi \Phi$. If $\gamma : (-\epsilon, \epsilon) \longrightarrow \Phi$ is any smooth curve with $\gamma(0) = \phi$, then let $\gamma'(0) \in \mathsf{T}_\phi \Phi$ be the velocity vector of γ at 0; if $\vec{0}_\psi \in \mathsf{T}_\psi \Psi$ is the zero vector, then $(\vec{0}_\psi, \gamma'(0))$ is an element of $\mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi)$. Let $\vec{0}_\phi$ be the zero vector in $\mathsf{T}_\phi \Phi$, and let $\vec{0}_{(\psi,\phi)} := (\vec{0}_\psi, \vec{0}_\phi) \in \mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi)$.

For every $(\psi, \phi) \in \Psi \times \Phi$, let $(\underset{(\psi,\phi)}{\vec{\preceq}})$ be a preorder on $\mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi)$ with the following property: If $\gamma : (-\epsilon, \epsilon) \longrightarrow \Phi$ is any smooth curve with $\gamma(0) = \phi$, such that $\phi \underset{\psi}{\preceq} \gamma(t)$ for all $t > 0$, then $\vec{0}_{(\psi,\phi)} \underset{(\psi,\phi)}{\vec{\preceq}} (\vec{0}_\psi, \gamma'(0))$. Intuitively, if $\vec{v} \in \mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi)$ and $\vec{v} \underset{(\psi,\phi)}{\vec{\succeq}} \vec{0}_{(\psi,\phi)}$, then this means that infinitesimal movement through the manifold $\Psi \times \Phi$ in the $\vec{v}$ direction is regarded as a net improvement, even if it involves a change of psychological state as well as physical state. In other words, we are allowed to make 'infinitesimal' interpersonal comparisons of well-being: comparisons between individuals whose psychologies are only infinitesimally different. This yields a wipo $(\underset{\sim}{\preceq})$ on $\Psi \times \Phi$, defined as follows:

$$\Big((\psi_0, \phi_0) \underset{\sim}{\preceq} (\psi_1, \phi_1)\Big) \iff \begin{pmatrix} \exists \text{ smooth path } \gamma : [0,1] \longrightarrow \Psi \times \Phi \text{ with } \gamma(0) = (\psi_0, \phi_0), \\ \gamma(1) = (\psi_1, \phi_1), \text{ and } \gamma'(t) \underset{\gamma(t)}{\vec{\succeq}} \vec{0}_{\gamma(t)} \text{ for all } t \in [0,1] \end{pmatrix}.$$

In other words, it is possible to move from (ψ_0, ϕ_0) to (ψ_1, ϕ_1) along a path which, at every instant, is regarded as an 'infinitesimal improvement'. We refer to γ as an *improvement path*.

The relation $(\underset{\sim}{\preceq})$ is not necessarily acyclic, unless further conditions are imposed on the system of order relations $\mathcal{X} = \{\underset{(\psi,\phi)}{\vec{\preceq}}\}_{(\psi,\phi) \in \Psi \times \Phi}$. The system $\mathcal{X}$ is *smooth* if there exists an open cover $\{\mathcal{O}_j\}_{j \in \mathcal{J}}$ of Ψ (for some indexing set $\mathcal{J}$), and for each $j \in \mathcal{J}$, a smooth function $u_j : \mathcal{O}_j \times \Phi \longrightarrow \mathbb{R}$ such that:

(Sm1) For each $j \in \mathcal{J}$, each $\psi_1, \psi_2 \in \mathcal{O}_j$, we have $u_j(\{\psi_1\} \times \Phi) = u_j(\{\psi_2\} \times \Phi)$.

(Sm2) For each $j \in \mathcal{J}$, each $(\psi, \phi) \in \mathcal{O}_j \times \Phi$, and each $\vec{v} \in \mathsf{T}_{(\psi,\phi)} (\Psi \times \Phi)$, if $\vec{v} \underset{(\psi,\phi)}{\vec{\succeq}} \vec{0}_{(\psi,\phi)}$, then $\nabla u_i(\psi, \phi)[\vec{v}] \geq 0$.

(Sm3) For any $j, k \in \mathcal{J}$, if $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, then u_j and u_k are 'ordinally equivalent' on their domain overlap: for all $\psi, \psi' \in \mathcal{O}_j \cap \mathcal{O}_k$ and all $\phi, \phi' \in \Phi$,, we have $u_i(\psi, \phi) \leq u_i(\psi', \phi')$ if and only if $u_j(\psi, \phi) \leq u_j(\psi', \phi')$.

Proposition 8.6 *If Ψ is simply connected and $\mathcal{X}$ is smooth, then $(\precsim)$ is a wipo.*

Conclusion

Under quite mild and plausible assumptions about interpersonal comparability, it is possible to define a *wipo* —an incomplete preorder on a space of psychophysical states —which makes possible substantive interpersonal comparisons of well-being (§2 and §8). This allows us to construct a *sprow*: a preorder on the space of social alternatives which, while still incomplete, is far more complete than the Pareto preorder (§3). If the wipo satisfies certain technical conditions, it can be represented by a set of ordinal utility functions (§4); we can then combine this ordinal utility representation with a classic social welfare order on $\mathbb{R}^{\mathcal{I}}$ to obtain 'welfarist' sprows (§5). The 'approximate egalitarian' sprow occupies a special place amongst these welfarist sprows (Theorem 5.6).

When we extend this model to choice under uncertainty, we find that every reasonable social order extends the 'approximate utilitarian' ordering (Theorem 6.5), which, under a mild topological hypothesis, also admits a 'welfarist' representation (Theorem 6.7 and Corollary 6.8). A slightly different approach to stochastic interpersonal comparisons leads to a profile-independent version of Harsanyi's Social Aggregation Theorem (Theorem 7.2).

In summary: approximate interpersonal comparisons are intuitively plausible, and admit precise mathematical representations, from which a nontrivial theory of distributive justice can be built. However, many questions remain unanswered. Is there a more direct connection between the wipos of §2 and the wipols of §6? Under what conditions do the wipos constructed in §8 satisfy the 'regularity' hypothesis of Theorem 4.2? Is this regularity hypothesis really necessary in Proposition 5.4? Are the converses of Theorem 5.6(b,c) true? Also, aside from the Suppes-Sen, approximate egalitarian, and approximate utilitarian preorders, what other sprow(l)s have natural characterizations in the framework we have developed? Finally, a practical question: how can we operationalize approximate interpersonal welfare comparisons, so that these social orderings can be applied in practice?

Appendix: Proofs

Proofs from §1-§3.

Proof of Lemma 1.1. (a) is clear from the definition.

(b) Let $x, x' \in \mathcal{X}$. If $x \preceq x'$, then $x \underset{\lambda}{\preceq} x'$ for all λ; and thus, $x \underset{M}{\preceq} x'$.

Suppose $x \prec x'$. Then $x \underset{\lambda}{\preceq} x'$ for all λ; and thus, $x \underset{M}{\preceq} x'$; we must show that $x \underset{M}{\not\succeq} x'$. By contradiction, suppose $x \underset{M}{\succeq} x'$. Then $x \underset{\lambda}{\succeq} x'$ for all λ, which means $x \succeq x'$, contradicting the hypothesis that $x \prec x'$.

(c) If $(\underset{2}{\preceq})$ either extends or refines $(\underset{1}{\preceq})$, then every pair in $\mathcal{X}$ which are $(\underset{1}{\preceq})$-comparable are also $(\underset{2}{\preceq})$-comparable; hence if $(\underset{1}{\preceq})$ is complete then $(\underset{2}{\preceq})$ is also complete. The second implication in (c) then follows from statement (1).

(d) "$\Longrightarrow$" Suppose $x \underset{2}{\prec} x'$. Either $x \underset{1}{\preceq} x'$ or $x \underset{1}{\succeq} x'$, or both (because $(\underset{1}{\preceq})$ is complete). But if $x \underset{1}{\succeq} x'$, then $x \underset{2}{\succeq} x'$ (because $(\underset{2}{\preceq})$ extends $(\underset{1}{\preceq})$); this contradicts the fact that $x \underset{2}{\prec} x'$. Thus, we must have $x \underset{1}{\preceq} x'$ and *not* $x \underset{1}{\succeq} x'$; hence $x \underset{1}{\prec} x'$, as desired.

On the other hand, if $x \underset{2}{\approx} x'$, then $x \underset{1}{\preceq} x'$ or $x \underset{1}{\succeq} x'$ (because $(\underset{1}{\preceq})$ is complete).

"$\Longleftarrow$" Suppose $x \underset{1}{\preceq} x'$. Either $x \underset{2}{\preceq} x'$ or $x \underset{2}{\succ} x'$ (because $(\underset{2}{\preceq})$ is complete). But if $x \underset{2}{\succ} x'$, then $x \underset{1}{\succ} x'$ (because $(\underset{1}{\preceq})$ refines $(\underset{2}{\preceq})$); this contradicts the fact that $x \underset{1}{\preceq} x'$. Thus, we must have $x \underset{2}{\preceq} x'$, as desired.

(e) "$\Longrightarrow$" Let $x \neq x'$. If $x \underset{2}{\prec} x'$, then $x \underset{2}{\preceq} x'$; hence $x \underset{1}{\preceq} x'$ (because $(\underset{1}{\preceq})$ extends $(\underset{2}{\preceq})$); hence $x \underset{1}{\prec} x'$ (because $x \neq x'$ and $(\underset{1}{\preceq})$ is strict).

"$\Longleftarrow$" Let $x \neq x'$. If $x \underset{2}{\preceq} x'$, then $x \underset{2}{\prec} x'$ (because $x \neq x'$ and $(\underset{2}{\preceq})$ is strict); hence $x \underset{1}{\prec} x'$ (because $(\underset{1}{\preceq})$ refines $(\underset{2}{\preceq})$); hence $x \underset{1}{\preceq} x'$.

(f) If $(\underset{1}{\preceq})$ either extends or refines $(\underset{2}{\preceq})$, then (e) says that $(\underset{1}{\preceq})$ both extends *and* refines $(\underset{2}{\preceq})$; then the second implication in (c) implies that $(\underset{1}{\preceq})$ is identical with $(\underset{2}{\preceq})$. □

Proof of Proposition 3.4. For the proofs of (a) and (b), let $(\boldsymbol{\psi}, \boldsymbol{\phi}), (\boldsymbol{\psi}', \boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$.

(a) ($\mathrm{Par}^{\trianglelefteq}$) Suppose $(\psi_i, \phi_i) \preceq (\psi'_i, \phi'_i)$ for all $i \in \mathcal{I}$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be the identity map. Then $(\psi_i, \phi_i) \preceq (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ all $i \in \mathcal{I}$; hence $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}')$.

($\mathrm{Anon}^{\trianglelefteq}$) Suppose $(\boldsymbol{\psi}', \boldsymbol{\phi}') = \sigma(\boldsymbol{\psi}, \boldsymbol{\phi})$ for some permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$. Then $(\psi_i, \phi_i) \approx (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ all $i \in \mathcal{I}$. Thus, $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\overset{\vartriangle}{\approx}} (\boldsymbol{\psi}', \boldsymbol{\phi}')$.

(b) Suppose $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}')$. We must show that $(\boldsymbol{\psi}, \boldsymbol{\phi}) \trianglelefteq (\boldsymbol{\psi}', \boldsymbol{\phi}')$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation such that $(\psi_i, \phi_i) \preceq (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ all $i \in \mathcal{I}$. Then $(\boldsymbol{\psi}, \boldsymbol{\phi}) \trianglelefteq \sigma(\boldsymbol{\psi}', \boldsymbol{\phi}') \overset{\vartriangle}{\approx} (\boldsymbol{\psi}', \boldsymbol{\phi}')$. (Here "$\overset{\vartriangle}{\approx}$" is by ($\mathrm{Anon}^{\trianglelefteq}$) and "$\trianglelefteq$" is by ($\mathrm{Par}^{\trianglelefteq}$), because $(\psi_i, \phi_i) \preceq (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ all $i \in \mathcal{I}$.) Thus, $(\boldsymbol{\psi}, \boldsymbol{\phi}) \trianglelefteq (\boldsymbol{\psi}', \boldsymbol{\phi}')$ by transitivity.

Suppose $(\trianglelefteq)$ also satisfies (SPar), and suppose $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\triangleleft} (\boldsymbol{\psi}', \boldsymbol{\phi}')$. We must show that $(\boldsymbol{\psi}, \boldsymbol{\phi}) \triangleleft (\boldsymbol{\psi}', \boldsymbol{\phi}')$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation such that $(\psi_i, \phi_i) \preceq (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ all $i \in \mathcal{I}$. We must have $(\psi_i, \phi_i) \prec (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ for some $i \in \mathcal{I}$, because if $(\psi_i, \phi_i) \approx (\psi'_{\sigma(i)}, \phi'_{\sigma(i)})$ for all $i \in \mathcal{I}$, then $(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\overset{\vartriangle}{\approx}} (\boldsymbol{\psi}', \boldsymbol{\phi}')$, contradicting the assumption that

$(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{s}{\triangleleft} (\boldsymbol{\psi}', \boldsymbol{\phi}')$. Thus $(\boldsymbol{\psi}, \boldsymbol{\phi}) \triangleleft \sigma(\boldsymbol{\psi}', \boldsymbol{\phi}') \overset{\triangle}{\approx} (\boldsymbol{\psi}', \boldsymbol{\phi}')$. (Here "$\overset{\triangle}{\approx}$" is by (Anon$^{\trianglelefteq}$) and "($\triangleleft$)" is by (SPar$^{\trianglelefteq}$).) Thus, $(\boldsymbol{\psi}, \boldsymbol{\phi}) \triangleleft (\boldsymbol{\psi}', \boldsymbol{\phi}')$ by transitivity.

(c) follows immediately from (Par$^{\trianglelefteq}$). To prove (d), it suffices to observe that, if every relation ($\underset{\lambda}{\trianglelefteq}$) satisfies (Par$^{\trianglelefteq}$) and (Anon$^{\trianglelefteq}$), then their intersection ($\trianglelefteq$) must also. □

Proof of Claim 3.6.* For any $\mathbf{b}, \mathbf{p} \in \mathcal{B}$, we have $\mathbf{b} \underset{æ,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{p}$ iff either (1) $b_1 \leq p_1$ and $b_2 \leq p_2$, or (2) $b_1 \leq p_1$ and $b_1 \leq p_2 - \delta$ or (3) $b_2 \leq p_2$ and $b_2 \leq p_1 - \delta$ (with one of the inequalities being strict in each case). Case (1) is false if and only if $\mathbf{b} \in \mathcal{P}$. So, suppose $\mathbf{b} \in \mathcal{P}$. For $k = 1, 2$, let $\overline{P}_k := \max\{p_k\ ;\ (p_1, p_2) \in \mathcal{P}\}$. Then

$$
\begin{aligned}
\Big(\mathbf{b} \text{ is } (\underset{æ,\boldsymbol{\psi}}{\blacktriangleleft})\text{-undominated}\Big) &\iff \Big(\forall\ \mathbf{p} \in \mathcal{P},\ \text{Case (2) is false and Case (3) is false}\Big) \\
\iff&\ \Big(\forall\ \mathbf{p} \in \mathcal{P},\ [b_1 \geq p_1 \text{ or } b_1 + \delta \geq p_2] \text{ and } [b_2 \geq p_2 \text{ or } b_2 + \delta \geq p_1]\Big) \\
\iff&\ \Big(\forall\ \mathbf{p} \in \mathcal{P},\ [(b_1 < p_1) \Rightarrow (b_1 + \delta \geq p_2)] \text{ and } [(b_2 < p_2) \Rightarrow (b_2 + \delta \geq p_1)]\Big) \\
\underset{(*)}{\iff}&\ \Big(\forall\ \mathbf{p} \in \mathcal{P},\ [(b_2 > p_2) \Rightarrow (b_1 + \delta \geq p_2)] \text{ and } [(b_1 > p_1) \Rightarrow (b_2 + \delta \geq p_1)]\Big) \\
\iff&\ \begin{pmatrix} \forall\ p_2 \leq \overline{P}_2,\ [(b_2 > p_2) \Rightarrow (b_1 + \delta \geq p_2)], \text{ while} \\ \forall\ p_1 \leq \overline{P}_1,\ [(b_1 > p_1) \Rightarrow (b_2 + \delta \geq p_1)] \end{pmatrix} \\
\iff&\ \Big(b_1 + \delta \geq b_2 \text{ and } b_2 + \delta \geq b_1\Big) \iff \Big(|b_1 - b_2| \leq \delta\Big).
\end{aligned}
$$

Here, $(*)$ is because $(b_1 < p_1) \Leftrightarrow (b_2 > p_2)$ for all distinct $\mathbf{b}, \mathbf{p} \in \mathcal{P}$. □

Proof of Theorem 4.2 and ancillary results.

The proof of Theorem 4.2 requires some technical results (Lemma 4.3) which will also be used in the proof of Proposition 5.4

A *universal utility function* is a function $U : \Psi \times \Phi \longrightarrow \mathbb{R}$ such that, for each $\psi \in \Psi$, the function $U(\psi, \bullet) : \Phi \longrightarrow \mathbb{R}$ represents the preference ordering ($\underset{\psi}{\preceq}$). That is:

$$
\text{for all } \phi_1, \phi_2 \in \Phi, \qquad \Big(\phi_1 \underset{\psi}{\preceq} \phi_2\Big) \iff \Big(U(\psi, \phi_1) \leq U(\psi, \phi_2)\Big). \tag{26}
$$

If ($\preceq$) is regular, then axiom (R1) guarantees the existence of a universal utility function. In particular, any possible hedometer h defines a universal utility function. However, not every universal utility function U is a possible hedometer for ($\preceq$). The problem is that U is not 'calibrated', so that there is no way to meaningfully compare the values of $U(\psi, \phi)$ and $U(\psi', \phi')$ if $\psi \neq \psi'$. A *hedometric calibration* for U and ($\preceq$) is a system $\boldsymbol{\kappa} = [\kappa_\psi]_{\psi \in \Psi}$, where

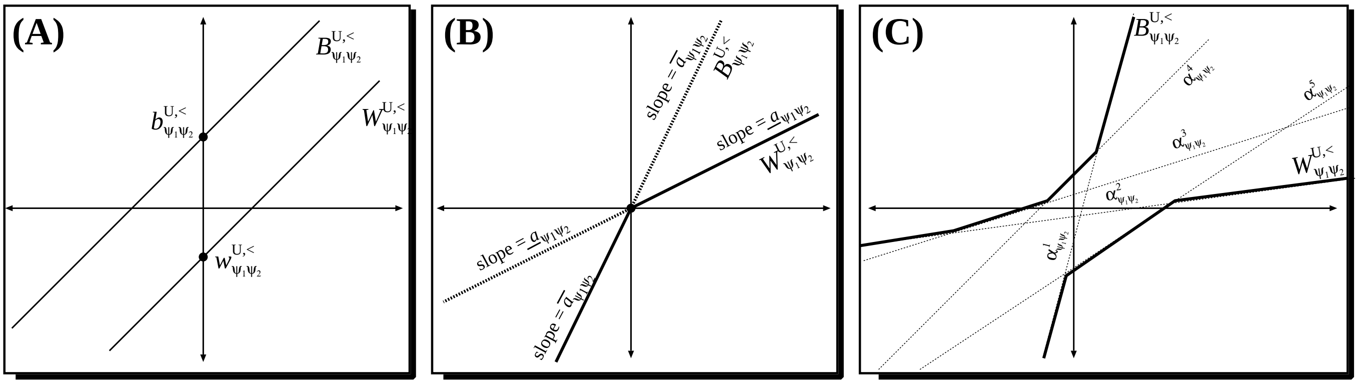

Figure 7: (A) The functions $B^{U,(\preceq)}_{\psi_1\psi_2}(r)$ and $W^{U,(\preceq)}_{\psi_1\psi_2}(r)$ from Example 4.2(a). Here, $b^{U,(\preceq)}_{\psi_1,\psi_2} := c \cdot d(\psi_1,\psi_2)^\gamma$ and $w^{U,(\preceq)}_{\psi_1,\psi_2} := -c \cdot d(\psi_1,\psi_2)^\gamma$. (B) The functions $B^{U,(\preceq)}_{\psi_1\psi_2}$ and $W^{U,(\preceq)}_{\psi_1\psi_2}$ from Example 4.2(b). Here, $\overline{a}_{\psi_1,\psi_2} := c^{d(\psi_1,\psi_2)}$ and $\underline{a}_{\psi_1,\psi_2} := c^{-d(\psi_1,\psi_2)}$. (C) $B^{U,(\preceq)}_{\psi_1\psi_2}$ and $W^{U,(\preceq)}_{\psi_1\psi_2}$ are polygonal functions obtained by taking the pointwise maximum and minimum over a set of affine increasing functions; see Example 4.2(c).

$\kappa_\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is increasing for each $\psi \in \Psi$, such that, for any $(\psi_1,\phi_1),(\psi_2,\phi_2) \in \Psi \times \Phi$, with $U(\psi_k,\phi_k) = r_k$ for $k = 1,2$:

$$\begin{array}{rcl} \Big((\psi_1,\phi_1) \preceq (\psi_2,\phi_2)\Big) & \Longrightarrow & \Big(\kappa_{\psi_1}(r_1) \leq \kappa_{\psi_2}(r_2)\Big) \\ \text{and} \quad \Big((\psi_1,\phi_1) \prec (\psi_2,\phi_2)\Big) & \Longrightarrow & \Big(\kappa_{\psi_1}(r_1) < \kappa_{\psi_2}(r_2)\Big). \end{array} \tag{27}$$

Let $\mathcal{C}\!\mathcal{A}\!\mathcal{L}(U,\preceq)$ be the set of all hedometric calibrations for U and $(\preceq)$.

Given any $\psi_1,\psi_2 \in \Psi$, we define the *comparison functions* $W^{U,(\preceq)}_{\psi_1\psi_2}, B^{U,(\preceq)}_{\psi_1\psi_2} : \mathbb{R} \longrightarrow \mathbb{R}$ as follows: for any $r_1 \in \mathbb{R}$,

$$B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \;:=\; \inf\left\{ \begin{array}{c} r_2\ ;\ \exists\ \phi_1,\phi_2 \in \Phi \text{ with } r_1 \leq U(\psi_1,\phi_1), \\ (\psi_1,\phi_1) \preceq (\psi_2,\phi_2), \text{ and } U(\psi_2,\phi_2) = r_2 \end{array}\right\}, \tag{28}$$

$$\text{and} \quad W^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \;:=\; \sup\left\{ \begin{array}{c} r_2\ ;\ \exists\ \phi_1,\phi_2 \in \Phi \text{ with } r_1 \geq U(\psi_1,\phi_1), \\ (\psi_1,\phi_1) \succeq (\psi_2,\phi_2), \text{ and } U(\psi_2,\phi_2) = r_2 \end{array}\right\}. \tag{29}$$

In words: $B^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$ represents the smallest value r_2 such that a U-utility of r_2 for psyche ψ_2 is *definitely* better than a U-utility of r_1 for psyche ψ_1. Likewise, $W^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$ represents the largest value r_2 such that a U-utility of r_2 for psyche ψ_2 is definitely *worse* than a U-utility of r_1 for psyche ψ_1. Observe that the sets on the right hand sides of equations (28) and (29) are nonempty by axiom (W2); thus, these supremums and infimums are always finite. Also note that, for any $\psi \in \Psi$, we have $B^{U,(\preceq)}_{\psi\psi} = \mathbf{Id} = W^{U,(\preceq)}_{\psi\psi}$ (where $\mathbf{Id}$ is the identity function on $\mathbb{R}$).

Example 4.2: Let $\Phi = \mathbb{R}$, as in §2.1. Then we can define $U : \Psi \times \mathbb{R} \longrightarrow \mathbb{R}$ by $U(\psi, r) = r$.

(a) In Example 2.1(a), $B^{U,(\preceq)}_{\psi_1\psi_2}(r) = r + c \cdot d(\psi_1,\psi_2)^\gamma$ and $W^{U,(\preceq)}_{\psi_1\psi_2}(r) = r - c \cdot d(\psi_1,\psi_2)^\gamma$; see Figure 7(A).

(b) In Example 2.1(b), we have

$$B^{U,(\preceq)}_{\psi_1\psi_2}(r) \;=\; \begin{cases} c^{d(\psi_1,\psi_2)}\cdot r & \text{if} \quad r\geq 0; \\ c^{-d(\psi_1,\psi_2)}\cdot r & \text{if} \quad r<0 \end{cases},$$

$$\text{while} \quad W^{U,(\preceq)}_{\psi_1\psi_2}(r) \;=\; \begin{cases} c^{-d(\psi_1,\psi_2)}\cdot r & \text{if} \quad r\geq 0; \\ c^{d(\psi_1,\psi_2)}\cdot r & \text{if} \quad r<0. \end{cases}.$$

See Figure 7(B). (Example 6.2 below derives a very similar structure of interpersonal utility comparisons, in the setting of lotteries).

(c) Let $h^1, h^2, \ldots, h^N : \Psi\times\mathbb{R}{\longrightarrow}\mathbb{R}$ be a collection of 'possible hedometers'; suppose we know that one of them is the 'true' hedometer, but we don't know which one. Thus, as in §2.2, we can define a wipo $(\underset{\mathcal{H}}{\preceq})$ on $\Psi\times\mathbb{R}$ by

$$\Big((\psi,r)\underset{\mathcal{H}}{\preceq}(\psi',r')\Big) \quad\iff\quad \Big(h^n(\psi,r)\;\leq\; h^n(\psi',r'), \text{ for all } n\in[1...N]\Big).$$

For all $n\in[1...N]$ and $\psi\in\Psi$, write $h^n(\psi,r)$ as $h^n_\psi(r)$ for all $r\in\mathbb{R}$; this yields an increasing function $h^n_\psi:\mathbb{R}{\longrightarrow}\mathbb{R}$. For all $\psi_1,\psi_2\in\Psi$, define $\alpha^n_{\psi_1,\psi_2} := (h^n_{\psi_2})^{-1}\circ h^n_{\psi_1}:\mathbb{R}{\longrightarrow}\mathbb{R}$.

Claim 1: *For all $r\in\mathbb{R}$, we have $B^{U,(\preceq)}_{\psi_1\psi_2}(r) = \max_{n\in[1...N]} \alpha^n_{\psi_1,\psi_2}(r)$, and $W^{U,(\preceq)}_{\psi_1\psi_2}(r) = \min_{n\in[1...N]} \alpha^n_{\psi_1,\psi_2}(r)$.*

Proof. For all $r_1,r_2\in\mathbb{R}$,

$$\Big((\psi_1,r_1)\underset{\mathcal{H}}{\preceq}(\psi_2,r_2)\Big) \quad\iff\quad \Big(h^n_{\psi_2}(r_2)\geq h^n_{\psi_1}(r_1), \text{ for all } n\in[1...N]\Big)$$
$$\underset{(*)}{\iff} \Big(r_2\geq \alpha^n_{\psi_1,\psi_2}(r_1), \text{ for all } n\in[1...N]\Big) \quad\iff\quad \Big(r_2\geq \max_{n\in[1...N]}\alpha^n_{\psi_1,\psi_2}(r_1)\Big).$$

Here, $(*)$ is because $h^n_{\psi_2}$ is an increasing function for all $n\in[1...N]$.

This shows that $B^{U,(\preceq)}_{\psi_1\psi_2}(r) = \max_{n\in[1...N]} \alpha^n_{\psi_1,\psi_2}(r)$. The proof for $W^{U,(\preceq)}_{\psi_1\psi_2}$ is similar. $\diamondsuit$ `Claim 1`

For example, if $h^n_\psi, \mathbb{R}{\longrightarrow}\mathbb{R}$ is an affine increasing function for all $\psi\in\Psi$ and $n\in\mathbb{N}$, then the functions $\alpha^n_{\psi_1,\psi_2}$ are also affine increasing; thus, $B^{U,(\preceq)}_{\psi_1\psi_2}$ and $W^{U,(\preceq)}_{\psi_1\psi_2}$ are polygonal functions as shown in Figure 7(C). $\diamondsuit$

Lemma 4.3 *Let $(\preceq)$ be a wipo on $\Psi\times\Phi$.*

- **(a)** *If $h\in\mathcal{H}_{\mathcal{ED}}(\preceq)$, and $f:\mathbb{R}{\longrightarrow}\mathbb{R}$ is increasing, then $f\circ h\in\mathcal{H}_{\mathcal{ED}}(\preceq)$.*

Let $U:\Psi\times\Phi{\longrightarrow}\mathbb{R}$ be a universal utility function.

- **(b)** *Let $\boldsymbol{\kappa}\in\mathcal{C}_{\mathcal{AC}}(U,\preceq)$. Define $\boldsymbol{\kappa}U:\Psi\times\Phi{\longrightarrow}\mathbb{R}$ by $\boldsymbol{\kappa}U(\psi,\phi):=\kappa_\psi[U(\psi,\phi)]$, for all $(\psi,\phi)\in\Psi\times\Phi$. Then $\boldsymbol{\kappa}U\in\mathcal{H}_{\mathcal{ED}}(\preceq)$.*

- **(c)** *For any $\psi_1,\psi_2\in\Psi$, the functions $B^{U,(\preceq)}_{\psi_1\psi_2}$ and $W^{U,(\preceq)}_{\psi_1\psi_2}$ are nondecreasing, and $B^{U,(\preceq)}_{\psi_1\psi_2}(r)\leq W^{U,(\preceq)}_{\psi_1\psi_2}(r)$ for all $r\in\mathbb{R}$.*

(d) *For any* $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$,

(d1) $\left(B^{U,(\preceq)}_{\psi_1\psi_2}[U(\psi_1, \phi_1)] < U(\psi_2, \phi_2)\right) \implies \left((\psi_1, \phi_1) \prec (\psi_2, \phi_2)\right)$; *and*

(d2) $\left(U(\psi_1, \phi_1) < W^{U,(\preceq)}_{\psi_2\psi_1}[U(\psi_2, \phi_2)]\right) \implies \left((\psi_1, \phi_1) \prec (\psi_2, \phi_2)\right)$.

Suppose $(\preceq)$ *is regular.*

(e) *For any* $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$, *the following are equivalent:*

[i] $B^{U,(\preceq)}_{\psi_1\psi_2}[U(\psi_1, \phi_1)] \leq U(\psi_2, \phi_2)$;

[ii] $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$;

[iii] $U(\psi_1, \phi_1) \leq W^{U,(\preceq)}_{\psi_2,\psi_1}[U(\psi_2, \phi_2)]$.

(f) *For any* $\psi_1, \psi_2, \psi_3 \in \Psi$ *and* $\phi_1 \in \Phi$, *if* $r_1 = U(\psi_1, \phi_1)$, *then*

$$B^{U,(\preceq)}_{\psi_1\psi_3}(r_1) \;\leq\; B^{U,(\preceq)}_{\psi_2\psi_3} \circ B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \quad \textit{and} \quad W^{U,(\preceq)}_{\psi_1\psi_3}(r_1) \;\geq\; W^{U,(\preceq)}_{\psi_2\psi_3} \circ W^{U,(\preceq)}_{\psi_1\psi_2}(r_1).$$

(g) *Fix* $\psi_0 \in \Psi$. *For all* $\psi \in \Psi$, *let* $\kappa_\psi := B^{U,(\preceq)}_{\psi,\psi_0}$. *Then* $[\kappa_\psi]_{\psi\in\Psi} \in \mathcal{C}_{\mathcal{AC}}(U, \preceq)$.

(h) *Fix* $\psi_0 \in \Psi$. *For all* $\psi \in \Psi$, *let* $\kappa_\psi := W^{U,(\preceq)}_{\psi,\psi_0}$. *Then* $[\kappa_\psi]_{\psi\in\Psi} \in \mathcal{C}_{\mathcal{AC}}(U, \preceq)$.

Proof. **(a)** and **(b)** follows immediately from the defining formulae (7), (26) and (27), while **(c)** follows from the defining formulae (28) and (29).

(d1) Let $r_1 := U(\psi_1, \phi_1)$ and $r_2 := B^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$. For any $\epsilon > 0$, definition (28) yields some $\phi'_1, \phi'_2 \in \Phi$ such that $r_1 \leq U(\psi_1, \phi'_1)$, $(\psi_1, \phi'_1) \preceq (\psi_2, \phi'_2)$, and $U(\psi_2, \phi'_2) = r'_2 \leq r_2 + \epsilon$. But if $U(\psi_1, \phi_1) = r_1 \leq U(\psi_1, \phi'_1)$, then $\phi_1 \underset{\psi_1}{\preceq} \phi'_1$ by formula (26), and hence $(\psi_1, \phi_1) \preceq (\psi_1, \phi'_1)$ by (W1). If $r_2 < U(\psi_2, \phi_2)$, then we can make ϵ small enough that $r_2 + \epsilon < U(\psi_2, \phi_2)$. Thus, $U(\psi_2, \phi'_2) < U(\psi_2, \phi_2)$, so $\phi'_2 \underset{\psi_2}{\prec} \phi_2$ by (26), and hence $(\psi_2, \phi'_2) \prec (\psi_2, \phi_2)$ by (W1). Putting it all together, we have

$$(\psi_1, \phi_1) \quad \preceq \quad (\psi_1, \phi'_1) \quad \preceq \quad (\psi_2, \phi'_2) \quad \prec \quad (\psi_2, \phi_2),$$

and hence $(\psi_1, \phi_1) \prec (\psi_2, \phi_2)$, by transitivity. This proves (d1); the proof of (d2) is similar.

Before proving (e), we need a brief digression about order topologies. Fix $\psi \in \Psi$. For all $\phi \in \Phi$, define[15]

$$\begin{aligned} (\phi, \infty)_\psi &:= \left\{\phi' \in \Phi \; ; \; \phi' \underset{\psi}{\succ} \phi\right\}; & (-\infty, \phi)_\psi &:= \left\{\phi' \in \Phi \; ; \; \phi' \underset{\psi}{\prec} \phi\right\}; \\ [\phi, \infty)_\psi &:= \left\{\phi' \in \Phi \; ; \; \phi' \underset{\psi}{\succeq} \phi\right\}; \quad \text{and} & (-\infty, \phi]_\psi &:= \left\{\phi' \in \Phi \; ; \; \phi' \underset{\psi}{\preceq} \phi\right\}. \end{aligned}$$

[15]Note that we here use this 'half-infinite interval' notation only because it is suggestive, not because we assume any of these sets are linearly ordered.

Let $\mathcal{T}_\psi$ be the $(\underset{\psi}{\preceq})$-*order topology* on Φ, generated by the basis of open sets $\{(\phi,\infty)_\psi\}_{\psi\in\Psi}\cup\{(-\infty,\phi)_\psi\}_{\psi\in\Psi}$. Next, for any $(\psi,\phi)\in\Psi\times\Phi$, we define

$$\begin{array}{rrcl} & ((\psi,\phi),\infty) & := & \{(\psi',\phi')\in\Psi\times\Phi\ ;\ (\psi,\phi)\prec(\psi',\phi')\}; \\ & [(\psi,\phi),\infty) & := & \{(\psi',\phi')\in\Psi\times\Phi\ ;\ (\psi,\phi)\preceq(\psi',\phi')\}; \\ & (-\infty,(\psi,\phi)) & := & \{(\psi',\phi')\in\Psi\times\Phi\ ;\ (\psi,\phi)\succ(\psi',\phi')\}; \\ \text{and} & (-\infty,(\psi,\phi)] & := & \{(\psi',\phi')\in\Psi\times\Phi\ ;\ (\psi,\phi)\succeq(\psi',\phi')\}. \end{array}$$

Finally, for any $(\psi_0,\phi_0)\in\Psi\times\Phi$ and $\psi\in\Psi$, we define

$$((\psi_0,\phi_0),\infty)_\psi \quad := \quad \{\phi\in\Phi\ ;\ (\psi,\phi)\succ(\psi_0,\phi_0)\} \quad = \quad \{\phi\in\Phi\ ;\ (\psi,\phi)\in((\psi_0,\phi_0),\infty)\}.$$

We define $[(\psi_0,\phi_0),\infty)_\psi$, $(-\infty,(\psi_0,\phi_0))_\psi$, and $(-\infty,(\psi_0,\phi_0)]_\psi$ similarly.

Claim 1: *Suppose* $(\preceq)$ *satisfies axiom* (R2) *of regularity. Then:*

(a) *For any* $(\psi_0,\phi_0)\in\Psi\times\Phi$ *and any* $\psi\in\Psi$, *the sets* $[(\psi_0,\phi_0),\infty)_\psi$ *and* $(-\infty,(\psi_0,\phi_0)]_\psi$ *are* $\mathcal{T}_\psi$*-closed.*

(b) *For any* $\psi\in\Psi$ *and* $\phi\in\Phi$*:*

[i] *There exists a net*[16] $\{\phi^\lambda\}_{\lambda\in\Lambda}\subset\Phi$ *which converges to* ϕ *in the* $\mathcal{T}_\psi$*-topology, such that* $\phi^\lambda\underset{\psi}{\prec}\phi$ *for all* $\lambda\in\Lambda$.

[ii] *There exists a net* $\{\phi^\lambda\}_{\lambda\in\Lambda}\subset\Phi$ *which converges to* ϕ *in the* $\mathcal{T}_\psi$*-topology, such that* $\phi\underset{\psi}{\prec}\phi^\lambda$ *for all* $\lambda\in\Lambda$.

Proof. (a) It suffices to show that the complements $[(\psi_0,\phi_0),\infty)_\psi^{\complement}$ and $(-\infty,(\psi_0,\phi_0)]_\psi^{\complement}$ are $\mathcal{T}_\psi$-open, which means they are unions of $\mathcal{T}_\psi$-basic open sets. In particular, we claim:

$$[(\psi_0,\phi_0),\infty)_\psi^{\complement} \quad = \quad \bigcup_{\phi'\in[(\psi_0,\phi_0),\infty)_\psi^{\complement}} (-\infty,\phi')_\psi. \tag{30}$$

To see "$\supseteq$" in eqn.(30), we must show, for any $\phi'\in[(\psi_0,\phi_0),\infty)_\psi^{\complement}$, that $(-\infty,\phi')_\psi\subseteq[(\psi_0,\phi_0),\infty)_\psi^{\complement}$. By contradiction, suppose there exists $\phi\in(-\infty,\phi')_\psi\cap[(\psi_0,\phi_0),\infty)_\psi$; then we would have $(\psi_0,\phi_0)\preceq(\psi,\phi)\prec(\psi,\phi')$, and hence $(\psi_0,\phi_0)\prec(\psi,\phi')$ by transitivity. But then $\phi'\in[(\psi_0,\phi_0),\infty)_\psi$, contradicting the fact that $\phi'\in[(\psi_0,\phi_0),\infty)_\psi^{\complement}$.

To see "$\subseteq$", let $\phi\in[(\psi_0,\phi_0),\infty)_\psi^{\complement}$. Then $(\psi,\phi)\not\succeq(\psi_0,\phi_0)$. Thus, either $(\psi,\phi)\prec(\psi_0,\phi_0)$, or $(\psi,\phi)\not\prec\!\!\!\not\succ(\psi_0,\phi_0)$.

[16] A *directed set* is a partially ordered set $(\Lambda,\leq)$ such that, for any $\lambda_1,\lambda_2\in\Lambda$, there exists some $\lambda\in\Lambda$ such that $\lambda_1\leq\lambda$ and $\lambda_2\leq\lambda$. In particular, any totally ordered set is a directed set. A Λ-*net* is a function from Λ into Φ, which we normally write in 'sequence' notation $\{\phi^\lambda\}_{\lambda\in\Lambda}$. This Λ-net *converges* to ϕ if, for every neighbourhood $\mathcal{N}\subseteq\Phi$ around ϕ, there is some λ_0 such that $\phi^\lambda\in\mathcal{N}$ for all $\lambda\geq\lambda_0$. For example, a convergent sequence is a convergent $\mathbb{N}$-net. If $\mathcal{C}\subseteq\Phi$, then $\mathcal{C}$ is closed if and only $\mathcal{C}$ contains the limit point of any convergent net of points in $\mathcal{C}$ (Willard, 2004, Thm. 11.7, p.75).

If $(\psi,\phi) \prec (\psi_0,\phi_0)$, then Axiom (R2)[i] says there is some $\phi' \in \Phi$ such that $(\psi,\phi) \prec (\psi,\phi') \prec (\psi_0,\phi_0)$. Thus, $\phi \underset{\psi}{\prec} \phi'$ by (W1), while $(\psi,\phi') \in (-\infty,(\psi_0,\phi_0)) \subseteq [(\psi_0,\phi_0),\infty)^{\complement}$. Thus, $\phi \in (-\infty,\phi')_\psi$ while $\phi' \in [(\psi_0,\phi_0),\infty)^{\complement}_\psi$. Thus, ϕ is in the union on the right side of eqn.(30).

If $(\psi,\phi) \not\approx (\psi_0,\phi_0)$, then Axiom (R2)[iii] says there is some $\phi' \in \Phi$ such that $(\psi,\phi) \prec (\psi,\phi') \not\approx (\psi_0,\phi_0)$. Thus, $\phi \underset{\psi}{\prec} \phi'$ by (W1), while $(\psi,\phi') \in [(\psi_0,\phi_0),\infty)^{\complement}$ because $(\psi,\phi') \not\succeq (\psi_0,\phi_0)$. Thus, $\phi \in (-\infty,\phi')_\psi$ while $\phi' \in [(\psi_0,\phi_0),\infty)^{\complement}_\psi$. Thus, ϕ is again in the union on the right side of eqn.(30).

A similar proof (using (R2)[ii,iii]) shows that $(-\infty,(\psi_0,\phi_0)]^{\complement}_\psi$ is also $\mathcal{T}_\psi$-open.

(b) [i] Let $\Lambda := (-\infty,\phi)_\psi$ with the ordering $(\underset{\psi}{\preceq})$. If $\lambda \in \Lambda$, then $\lambda \underset{\psi}{\prec} \phi$ and hence $(\psi,\lambda) \prec (\psi,\phi)$ (by axiom (W1)). Thus, (R2)[ii] says there exists some $\phi^\lambda \in \Phi$ such that $(\psi,\lambda) \prec (\psi,\phi^\lambda) \prec (\psi,\phi)$, and hence $\lambda \underset{\psi}{\prec} \phi^\lambda \underset{\psi}{\prec} \phi$.

We now have a net $\{\phi^\lambda\}_{\lambda\in\Lambda}$ such that $\phi^\lambda \underset{\psi}{\prec} \phi$ for all $\lambda \in \Lambda$. To see that this net $\mathcal{T}_\psi$-converges to ϕ, let $\mathcal{N} \subseteq \Phi$ be a $\mathcal{T}_\psi$-neighbourhood around ϕ. There exist $\lambda_0,\kappa_0 \in \Phi$ with $\lambda_0 \underset{\psi}{\prec} \phi \underset{\psi}{\prec} \kappa_0$ such that, if $(\lambda_0,\kappa_0) := \left\{\phi' \in \Phi \;;\; \lambda_0 \underset{\psi}{\prec} \phi' \underset{\psi}{\prec} \kappa_0\right\}$, then $(\lambda_0,\kappa_0) \subseteq \mathcal{N}$. Now, for all $\lambda \in \Lambda$, if $\lambda_0 \underset{\psi}{\preceq} \lambda$, then $\lambda_0 \underset{\psi}{\prec} \phi^\lambda \underset{\psi}{\prec} \kappa_0$ [because $\lambda \underset{\psi}{\prec} \phi^\lambda \underset{\psi}{\prec} \phi$], so $\phi^\lambda \in (\lambda_0,\kappa_0)$, so $\phi^\lambda \in \mathcal{N}$, as desired.

The proof of [ii] is similar (using (R2)[i]). ◇ `Claim 1`

Proof of **(e)** "[ii]$\Longrightarrow$[i]" is true by defining formula (28).

To see "[i]$\Longrightarrow$[ii]", let $r_1 := U(\psi_1,\phi_1)$ and $r_2 := B^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$. If $r_2 < U(\psi_2,\phi_2)$, then (d) implies that $(\psi_1,\phi_1) \prec (\psi_2,\phi_2)$. So, suppose $r_2 = U(\psi_2,\phi_2)$.

To show $(\psi_1,\phi_1) \preceq (\psi_2,\phi_2)$, we must show that $\phi_2 \in [(\psi_1,\phi_1),\infty)_{\psi_2}$. Claim 1(b)[ii] yields a net $\{\phi_2^\lambda\}_{\lambda\in\Lambda} \subset \Phi$ converging to ϕ_2 in the $\mathcal{T}_{\psi_2}$-topology, such that $\phi_2 \underset{\psi_2}{\prec} \phi_2^\lambda$ for all $\lambda \in \Lambda$. Thus, $U(\psi_2,\phi_2) < U(\psi_2,\phi_2^\lambda)$ for all $\lambda \in \Lambda$, by (26). Thus, $r_2 < U(\psi_2,\phi_2^\lambda)$ for all $\lambda \in \Lambda$, so (d) says that $(\psi_1,\phi_1) \prec (\psi_2,\phi_2^\lambda)$ for all $\lambda \in \Lambda$. Thus, $\phi_2^\lambda \in [(\psi_1,\phi_1),\infty)_{\psi_2}$ for all $\lambda \in \Lambda$. Now, $[(\psi_1,\phi_1),\infty)_{\psi_2}$ is $\mathcal{T}_{\psi_2}$-closed by Claim 1(a). But $\mathcal{T}_{\psi_2}\text{-}\lim_{\lambda\in\Lambda} \phi_2^\lambda = \phi_2$, so $\phi_2 \in [(\psi_1,\phi_1),\infty)_{\psi_2}$, which means $(\psi_2,\phi_2) \succeq (\psi_1,\phi_1)$, as desired.

The proof of '[ii] $\Longleftrightarrow$ [iii]" is similar.

(f) (by contradiction) Suppose $B^{U,(\preceq)}_{\psi_1\psi_3}(r_1) > B^{U,(\preceq)}_{\psi_2\psi_3} \circ B^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$. Let $r_2 := B^{U,(\preceq)}_{\psi_1\psi_2}(r_1)$ and $r_3 := B^{U,(\preceq)}_{\psi_2\psi_3}(r_2)$. Thus, for any $\epsilon > 0$, there exist $\phi_2,\phi_3 \in \Phi$ such that $r_2 \le U(\psi_2,\phi_2)$, $(\psi_2,\phi_2) \preceq (\psi_3,\phi_3)$, and $U(\psi_3,\phi_3) \le r_3 + \epsilon$. Let $r_3' := B^{U,(\preceq)}_{\psi_1\psi_3}(r_1)$. By hypothesis, $r_3 < r_3'$; thus, we can make ϵ small enough that $r_3 + \epsilon < r_3'$, and hence $U(\psi_3,\phi_3) < r_3'$.

Now, $U(\psi_1,\phi_1) = r_1$, so $B^{U,(\preceq)}_{\psi_1\psi_2}(U(\psi_1,\phi_1)) = B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) = r_2 \le U(\psi_2,\phi_2)$, so (e) implies that $(\psi_1,\phi_1) \preceq (\psi_2,\phi_2)$. But $(\psi_2,\phi_2) \preceq (\psi_3,\phi_3)$ by definition, hence $(\psi_1,\phi_1) \preceq (\psi_3,\phi_3)$

by transitivity. Now, $U(\psi_1, \phi_1) = r_1$ and $U(\psi_3, \phi_3) < r'_3$; thus, definition (28) says that $B^{U,(\preceq)}_{\psi_1\psi_3}(r_1) < r'_3$. But $r'_3 = B^{U,(\preceq)}_{\psi_1\psi_3}(r_1)$. Contradiction.

The proof for $W^{U,(\preceq)}_{\psi_1\psi_3}$ is similar.

(g) Let $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$ and let $r_k := U(\psi_k, \phi_k)$ for $k = 1, 2$. We must verify defining formula (27).

If $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$, then $B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \leq r_2$ by definition (28). Thus,

$$\kappa_{\psi_1}(r_1) \quad := \quad B^{U,(\preceq)}_{\psi_1,\psi_0}(r_1) \underset{(*)}{\leq} B^{U,(\preceq)}_{\psi_2,\psi_0} \circ B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \underset{(\dagger)}{\leq} B^{U,(\preceq)}_{\psi_2,\psi_0}(r_2) \quad =: \quad \kappa_{\psi_2}(r_2), \tag{31}$$

so $\kappa_{\psi_1}(r_1) \leq \kappa_{\psi_2}(r_2)$, as desired. Here, $(*)$ is by part (f), and $(\dagger)$ is because $B^{U,(\preceq)}_{\psi_2,\psi_0}$ is nondecreasing by part (c), while $B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \leq r_2$.

If $(\psi_1, \phi_1) \prec (\psi_2, \phi_2)$, then (R2)[ii] says there exists $\phi'_2 \in \Phi$ such that $(\psi_1, \phi_1) \prec (\psi_2, \phi'_2) \prec (\psi_2, \phi_2)$. Thus, if $r'_2 := U(\psi_2, \phi'_2)$, then $B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \leq r'_2$ by definition (28). Thus,

$$\kappa_{\psi_1}(r_1) \quad := \quad B^{U,(\preceq)}_{\psi_1,\psi_0}(r_1) \underset{(*)}{\leq} B^{U,(\preceq)}_{\psi_2,\psi_0} \circ B^{U,(\preceq)}_{\psi_1\psi_2}(r_1) \underset{(\dagger)}{\leq} B^{U,(\preceq)}_{\psi_2,\psi_0}(r'_2) \underset{(\ddagger)}{<} B^{U,(\preceq)}_{\psi_2,\psi_0}(r_2) \quad =: \quad \kappa_{\psi_2}(r_2),$$

so $\kappa_{\psi_1}(r_1) < \kappa_{\psi_2}(r_2)$, as desired. Here, $(*)$ and $(\dagger)$ are justified exactly as in (31). To see $(\ddagger)$, start with the fact that $(\psi_2, \phi'_2) \prec (\psi_2, \phi_2)$, and use (R3)[i] to get some $\phi'_0 \in \Phi$ such that $(\psi_2, \phi'_2) \preceq (\psi_0, \phi'_0)$ while $(\psi_2, \phi_2) \not\preceq (\psi_0, \phi'_0)$. Thus, $B^{U,(\preceq)}_{\psi_2,\psi_0}(r'_2) \leq U(\psi_0, \phi'_0)$ by definition (28) [because $r'_2 = U(\psi_2, \phi'_2)$ and $(\psi_2, \phi'_2) \preceq (\psi_0, \phi'_0)$], while $U(\psi_0, \phi'_0) < B^{U,(\preceq)}_{\psi_2,\psi_0}(r_2)$ by applying the contrapositive of part (e) [because $r_2 = U(\psi_2, \phi_2)$ and $(\psi_2, \phi_2) \not\preceq (\psi_0, \phi'_0)$].

This proves (g). The proof of (h) is similar (using (R3)[ii]). $\square$

Remark. To prove Lemma 4.3(e) [and hence, Lemma 4.3(f)], all we really need is some topology on Φ which has the properties listed in Claim 1. Axiom (R2) simply ensures that the $(\underset{\psi}{\preceq})$-order topology on Φ has these properties. Axiom (R3) is only used to obtain the strict inequality $(\ddagger)$ while proving Lemma 4.3(g) [Lemma 4.3(c) by itself is not sufficient].

Proof of Theorem 4.2. Let $(\preceq)$ be a wipo on $\Psi \times \Phi$. We must verify the equivalence (8). Let $(\psi_1, \phi_1), (\psi_2, \phi_2) \in \Psi \times \Phi$. Statement (8) "$\Longrightarrow$" is true by definition (7).

To prove statement (8) "$\Longleftarrow$", Let $U : \Psi \times \Phi \longrightarrow \mathbb{R}$ be any universal utility function (which exists by (R1)). Define $h(\psi, \phi) := B^{U,(\preceq)}_{\psi,\psi_2}[U(\psi, \phi)]$ for all $(\psi, \phi) \in \Psi \times \Phi$, then h is a hedometer by Lemma 4.3(b,g). (This also implies that $\mathcal{H}_{\mathcal{ED}}(\preceq) \neq \emptyset$, as claimed).

If $h(\psi_1, \phi_1) \leq h(\psi_2, \phi_2)$, then $B^{U,(\preceq)}_{\psi_1\psi_2}[U(\psi_1, \phi_1)] \leq B^{U,(\preceq)}_{\psi_2,\psi_2}[U(\psi_2, \phi_2)] = U(\psi_2, \phi_2)$. Thus Lemma 4.3(e) says that $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$. $\square$

Proofs from §5.

Proof of Proposition 5.2. "$\Longleftarrow$" $(\underset{h}{\trianglelefteq})$ satisfies (Anon$^{\trianglelefteq}$) because $(\underline{\blacktriangleleft})$ satisfies (Anon$^{\underline{\blacktriangleleft}}$). If $h \in \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$, then $(\underset{h}{\trianglelefteq})$ satisfies (Par$^{\trianglelefteq}$) because $(\underline{\blacktriangleleft})$ satisfies (Par$^{\underline{\blacktriangleleft}}$).

"$\Longrightarrow$" (by contradiction) If $(\underline{\blacktriangleleft})$ is not a SWO, then $(\underline{\blacktriangleleft})$ must violate axiom (Anon$^{\underline{\blacktriangleleft}}$); it follows that $(\underset{h}{\trianglelefteq})$ violates (Anon$^{\trianglelefteq}$).

Suppose $h \notin \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$. Then there exist $(\psi, \phi), (\psi', \phi') \in \Psi \times \Phi$ such that $(\psi, \phi) \preceq (\psi', \phi')$, but $h(\psi, \phi) > h(\psi', \phi')$. Let $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ and $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ be the worlds such that $(\psi_i^1, \phi_i^1) = (\psi, \phi)$ for all $i \in \mathcal{I}$, while $(\psi_i^2, \phi_i^2) = (\psi', \phi')$ for all $i \in \mathcal{I}$. Then $h(\psi_i^1, \phi_i^1) > h(\psi_i^2, \phi_i^2)$ for all $i \in \mathcal{I}$, so $\mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \blacktriangleright \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ by (Par$^{\underline{\blacktriangleleft}}$); hence $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{h}{\triangleright} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. But $(\psi_i^1, \phi_i^1) \preceq (\psi_i^2, \phi_i^2)$ for all $i \in \mathcal{I}$, so (Par$^{\trianglelefteq}$) requires that $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{h}{\trianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Contradiction. □

Proof of Proposition 5.4. "$\Longrightarrow$" Let $f : \mathcal{I} \longrightarrow \mathcal{I}$ be such that $(\psi_{f(i)}^1, \phi_{f(i)}^1) \preceq (\psi_i^2, \phi_i^2)$ for all $i \in \mathcal{I}$. Let $h \in \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$. Find $i^* \in \mathcal{I}$ such that $h(\psi_{i^*}^2, \phi_{i^*}^2) = \min_{i \in \mathcal{I}} h(\psi_i^2, \phi_i^2)$. If $j^* = f(i^*)$, then $(\psi_{j^*}^1, \phi_{j^*}^1) \preceq (\psi_{i^*}^2, \phi_{i^*}^2)$, so $h(\psi_{j^*}^1, \phi_{j^*}^1) \leq h(\psi_{i^*}^2, \phi_{i^*}^2)$; hence

$$\min_{j \in \mathcal{I}} h(\psi_j^1, \phi_j^1) \quad \leq \quad h(\psi_{j^*}^1, \phi_{j^*}^1) \quad \leq \quad h(\psi_{i^*}^2, \phi_{i^*}^2) \quad = \quad \min_{i \in \mathcal{I}} h(\psi_i^2, \phi_i^2),$$

as desired. This works for all $h \in \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$.

"$\Longleftarrow$" (by contrapositive) Suppose $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underset{\text{æ}}{\ntrianglelefteq} (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Then there is some $i^* \in \mathcal{I}$ such that, for every $j \in \mathcal{I}$, $(\psi_j^1, \phi_j^1) \npreceq (\psi_{i^*}^2, \phi_{i^*}^2)$. Let $U : \Psi \times \Phi \longrightarrow \mathbb{R}$ be a universal utility function (which exists by axiom (R1)). For all $(\psi, \phi) \in \Psi \times \Phi$, define $h(\psi, \phi) := B_{\psi, \psi_{i^*}^2}^{U,(\preceq)}[U(\psi, \phi)]$; then $h \in \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$ by Lemma 4.3(b,g). Then for all $j \in \mathcal{I}$,

$$\begin{aligned} h(\psi_j^1, \phi_j^1) \quad &:= \quad B_{\psi_j^1 \psi_{i^*}^2}^{U,(\preceq)}\left[U(\psi_j^1, \phi_j^1)\right] \quad \underset{(*)}{>} \quad U(\psi_{i^*}^2, \phi_{i^*}^2) \\ &\underset{(\dagger)}{=} \quad B_{\psi_{i^*}^2 \psi_{i^*}^2}^{U,(\preceq)}\left[U(\psi_{i^*}^2, \phi_{i^*}^2)\right] \quad =: \quad h(\psi_{i^*}^2, \phi_{i^*}^2). \end{aligned} \tag{32}$$

Here, $(\dagger)$ is because $B_{\psi_{i^*}^2 \psi_{i^*}^2}^{U,(\preceq)}(r) = r$ for all $r \in \mathbb{R}$. To see $(*)$: suppose $B_{\psi_j^1 \psi_{i^*}^2}^{U,(\preceq)}\left[U(\psi_j^1, \phi_j^1)\right] \leq U(\psi_{i^*}^2, \phi_{i^*}^2)$; then Lemma 4.3(e) says that $(\psi_j^1, \psi_j^1) \preceq (\psi_{i^*}^2, \phi_{i^*}^2)$, contradicting the definition of i^*. We now have

$$\min_{j \in \mathcal{I}} h(\psi_j^1, \phi_j^1) \quad \underset{(*)}{>} \quad h(\psi_{i^*}^2, \phi_{i^*}^2) \quad \geq \quad \min_{i \in \mathcal{I}} h(\psi_{i^*}^2, \phi_{i^*}^2),$$

where $(*)$ is by eqn.(32) because $\mathcal{I}$ is finite. Thus, it is not true that $\min_{i \in \mathcal{I}} h(\psi_i^1, \phi_i^1) \leq \min_{i \in \mathcal{I}} h(\psi_i^2, \phi_i^2)$ for all $h \in \mathcal{H}_{\mathcal{E}\mathcal{D}}(\preceq)$. □

Proof of Theorem 5.6 and ancillary results.

For any $f : \mathbb{R}\longrightarrow\mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, define $\mathbf{f}(\mathbf{r}) := \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$, where $r'_i := f(r_i)$ for all $i \in \mathcal{I}$. Recall the axiom of *Ordinal Level Comparability* for a SWO:

(OLC) For any increasing $f : \mathbb{R}\longrightarrow\mathbb{R}$ and $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^{\mathcal{I}}$: $\Big(\mathbf{r}^1 \trianglelefteq \mathbf{r}^2\Big) \iff \Big(\mathbf{f}(\mathbf{r}^1) \trianglelefteq \mathbf{f}(\mathbf{r}^2)\Big)$.

Lemma 5.8 *Let* $(\preceq)$, $(\trianglelefteq)$ *and* $(\unlhd)$ *be as in* Theorem 5.6. *Then*

$$\Big((\unlhd) \textit{ is minimally decisive}\Big) \quad \iff \quad \Big((\trianglelefteq) \textit{ satisfies } \text{(OLC)}\Big).$$

Proof. For any $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^{\mathcal{I}}$, the *rank structure* of the pair $(\mathbf{r}^1, \mathbf{r}^2)$ is the complete order $(\underline{\lessdot})$ on $\{1,2\} \times \mathcal{I}$ defined as follows: for all $n, m \in \{1,2\}$ and $i, j \in \mathcal{I}$, $(n,i)\underline{\lessdot}(m,j)$ if and only if $r_i^n \leq r_j^m$.

Claim 1: *Let* $\mathbf{r}^1, \mathbf{r}^2, \mathbf{s}^1, \mathbf{s}^2 \in \mathbb{R}^{\mathcal{I}}$. *If* $(\mathbf{r}^1, \mathbf{r}^2)$ *has the same rank structure as* $(\mathbf{s}^1, \mathbf{s}^2)$, *then there exists some increasing function* $f : \mathbb{R}\longrightarrow\mathbb{R}$ *with* $\mathbf{s}^1 = \mathbf{f}(\mathbf{r}^1)$ *and* $\mathbf{s}^2 = \mathbf{f}(\mathbf{r}^2)$.

Proof. Let $\mathcal{R} := \{r_i^1\}_{i\in\mathcal{I}} \cup \{r_i^2\}_{i\in\mathcal{I}}$ and $\mathcal{S} := \{s_i^1\}_{i\in\mathcal{I}} \cup \{s_i^2\}_{i\in\mathcal{I}}$. Define $f : \mathcal{R}\longrightarrow\mathcal{S}$ by $f(r_i^k) := s_i^k$. If $(\mathbf{r}^1, \mathbf{r}^2)$ has the same rank structure as $(\mathbf{s}^1, \mathbf{s}^2)$, then f is well-defined and order-preserving. Thus, we can extend f to an increasing function $f : \mathbb{R}\longrightarrow\mathbb{R}$, with $\mathbf{s}^1 = \mathbf{f}(\mathbf{r}^1)$ and $\mathbf{s}^2 = \mathbf{f}(\mathbf{r}^2)$. $\diamond$ `Claim 1`

Claim 2: *Let* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ *be fully* $(\preceq)$*-comparable. If* $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$, $\mathbf{r}^1 := \mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$, *and* $\mathbf{r}^2 := \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$, *then the rank structure of* $(\mathbf{r}^1, \mathbf{r}^2)$ *is the same as the rank structure of* $((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2))$.

Proof. This follows immediately from the two formulae (7) defining 'hedometer'. $\diamond$ `Claim 2`

"$\Longrightarrow$" (by contrapositive) Suppose $(\trianglelefteq)$ violates (OLC). Then there exists some $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^{\mathcal{I}}$ and increasing $g : \mathbb{R}\longrightarrow\mathbb{R}$ such that $\mathbf{r}^1 \trianglelefteq \mathbf{r}^2$ but $\mathbf{g}(\mathbf{r}^1) \blacktriangleright \mathbf{g}(\mathbf{r}^2)$.

Claim 3: *There exist some fully* $(\preceq)$*-comparable* $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ *and* $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$ *such that* $\mathbf{h}(\psi_1, \phi_1) = \mathbf{r}^1$ *and* $\mathbf{h}(\psi_2, \phi_2) = \mathbf{r}^2$.

Proof. Axiom (MR) says that we can find some fully $(\preceq)$-comparable $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ such that the rank structure of $((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2))$ is the same as the rank structure of $(\mathbf{r}^1, \mathbf{r}^2)$. Let $h' \in \mathcal{H}_{\mathcal{ED}}(\preceq)$ be any hedometer, let $\mathbf{s}^1 := \mathbf{h}'(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$, and $\mathbf{s}^2 := \mathbf{h}'(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Then Claim 2 says the rank structure of $(\mathbf{s}^1, \mathbf{s}^2)$ is the same as that of of $((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2))$, and thus, the same as that of $(\mathbf{r}^1, \mathbf{r}^2)$. Thus, Claim 1 says there is an increasing function $f : \mathbb{R}\longrightarrow\mathbb{R}$, with $\mathbf{r}^1 = \mathbf{f}(\mathbf{s}^1)$ and $\mathbf{r}^2 = \mathbf{f}(\mathbf{s}^2)$. Let $h := f \circ h'$; then $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$ by Lemma 4.3(a), and $\mathbf{r}^1 := \mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$, and $\mathbf{r}^2 := \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$, as desired. $\diamond$ `Claim 3`

Now, let $h'' := g \circ h$; then $h'' \in \mathcal{H}_{\mathcal{ED}}(\preceq)$ by Lemma 4.3(a), $\mathbf{h}''(\psi_1, \phi_1) = \mathbf{g}(\mathbf{r}^1)$ and $\mathbf{h}''(\psi_2, \phi_2) = \mathbf{g}(\mathbf{r}^2)$. But $\mathbf{r}^1 \trianglelefteq \mathbf{r}^2$, while $\mathbf{g}(\mathbf{r}^1) \blacktriangleright \mathbf{g}(\mathbf{r}^2)$. Checking definition (9), we see that neither $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \trianglelefteq (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$ nor $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \trianglelefteq (\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$. Thus, $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ is not $(\trianglelefteq)$-comparable to $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$; hence $(\trianglelefteq)$ is not minimally decisive.

"$\Longleftarrow$" Suppose $(\underline{\blacktriangleleft})$ satisfies (OLC).

Claim 4: *Let* $\mathbf{r}^1, \mathbf{r}^2, \mathbf{s}^1, \mathbf{s}^2 \in \mathbb{R}^{\mathcal{I}}$. *If* $(\mathbf{r}^1, \mathbf{r}^2)$ *has the same rank structure as* $(\mathbf{s}^1, \mathbf{s}^2)$, *and* $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, *then* $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$.

Proof. Claim 1 says there is an increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $\mathbf{s}^1 = \mathbf{f}(\mathbf{r}^1)$ and $\mathbf{s}^2 = \mathbf{f}(\mathbf{r}^2)$. Thus, if $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, then $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$, because $(\underline{\blacktriangleleft})$ satisfies (OLC). $\diamondsuit$ `Claim 4`

Let $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ be fully $(\preceq)$-comparable and let $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$. Let $\mathbf{r}^1 := \mathbf{h}(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ and $\mathbf{r}^2 := \mathbf{h}(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Since $(\underline{\blacktriangleleft})$ is a complete ordering of $\mathbb{R}^{\mathcal{I}}$, we have either $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$ or $\mathbf{r}^2 \underline{\blacktriangleleft} \mathbf{r}^1$. Without loss of generality, assume $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$.

Claim 5: *For all* $h' \in \mathcal{H}_{\mathcal{ED}}(\preceq)$, *we have* $\mathbf{h}'(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \underline{\blacktriangleleft} \mathbf{h}'(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$.

Proof. Let $\mathbf{s}^1 := \mathbf{h}'(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$, and $\mathbf{s}^2 := \mathbf{h}'(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Claim 2 says the rank structure of $(\mathbf{s}^1, \mathbf{s}^2)$ is the same as that of $((\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2))$, which is in turn the same as that of $(\mathbf{r}^1, \mathbf{r}^2)$. Thus, if $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, then Claim 4 implies that $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$. $\diamondsuit$ `Claim 5`

Combining Claim 5 with defining formula (9), we see that $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1) \trianglelefteq (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. Thus, $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1)$ is $(\trianglelefteq)$-comparable to $(\boldsymbol{\psi}^2, \boldsymbol{\phi}^2)$. This argument works for any $(\boldsymbol{\psi}^1, \boldsymbol{\phi}^1), (\boldsymbol{\psi}^2, \boldsymbol{\phi}^2) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ which are fully $(\preceq)$-comparable. Thus, $(\trianglelefteq)$ is minimally decisive. $\square$

Consider the following version of the 'minimal equity' property for a SWO $(\underline{\blacktriangleleft})$.

(MinEq$^{\underline{\blacktriangleleft}}$) There exist $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$ and $i, j \in \mathcal{I}$ such that:

- **(q1$^{\underline{\blacktriangleleft}}$)** $r_i < r'_i \leq r'_j < r_j$;
- **(q2$^{\underline{\blacktriangleleft}}$)** $r_i \leq r_k = r'_k$ for all $k \in \mathcal{I} \setminus \{i, j\}$; and
- **(q3$^{\underline{\blacktriangleleft}}$)** $\mathbf{r} \underline{\blacktriangleleft} \mathbf{r}'$.

Lemma 5.9 *Let* $(\preceq)$, $(\underline{\blacktriangleleft})$ *and* $(\trianglelefteq)$ *be as in* Theorem 5.6. *If* $(\trianglelefteq)$ *is satisfies* (MinEq$^{\trianglelefteq}$), *then* $(\underline{\blacktriangleleft})$ *satisfies* (MinEq$^{\underline{\blacktriangleleft}}$).

Proof. Find $(\boldsymbol{\psi}, \boldsymbol{\phi}), (\boldsymbol{\psi}', \boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ satisfying conditions (q1$^{\trianglelefteq}$)-(q3$^{\trianglelefteq}$) in axiom (MinEq$^{\trianglelefteq}$). Let $h \in \mathcal{H}_{\mathcal{ED}}(\preceq)$, let $\mathbf{r} := \mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi})$, and let $\mathbf{r}' := \mathbf{h}(\boldsymbol{\psi}', \boldsymbol{\phi}')$. We claim that $\mathbf{r}$ and $\mathbf{r}'$ satisfy conditions (q1$^{\underline{\blacktriangleleft}}$)-(q3$^{\underline{\blacktriangleleft}}$).

(q1$^{\underline{\blacktriangleleft}}$): We have $(\psi_i, \phi_i) \prec (\psi'_i, \phi'_i) \preceq (\psi'_j, \phi'_j) \prec (\psi_j, \phi_j)$, by (q1$^{\trianglelefteq}$); thus, the two formulae (7) (defining 'hedometer') imply that $r_i < r'_i \leq r'_j < r_j$.

(q2$^{\underline{\blacktriangleleft}}$): For all $k \in \mathcal{I} \setminus \{i, j\}$, we have $(\psi_i, \phi_i) \preceq (\psi_k, \phi_k) \approx (\psi'_k, \phi'_k)$ by (q2$^{\trianglelefteq}$); thus, the two formulae (7) imply that $r_i \leq r_k = r'_k$.

(q3$^{\underline{\blacktriangleleft}}$): We have $(\boldsymbol{\psi}, \boldsymbol{\phi}) \trianglelefteq (\boldsymbol{\psi}', \boldsymbol{\phi}')$ by (q3$^{\trianglelefteq}$), so formula (9) requires that $\mathbf{r} \underline{\blacktriangleleft} \mathbf{r}'$. $\square$

Lemma 5.10 *Let* $(\underset{1}{\underline{\blacktriangleleft}})$ *and* $(\underset{2}{\underline{\blacktriangleleft}})$ *be two SWOs on* $\mathbb{R}^{\mathcal{I}}$*, and suppose* $(\underset{1}{\underline{\blacktriangleleft}})$ *refines* $(\underset{2}{\underline{\blacktriangleleft}})$*. If* $(\underset{1}{\underline{\blacktriangleleft}})$ *satisfies* (MinEq$^{\underline{\blacktriangleleft}}$)*, then* $(\underset{2}{\underline{\blacktriangleleft}})$ *also satisfies* (MinEq$^{\underline{\blacktriangleleft}}$)*.*

Proof. Suppose $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$ satisfy the conditions of (MinEq$^{\underline{\blacktriangleleft}}$) for $(\underset{1}{\underline{\blacktriangleleft}})$. Then $\mathbf{r} \underset{1}{\underline{\blacktriangleleft}} \mathbf{r}'$. If $(\underset{1}{\underline{\blacktriangleleft}})$ refines $(\underset{2}{\underline{\blacktriangleleft}})$, then Lemma 1.1(d) says $(\underset{2}{\underline{\blacktriangleleft}})$ extends $(\underset{1}{\underline{\blacktriangleleft}})$. Thus, $\mathbf{r} \underset{2}{\underline{\blacktriangleleft}} \mathbf{r}'$. Thus $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$ also satisfy the conditions of (MinEq$^{\underline{\blacktriangleleft}}$) for $(\underset{2}{\underline{\blacktriangleleft}})$. □

Lemma 5.11 *Let* $(\preceq)$ *be a wipo and let* $(\underset{1}{\underline{\blacktriangleleft}})$ *and* $(\underset{2}{\underline{\blacktriangleleft}})$ *be two SWOs on* $\mathbb{R}^{\mathcal{I}}$*. Let* $(\underset{k}{\trianglelefteq})$ *be the* $(\preceq, \underset{k}{\underline{\blacktriangleleft}})$*-welfarist sprow for* $k = 1, 2$*. If* $(\underset{2}{\underline{\blacktriangleleft}})$ *extends* $(\underset{1}{\underline{\blacktriangleleft}})$*, then* $(\underset{2}{\trianglelefteq})$ *extends* $(\underset{1}{\trianglelefteq})$*.*

Proof. Let $(\boldsymbol{\psi}, \boldsymbol{\phi}), (\boldsymbol{\psi}', \boldsymbol{\phi}') \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. Then

$$\begin{aligned}
\Big((\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{1}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}')\Big) &\iff \Big(\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{1}{\underline{\blacktriangleleft}} \mathbf{h}(\boldsymbol{\psi}', \boldsymbol{\phi}') \text{ for all } h \in \mathcal{H}_{\mathcal{SD}}(\preceq)\Big) \\
&\underset{(*)}{\Longrightarrow} \Big(\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{2}{\underline{\blacktriangleleft}} \mathbf{h}(\boldsymbol{\psi}', \boldsymbol{\phi}') \text{ for all } h \in \mathcal{H}_{\mathcal{SD}}(\preceq)\Big) \\
&\iff \Big((\boldsymbol{\psi}, \boldsymbol{\phi}) \underset{2}{\trianglelefteq} (\boldsymbol{\psi}', \boldsymbol{\phi}')\Big),
\end{aligned}$$

where $(*)$ is because $(\underset{2}{\underline{\blacktriangleleft}})$ extends $(\underset{1}{\underline{\blacktriangleleft}})$. □

(Note that the proof of Lemma 5.11 breaks down if we replace 'extends' with 'refines'.)

Proof of Theorem 5.6. (a) Lemma 5.8 says that $(\trianglelefteq)$ is minimally decisive if and only if $(\underline{\blacktriangleleft})$ satisfies (OLC). However, a well-known result of Hammond (1976) says that $(\underline{\blacktriangleleft})$ satisfies (OLC) if and only if $(\underline{\blacktriangleleft})$ refines the rank-k dictatorship SWO $(\underset{k}{\underline{\blacktriangleleft}})$ for some $k \in [1...I]$ (Moulin, 1988, Thm 2.4, page 40).

(b) From (a) we know that $(\underline{\blacktriangleleft})$ refines some rank-k dictatorship $(\underset{k}{\underline{\blacktriangleleft}})$. If $(\trianglelefteq)$ satisfies (MinEq$^{\trianglelefteq}$), then Lemma 5.9 says that $(\underline{\blacktriangleleft})$ satisfies (MinEq$^{\underline{\blacktriangleleft}}$). Thus, Lemma 5.10 says that $(\underset{k}{\underline{\blacktriangleleft}})$ also satisfies (MinEq$^{\underline{\blacktriangleleft}}$). But the only rank-$k$ dictatorship which satisfies (MinEq$^{\underline{\blacktriangleleft}}$) is the egalitarian SWO $(\underset{e}{\underline{\blacktriangleleft}})$. Thus, $(\underset{k}{\underline{\blacktriangleleft}})$ is $(\underset{e}{\underline{\blacktriangleleft}})$, so $(\underline{\blacktriangleleft})$ refines $(\underset{e}{\underline{\blacktriangleleft}})$. Then Lemma 1.1(d) says $(\underset{e}{\underline{\blacktriangleleft}})$ extends $(\underline{\blacktriangleleft})$. Then Lemma 5.11 says $(\underset{\text{æ}}{\trianglelefteq})$ extends $(\trianglelefteq)$.

(c,d) Suppose $(\trianglelefteq)$ either extends or refines $(\underset{\text{æ}}{\trianglelefteq})$; we will show that $(\trianglelefteq)$ is minimally decisive and satisfies (MinEq).

Minimally Decisive. $(\underset{\text{æ}}{\trianglelefteq})$ is minimally decisive by Example 5.5. Thus, if $(\trianglelefteq)$ extends or refines $(\underset{\text{æ}}{\trianglelefteq})$, then $(\trianglelefteq)$ is also minimally decisive (because $(\trianglelefteq)$ can compare any pair of worlds which $(\underset{\text{æ}}{\trianglelefteq})$ can compare).

Minimal Equity. Using axiom (MR), find fully $(\preceq)$-comparable worlds $(\boldsymbol{\psi},\boldsymbol{\phi}),(\boldsymbol{\psi}',\boldsymbol{\phi}')\in\Psi^{\mathcal{I}}\times\Phi^{\mathcal{I}}$ with individuals $i,j\in\mathcal{I}$ such that $(\psi_i,\phi_i)\prec(\psi'_i,\phi'_i)\preceq(\psi'_j,\phi'_j)\prec(\psi_j,\phi_j)$ and also $(\psi_i,\phi_i)\prec(\psi_k,\phi_k)\approx(\psi'_k,\phi'_k)$ for all $k\in\mathcal{I}\setminus\{i,j\}$. Thus, $(\boldsymbol{\psi},\boldsymbol{\phi}),(\boldsymbol{\psi}',\boldsymbol{\phi}')$ satisfy conditions (q1$^{\trianglelefteq}$) and (q2$^{\trianglelefteq}$) in the definition of (MinEq$^{\trianglelefteq}$). Also, $(\boldsymbol{\psi},\boldsymbol{\phi})\underset{æ}{\trianglelefteq}(\boldsymbol{\psi}',\boldsymbol{\phi}')$ (define $f:\mathcal{I}\longrightarrow\mathcal{I}$ by $f(k)=i$ for all $k\in\mathcal{I}$). However, $(\boldsymbol{\psi},\boldsymbol{\phi})\underset{æ}{\not\trianglerighteq}(\boldsymbol{\psi}',\boldsymbol{\phi}')$(because $(\psi_i,\phi_i)\prec(\psi'_k,\phi'_k)$ for all $k\in\mathcal{I}$). Thus, $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)\underset{æ}{\triangleleft}(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$. Thus, if $(\trianglelefteq)$ either extends or refines $(\underset{æ}{\trianglelefteq})$, then $(\boldsymbol{\psi}^1,\boldsymbol{\phi}^1)\trianglelefteq(\boldsymbol{\psi}^2,\boldsymbol{\phi}^2)$. Thus, $(\boldsymbol{\psi},\boldsymbol{\phi}),(\boldsymbol{\psi}',\boldsymbol{\phi}')$ also satisfy condition (q3$^{\trianglelefteq}$); hence $(\trianglelefteq)$ satisfies (MinEq$^{\trianglelefteq}$).

So, if $(\trianglelefteq)$ refines $(\underset{æ}{\trianglelefteq})$, then $(\trianglelefteq)$ is minimally decisive and satisfies (MinEq); this proves (c). On the other hand, if $(\trianglelefteq)$ *extends* $(\underset{æ}{\trianglelefteq})$, then part (b) implies that $(\underset{æ}{\trianglelefteq})$ also extends $(\trianglelefteq)$, which means they must be equal. This proves (d). □

Proofs from §6.

Proof of Claim 6.2.* For any $\mathbf{w}\in\mathcal{W}$, we have

$$\sum_{i\in\mathcal{I}}w_i\cdot h_0(\psi_i,\rho'_i)-\sum_{i\in\mathcal{I}}w_i\cdot h_0(\psi_i,\rho_i)\quad=\quad\Big(w_1r'_1+w_2r'_2\Big)-\Big(w_1r_1+w_2r_2\Big)$$
$$=\quad w_1\cdot(r'_1-r_1)+w_2\cdot(r'_2-r_2)\quad=\quad w_2\cdot\left(\left(\frac{w_1}{w_2}\right)\cdot(r'_1-r_1)+(r'_2-r_2)\right).$$

Thus, statement (19) becomes

$$\Big(\boldsymbol{\rho}\underset{u,\psi}{\trianglelefteq}\boldsymbol{\rho}'\Big)\quad\iff\quad\left(\left(\tfrac{w_1}{w_2}\right)\cdot(r'_1-r_1)+(r'_2-r_2)\geq 0,\text{ for all }\mathbf{w}\in\mathcal{W}\right)$$
$$\iff\quad\begin{pmatrix}\text{either (A)}\quad(r'_1-r_1)\geq 0\text{ and }(r'_2-r_2)\geq 0;\text{ or}\\ \text{(B)}\quad(r'_1-r_1)\geq 0\geq(r'_2-r_2)\text{ and }\underline{A}\cdot(r'_1-r_1)\geq(r_2-r'_2);\text{ or}\\ \text{(C)}\quad(r'_2-r_2)\geq 0\geq(r'_1-r_1)\text{ and }(r'_2-r_2)\geq\overline{A}\cdot(r_1-r'_1).\end{pmatrix}\qquad(33)$$

If $S:=\frac{r'_2-r_2}{r'_1-r_1}$, then condition (B) in statement (33) is equivalent to $r'_1\geq r_1,\ r'_2\leq r_2$ and $S\geq-\underline{A}$. Meanwhile, condition (C) is equivalent to $r'_1\leq r_1,\ r'_2\geq r_2$, and $S\leq-\overline{A}$.

Thus, the right side of statement (33) is equivalent to the right side of statement (20). □

Proof of Theorem 6.5. Let $(\trianglelefteq)$ be a $(\preceq)$-sprowl on $\mathfrak{P}^{\otimes\mathcal{I}}$. Let $\boldsymbol{\rho},\boldsymbol{\rho}'\in\mathfrak{P}^{\otimes\mathcal{I}}$ be two world-lotteries. We must show that $(\boldsymbol{\rho}\underset{u}{\trianglelefteq}\boldsymbol{\rho}')\Longrightarrow(\boldsymbol{\rho}\trianglelefteq\boldsymbol{\rho}')$, and $(\boldsymbol{\rho}\underset{u}{\triangleleft}\boldsymbol{\rho}')\Longrightarrow(\boldsymbol{\rho}\triangleleft\boldsymbol{\rho}')$.

Without loss of generality, suppose $\mathcal{I}=[1....I]$, and define the permutation $\sigma:\mathcal{I}\longrightarrow\mathcal{I}$ by $\sigma(i):=(i+1)\bmod I$. Define $\widehat{\boldsymbol{\rho}}:=\dfrac{1}{I}\displaystyle\sum_{n=0}^{I-1}\sigma^n(\boldsymbol{\rho})$ and $\widehat{\boldsymbol{\rho}'}:=\dfrac{1}{I}\displaystyle\sum_{n=0}^{I-1}\sigma^n(\boldsymbol{\rho}')$. Then

$$\boldsymbol{\rho}\quad=\quad\frac{1}{I}\sum_{n=0}^{I-1}\boldsymbol{\rho}\quad\overset{\vartriangle}{\approx}\quad\frac{1}{I}\sum_{n=0}^{I-1}\sigma^n(\boldsymbol{\rho})\quad=\quad\widehat{\boldsymbol{\rho}}.\qquad(34)$$

Here "$\overset{\triangle}{\approx}$" is by I-fold application of axiom (Lin$^{\trianglelefteq}$), because $\boldsymbol{\rho} \overset{\triangle}{\approx} \sigma^n(\boldsymbol{\rho})$ for all $n \in \mathbb{N}$, by axiom (Anon$^{\trianglelefteq}$). By a similar argument, $\boldsymbol{\rho}' \overset{\triangle}{\approx} \widehat{\boldsymbol{\rho}}'$. Meanwhile, for all $i \in \mathcal{I}$, we have

$$\hat{\rho}_i \quad = \quad \overline{\rho} \qquad \text{and} \qquad \hat{\rho}'_i \quad = \quad \overline{\rho}', \tag{35}$$

where $\overline{\rho}$ and $\overline{\rho}'$ are the per capita average lotteries of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$, as defined in eqn.(14). Thus,

$$\begin{aligned}
\left(\boldsymbol{\rho} \underset{u}{\trianglelefteq} \boldsymbol{\rho}'\right) \;&\underset{(*)}{\Longleftrightarrow}\; \left(\overline{\rho} \preceq \overline{\rho}'\right) \;\underset{(\dagger)}{\Longleftrightarrow}\; \left(\hat{\rho}_i \preceq \hat{\rho}'_i \text{ for all } i \in \mathcal{I}\right) \\
&\underset{(\ddagger)}{\Longrightarrow}\; \left(\widehat{\boldsymbol{\rho}} \trianglelefteq \widehat{\boldsymbol{\rho}'}\right) \underset{(\diamond)}{\Longleftrightarrow} \left(\boldsymbol{\rho} \trianglelefteq \boldsymbol{\rho}'\right). \\
\text{Likewise,} \quad \left(\boldsymbol{\rho} \underset{u}{\triangleleft} \boldsymbol{\rho}'\right) \;&\underset{(*)}{\Longleftrightarrow}\; \left(\overline{\rho} \prec \overline{\rho}'\right) \;\underset{(\dagger)}{\Longleftrightarrow}\; \left(\hat{\rho}_i \prec \hat{\rho}'_i \text{ for all } i \in \mathcal{I}\right) \\
&\underset{(\ddagger)}{\Longrightarrow}\; \left(\widehat{\boldsymbol{\rho}} \triangleleft \widehat{\boldsymbol{\rho}'}\right) \underset{(\diamond)}{\Longleftrightarrow} \left(\boldsymbol{\rho} \triangleleft \boldsymbol{\rho}'\right).
\end{aligned}$$

Here, $(*)$ is by defining formula (15), $(\dagger)$ is by eqn.(35), $(\ddagger)$ is by axiom (Par$^{\trianglelefteq}$), and $(\diamond)$ is by eqn.(34) and the transitivity of $(\trianglelefteq)$. □

The proof of Theorem 6.7 depends on Theorem 6.9, which in turn requires two lemmas.

Lemma 6.10 *Let $(\preceq)$ be a preorder on $\mathfrak{P}$ satisfying axioms* (Lin$^{\preceq}$) *and* (Cont$^{\preceq}$). *For all $\rho, \rho'_1, \rho'_2 \in \mathfrak{P}$, and all $s, s' \in (0,1)$ such that $s + s' = 1$,*

$$\Big((s\rho + s'\rho'_1) \succeq (s\rho + s'\rho'_2)\Big) \quad \Longrightarrow \quad \Big(\rho'_1 \succeq \rho'_2\Big).$$

Proof. The proof is exactly the same as (Shapley and Baucells, 1998, Lemma 1.2) or (Dubra et al., 2004, Lemma 1, §4). (The proof does not need Dubra et al.'s standing hypothesis that $\mathfrak{P}$ is the space of Borel probability measures on a compact metric space.) □

Now, let $\mathfrak{V} := \{\nu \in \mathfrak{M} \,;\, \nu[\mathcal{X}] = 0\}$. For any $\rho \in \mathfrak{P}$, define:

$$\mathcal{P}(\rho) \quad := \quad \{\nu \in \mathfrak{V} \,;\, \nu + \rho \in \mathfrak{P}\} \quad \text{and} \quad \mathcal{B}(\rho) \quad := \quad \{\nu \in \mathcal{P}(\rho) \,;\, \rho \preceq \nu + \rho\}.$$

Lemma 6.11 *Let $\mathfrak{P} \subseteq \mathfrak{M}$ and $(\preceq)$ be as in Theorem 6.9. There exists a closed, convex cone $\overline{\mathcal{B}} \subset \mathfrak{V}$ such that, for all $\rho \in \mathfrak{P}$, we have $\mathcal{B}(\rho) \;=\; \mathcal{P}(\rho) \cap \overline{\mathcal{B}}$.*

Proof. For any $\rho \in \mathfrak{P}$, let $\widetilde{\mathcal{B}}(\rho) := \{r\,\beta \,;\, r \geq 0,\ \beta \in \mathcal{B}(\rho)\}$; thus $\widetilde{\mathcal{B}}(\rho)$ is the cone in $\mathfrak{M}$ generated by $\mathcal{B}(\rho)$.

Claim 1: *For all $\rho \in \mathfrak{P}$, the set $\mathcal{B}(\rho)$ is closed and convex.*

Proof. (*Convex*) Let $\beta, \beta' \in \mathcal{B}(\rho)$ and let $s, s' \in [0,1]$ with $s + s' = 1$. We must show that $s\beta + s'\beta' \in \mathcal{B}(\rho)$. But $\rho = s\rho + s'\rho$. Thus,

$$\rho+(s\beta+s'\beta') \quad = \quad (s\rho+s'\rho)+(s\beta+s'\beta') \quad = \quad s(\rho+\beta)+s'(\rho+\beta') \quad \underset{(*)}{\succeq} \quad s\rho+s'\rho \quad = \quad \rho.$$

Here, $(*)$ is by axiom (Lin$^{\preceq}$), because $\rho + \beta \succeq \rho$ and $\rho + \beta' \succeq \rho$ by hypothesis.

(*Closed*) Let $\{\beta_n\}_{n=1}^{\infty} \subset \mathcal{B}(\rho)$. Suppose $\beta_n \xrightarrow[n\to\infty]{} \beta$; we must show that $\beta \in \mathcal{B}(\rho)$. Now, $(\rho + \beta_n) \xrightarrow[n\to\infty]{} \rho + \beta$, and $\rho + \beta_n \in \mathfrak{P}$ for all $n \in \mathbb{N}$; hence $\rho + \beta \in \mathfrak{P}$ because $\mathfrak{P}$ is closed; hence $\beta \in \mathcal{P}(\rho)$. Next, $\rho + \beta_n \succeq \rho$ for all $n \in \mathbb{N}$; hence $\rho + \beta \succeq \rho$ by axiom (Cont$^{\preceq}$); thus, $\beta \in \mathcal{B}(\rho)$, as desired. ◇ `Claim 1`

It follows from Claim 1 that $\widetilde{\mathcal{B}}(\rho)$ is also closed and convex.

Claim 2: *For all $\rho \in \mathfrak{P}$, $\mathcal{B}(\rho) = \mathcal{P}(\rho) \cap \widetilde{\mathcal{B}}(\rho)$.*

Proof. Clearly, $\mathcal{B}(\rho) \subseteq \mathcal{P}(\rho) \cap \widetilde{\mathcal{B}}(\rho)$ because $\mathcal{B}(\rho) \subseteq \mathcal{P}(\rho)$ and $\mathcal{B}(\rho) \subseteq \widetilde{\mathcal{B}}(\rho)$ by definition. To see that $\mathcal{B}(\rho) \supseteq \mathcal{P}(\rho) \cap \widetilde{\mathcal{B}}(\rho)$, let $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho)$; then $\widetilde{\beta} = r\,\beta$ for some $r \geq 0$ and some $\beta \in \mathcal{B}(\rho)$. If $r\beta \in \mathcal{P}(\rho)$, then we must show that $r\beta \in \mathcal{B}(\rho)$. There are two cases: either $r \leq 1$ or $r > 1$.

Case ($r \leq 1$). First note that $\mathbf{0} \in \mathcal{B}(\rho)$ because $\rho + \mathbf{0} = \rho \succeq \rho$. Thus, $r\,\beta = r\,\beta + (1-r)\mathbf{0} \in \mathcal{B}(\rho)$ by Claim 1.

Case ($r > 1$). If $r\,\beta \in \mathcal{P}(\rho)$, then $\rho + r\,\beta \in \mathfrak{P}$. We must show that $\rho + r\,\beta \succeq \rho$. We have

$$\left(1 - \frac{1}{r}\right)\rho \;+\; \frac{1}{r}(\rho + r\beta) \quad = \quad \rho + \beta \quad \succeq \quad \rho \quad = \quad \left(1 - \frac{1}{r}\right)\rho \;+\; \frac{1}{r}\rho.$$

Thus, if we set $\rho_1' := (\rho + r\beta)$, $\rho_2' := \rho$, $s' := \frac{1}{r}$, and $s := 1 - s'$ in Lemma 6.10, we conclude that $\rho + r\beta \succeq \rho$. Thus, $r\,\beta \in \mathcal{B}(\rho)$, as desired. ◇ `Claim 2`

Claim 3: *Let $\rho_1, \rho_2 \in \mathfrak{P}$. Let $\rho = s_1\rho_1 + s_2\rho_2$ for some $s_1, s_2 \in [0,1]$ with $s_1 + s_2 = 1$.*

(a) *If $\beta_1 \in \mathcal{P}(\rho_1)$ and $\beta_2 \in \mathcal{P}(\rho_2)$, then $(s_1\beta_1 + s_2\beta_2) \in \mathcal{P}(\rho)$.*

(b) *Thus, $\mathcal{P}(\rho_1) \cap \mathcal{P}(\rho_2) \subseteq \mathcal{P}(\rho)$.*

(c) *If $\beta_1 \in \mathcal{B}(\rho_1)$ and $\beta_2 \in \mathcal{B}(\rho_2)$, then $(s_1\beta_1 + s_2\beta_2) \in \mathcal{B}(\rho)$.*

(d) *Thus, $\mathcal{B}(\rho_1) \cap \mathcal{B}(\rho_2) \subseteq \mathcal{B}(\rho)$.*

(e) *Finally, $\mathcal{B}(\rho_1) \cap \mathcal{P}(\rho_2) \subseteq \mathcal{B}(\rho)$.*

Proof. (a) $(s_1\beta_1 + s_2\beta_2) + \rho = (s_1\beta_1 + s_2\beta_2) + (s_1\rho_1 + s_2\rho_2) = s_1(\beta_1 + \rho_1) + s_2(\beta_2 + \rho_2) \in \mathfrak{P}$, because $(\beta_1 + \rho_1) \in \mathfrak{P}$ and $(\beta_2 + \rho_2) \in \mathfrak{P}$, because $\beta_1 \in \mathcal{P}(\rho_1)$ and $\beta_2 \in \mathcal{P}(\rho_2)$. Thus, $(s_1\beta_1 + s_2\beta_2) \in \mathcal{P}(\rho)$.

(b) Suppose $\beta \in \mathcal{P}(\rho_1) \cap \mathcal{P}(\rho_2)$. Set $\beta_1 = \beta_2 = \beta$ in (a) to conclude $\beta \in \mathcal{P}(\rho)$.

(c) $(s_1\beta_1 + s_2\beta_2) + \rho = (s_1\beta_1 + s_2\beta_2) + (s_1\rho_1 + s_2\rho_2) = s_1(\beta_1 + \rho_1) + s_2(\beta_2 + \rho_2) \succeq s_1\rho_1 + s_2\rho_2 = \rho$, where "$\succeq$" is by axiom (Lin$^{\preceq}$), because $\beta_1 + \rho_1 \succeq \rho_1$ and $\beta_2 + \rho_2 \succeq \rho_2$, because $\beta_1 \in \mathcal{B}(\rho_1)$ and $\beta_2 \in \mathcal{B}(\rho_2)$. Thus, $(s_1\beta_1 + s_2\beta_2) \in \mathcal{B}(\rho)$.

(d) Suppose $\beta \in \mathcal{B}(\rho_1) \cap \mathcal{B}(\rho_2)$. Set $\beta_1 = \beta_2 = \beta$ in (c) to conclude $\beta \in \mathcal{B}(\rho)$.

(e) Let $\beta \in \mathcal{B}(\rho_1)$. Set $\beta_1 = \beta$ and $\beta_2 = \mathbf{0}$ in (c) to get $s_1\beta \in \mathcal{B}(\rho)$. Thus, $\beta \in \widetilde{\mathcal{B}}(\rho)$. However, if $\beta \in \mathcal{B}(\rho_1)$, then $\beta \in \mathcal{P}(\rho_1)$. Since also $\beta \in \mathcal{P}(\rho_2)$, part (b) implies that $\beta \in \mathcal{P}(\rho)$. Thus, $\beta \in \widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho)$, so $\beta \in \mathcal{B}(\rho)$ by Claim 2. ◇ `Claim 3`

Claim 4: *For all $\rho, \rho' \in \mathfrak{P}$, we have $\widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho') \subseteq \mathcal{B}(\rho')$.*

Proof. Let $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho')$. Then $\widetilde{\beta} = r\beta$ for some $r \geq 0$ and $\beta \in \mathcal{B}(\rho)$. If $r \leq 1$ then $\widetilde{\beta} \in \mathcal{B}(\rho)$ by Claim 1, in which case we redefine $\beta := \widetilde{\beta}$ and $r := 1$. Otherwise, if $r > 1$, then $\beta = (1/r)\widetilde{\beta}$, so $\beta \in \mathcal{P}(\rho')$ because $\widetilde{\beta} \in \mathcal{P}(\rho')$, and $\mathcal{P}(\rho')$ is convex and contains $\mathbf{0}$. Either way, $\beta \in \mathcal{B}(\rho) \cap \mathcal{P}(\rho')$.

For all $n \in \mathbb{N}$, let $\rho_n := \frac{1}{n}\rho + (1 - \frac{1}{n})\rho'$. Then $\rho_n \xrightarrow[n\to\infty]{} \rho'$, and for all $n \in \mathbb{N}$, we have $\beta \in \mathcal{B}(\rho_n)$ by Claim 3(e), because $\beta \in \mathcal{B}(\rho) \cap \mathcal{P}(\rho')$. Thus, $\beta + \rho_n \succeq \rho_n$ for all $n \in \mathbb{N}$. But $(\beta + \rho_n) \xrightarrow[n\to\infty]{} (\beta + \rho')$ because $\rho_n \xrightarrow[n\to\infty]{} \rho'$. Thus, axiom (Cont$^{\preceq}$) implies that $\beta + \rho' \succeq \rho'$; hence $\beta \in \mathcal{B}(\rho')$.

Now, $\widetilde{\beta} = r\,\beta$, so $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho')$. But $\widetilde{\beta} \in \mathcal{P}(\rho')$ also, so Claim 2 says that $\widetilde{\beta} \in \mathcal{B}(\rho')$.

This argument works for all $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho')$; thus $\widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho') \subseteq \mathcal{B}(\rho')$. ◇ `Claim 4`

Claim 5: *Let $\widetilde{\mathcal{B}} := \bigcup_{\rho \in \mathfrak{P}} \widetilde{\mathcal{B}}(\rho)$. Then $\widetilde{\mathcal{B}}$ is a convex cone in $\mathfrak{V}$.*

Proof. Clearly, for any $\beta \in \widetilde{\mathcal{B}}$ and $r \geq 0$, we have $r\beta \in \widetilde{\mathcal{B}}$ as well. It remains to show that $\widetilde{\mathcal{B}}$ is convex. Let $\widetilde{\beta}_1, \widetilde{\beta}_2 \in \widetilde{\mathcal{B}}$ be nonzero, and let $\widetilde{\beta} = s_1\widetilde{\beta}_1 + s_2\widetilde{\beta}_2$ for some $s_1, s_2 \in [0, 1]$ with $s_1 + s_2 = 1$. We must show that $\widetilde{\beta} \in \widetilde{\mathcal{B}}$ also.

Now, for $k = 1, 2$ we have $\widetilde{\beta}_k \in \widetilde{\mathcal{B}}(\rho_k)$ for some $\rho_k \in \mathfrak{P}$. Let $\rho = s_1\rho_1 + s_2\rho_2$; we will show that $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho)$, which will imply that $\widetilde{\beta} \in \widetilde{\mathcal{B}}$, as desired.

For $k = 1, 2$, if $\widetilde{\beta}_k \in \widetilde{\mathcal{B}}(\rho_k)$, then $\widetilde{\beta}_k = r_k\,\beta_k$ for some $\beta_k \in \mathcal{B}(\rho_k)$ and $r_k > 0$. Let $r := \max\{r_1, r_2\}$. Then $r > 0$, and $\frac{1}{r}\,\widetilde{\beta}_k = \frac{r_k}{r}\,\beta_k \in \mathcal{B}(\rho_k)$ for $k = 1, 2$ (because $\mathcal{B}(\rho_k)$ is convex by Claim 1, and it contains β_k and $\mathbf{0}$). Thus, $\frac{1}{r}\,\widetilde{\beta} = \frac{1}{r}\,(s_1\widetilde{\beta}_1 + s_2\widetilde{\beta}_2) = s_1(\frac{1}{r}\,\widetilde{\beta}_1) + s_2(\frac{1}{r}\,\widetilde{\beta}_2)$ is in $\mathcal{B}(\rho)$ by Claim 3(c). Thus, $\widetilde{\beta} \in \widetilde{\mathcal{B}}(\rho)$, as claimed. ◇ `Claim 5`

Let $\overline{\mathcal{B}}$ be the closure of $\widetilde{\mathcal{B}}$. Claim 5 implies that $\overline{\mathcal{B}}$ is a closed, convex cone in $\mathfrak{V}$.

Claim 6: *For all $\rho' \in \mathfrak{P}$, we have $\overline{\mathcal{B}} \cap \mathcal{P}(\rho') = \mathcal{B}(\rho')$.*

Proof. "$\supseteq$" By definition, $\mathcal{B}(\rho') \subseteq \widetilde{\mathcal{B}}(\rho') \subseteq \widetilde{\mathcal{B}} \subseteq \overline{\mathcal{B}}$ and $\mathcal{B}(\rho') \subseteq \mathcal{P}(\rho')$, so $\mathcal{B}(\rho') \subseteq \overline{\mathcal{B}} \cap \mathcal{P}(\rho')$.

"$\subseteq$" We have $\widetilde{\mathcal{B}} \cap \mathcal{P}(\rho') = \bigcup_{\rho \in \mathfrak{P}} \left(\widetilde{\mathcal{B}}(\rho) \cap \mathcal{P}(\rho')\right) \subseteq \mathcal{B}(\rho')$ by Claim 4. Taking the closure of both sides, we get $\overline{\mathcal{B}} \cap \mathcal{P}(\rho') \subseteq \mathcal{B}(\rho')$, because $\mathcal{B}(\rho')$ is closed by Claim 1. ◇ `Claim 6`

$\square$

Proof of Theorem 6.9. For any measurable $f : \mathcal{X}\longrightarrow\mathbb{R}$, define $f^*(\overline{\mathcal{B}}) := \{f^*(\beta)\ ;\ \beta \in \overline{\mathcal{B}}\}$. Then $f^*(\overline{\mathcal{B}})$ is a cone in $\mathbb{R}$ (because $\overline{\mathcal{B}}$ is a cone and f^* is linear); hence either $f^*(\overline{\mathcal{B}}) = [0,\infty)$ or $f^*(\overline{\mathcal{B}}) = (-\infty, 0]$ or $f^*(\overline{\mathcal{B}}) = \mathbb{R}$.

Let $\mathcal{U} := \{u : \mathcal{X}\longrightarrow\mathbb{R}\ ;\ \text{measurable, } u^*(\beta) \geq 0,\ \ \forall\ \beta \in \overline{\mathcal{B}}\}$.

Claim 1: $\overline{\mathcal{B}} = \{\nu \in \mathfrak{V}\ ;\ u^*(\nu) \geq 0,\ \forall\ u \in \mathcal{U}\}$.

Proof. Let $\overline{\mathcal{B}}' := \{\nu \in \mathfrak{V}\ ;\ u^*(\nu) \geq 0,\ \forall\ u \in \mathcal{U}\}$. Then $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}}'$ by definition of $\mathcal{U}$. To see that $\overline{\mathcal{B}} \supseteq \overline{\mathcal{B}}'$, suppose $\nu \notin \overline{\mathcal{B}}$. Now, $\overline{\mathcal{B}}$ is closed and convex, while $\mathfrak{M}$ is convex-separable, so there exists some measurable function $f : \mathcal{X}\longrightarrow\mathbb{R}$ such that $f^*(\nu) < \inf\big(f^*(\overline{\mathcal{B}})\big)$. It follows that $f^*(\overline{\mathcal{B}}) = [0,\infty)$, and $f^*(\nu) < 0$. Thus, $f \in \mathcal{U}$ and $f^*(\nu) < 0$, so that $\nu \notin \overline{\mathcal{B}}'$. $\diamond$ `Claim 1`

Thus, for any $\rho_1, \rho_2 \in \mathfrak{P}$, if $\nu = \rho_2 - \rho_1 \in \mathfrak{V}$, then $\nu \in \mathcal{P}(\rho_1)$, and we have

$$\Big(\rho_1 \preceq \rho_2\Big) \underset{(*)}{\iff} \Big(\nu \in \mathcal{B}(\rho_1)\Big) \underset{(\dagger)}{\iff} \Big(\nu \in \overline{\mathcal{B}}\Big) \underset{(\ddagger)}{\iff} \Big(u^*(\nu) \geq 0,\ \forall\ u \in \mathcal{U}\Big)$$
$$\underset{(\diamond)}{\iff} \Big(u^*(\rho_2) - u^*(\rho_1) \geq 0,\ \forall\ u \in \mathcal{U}\Big),$$

as desired. Here, $(*)$ is by definition of $\mathcal{B}(\rho_1)$, $(\dagger)$ is by Lemma 6.11, and $(\ddagger)$ is by Claim 1. Finally, $(\diamond)$ is because $\nu := \rho_2 - \rho_1$ and u^* is linear. $\square$

Proof of Theorem 6.7. "$\Longrightarrow$" follows from the definition of $\mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)$.

"$\Longleftarrow$" Theorem 6.9 says that there is a set $\mathcal{U}$ of measurable functions from $\Psi \times \Phi$ to $\mathbb{R}$ such that, for all $\rho, \rho' \in \mathfrak{P}$,

$$\Big(\rho \preceq \rho'\Big) \iff \Big(u(\rho) \leq u(\rho') \text{ for all } u \in \mathcal{U}\Big). \tag{36}$$

Clearly, any element of $\mathcal{U}$ is a lottery hedometer for $(\preceq)$; hence $\mathcal{U} \subseteq \mathcal{H}_{\mathcal{ED}}(\preceq)$. Thus, the "$\Longleftarrow$" direction of eqn.(36) implies the "$\Longleftarrow$" direction of the theorem. $\square$

Proof of Corollary 6.8. Let $\overline{\rho}$ and $\overline{\rho}'$ be the per capita average lotteries of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$, as defined in eqn.(14). For any measurable $h : \Psi \times \Phi\longrightarrow\mathbb{R}$, we have

$$h^*(\overline{\rho}) \quad = \quad \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} h^*(\rho_i) \quad \text{and} \quad h^*(\overline{\rho}') \quad = \quad \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} h^*(\rho_i'), \tag{37}$$

because the function $h^* : \mathfrak{P}\longrightarrow\mathbb{R}$ is linear. Thus,

$$\Big(\boldsymbol{\rho} \underset{u}{\trianglelefteq} \boldsymbol{\rho}'\Big) \underset{(*)}{\iff} \Big(\overline{\rho} \preceq \overline{\rho}'\Big) \underset{(\dagger)}{\iff} \Big(h^*(\overline{\rho}) \leq h^*(\overline{\rho}'), \text{ for all } h \in \mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)\Big)$$
$$\underset{(\diamond)}{\iff} \left(\sum_{i\in\mathcal{I}} h^*(\rho_i) \leq \sum_{i\in\mathcal{I}} h^*(\rho_i'), \text{ for all } h \in \mathcal{H}^{\text{lot}}_{\mathcal{ED}}(\preceq)\right),$$

as desired. Here, $(*)$ is by defining formula (15); $(\dagger)$ is by Theorem 6.7, and $(\diamond)$ is by eqn.(37). $\square$

Proofs from §7.

Proof of Lemma 7.1. Fix $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$. Define the function $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})} : \mathfrak{S} \longrightarrow \mathbb{R}$ by $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\mathcal{S}] := U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi}) \cdot \pi[\mathcal{S}]$, for all $\mathcal{S} \in \mathfrak{S}$. Axiom (Bayes) says that $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}$ is countably additive (i.e. $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\bigsqcup_{n=1}^{\infty} \mathcal{S}_n] = \sum_{n=1}^{\infty} \mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\mathcal{S}_n]$); hence it is a sigma-finite signed measure (because π is a probability measure and $|U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi})| < \infty$ for all $\mathcal{S} \in \mathfrak{S}$). Clearly, $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}$ is absolutely continuous relative to ρ [i.e. $(\rho[\mathcal{S}] = 0) \Longrightarrow (\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\mathcal{S}] = 0)$]. Thus, the Radon-Nikodym Theorem (Conway, 1990, Thm.C.7, p.380) says there is a $\mathfrak{S}$-measurable function $f_{(\boldsymbol{\psi},\boldsymbol{\phi})} : \Omega \longrightarrow \mathbb{R}$ such that $\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\mathcal{S}] = \int_{\mathcal{S}} f_{(\boldsymbol{\psi},\boldsymbol{\phi})}\, d\rho$ for all $\mathcal{S} \in \mathfrak{S}$. Now define $U : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \times \Omega \longrightarrow \mathbb{R}$ by $U_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) := f_{(\boldsymbol{\psi},\boldsymbol{\phi})}(\omega)$, for all $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ and $\omega \in \Omega$. Then for any $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ and $\mathcal{S} \in \mathfrak{S}$, we have

$$U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi}) \quad = \quad \frac{\mu_{(\boldsymbol{\psi},\boldsymbol{\phi})}[\mathcal{S}]}{\pi[\mathcal{S}]} \quad = \quad \frac{1}{\pi[\mathcal{S}]} \int_{\mathcal{S}} f_{(\boldsymbol{\psi},\boldsymbol{\phi})}\, d\rho \quad = \quad \frac{1}{\pi(\mathcal{S})} \int_{\mathcal{S}} U_\omega(\boldsymbol{\psi},\boldsymbol{\phi})\, d\pi[\omega],$$

which yields eqn.(22). □

Proof of Theorem 7.2. For any $\omega \in \Omega$, if the vNM preference relation $(\underset{\omega}{\trianglelefteq})$ satisfies (Par), then Harsanyi's SAT implies that $(\underset{\omega}{\trianglelefteq})$ can be represented as maximizing the expected value of a vNM utility function $\widetilde{U}_\omega : \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \longrightarrow \mathbb{R}$ of the form:

$$\widetilde{U}_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \quad := \quad \sum_{i\in\mathcal{I}} c_i^\omega \cdot h_i^\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \quad = \quad \sum_{i\in\mathcal{I}} c_i^\omega \cdot H(\psi_i,\phi_i,\omega), \quad \text{for all } (\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}},$$

for some nonnegative constants $\{c_i^\omega\}_{i\in\mathcal{I}} \subset \mathbb{R}_+$. Axiom (Nonindiff) says at least one these constants is nonzero, while (Anon) implies that they must all be equal; hence we can assume without loss of generality that $c_i^\omega = 1$ for all $i \in \mathcal{I}$, so that $\widetilde{U}_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) = \sum_{i\in\mathcal{I}} H(\psi_i,\phi_i,\omega)$ for all $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ and $\omega \in \Omega$.

Now, U_ω and $\widetilde{U}_\omega$ represent the same vNM preference relation $(\underset{\omega}{\trianglelefteq})$, so there exist constants $a(\omega) > 0$ and $b(\omega) \in \mathbb{R}$ such that $U_\omega = a(\omega)\,\widetilde{U}_\omega + b(\omega)$. That is:

$$U_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \;=\; b(\omega) + a(\omega) \sum_{i\in\mathcal{I}} H(\psi_i,\phi_i,\omega), \quad \text{for all } (\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \text{ and } \omega \in \Omega.$$

Axiom (Welf) then implies that $a(\omega_1) = a(\omega_2)$ and $b(\omega_1) = b(\omega_2)$ for all $\omega_1, \omega_2 \in \Omega$. Thus, there are constants $a > 0$ and $b \in \mathbb{R}$ such that

$$U_\omega(\boldsymbol{\psi},\boldsymbol{\phi}) \;=\; b + a \sum_{i\in\mathcal{I}} H(\psi_i,\phi_i,\omega), \quad \text{for all } (\boldsymbol{\psi},\boldsymbol{\phi}) \in \Psi^{\mathcal{I}} \times \Phi^{\mathcal{I}} \text{ and } \omega \in \Omega. \tag{38}$$

Substituting (38) into (22), we get:

$$\begin{aligned} U_{\mathcal{S}}(\boldsymbol{\psi},\boldsymbol{\phi}) \quad &= \quad \frac{1}{\pi(\mathcal{S})} \int_{\mathcal{S}} \left(b + a \sum_{i\in\mathcal{I}} H(\psi_i,\phi_i,\omega) \right) d\pi[\omega] \\ &= \quad b + a \sum_{i\in\mathcal{I}} \frac{1}{\pi(\mathcal{S})} \int_{\mathcal{S}} H(\psi_i,\phi_i,\omega)\, d\pi[\omega] \quad = \quad b \;+\; a \sum_{i\in\mathcal{I}} \overline{h}_{\mathcal{S}}(\psi_i,\phi_i), \end{aligned} \tag{39}$$

and where $\overline{h}_{\mathcal{S}}$ is defined as in eqn.(23). But clearly the vNM utility function $U_{\mathcal{S}}$ in eqn.(39) is equivalent to the vNM utility function $\overline{U}_{\mathcal{S}}$ in eqn.(24). □

Proofs from §8.

Proof of Proposition 8.2. Clearly, $(\underset{\&}{\preceq})$ is reflexive. We must show that $(\underset{\&}{\preceq})$ is transitive and satisfies properties (W1) and (W2).

Transitive. Suppose $(\psi_1, \phi_1)\underset{\&}{\preceq}(\psi_2, \phi_2)$ and $(\psi_2, \phi_2)\underset{\&}{\preceq}(\psi_3, \phi_3)$. We must show that $(\psi_1, \phi_1)\underset{\&}{\preceq}(\psi_3, \phi_3)$.

We have $(\psi_2, \phi_2)\underset{\psi_2,\psi_3}{\preceq}(\psi_3, \phi_3)$, and $(\psi_2, \phi_2)\underset{\psi_3,\psi_2}{\preceq}(\psi_3, \phi_3)$, while $(\psi_1, \phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2, \phi_2)$, so consistency requires that $(\psi_1, \phi_1)\underset{\psi_1,\psi_3}{\preceq}(\psi_3, \phi_3)$.

Likewise, $(\psi_2, \phi_2)\underset{\psi_1,\psi_2}{\succeq}(\psi_1, \phi_1)$, and $(\psi_2, \phi_2)\underset{\psi_2,\psi_1}{\succeq}(\psi_1, \phi_1)$, while $(\psi_3, \phi_3)\underset{\psi_3,\psi_2}{\succeq}(\psi_2, \phi_2)$, so consistency requires that $(\psi_3, \phi_3)\underset{\psi_3,\psi_1}{\succeq}(\psi_1, \phi_1)$.

Thus, $(\psi_1, \phi_1)\underset{\psi_1,\psi_3}{\preceq}(\psi_3, \phi_3)$ and $(\psi_1, \phi_1)\underset{\psi_3,\psi_1}{\preceq}(\psi_3, \phi_3)$, so $(\psi_1, \phi_1)\underset{\&}{\preceq}(\psi_3, \phi_3)$, as desired.

(W1) Fix $\psi \in \Psi$ and $\phi, \phi' \in \Phi$, with $\phi_1 \underset{\psi}{\succ} \phi_1'$. By hypothesis, $(\underset{\psi,\psi}{\preceq})$ is a wipo on $\{\psi\} \times \Phi$, so it agrees with $(\underset{\psi}{\preceq})$. Thus, $(\psi, \phi)\underset{\psi,\psi}{\succ}(\psi, \phi')$; hence applying definition (25) (with $\psi_1 = \psi_2 = \psi$) we conclude that $(\psi_1, \phi_1)\underset{\&}{\preceq}(\psi_1, \phi_1)$.

(W2) Fix $\psi_1, \psi_2 \in \Psi$ and $\phi_1 \in \Phi$. The relation $(\underset{\psi_1,\psi_2}{\preceq})$ is a wipo, so it satisfies (W2), so there is some $\phi_2' \in \Phi$ such that $(\psi_1, \phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2, \phi_2')$. Likewise, $(\underset{\psi_2,\psi_1}{\preceq})$ satisfies (W2), so there is some $\phi_2'' \in \Phi$ such that $(\psi_1, \phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2, \phi_2'')$. Let ϕ_2 be the $(\underset{\psi_2}{\preceq})$-maximum of $\{\phi_2', \phi_2''\}$ (well-defined because $(\underset{\psi_2}{\preceq})$ is a complete order of Φ). Then we have $(\psi_1, \phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2, \phi_2)$ and $(\psi_1, \phi_1)\underset{\psi_2,\psi_1}{\preceq}(\psi_2, \phi_2)$, and hence $(\psi_1, \phi_1)\underset{\&}{\preceq}(\psi_2, \phi_2)$.

Through an identical construction, we can obtain some $\phi_2 \in \Phi$ such that $(\psi_1, \phi_1)\underset{\&}{\succeq}(\psi_2, \phi_2)$. This works for all $\psi_1, \psi_2 \in \Psi$ and $\phi_1 \in \Phi$; thus, $(\underset{\&}{\preceq})$ satisfies (W2). □

To prove Proposition 8.5 we need some technical preliminaries. A ***preference chain*** is a sequence $(\psi_1, \phi_1)\underset{\psi_1}{\preceq}(\psi_2, \phi_2)\underset{\psi_2}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N, \phi_N)$. Clearly, the underlying sequence $\boldsymbol{\psi} = (\psi_0, \psi_1, \psi_2, \ldots, \psi_N)$ must be an $\mathfrak{N}$-chain; in this case we say that $\boldsymbol{\psi}$ ***carries*** a preference chain between (ψ_1, ϕ_1) and (ψ_N, ϕ_N).

Lemma 8.7 *Suppose $(\underset{\psi}{\preceq})$ is indifference-connected for all $\psi \in \Psi$. Suppose $\boldsymbol{\psi}$ carries a preference chain between (ψ_1, ϕ_1) and (ψ_N, ϕ_N). If $\boldsymbol{\psi}$ is homotopic to $\boldsymbol{\psi}'$, then $\boldsymbol{\psi}'$ also carries a preference chain between (ψ_1, ϕ_1) and (ψ_N, ϕ_N).*

Proof. It suffices to prove this when $\boldsymbol{\psi} \underset{\epsilon}{\simeq} \boldsymbol{\psi}'$ (the general case follows by induction).

First suppose $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n, \psi_{n+1}, \cdots, \psi_N)$ carries the preference chain $(\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$. Suppose $\psi_{n+1} \in \mathcal{N}_{\psi_{n-1}}$ and $\boldsymbol{\psi}' := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_{n+1}, \cdots, \psi_N)$. Then $(\psi_n, \phi_n) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1})$, because $(\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1})$ and $\psi_n, \psi_{n+1} \in \mathcal{N}_{\psi_{n-1}} \cap \mathcal{N}_{\psi_n}$, and $(\underset{\psi_n}{\preceq})$ agrees with $(\underset{\psi_{n-1}}{\preceq})$ on $(\mathcal{N}_{\psi_{n-1}} \cap \mathcal{N}_{\psi_n}) \times \Phi$ by (RO). Thus, $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1})$, because $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_n, \phi_n)$ and $(\underset{\psi_{n-1}}{\preceq})$ is transitive. Thus, we get a preference chain $(\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$ supported on $\boldsymbol{\psi}'$, as desired.

Now suppose $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_{n+1}, \cdots, \psi_N)$ carries the preference chain $(\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$, and suppose $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n, \psi_{n+1}, \cdots, \psi_N)$, for some $\psi_n \in \mathcal{N}_{\psi_{n-1}}$ such that $\psi_{n+1} \in \mathcal{N}_{\psi_n}$. Since $(\underset{\psi_{n-1}}{\preceq})$ is indifference-connected, we can find some $\phi_n \in \Phi$ such that $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\approx} (\psi_n, \phi_n)$. Thus, $(\psi_n, \phi_n) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1})$ because $(\underset{\psi_{n-1}}{\preceq})$ is transitive. Thus, $(\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1})$ because $\psi_n, \psi_{n+1} \in \mathcal{N}_{n-1} \cap \mathcal{N}_n$, and $(\underset{\psi_n}{\preceq})$ agrees with $(\underset{\psi_{n-1}}{\preceq})$ on $(\mathcal{N}_{n-1} \cap \mathcal{N}_n) \times \Phi$ by (RO). Thus, we get a preference chain $(\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\approx} (\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$ supported on $\boldsymbol{\psi}'$, as desired. □

Lemma 8.8 *Suppose $\mathfrak{N}$ is a simply connected, and for all $\psi \in \Psi$, suppose the relation $(\underset{\psi}{\preceq})$ is indifference-connected. Then the system $\{\underset{\psi}{\preceq}\}_{\psi \in \Psi}$ admits no preference cycles.*

Proof. Suppose $(\psi_0, \phi_0) \underset{\psi_0}{\preceq} (\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{N-2}}{\preceq} (\psi_{N-1}, \phi_{N-1}) \underset{\psi_{N-1}}{\preceq} (\psi_0, \phi_0') \underset{\psi_0}{\prec} (\psi_0, \phi_0)$ is a preference cycle. Let $\psi_N := \psi_0$ and $\phi_N := \phi_0'$. Then $\boldsymbol{\psi} := (\psi_0, \ldots, \psi_N)$ is a closed $\mathfrak{N}$-chain carrying the preference chain $\boldsymbol{\xi} := [(\psi_0, \phi_0) \underset{\psi_0}{\preceq} (\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)]$. Since Ψ is simply connected, the chain $\boldsymbol{\psi}$ is homotopic to a trivial chain $(\psi_0, \psi_0, \ldots, \psi_0)$, and by Lemma 8.7, this homotopy transforms the preference chain $\boldsymbol{\xi}$ into a preference chain $(\psi_0, \phi_0) \underset{\psi_0}{\preceq} (\psi_0, \hat{\phi}_1) \underset{\psi_0}{\preceq} (\psi_0, \hat{\phi}_2) \underset{\psi_0}{\preceq} \cdots \underset{\psi_0}{\preceq} (\psi_0, \hat{\phi}_{N-1}) \underset{\psi_0}{\preceq} (\psi_N, \phi_N) = (\psi_0, \phi_0')$. Thus, we have $(\psi_0, \phi_0) \underset{\psi_0}{\preceq} (\psi_0, \phi_0')$ because $(\underset{\psi_0}{\preceq})$ is transitive. But this contradicts our hypothesis that $(\psi_0, \phi_0') \underset{\psi_0}{\prec} (\psi_0, \phi_0)$.

By contradiction, no such preference cycle can exist. □

Proof of Proposition 8.5. Let $(\underset{RO}{\preceq})$ be the join of $\{\underset{\psi}{\preceq}\}_{\psi \in \Psi}$. Then $(\underset{RO}{\preceq})$ is a preorder on $\Psi \times \Phi$.

(b) Lemma 8.8 implies that $(\underset{RO}{\preceq})$ satisfies axiom (W1). It remains only to show that $(\underset{RO}{\preceq})$ is a complete order (and hence, satisfies (W2)).

Let $(\psi,\phi),(\psi',\phi')\in\Psi\times\Phi$; we must show these two points are comparable. Since $\mathfrak{N}$ chain-connects Ψ, there is an $\mathfrak{N}$-chain $\psi=\psi_0,\psi_1,\psi_2,\ldots,\psi_N=\psi'$ connecting ψ to ψ'. Now, for all $n\in[0...N)$ the relation $(\underset{\psi_n}{\preceq})$ is indifference-connected, so we can construct an indifference chain $(\psi,\phi)=(\psi_0,\phi_0)\underset{\psi_0}{\approx}(\psi_1,\phi_1)\underset{\psi_1}{\approx}\cdots\underset{\psi_{N-1}}{\approx}(\psi_N,\phi_N)=(\psi',\phi_N)$, for some $\phi_N\in\Phi$. Thus, $(\psi,\phi)\underset{RO}{\approx}(\psi',\phi_N)$. But $(\underset{\psi'}{\preceq})$ is a complete ordering of Φ, so either $\phi_N\underset{\psi'}{\preceq}\phi'$ or $\phi_N\underset{\psi'}{\succeq}\phi'$; thus, either $(\psi',\phi_N)\underset{RO}{\preceq}(\psi',\phi')$ or $(\psi',\phi_N)\underset{RO}{\succeq}(\psi',\phi')$; thus, either $(\psi,\phi)\underset{RO}{\preceq}(\psi',\phi')$ or $(\psi,\phi)\underset{RO}{\succeq}(\psi',\phi')$, because $(\psi,\phi)\underset{RO}{\approx}(\psi',\phi_N)$ and $(\underset{RO}{\preceq})$ is transitive by construction.

(a) We must show that $(\preceq)$ satisfies (W1) and (W2).

(W1) By hypothesis, we can extend each local relation $(\underset{\psi}{\preceq})$ to some indifference-connected relation $(\underset{\psi}{\hat{\preceq}})$, such that the system $\{\mathcal{N}_\psi,\underset{\psi}{\hat{\preceq}}\}_{\psi\in\Psi}$ still satisfies axiom (RO). Now apply part (b) to $\{\underset{\psi}{\hat{\preceq}}\}_{\psi\in\Psi}$ to obtain a global wipo $(\underset{RO}{\hat{\preceq}})$. If $(\underset{RO}{\preceq})$ is the join of $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$, then $(\underset{RO}{\hat{\preceq}})$ extends $(\underset{RO}{\preceq})$. Thus, for each $\psi\in\Psi$, the relation $(\underset{RO}{\preceq})$ agrees with $(\underset{\psi}{\preceq})$ on $\{\psi\}\times\Phi$, because $(\underset{RO}{\hat{\preceq}})$ agrees with $(\underset{\psi}{\preceq})$ on $\{\psi\}\times\Phi$, by part (b).

(W2) Let $\psi,\psi'\in\Psi$ and $\phi\in\Phi$; we must find some $\phi'\in\Phi$ such that $(\psi,\phi)\underset{RO}{\preceq}(\psi',\phi')$. Let $\psi=\psi_0,\psi_1,\ldots,\psi_N=\psi'$ be an $\mathfrak{N}$-chain (this exists because $\mathfrak{N}$ chain-connects Ψ). There exists $\phi_1\in\Phi$ with $(\psi,\phi)\underset{\psi}{\preceq}(\psi_1,\phi_1)$, because $(\underset{\psi}{\preceq})$ is a wipo on $\mathfrak{N}_\psi\times\Phi$. Next, there exists $\phi_2\in\Phi$ with $(\psi_1,\phi_1)\underset{\psi_1}{\preceq}(\psi_2,\phi_2)$, because $(\underset{\psi_1}{\preceq})$ is a wipo on $\mathfrak{N}_{\psi_1}\times\Phi$. Proceeding inductively, we obtain a preference chain $(\psi,\phi)\underset{\psi_0}{\preceq}(\psi_1,\phi_1)\underset{\psi_1}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N,\phi_N)$. Let $\phi':=\phi_N$; then $(\psi,\phi)\underset{RO}{\preceq}(\psi',\phi')$. It follows that $(\underset{RO}{\preceq})$ is a wipo. $\square$

Proof of Proposition 8.6. For every $\psi\in\Psi$, find some $j\in\mathcal{J}$ with $\psi\in\mathcal{O}_j$. The open set $\mathcal{O}_j$ contains an open ball around ψ, and if this open ball is small enough, it is simply connected (because Ψ is a manifold). Thus, let $\mathcal{N}_\psi\subset\mathcal{O}_j$ be some simply connected open neighbourhood of ψ, and let u_ψ be the restriction of u_j to $\mathcal{N}_\psi\times\Phi$. This yields a simply connected neighbourhood system $\mathfrak{N}=\{\mathcal{N}_\psi\}_{\psi\in\Psi}$, as in Example 8.4(a).

For every $\psi\in\Psi$, define a 'local' wipo $(\underset{\psi}{\preceq})$ on $\mathcal{N}_\psi\times\Phi$ as follows: for all $(\nu_0,\phi_0),(\nu_1,\phi_1)\in\mathcal{N}_\psi\times\Phi$,

$$\Big((\nu_0,\phi_0)\underset{\psi}{\preceq}(\nu_1,\phi_1)\Big)\iff\begin{pmatrix}\exists\text{ improvement path }\gamma:[0,1]\longrightarrow\mathcal{N}_\psi\times\Phi\\ \text{with }\gamma(0)=(\nu_0,\phi_0)\text{ and }\gamma(1)=(\nu_1,\phi_1)\end{pmatrix}.$$

Thus, $(\underset{\rightsquigarrow}{\preceq})$ is obtained by taking the join of all the local wipos $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$, exactly as in the definition of $(\underset{RO}{\preceq})$ in §8.3. Thus, it suffices to show that the system $\{\underset{\psi}{\preceq}\}_{\psi\in\Psi}$ is consistent, and then invoke Proposition 8.5(a).

Let $\psi \in \Psi$. Define $(\underset{\psi}{\hat{\preceq}})$ on $\mathcal{N}_\psi \times \Phi$ as follows: for all $(\nu_0, \phi_0), (\nu_1, \phi_1) \in \mathcal{N}_\psi \times \Phi$,

$$\Big((\nu_0, \phi_0) \underset{\psi}{\hat{\preceq}} (\nu_1, \phi_1)\Big) \iff \Big(u_\psi(\nu_0, \phi_0) \leq u_\psi(\nu_1, \phi_1)\Big). \tag{40}$$

Clearly, $(\underset{\psi}{\hat{\preceq}})$ is a complete order on $\mathcal{N}_\psi \times \Phi$. Axiom (Sm1) ensures that $(\underset{\psi}{\hat{\preceq}})$ is indifference-connected.

Claim 1: *The system $\{\mathcal{N}_\psi, \underset{\psi}{\hat{\preceq}}\}_{\psi \in \Psi}$ satisfies property* (RO) *from §8.3.*

Proof. Let $\psi_1, \psi_2 \in \Psi$. Suppose $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2} \neq \emptyset$, and the relations $(\underset{\psi_1}{\hat{\preceq}})$ and $(\underset{\psi_2}{\hat{\preceq}})$ are defined by (40). Suppose u_{ψ_1} is the restriction of u_j to $\mathcal{N}_{\psi_1}$ and u_{ψ_2} is the restriction of u_k to $\mathcal{N}_{\psi_2}$, for some $j, k \in \mathcal{J}$. Thus, $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$ (since it contains $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2}$), and then property (Sm3) ensures that $(\underset{\psi_1}{\hat{\preceq}})$ and $(\underset{\psi_2}{\hat{\preceq}})$ agree on $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2}$. ◇ `Claim 1`

Claim 2: *For any $\psi \in \Psi$, the preorder $(\underset{\psi}{\hat{\preceq}})$ extends $(\underset{\psi}{\preceq})$.*

Proof. Let $(\nu_0, \phi_0), (\nu_1, \phi_1) \in \mathcal{N}_\psi \times \Phi$, with $(\nu_0, \phi_0) \underset{\psi}{\preceq} (\nu_1, \phi_1)$; we must show that $(\nu_0, \phi_0) \underset{\psi}{\hat{\preceq}} (\nu_1, \phi_1)$. But if $(\nu_0, \phi_0) \underset{\psi}{\preceq} (\nu_1, \phi_1)$, then there is some improvement path $\gamma : [0,1] \longrightarrow \mathcal{N}_\psi \times \Phi$ with $\gamma(0) = (\nu_0, \phi_0)$ and $\gamma(1) = (\nu_1, \phi_1)$. Thus,

$$u_\psi(\nu_1, \phi_1) \quad = \quad u_\psi \circ \gamma(1) \underset{(*)}{=} u_\psi \circ \gamma(0) + \int_0^1 (u_\psi \circ \gamma)'(t)\, dt \underset{(\dagger)}{\geq} u_\psi \circ \gamma(0) \quad = \quad u_\psi(\nu_0, \phi_0),$$

so $(\nu_0, \phi_0) \underset{\psi}{\hat{\preceq}} (\nu_1, \phi_1)$, as desired.

Here, $(*)$ is the Fundamental Theorem of Calculus. Inequality $(\dagger)$ is because $(u_\psi \circ \gamma)'(t) \underset{(c)}{=} \nabla u_\psi(\gamma(t))[\gamma'(t)] \underset{(\diamond)}{\geq} 0$ for all $t \in [0,1]$. Here, (c) is by the Chain Rule, and $(\diamond)$ is by (Sm2) and the fact that $\gamma'(t) \underset{\gamma(t)}{\vec{\succeq}} \vec{0}_{\gamma(t)}$ for all $t \in [0,1]$ (because γ is an improvement path). ◇ `Claim 2`

Thus, the system $\{\underset{\psi}{\preceq}\}_{\psi \in \Psi}$ is consistent, so Proposition 8.5(a) implies that $(\underset{\leadsto}{\preceq})$ is a wipo.
□

Remark. In the proof of Proposition 8.6, the inequality $u_\psi(\nu_0, \phi_0) \leq u_\psi(\nu_1, \phi_1)$ is necessary, but *not sufficient* to conclude that $(\nu_0, \phi_0) \underset{\leadsto}{\preceq} (\nu_1, \phi_1)$. Thus, assuming the existence of a function $u_\psi : \mathcal{N}_\psi \times \Phi \longrightarrow \mathbb{R}$ is *not* tantamount to assuming some 'local' form of 'ordinal, fully comparable' utility functions —it is a much weaker assumption.

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