A Theory of Continuum Economies with Independent Shocks and Matchings

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"Entities should not be multiplied beyond necessity."

Occam's razor

Numerous economic models employ a continuum of negligible agents with a sequence of idiosyncratic shocks and random matchings. Several attempts have been made to build a rigorous mathematical justification for such models, but these attempts have left many questions unanswered. In this paper, we develop a discrete time framework in which the major, desirable properties of idiosyncratic shocks and random matchings hold. The agents live on a probability space, and the probability distribution for each agent is naturally replaced by the population distribution. The novelty of this approach is in the assumption of unknown identity. Each agent believes that initially he was randomly and uniformly placed on the agent space, i.e., the agent's identity (the exact location on the agent space) is unknown to the agent.

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1. Introduction

The Problem

A large number of models in monetary theory, game theory, evolution theory, etc., consider negligible agents that experience idiosyncratic shocks and randomly meet each other. Aliprantis, Camera, and Puzzello [3], Alós-Ferrer [5], Boylan [7], and Duffie and Sun [9] review multiple examples of such models. The assumptions in these models are often made in the spirit of the Law of Large Numbers. For example, it is often assumed that the sample distribution of shocks does not depend on the agent subset and equals the probability distribution (No Aggregate Uncertainty property). Another usual assumption requires the fraction of the agents from one set who meet agents from another set to be equal the measure of the second set (Mixing property).[3]

Some economists have pointed out serious contradictions among standard assumptions about idiosyncratic shocks and random matchings. One of the most famous contradictions was described by Feldman and Gilles [10], who showed that for the unit interval of agents with Borel σ-algebra, it is impossible to simultaneously satisfy No Aggregate Uncertainty property on all measurable agent subsets.[4] Judd [19] proved that the measure of the realizations for which the sample distribution function of idiosyncratic shocks does not exist on the whole agent space has inner measure zero and outer measure one. The second contradiction was described by McLennan and Sonnenschein [21]. The authors noticed that a measure preserving matching cannot simultaneously be mixing on all pairs of measurable agent subsets.

Notwithstanding the contradictions in the standard assumptions, the economic models that employ these assumptions are still widely used. By using idiosyncratic shocks and random matchings on a space of negligible agents, one wants to achieve two important goals. The first is to eliminate an agent's influence on the aggregate characteristics of the economy; this is achieved by using negligible agents. The second is to provide an analogue

[3]The term "meeting" will be used only with respect to one agent meeting another. A one-time process of all the agents being paired up with each other will be called a "matching."

[4]Contradictions mentioned in this paragraph will be considered in detail in Subsection 2.3. "Standard Approach Inconsistencies."

of the Law of Large Numbers with respect to agent attributes, shocks, and meetings. Although the Law of Large Numbers holds increasingly large finite populations, a naïve replacement of a large but finite agent space with a space of negligible agents creates multiple problems arising from significant differences in these spaces. As an example of such a difference, every function is measurable on a finite space with the natural discrete σ-algebra, whereas a σ-algebra of an infinite agent space might seriously restrict the set of measurable functions.

In this paper, we suggest a new approach to resolve the conflicting issues of the standard assumptions of idiosyncratic shocks and random matchings on a space of negligible agents. Using this approach, we build a mathematically valid discrete time model of shocks and matchings that are independent from the history. The construction justifies the use of numerous existing economic models. Obviously, it is almost impossible to build a universal solution, however, the new approach lays out a foundation for many other models, depending on the properties required.

Existing Solutions

There have been three distinct approaches to the idiosyncratic shocks problem. In the first approach, a continuum economy is approximated either with a finite or countable set of agents. Using this approach, Feldman and Gilles [10] showed the existence of a finitely additive measure on the agent space such that the Law of Large Numbers-like properties are satisfied. Al-Najjar [1] considered finite but increasingly large economies in which the continuum-like Law of Large Numbers holds for any subinterval of the $[0, 1]$ agent set. Instead of the Lebesgue integral, Uhlig [23] used the Pettis integral, which captures the idea of a normalized countable sum of shocks and thus gives the desired properties.

Hammond and Sun in [17] and [18] studied the behavior of aggregate shocks of a sequence of the agents randomly chosen from a continuum population. They found the Monte Carlo limit measure of the shocks, thus connecting the properties of continuum populations with the properties of random countable subsets. Al-Najjar [2] discussed the equivalence of large discrete and continuum population games.

The approach of finite or countable economies can also be applied to random matchings. Gilboa and Matsui [13] considered a continuous time model with a countable population of individuals, each of whom meets someone only once during his lifetime. The authors satisfied No Aggregate Uncertainty property for any measurable agent subset. Boylan [7] considered simultaneous matching of countably many agents and formulated the Law of Large Numbers-like properties with respect to a finite set of agent types. In [8] Boylan discussed the limit properties of random matchings in discrete time games with a finite number of agents as the number of agents increases to infinity and the time grid becomes finer.

In the second approach, the σ-algebra is extended to satisfy desired properties. For any arbitrary iid random variables on the unit interval of agents Judd [19] built an extension of the sample space so that there is no aggregate uncertainty on the whole agent space. Green [14] endowed the unit interval of agents with a σ-algebra richer than the Borel σ-algebra, and constructed a family of iid random variables with no aggregate uncertainty on any subinterval. Sun [22] used hyperfinite Loeb spaces to demonstrate a similar result for an arbitrary agent space.

Duffie and Sun [9] applied the approach of extending the σ-algebra to random matchings. They employed hyperfinite Loeb spaces to construct a random matching with respect to a finite set of agent types. The authors built an agent space and a matching satisfying the Law of Large Numbers-like properties. The matching is independent in types with respect to any agent types assignment.

The third approach was introduced by Feldman and Gilles [10]. In this approach, the main property of idiosyncratic shocks—independence—is relaxed. Developing the idea of dependent shocks, Alós-Ferrer [5] considered a randomly rotated circumference of a continuum of agents to construct a dependent random matching. The rotated circumference is naturally mapped onto the original one. The shock of an agent is the shock that was originally at the point of the circumference before the rotation. The matching satisfies No Aggregate Uncertainty property with respect to a finite set of types, except for independence of the matches. In [6], Alós-Ferrer extended the result to multiple populations.

Aliprantis, Camera, and Puzzello in [3] and [4] did not suggest any solution to the problem of idiosyncratic shocks or random matching, but did build a set-theoretical foundation for a matching. They also developed a theory of anonymity. Anonymity, which plays an important role in some matching models, requires that an agent and his partner have non-intersecting sets of previous partners.[5] Thus, meeting agents cannot meet again in the future, directly or through their partners, and therefore do not need to act strategically with respect to their current partners. Previous partners can be defined in several different ways; this leads to different definitions of anonymity. Aliprantis, Camera, and Puzzello constructed a sequence of anonymous matchings for a countable population. At the same time, the role of anonymity in randomness of the matchings was not discussed.

Why Do We Need a Further Solution?

Although multiple remedies have been suggested for the problems of idiosyncratic shocks and random matchings, no solution is close to being perfect. Finite or countable approximations do not provide an infinitely additive measure and therefore might cause problems with integration: the integral might not coincide with the Lebesgue integral. Correlated shocks and meetings lack one of the most important properties of randomness across the agents: independence.

Duffie and Sun [9] found a random matching with respect to a finite set of agent types. Due to a finite set of agent types, the Law of Large Numbers-like properties use simple form formulations, with the clauses “for all random realizations” and “for all agents sets” rearranged. At the same time, multiple economic models require infinite, and sometimes continuum, sets of agent types.[6] Due to this requirement of infinite or continuum agent types, Duffie and Sun's solution cannot be applied to such models.

Finally, to our knowledge, no paper simultaneously incorporates both idiosyncratic shocks and random matchings. Thus, it is difficult if not impossible to apply any of the existing solutions to economic models with idiosyncratic shocks and random matchings.

The Main Idea

The existing solutions explicitly or implicitly assume that the agents know their identities. Identity is the agent's location on the agent space. Thus, an agent knows his identity if he correctly associates himself with the corresponding element of the agent space (identity). Because of this assumption of known identity, the standard approach requires a sample space that allows us to model uncertainty the agents face about their

[5]See, for example, Green and Zhou [15].

[6]For example, the models with continuum money holdings, see Green and Zhou [15], or models with continuum action sets, see Gale [12].

future shocks and meetings. Therefore, two separate spaces are needed: agent space and sample space.

However, the requirement of identity knowledge is often excessive and has no influence on the results. What the models really require is that the agents initially be assigned some attributes, like endowment, products they consume, etc. In assigning these attributes, there is no need to distinguish the agents based on their identities. In a symmetric equilibrium, the agents do not use identity knowledge in choosing their actions. Thus, the use of a symmetric equilibrium also implies that the agents do not know their identities.

Employing the idea of unknown identity, we eliminate the sample space and consider only one space—the identity/agent space, which is essentially a probability space.[7] The randomness the agents face comes from their unknown identities. Each agent believes that he was randomly and uniformly placed on the identity space, as if he were randomly assigned an identity. The shocks and meetings are some predetermined functions of the identities and are known to everyone. However, the agents do not know their identities and therefore perceive future shocks and meetings as random. Based on the previous shocks and meetings, the agents update their identities beliefs using the Bayesian rule. The updated beliefs about identities still allow uncertainty about future events the agents face.

One might think about an identity as being comprised of agent attributes and a sequence of predetermined events (shocks, meetings) that will happen to the agent. Initially, the agent does not know his identity (attributes and the sequence of events), although there is no randomness in assigning identities to the agents. This is why the identity space is the same as the agent space. As time goes by, every agent learns a bigger and bigger part of this sequence of events. However, based on the known part of the sequence, he cannot efficiently predict events that have not yet had happened.

The difference between the standard and new approaches can be seen through the following example. Suppose that we have six agents, and the shocks take values from set $\{1, 2, 3, 4, 5, 6\}$. In the standard approach, each agent is represented by a die, and to determine the shocks, we have to throw all six dice, one per agent. An agent's shock is

[7]Eliminating the sample space as an unnecessary detail explains Occam's razor in the epigraph of this paper.

the number on his die. Thus, in the standard approach, the shocks can be the same, or all agents can have different shocks. In the new approach, there are also six dice (identities), however, each die has only one number on it: $1, 2, 3, 4, 5$, or 6, a die per each number. The agents are randomly assigned to the dice on a one-to-one basis. The shock is the number on the agent's die, and different agents always have different shocks. Thus, in the standard approach, there is aggregate uncertainty; the average shock might be different for different realizations. In the new approach, there is no aggregate uncertainty, because the sample distribution of the shocks is always the same.

The approach of unknown identities is based on Kolmogorov's definition of randomness, see Kolmogorov [20]. Namely, random variables—shocks—are some measurable functions on the agent space. The agent's identity is not known to the agent; the agent believes that his identity is uniformly distributed on the agent space. With time, the agent updates his identity belief based on the events observed.

The Results

The unknown identity assumption allows us to redefine shocks and matchings. In order to define independence of shocks and matchings, we employ the concept of history, i.e., all events that the agent has observed on his own (i.e., his own shocks) or through communication during the meetings (shocks of other agents). Shocks and matchings are independent from the history if the agents cannot make any informative conclusion about the future based on their histories. Theorem 1 demonstrates that anonymity is required for the matchings to be independent from the history.

In Theorem 2 we construct a space of negligible agents and a sequence of shocks and anonymous matchings on this agent space that are independent from the history. To achieve this goal, we combine the discrete approach for a countable population (Aliprantis, Camera, and Puzzello [3] and [4]) with measure- and probability-theory tools. The Law of Large Numbers-like properties on this new space hold with respect to the σ-algebra generated by the histories.

The new approach allows us to combine in one model both idiosyncratic shocks and random matchings. The Law of Large Numbers-like properties hold in every important aspect for both shocks and matchings for any history-generated agent subset. The new approach is important for applications as it incorporates several features the previous solutions failed to justify.

Plan of the Paper

The rest of the paper is organized as follows. Section 2 discusses the standard approach to idiosyncratic shocks and random matchings and shows the problems arising from using this approach. Section 3 provides new definitions of independent shocks and matchings and discusses the connection between these two objects and anonymity. Section 4 proves the existence of independent shocks and matchings. Section 5 is our conclusion.

2. Standard Approach

In this section, we discuss what is usually understood by idiosyncratic shocks and random matchings. Particular assumptions vary from paper to paper and are often vaguely formulated. Thus, we give here the most general setup as we understand it when the agents first experience shocks and then meet pairwise with each other. Some parts of this setup are given in multiple papers on idiosyncratic shocks and random matchings. These papers include Alós-Ferrer [5], Boylan [7], Duffie and Sun [9], Feldman and Gilles [10], Gale [12], Green [14], Judd [19], McLennan and Sonnenschein [21], and Sun [22]. For simplicity and without loss of generality, time is not taken into account in this section. After defining the properties, we discuss their possible alternative formulations and the problems which arise in this standard approach.

2.1. The Setup

Let A be the agent space with σ-algebra $\mathcal{A}$ and probability measure μ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the sample space. Measure μ is assumed to be atomless, i.e., the agents are negligible. Every agent $a \in A$ first experiences shock ξ_a and then meets another agent in accordance with some rule $\mathbf{M}_a$.[8] The rest of the assumptions describe those two objects, ξ and $\mathbf{M}$, and fall into three different categories: assumptions about shocks ξ_a (shocks are

[8] The singular form "shock" will be used to refer to one random variable, or to refer to its realization for one agent. Plural "shocks" will be used to refer to many random variables, or to realizations for different agents.

idiosyncratic), assumptions about matching $\mathbf{M}$ (meetings are random), and assumptions about joint properties of shocks and matching.

Idiosyncratic Shocks

An *idiosyncratic shock* is a function $\xi : A \times \Omega \to \mathbb{R}$ (denoted also as $\xi_a : \Omega \to \mathbb{R}$ or $\xi_\omega : A \to \mathbb{R}$) with the following properties:

A1. *Measurability*: for any $a \in A$ function $\xi_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ function $\xi_\omega(\cdot)$ is $\mathcal{A}$-measurable;[9]

A2. *Identical Distribution*: shock $\xi_a(\cdot)$ has the same cdf $F(\cdot)$ for every agent $a \in A$;

A3. *Independence*: for any different agents $a_1, a_2, \ldots, a_l \in A$ corresponding shocks $\xi_{a_1}(\cdot)$, $\xi_{a_2}(\cdot), \ldots, \xi_{a_l}(\cdot)$ are independent;

A4. *No Aggregate Uncertainty*: for any random realization the sample cdf of the shocks equals $F(\cdot)$ on any measurable agent subset,[10]

$$\forall \omega \in \Omega,\ \forall B \in \mathcal{A},\ \forall x \in \mathbb{R} \qquad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = F(x)\mu(B).$$

Random Matching

A *random matching*, which determines whom everyone meets for each $\omega \in \Omega$, is a mapping $\mathbf{M} : A \times \Omega \to A$ (also denoted as $\mathbf{M}_a : \Omega \to A$ or $\mathbf{M}_\omega : A \to A$) with the following properties:

B1. *No Agent Is Idle*: no agent meets himself,

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(a) \neq a;$$

B2. *Involution*: the partner's partner is the agent himself,

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(\mathbf{M}_\omega(a)) = a;$$

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[9] By measurability we mean separate measurability, in contrast to joint measurability.

[10] We formulate here No Aggregate Uncertainty property with respect to the distribution. Some authors formulate it with respect to the average, see Feldman and Gilles [10]:

$$\int_B \xi_\omega(a)\, d\mu(a) = \mu(B) \int_{\mathbb{R}} x\, dF(x) \qquad \forall \omega \in \Omega, \forall B \in \mathcal{A}.$$

B3. *Measurability*: for any $a \in A$ operator $\mathbf{M}_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ operator $\mathbf{M}_\omega(\cdot)$ is $\mathcal{A}$-measurable;[11]

B4. *Measure Preserving*: for any $a \in A$ operator $\mathbf{M}_a(\cdot)$ does not change measure of any measurable agent subset,

$$\forall \omega \in \Omega,\ \forall B \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B)) = \mu(B);$$

B5. *Uniform Distribution*: probability of meeting an agent from some subset equals the measure of this subset,

$$\forall a \in A,\ \forall B \in \mathcal{A} \qquad \mathbf{P}(\{\omega : \mathbf{M}_a(\omega) \in B\}) = \mu(B);$$

B6. *Independence*: for any different agents $a_1, a_2, \ldots, a_l \in A$ their partners $\mathbf{M}_{a_1}(\cdot)$, $\mathbf{M}_{a_2}(\cdot), \ldots, \mathbf{M}_{a_l}(\cdot)$ are independent;

B7. *Mixing*: the fraction of the agents from one subset who meet agents from another subset equals the measure of the second subset,

$$\forall \omega \in \Omega,\ \forall B_1, B_2 \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2).$$

In addition to properties A1-A4 and B1-B7, some independence is usually assumed between the shocks and matching. Although different papers use different and often vague independence formulations, these formulations have one element in common: independence of an agent's future from the current/past events.

Example 1. To illustrate how the standard approach works, consider the following example. For simplicity of the example, the agent space is discrete and consists of eight elements, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. Each agent a experiences shock $\xi_a : \Omega \to \{-1, 1\}$, $\mathbf{P}(\xi_a(\omega) = 1) = \mathbf{P}(\xi_a(\omega) = -1) = 1/2$; shocks are independent across the agents. The matching rule is the following. Consider all deterministic one-to-one matchings satisfying properties B1 (No Agent Is Idle) and B2 (Involution). Consider a random matching in which all deterministic matchings are equally probable. We assume that the agents are matched independently of the shocks.

In this example, properties A1 and B3 (Measurability), A2 (Identical Distribution), A3 (Independence), B1 (No Agent Is Idle), B2 (Involution), and B4 (Measure Preserving)

[11] See Footnote 9 about separate and joint measurability.

hold by construction. However, properties A4 (No Aggregate Uncertainty), B5 (Uniform Distribution), B6 (Independence), and B7 (Mixing) do not hold. For example, property B5 (Uniform Distribution) does not hold, because no agent can meet himself, and thus the distribution of Agent a_1's partners differs from the distribution of Agent a_2's partners; property B6 (Independence) does not hold, because Agent a_1 meeting Agent a_2 infers Agent a_2 meeting Agent a_1.■

2.2. Different Formulations of the Properties

In defining idiosyncratic shocks and random matching, we used "for all $\omega \in \Omega$" formulations, which are essentially similar to "for almost all $\omega \in \Omega$" formulations. However, some properties dealing with agent subsets allow multiple variations. For example, consider property A4 (No Aggregate Uncertainty). The following alternative formulations are possible:

A4'. For any measurable agent subset, there is no aggregate uncertainty on it with probability one, i.e., for any $B \in \mathcal{A}$, for almost all $\omega \in \Omega$ holds

$$\forall x \in \mathbb{R} \qquad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x);$$

A4". With probability one, there is no aggregate uncertainty on any measurable agent subset, i.e., for almost all $\omega \in \Omega$, for any $B \in \mathcal{A}$ holds

$$\forall x \in \mathbb{R} \qquad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x).$$

The difference between A4' and A4" formulations is in the location of the clause "for any $B \in \mathcal{A}$" with respect to the clause "for almost all $\omega \in \Omega$." In A4', we first fix the agent subset, and then we say that for almost all realizations, there is no aggregate uncertainty on this subset. In A4", for almost all realizations, there is no aggregate uncertainty on any measurable agent subset. Obviously, A4' follows from A4", but not vice versa. Similar to A4' and A4", different formulations can be used for other properties, like B4 (Measure Preserving), B5 (Uniform Distribution), etc.

To illustrate the difference between A4' and A4" formulations, consider the following example.[12] Take a sequence $\{\tau_i : \Omega \to [0,1]\}_{i \in \mathbb{N}}$ of uniformly distributed on $[0,1]$ independent random variables. For any set $B \subset [0,1]$ define $\nu_\omega(B) = \lim_{n\to\infty} \frac{\#\{i<n : \tau_i(\omega) \in B\}}{n}$, if

[12]This example has the same idea as the one given by Al-Najjar [1], Footnote 17.

it exists. From the Strong Law of Large Numbers it follows that for any B with measure $\mu(B)$, with probability one $\nu_\omega(B)$ exists and $\nu_\omega(B) = \mu(B)$, or A4'-type property holds. However, for any particular ω, sequence $\tau_i(\omega)$ consists of a countable number of elements, therefore there exist measurable sets B such that $\nu_\omega(B) \neq \mu(B)$, for example

$$B = [0,1] \setminus \cup_{i\in\mathbb{N}}\{\tau_i(\omega)\},$$

i.e., A4"-type property does not hold.

2.3. Standard Approach Inconsistencies

Although the standard approach with A4"-type formulations looks intuitively natural, it contains multiple contradictions. We consider two famous problems one can face while using this approach. The examples show that idiosyncratic shocks or random matching with A4"-type properties do not exist.

There Always Exists an Agent Subset with Aggregate Uncertainty

Proposition 1 of Feldman and Gilles [10] shows that for the unit interval of agents with Borel σ-algebra and Bernoulli shocks, for any realization, there always exists a measurable agent subset with aggregate uncertainty. This result can be generalized. Consider an agent space. For an arbitrary $\omega \in \Omega$ take some m and define $B_m = \{a \in A : \xi_\omega(a) < m\}$. If property A4" holds, then for any non-degenerate distribution of shocks $\xi_\omega(\cdot)$, we can always choose m so that $\mu(B_m) \in (0,1)$. Denoting the average shock on $B \in \mathcal{A}$ as $\mathbf{E}_B\xi_\omega(\cdot)$, we have $\mathbf{E}_{B_m}\xi_\omega(\cdot) < \mathbf{E}_{\overline{B}_m}\xi_\omega(\cdot)$. The last inequality means that aggregate uncertainty exists either on B_m, or on $\overline{B}_m$, or on both B_m and $\overline{B}_m$.

Thus, it is impossible to achieve No Aggregate Uncertainty property simultaneously on all measurable agent subsets for any $\omega \in \Omega$, i.e., property A4" cannot be satisfied.

Measure Preserving Matching Cannot Be Mixing

Property B7 (Mixing) says that for any realization $\omega \in \Omega$ exactly $\mu(B_2)$ fraction of the agents from subset B_1 are matched with the agents from subset B_2:

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2).$$

Take B_1 such that $\mu(B_1) \in (0,1)$ and $B_2 \equiv \mathbf{M}_\omega(B_1)$.[13] Then

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1) > \mu(B_1)^2 = \mu(B_1)\mu(B_2),$$

i.e., for any $\omega \in \Omega$, properties B4 (Measure Preserving) and B7 (Mixing) do not hold simultaneously on all measurable pairs of agent subsets.

3. Independent Shocks and Matchings

In the previous section, we showed that idiosyncratic shocks and random matchings do not exist in the standard approach. This section defines shocks and matchings using unknown identities. We show how the standard properties relate to the new ones. Throughout the section, we construct an example to illustrate new definitions.

3.1. Basic Definitions

Consider a set of identities A with σ-algebra $\mathcal{A}$ and atomless probability measure $\mu : \mathcal{A} \to [0,1]$. The same space serves as the agent space. Each agent α has general knowledge about $(A, \mathcal{A}, \mu)$. However, the agent does not know his identity $a(\alpha) \in A$—the location on the identity space. The uncertainty an agent faces comes from his unknown identity: he believes that the identity was drawn at random from A: for any $B \in \mathcal{A}$ holds $\mathbf{P}_\alpha(a(\alpha) \in B) = \mu(B)$, as if the agent was randomly placed on the identity space. Without loss of generality, for the rest of the paper we will identify agent α and his identity $a(\alpha)$.

Example 2. Although we assume that the agent space consists of an uncountable number of negligible agents, we consider a simplified example of a discrete space of eight agents,

$$A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\},$$

with discrete σ-algebra $\mathcal{A}$ and counting measure $\mu(\cdot)$. Each agent knows that there exist exactly eight agents; however, no agent knows who he is among these eight agents. Every agent a believes that he was randomly and uniformly placed on the agent space, therefore for any $i = 1, 2, \ldots, 8$ holds $\mathbf{P}(a = a_i) = 1/8$.■

Definition. A *shock* is a random variable $\xi : A \to Y$ for some measurable space (Y, Σ).

[13] This example was taken from McLennan and Sonnenschein [21].

For the rest of the paper, we assume that $Y = \mathbb{R}^m$ with Σ being the Borel σ-algebra on $\mathbb{R}^m$. Using this multidimensional space Y, one is able to model several real-valued shocks at every period of time. Although we use the same multidimensional real-valued shock space (Y, Σ) for every period of time, this is clearly not a limitation.

Such a definition of a shock replaces property A1 (Measurability), because a random variable is measurable by definition. Unknown identity assumption replaces property A2 (Identical Distribution), because the agents become ex ante identical in their beliefs about identities. Unknown identity assumption together with atomless measure assumption replaces property A3 (Independence), because independently chosen agents have independent identities.

Example 2 (continued). Consider the following shocks (see Fig. 1 below):

$$\xi(a_1) = \xi(a_3) = \xi(a_5) = \xi(a_6) = 1, \quad \xi(a_2) = \xi(a_4) = \xi(a_7) = \xi(a_8) = -1.$$

Each agent knows the mapping of the identities into shocks, however, each agent believes that he will experience shock -1 or 1 with probability $1/2$ because

$$\mathbf{P}(a : \xi(a) = 1) = \mathbf{P}(a \in \{a_1, a_3, a_5, a_6\}) = 4/8 = 1/2,$$

$$\mathbf{P}(a : \xi(a) = -1) = \mathbf{P}(a \in \{a_2, a_4, a_7, a_8\}) = 4/8 = 1/2. \blacksquare$$

Definition of a matching, in addition to measurability as in the definition of a shock, requires some additional properties.

Definition. A *matching* is an operator $\mathbf{M} : A \to A$ with the following properties:

C1. *No Agent Is Idle*: no agent meets himself,

$$\forall a \in A \qquad \mathbf{M}(a) \neq a;$$

C2. *Involution*: the partner's partner is the agent himself,

$$\forall a \in A \qquad \mathbf{M}(\mathbf{M}(a)) = a;$$

C3. *Measurability*: $\mathbf{M}$ maps measurable sets into measurable sets,

$$\forall B \in \mathcal{A} \qquad \mathbf{M}(B) \in \mathcal{A};$$

C4. *Measure Preserving*: $\mathbf{M}$ does not change the measure of a measurable set,

$$\forall B \in \mathcal{A} \qquad \mu(\mathbf{M}(B)) = \mu(B).$$

Unknown identity assumption replaces property B5 (Uniform Distribution), because all agents become ex ante identical; unknown identity assumption along with atomless measure assumption replace property B6 (Independence), because partners of independently chosen agents are independent. Properties C1-C4 (No Agent Is Idle, Involution, Measurability, and Measure Preserving) are equivalent to corresponding properties B1-B4. Property C2 (Involution) gives existence of matching $\mathbf{M}^{-1} \equiv \mathbf{M}$. Because of property C1 (No Agent Is Idle), a composition of two arbitrary matchings need not be a matching. To see this, notice that in a composition of a matching with itself, every agent is matched with himself, which contradicts C1 (No Agent Is Idle).

Example 2 (continued). Consider the following matching $\mathbf{M}$ satisfying properties C1 and C2 (see Fig. 1): Agent a_1 meets with Agent a_5, Agent a_2 meets with Agent a_6, Agent a_3 meets with Agent a_7, and Agent a_4 meets with Agent a_8. This matching $\mathbf{M}$ satisfies properties C3 (Measurability) and C4 (Measure Preserving), because it is a one-to-one mapping on a finite agent space with the discrete σ-algebra and counting measure.■

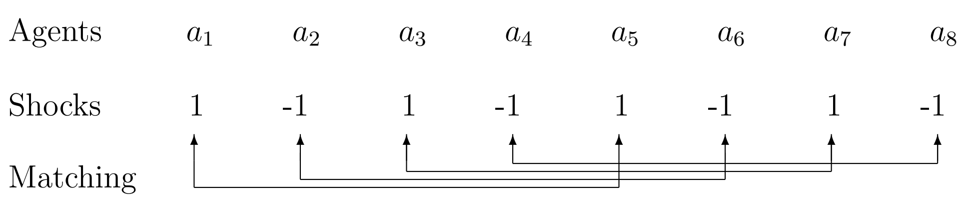

Figure 1. Shocks and matchings for Example 2.

Before proceeding to independent shocks and matchings, we need to define history to capture the idea of independence of the future from current and past events.

3.2. History

Time is discrete, $t \in \mathbb{T} \subseteq \mathbb{Z}$. At the beginning of each period t, the agents experience shocks $\xi_t(a)$, and at the end of each period they meet in accordance with matching $\mathbf{M}_t$.

We denote agent a's history measured right before the matching (or after the current period shock) by $H_t : A \to \mathcal{H}_t$, and right before the shock (or after the previous period matching) by $H'_t : A \to \mathcal{H}'_t$, where $\mathcal{H}_t$ and $\mathcal{H}'_t$ stand for the corresponding spaces of all histories with σ-algebras. The history evolves in accordance with transition functions $h_t : \mathcal{H}'_t \times Y \to \mathcal{H}_t$ and $h'_t : \mathcal{H}_{t-1} \times \mathcal{H}_{t-1} \to \mathcal{H}'_t$ such that $H_t(a) = h_t(H'_t(a), \xi(a))$ and $H'_t(a) = h'_t(H_{t-1}(a), H_{t-1}(\mathbf{M}_{t-1}(a))$. These transition functions, which are assumed to be jointly measurable, determine how much the agents remember from the past and how much information they share during the meetings. We assume that the agent's own shocks and information sharing during the meetings are the only two possible sources for the history:

$$\sigma(H_t(\cdot)) \subseteq \sigma\left(\{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(\cdot)\}_{t_0 \le t_1 < \ldots < t_l < t, l \ge 1}, \{\xi_{t_0}(\cdot)\}_{t_0 \le t}\right); \tag{1}$$

$$\sigma(H'_t(\cdot)) \subseteq \sigma\left(\{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(\cdot)\}_{t_0 \le t_1 < \ldots < t_l < t, l \ge 1}, \{\xi_{t_0}(\cdot)\}_{t_0 < t}\right). \tag{2}$$

Definition. The *maximal history* $\tilde{H}_t(\cdot)$, $\tilde{H}'_t(\cdot)$ is the history with transition functions

$$\tilde{H}_t(a) = h_M(\tilde{H}'_t(a), \xi(a)) \equiv (\tilde{H}'_t(a), \xi(a));$$

$$\tilde{H}'_t(a) = h'_M(\tilde{H}_{t-1}(a), \tilde{H}_{t-1}(\mathbf{M}_{t-1}(a)) \equiv (\tilde{H}_{t-1}(a), \tilde{H}_{t-1}(\mathbf{M}_{t-1}(a)).$$

Thus, under the maximal history, the agents share their full histories during the meetings and do not forget anything from the past (their own shocks and the their partners's histories).

Example 3. Suppose in a 3-period model the agents meet in accordance with Fig. 2, where the arrows represent meetings and circles represent shocks. Thus, Agent 1 meets Agent 5 in period 1, Agent 3 in period 2, and Agent 2 in period 3.

For the maximal history, at the end of period 3 (after the matching) Agent 1 knows his shocks at periods 1, 2, and 3 (known to Agent 1 shocks at the end of period 3 are denoted by large circles), Agent 2's shocks at periods 1, 2, and 3 (because they met at period 3), Agent 3's and Agent 4's shocks at periods 1 and 2 (because Agent 1 met with Agent 3 and Agent 2 met with Agent 4 at period 2), and Agent 5's, Agent 6's, Agent 7's, and Agent 8's shocks at period 1. At the same time, Agent 1 at the end of period 3 does not know, for example, Agent 3's shock at period 3.■

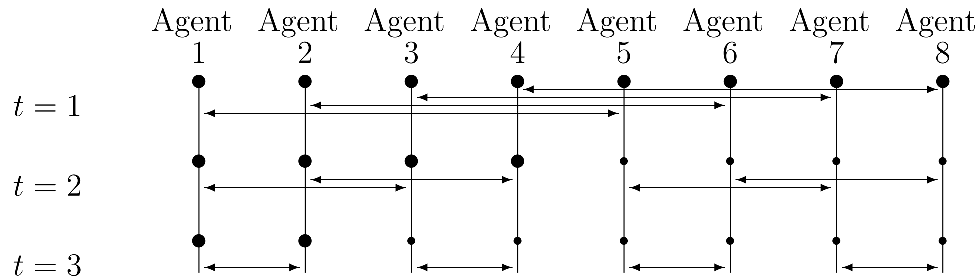

FIGURE 2. Shocks and matchings for Example 3.

Let $\mathcal{A}_t$ be the minimal σ-algebra in which history $H_t(a)$ is a measurable function: $\mathcal{A}_t = \sigma(H_t(\cdot))$, and $\mathcal{A}'_t$ be the minimal σ-algebra in which history $H'_t(a)$ is a measurable function: $\mathcal{A}'_t = \sigma(H'_t(\cdot))$. From formulas (1) and (2) follows that $\mathcal{A}_t, \mathcal{A}'_t \subseteq \mathcal{A}$.[14]

Example 2 (continued). Consider maximal history (see Fig. 1). Before the shock, σ-algebra $\tilde{\mathcal{A}}'_1 = \{\emptyset, A\}$ is trivial, and before the matching σ-algebra (see Fig. 1)

$$\tilde{\mathcal{A}}_1 = \sigma(\{a : \xi(a) = -1\}, \{a : \xi(a) = 1\}) = \{\emptyset, \{a_1, a_3, a_5, a_6\}, \{a_2, a_4, a_7, a_8\}, A\}.$$

After the matching, σ-algebra

$$\tilde{\mathcal{A}}'_2 = \sigma(\tilde{\mathcal{A}}_1, \mathbf{M} \circ \tilde{\mathcal{A}}_1) = \sigma(\{a_1, a_5\}, \{a_2, a_7\}, \{a_3, a_6\}, \{a_4, a_8\}).$$

■

3.3. Independence

Property B7 (Mixing) was not correctly defined due to the fact that no matching can be mixing on the whole nontrivial σ-algebra $\mathcal{A}$, as we showed in subsection 2.3. "Standard Approach Inconsistencies." However, if we consider a family of σ-algebras $\{\mathcal{A}_t\}$ generated by the histories, the sequence of matchings $\{\mathbf{M}_t\}$ might be mixing in the sense of $\mathbf{M}_t$ being mixing on $\mathcal{A}_t$ for each t. In the same way, at time t there might be no future aggregate uncertainty simultaneously on all agent subsets from the history-generated σ-algebra $\mathcal{A}'_t$. In this subsection, we define independence of shocks and matchings from the history.

[14]Inclusion $\mathcal{A}_t, \mathcal{A}'_t \subseteq \mathcal{A}$ does not simply follow from measurability of transition functions and shocks and matchings, because there might be no "start time," and therefore some knowledge might not be related to shocks and matchings; this extra knowledge might have always existed.

Definition. Shock ξ_t is *independent from the history* if

C5. *Shock Independence*: shock ξ_t is independent from σ-algebra $\mathcal{A}'_t \equiv \sigma(H'_t(\cdot))$.

Property C5 (Shock Independence) replaces property A4 (No Aggregate Uncertainty), because a shock is independent from the history-generated σ-algebra, and the fraction of the agents getting their shocks within some measurable set does not depend on the history.

To define a history-independent mixing property, we first need independence of a matching from a σ-algebra. The definition is similar to the definition of the independence of a random variable from a σ-algebra.

Definition. Matching $\mathbf{M}$ is *independent* from σ-algebra $\mathcal{C}$ if σ-algebras $\mathcal{C}$ and $\mathbf{M} \circ \mathcal{C}$ are independent.

Example 2 (continued). It is easy to verify that σ-algebras $\tilde{\mathcal{A}}_1$ and $\mathbf{M}(\tilde{\mathcal{A}}_1)$ are independent. Indeed, consider, for example, sets $B = \{a_1, a_3, a_5, a_6\} \in \tilde{\mathcal{A}}_1$ and $C = \{a_5, a_7, a_1, a_2\} \in \mathbf{M}(\tilde{\mathcal{A}}_1)$. Then

$$\mu(B \cap C) = \mu(\{a_1, a_5\}) = 1/4 = 1/2 * 1/2 = \mu(B)\mu(C).$$

Thus, matching $\mathbf{M}$ is independent from σ-algebra $\tilde{\mathcal{A}}_1 \equiv \sigma(\xi)$.■

Definition. Matching $\mathbf{M}_t$ is *independent from the history* if

C6. *Matching Independence*: matching $\mathbf{M}_t$ is independent from σ-algebra $\mathcal{A}_t \equiv \sigma(H_t(\cdot)))$.

Property C6 (Matching Independence) is a reformulation of Property B7 (Mixing). It says that for any $B, C \in \mathcal{A}_t$ events B and $\mathbf{M}_t(C)$ are independent, or

$$\mu(B \cap \mathbf{M}_t(C)) = \mu(B)\mu(\mathbf{M}_t(C)) = \mu(B)\mu(C).$$

If an agent knows that his history belongs to B, then his partner's history belongs to $\mathbf{M}(C)$ with probability $\mu(C)$. In other words, the agent's belief about partner's history does not depend on the agent's own history.

Properties C5 and C6 (Shock and Matching Independence) replace joint independence of matchings and shocks in the standard approach. Both definitions require the current event (own shock or partner's history) to be independent from the agent's past. For the matching, the past is $H_t(\cdot)$ and the event is $H_t(\mathbf{M}(\cdot))$—the partner's history. For the shock, the past is $H'_t(\cdot)$ and the event is $\xi_t(\cdot)$—the shock. Consequently, independent shocks and matchings defined above are equivalent to what is assumed by idiosyncratic shocks and random matchings in the standard approach.

Later in this paper, we will say that shocks and matchings are independent. By independence of the shocks and matchings, we will mean that shocks and matchings are independent from the history.

Table 1 summarizes correspondence between the standard and new properties.

Old Property	**New Property**
Shocks	
A1 (Measurability)	Shock definition
A2 (Identical Distribution)	Unknown identity
A3 (Independence)	Unknown identity, atomless measure
A4 (No Aggregate Uncertainty)	C5 (Shock Independence)
Matchings	
B1 (No Agent Is Idle), B2 (Involution), B3 (Measurability), B4 (Measure Preserving)	C1 (No Agent Is Idle), C2 (Involution), C3 (Measurability), C4 (Measure Preserving)
B5 Uniform Distribution	Unknown identity
B6 Independence	Unknown identity, atomless measure
B7 Mixing	C6 (Matching Independence)
Joint Independence	
Independence of shocks and matchings	C5 (Shock Independence), C6 (Matching Independence)

TABLE 1. Correspondence of the Properties.

Both independent shocks and matchings are defined through a particular history, which determines σ-algebras $\mathcal{A}'_t$ and A_t. The same shocks and matchings can be either independent or not for different histories. All histories can be partially ordered based on the amount of information they contain. The maximal history, under which the agents remember everything from the past and share complete information during the meetings, bears the maximal amount of information and thus is superior to any other history. If the shocks and matchings are independent for some history, they stay independent for any function of that history, because any functions of independent random variables are independent, too. If shocks and matchings are not independent for some history, they obviously are not independent for a superior history.

3.4. Anonymity in Matchings

The anonymity property, which requires any two agents to meet at most once, directly or through their partners, plays an important role in many economic models (see, for example, Green and Zhou [15]). This property is needed to ensure that the current action of an agent can not influence the behavior of his future partners. In this subsection, we establish a connection between independence of matchings and anonymity.

We define strongly anonymous matchings similar to Aliprantis, Camera, and Puzzello [3]. Namely, let

$$\Pi_t(a) = \bigcup_{t_1<t_2<\ldots<t_l<t, l\geq 0} \{\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a)\}$$

be the set of all agent a's previous direct and indirect partners before the matching at time t, including agent a himself. (For $l = 0$ we assume that $\mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(a) = a$.) By direct partners we mean all agents with whom agent a met directly:

$$\bigcup_{t_1<t} \{\mathbf{M}_{t_1}(a)\},$$

and by indirect partners we mean all agents with whom agent a met through other agents:

$$\bigcup_{t_1<t_2<\ldots<t_l<t, l\geq 2} \{\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a)\}.$$

Sets $\{\Pi_t(a)\}_{a\in A}$ determine if matching $\mathbf{M}_t$ is strongly anonymous.

Definition. Matching $\mathbf{M}_t$ is *strongly anonymous* if no meeting agents have common direct or indirect partners:

$$\forall a \in A \qquad \Pi_t(a) \cap \Pi_t(\mathbf{M}_t(a)) = \emptyset.$$

Example 3 (continued). The matchings in the matching scheme depicted at Fig. 2 are strongly anonymous for the maximal history. For example, $\Pi_1(a_1) = \{a_1\}$, $\Pi_2(a_1) =$

$\{a_1, a_5\}$, and $\Pi_3(a_1) = \{a_1, a_3, a_5, a_7\}$. Thus,

$$\Pi_3(a_1) \cap \Pi_3(\mathbf{M}_3(a_1)) = \emptyset$$

because $\Pi_3(\mathbf{M}(a_1)) = \Pi_3(a_2) = \{a_2, a_4, a_6, a_8\}$.■

The concept of a strongly anonymous matching is very strict; we can allow a possibility of common partners if it has zero probability. To do this, we introduce μ-strongly anonymous matchings.

Definition. Matching $\mathbf{M}_t$ is *μ-strongly anonymous* if, with probability one, the meeting agents do not have common direct or indirect partners:

$$\mu\left(\{a : \Pi_t(a) \cap \Pi_t(\mathbf{M}_t(a)) = \emptyset\}\right) = 1.$$

The following theorem demonstrates that a matching independent from the maximal history should be μ-strongly anonymous. If the history differs from the maximal history, the requirement of μ-strong anonymity can be weakened, because any other history is a function of the maximal history.

Theorem 1. *Let shocks $\xi_t \in \mathbb{R}^m$ have at least one continuous component. Then any independent from the maximal history matching $\mathbf{M}_t$ is μ-strongly anonymous.*

Proof of Theorem 1. Fix time period t. We can consider only continuous component for each of the shocks. Let B be the set of all agents who have previous direct or indirect partners in common with their matches, $B \equiv \{a : \Pi_t(a) \cap \Pi_t(\mathbf{M}_t(a)) \neq \emptyset\}$. Thus,

$$B = \bigcup_{\substack{t_1 < t_2 < \ldots < t_l < t, l \geq 0 \\ t'_1 < t'_2 < \ldots < t'_{l'} < t, l' \geq 0}} B^{t_1 t_2 \ldots t_l}_{t'_1 t'_2 \ldots t'_{l'}},$$

where

$$B^{t_1 t_2 \ldots t_l}_{t'_1 t'_2 \ldots t'_{l'}} = \left\{a \in A : \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_{l'}}(\mathbf{M}_t(a))\right\}.$$

Denote $\eta^{t_1 t_2 \ldots t_l}_{t_0}(a) = \xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a)$. Obviously,

$$B^{t_1 t_2 \ldots t_l}_{t'_1 t'_2 \ldots t'_{l'}} \subseteq \left\{a \in A : \eta^{t_1 t_2 \ldots t_l}_{\min(t_1, t'_1)}(a) = \eta^{t'_1 t'_2 \ldots t'_{l'}}_{\min(t_1, t'_1)}(\mathbf{M}_t(a))\right\}.$$

Matching $\mathbf{M}_t$ is independent from the maximal history, therefore $\mathbf{M}_t$ is independent from $\mathcal{A}_t$, and random variables $\eta^{t_1 t_2 \ldots t_l}_{\min(t_1, t'_1)}$ and $\eta^{t'_1 t'_2 \ldots t'_{l'}}_{\min(t_1, t'_1)} \circ \mathbf{M}_t$ are independent as $\eta^{t_1 t_2 \ldots t_l}_{\min(t_1, t'_1)}$ is

$\mathcal{A}_t$-measurable and $\eta^{t_1' t_2' \ldots t_{l'}'}_{\min(t_1,t_1')} \circ \mathbf{M}_t$ is $\mathbf{M} \circ \mathcal{A}_t$-measurable. From Feller [11] Theorem V.4.4. follows that

$$\mu \left(\left\{ a : \eta^{t_1 t_2 \ldots t_l}_{\min(t_1,t_1')}(a) = \eta^{t_1' t_2' \ldots t_{l'}'}_{\min(t_1,t_1')}(\mathbf{M}_t(a)) \right\} \right) = 0.^{15}$$

Therefore, measure of each $B^{t_1 t_2 \ldots t_l}_{t_1' t_2' \ldots t_{l'}'}$ equals zero. As B is a union of a countable number of sets $B^{t_1 t_2 \ldots t_l}_{t_1' t_2' \ldots t_{l'}'}$, set B has measure zero, too, i.e., matching $\mathbf{M}_t$ is μ-strongly anonymous.■

Theorem 1 states that a sequence of independent matchings should be μ-strongly anonymous, i.e., μ-strong anonymity follows from independence of the matchings. The opposite statement generally is not true, i.e., independence does not follow from strong anonymity because, if the matchings are not independent, the meeting agents might have dependent shocks.[16]

3.5. An Example of an Economic Model with Idiosyncratic Shocks and Random Matchings

In this subsection, we provide an example of an economic model that uses idiosyncratic shocks and random matchings, and discuss the assumptions the authors make. We demonstrate that the existing solutions for idiosyncratic shocks and random matchings problem do not fit the assumptions of this model, whereas the new approach easily justifies them.

Green and Zhou [15] describe the following discrete time model of infinitely lived agents, $t = 0, 1, \ldots$ The agents of measure one are uniformly and independently assigned types from interval $(0, 1]$. There is a perfectly divisible fiat money in the economy. The money

[15]Theorem V.4.4. of Feller [11] states that a convolution of two distributions, one of which is continuous, is also continuous, i.e., it does not have any mass points. The distribution of the difference of two independent random variables is the convolution of the distributions of the first random variable and (minus) the second random variable. Thus, the difference of two independent random variables, one of which is continuous, equals zero with probability zero, and therefore one random variable equals another with probability zero.

[16]In the standard approach with fixed matching, for any two agents with given different sets of previous (direct and indirect) partners, the two σ-algebras generated by independent shocks of the two meeting agents' partners are independent. However, this result cannot be extended to a random matching, where the sets of partners might depend on the realization of the sample space. To see this, consider a one-period model in which the agents first experience some shocks, and then two agents with the highest shocks and two agents with the lowest shocks meet. Although the matching is anonymous (there is only one period), the meeting agents obviously have dependent histories (shocks).

is randomly distributed among the agents in the beginning. Type i agent can consume goods $j \in \left[i, i + \frac{1}{2}\right]$ (mod 1). Each period, an agent with type i produces one unit of nonstorable good i and randomly meets another agent. During the meeting, the agent observes his partner's type. Because only one agent in the pair can consume the good produced by the other agent, one of the agents in each pair is the seller and the other one is the buyer. The buyer submits his bid, and the seller submits his offer. Both the bid and the offer include a price and a quantity. The buyer gets the quantity he bid at the price the seller offered if the bid quantity does not exceed the offer quantity and the offer price does not exceed the bid price.

The assumptions in the model are made in the spirit of the Law of Large Numbers. In particular, the sample distributions of the matchings are the same for all realizations. Also, all buyers and sellers face the same distributions of corresponding offers and bids. The encounters are assumed to be independent across time. An agent's history consists of all previous encounters. The authors consider a symmetric equilibrium in which the agents are anonymous, and the strategies depend on their histories and initial attributes (types and money holdings) only. The matchings are random in the sense that the partner's offer/bid is independent from the agent's history.

The random matching solution suggested by Duffie and Sun [9], which suites Green and Zhou's setup better than any other existing solution, is not applicable here for several reasons. The most important of these is that the Mixing property in Green and Zhou's paper is formulated with respect to the bids and offers, which, based on the equilibrium definition, depend on the initial attributes and agents' histories. The initial attributes are chosen from an uncountable set, and the bids and offers are similarly chosen from uncountable sets. Therefore, the number of bids- and offers-generated sets of agents can be uncountable. At the same time, Duffie and Sun's solution deals with A4'-type properties, thus it can satisfy Mixing property at most on a countable collection of the agent subsets. In addition to Mixing property reason, other issues were not addressed in the Duffie and Sun's solution. These issues include real-valued attributes (types and money holdings), real-valued bids and offers, history-dependent strategies, and possible strategies randomization.

Thus, A4'-type properties considered in the literature on idiosyncratic shocks and random matchings do not fit the models such as Green and Zhou's [15], in which history-generated σ-algebra of the agents might be uncountable. A4"-type properties would resolve the conflicting issues with all history-generated σ-algebras, however, as we showed earlier, the standard approach with A4"-type properties has inner contradictions. All other solutions one can find in the literature also fail to satisfy what we believe the economic models with idiosyncratic shocks and random matchings often require.

To justify the assumptions of this model in the new approach, notice that initial money holdings and agent types can be modeled as initial independent random variables. The offers/bids are corresponding random variables for each period of time. The Law of Large Numbers-type properties hold with respect to history generated (through initial characteristics, shocks, bids/offers, and matchings) agent subsets. The problem of the meeting of two agents with the same type can be resolved by excluding such agents along with those who directly or indirectly meet them (such agent subset has measure zero). Thus, the new approach provides a mathematical justification of Green and Zhou's model, whereas the standard approach fails to do so.

4. Existence

The following theorem constitutes the main result of the paper. It establishes the existence of an agent space with a sequence of independent shocks and matchings for the maximal history. As any other history is a function of the maximal history, the result of the theorem holds for any history.

Theorem 2. *For any m-dimensional cdf $F(\cdot)$, there exists a probability space $(A, \mathcal{A}, \mu)$ with a continuum of atomless agents, and a sequence of shocks $\{\xi_t\}_{t\in\mathbb{T}}$ with cdf $F(\cdot)$ and strongly anonymous matchings $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$, $\mathbb{T} \subseteq \mathbb{Z}$. Both shocks and matchings are independent from the maximal history.*

To prove Theorem 2, we first construct a probability space $(\Theta, \mathcal{Q}, \nu)$ with a sufficient number of random independent variables. Then, we create countably many copies of space Θ and allocate random variables to these copies. The agent space we are looking for is the union of all the copies. At each period of time, the shock is a random variable on the corresponding copy of the original probability space Θ, and the agents from one copy meet corresponding agents from another copy based on the recursive block-partition suggested by Aliprantis, Camera, and Puzzello [4]. Finally, we prove that the shocks and matchings constructed are independent.

Although for some $F(\cdot)$ it might be possible to find a probability space $(A, \mathcal{A}, \mu)$ with different from a continuum number of agents (finite, countable, or more than continuum) with independent shocks and matchings, Theorem 2 says that for any $F(\cdot)$ there at least exists a space with exactly continuum of agents, where $F(\cdot)$ need not be continuous, but can also be discrete, or singular, or a mix of all three types.

In the proof, we construct the shocks and matchings simultaneously with the agent space, which includes σ-algebra and probability measure. Thus, the agent space is not arbitrary, and for a randomly chosen agent space independent shocks and matchings might not exist.

The rest of the section is devoted to the formal proof of Theorem 2.

Proof of Theorem 2.

For simplicity and without loss of generality we assume that $\mathbb{T} = \mathbb{Z}$.

Agent Space

By Kolmogorov theorem (see Wentzell [24]), there exists a probability space $(\Theta, \mathcal{Q}, \nu)$ with a countable number of independent random variables $\{\xi_t^i\}_{i \in \mathbb{N}, t \in \mathbb{T}}$, each of which is distributed in accordance with m-dimensional cdf $F(\cdot)$. Consider spaces A_i, $i \in \mathbb{N}$, of which every one is an exact copy of $(\Theta, \mathcal{Q}, \nu)$. Let functions S_i naturally map Θ onto sets A_i (see Fig. 3). Define agent space $A = A_0 \sqcup A_1 \sqcup A_2 \sqcup \ldots$ and function $s(a)$ — the space number to which $a \in A$ belongs: $a \in A_{s(a)}$. By $S^{-1}(a)$ we understand corresponding $S_i^{-1}(a)$, where $i = s(a)$.

Random variables ξ_t^i have only two indexes, $i \in \mathbb{N}$ and $t \in \mathbb{T}$. As a result, by Kolmogorov theorem, we can take $[0,1]^{|\mathbb{T}|*|\mathbb{N}|*m}$ as space Θ (see Wentzell [24]), and $\bigsqcup_{i=0}^{\infty}[0,1]^{|\mathbb{T}|*|\mathbb{N}|*m}$ as agent space A. Using cardinal arithmetic (see Halmos [16]), one can show that

$$|A| = \left| \bigsqcup_{i=0}^{\infty}[0,1]^{|\mathbb{T}|*|\mathbb{N}|*m} \right| = |[0,1]|,$$

i.e., agent space A we constructed has a continuum of agents.

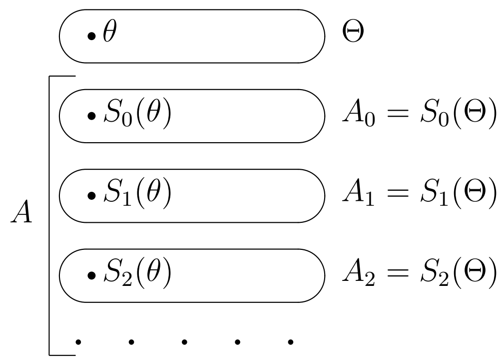

FIGURE 3. Agent space A construction.

Shocks

Define shocks $\xi_t : A \to \mathbb{R}$ in the following way: $\xi_t|A_i = \xi_t^i$. Variables $\xi_t(\cdot)$ consist of different components $\xi_t^i(S^{-1}(\cdot))$, depending on the space A_i to which the argument belongs. We also can write

$$\xi_t(a) = \xi_t^{s(a)}(S^{-1}(a)). \tag{3}$$

Fig. 4 illustrates the construction.

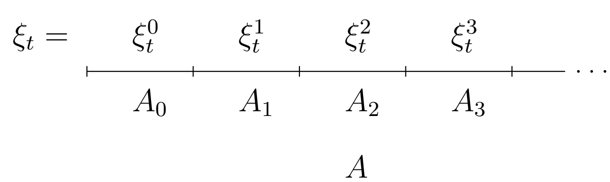

FIGURE 4. Construction of shock ξ_t.

After defining σ-algebra and measure later in this section, we will demonstrate that the shocks are measurable and have cdf $F(\cdot)$.

Matchings

Consider any bijective index $k(\cdot) : \mathbb{T} \to \mathbb{N}$ (for example, $k(0) = 0$, $k(1) = 1$, $k(-1) = 2, \dots$). The index is needed because time is indexed at most with the set of integer numbers, and probability spaces A_i in the definition of the agent space are indexed with the set of natural numbers. With the index, we are able to replace the time space with the space of natural numbers. Which of the two equal concepts is used, time or index, will be clear from each formula.

The matchings scheme is represented in Fig. 5. At time $k = 0$, agents from A_0 meet with the corresponding agents (based on the natural mapping) from A_1, agents from A_2 meet with A_3, agents from A_4 meet with A_5, and so on. At time $k = 1$, agents from $\{A_0, A_1\}$ meet with the corresponding agents from $\{A_2, A_3\}$, agents from $\{A_4, A_5\}$ meet with $\{A_6, A_7\}$, and so on. At time $k = 2$, agents from $\{A_0, A_1, A_2, A_3\}$ meet with the corresponding agents from $\{A_4, A_5, A_6, A_7\}$, and so on. In other words, if we denote the binary expansion of i by $\overline{\ldots x_2 x_1 x_0}$, then at time period with index k agents from $A_{\overline{\ldots x_k \ldots x_1 x_0}}$ meet with the corresponding agents from $A_{\overline{\ldots \overline{x}_k \ldots x_1 x_0}}$, where $\overline{x}_k = 1 - x_k$. The matchings we defined obviously satisfy properties C1 (No Agent Is Idle) and C2 (Involution). Properties C3 (Measurability) and C4 (Measure Preserving) will be satisfied by the definitions of σ-algebra and measure.

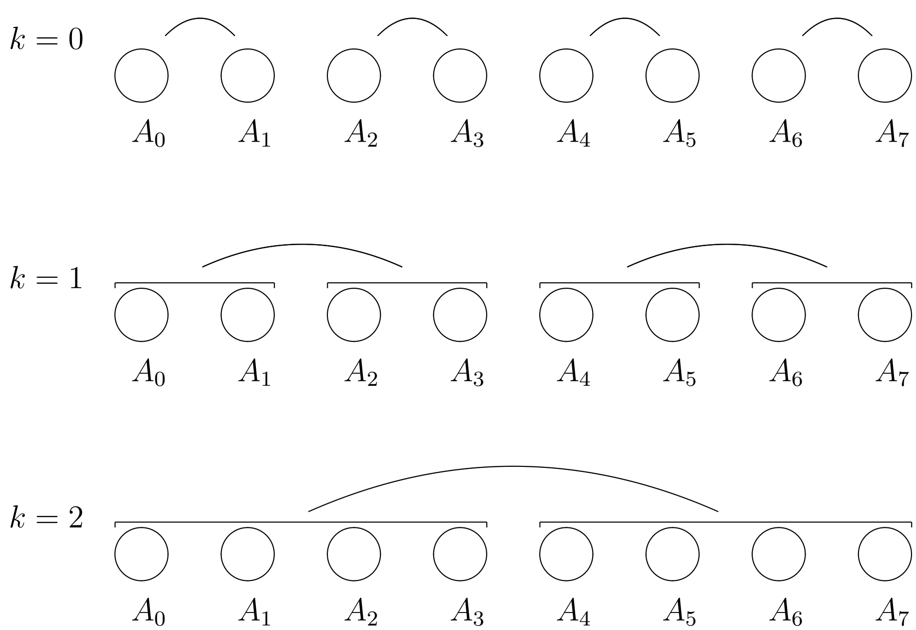

Figure 5. Matchings scheme.

For any $\theta \in \Theta$, agents $\{S_i(\theta)\}_{i \in \mathbb{N}}$ meet in accordance with the recursive block-partition from Aliprantis, Camera, Puzzello [4]. Theorem 13 of the same paper states that such matchings are strongly anonymous. Thus, our matchings $\{M_t\}_{t\in\mathbb{T}}$ are strongly anonymous, too.

For $a' = \mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a)$, one can see that the binary expansion of $s(a')$ of the copy of the space to which a' belongs differs from the binary expansion of $s(a)$ in digits $k_1, k_2, \ldots, k_l$. Therefore,

$$a' = S_{G_{k_1 k_2 \ldots k_l}(s(a))}(S^{-1}(a)), \tag{4}$$

where $G_{k_1 k_2 \ldots k_l}(\cdot)$ is the function of changing $k_1, k_2, \ldots, k_l$-th digits of the binary expansion of the argument. For any permutation of indexes $k_1, k_2, \ldots, k_l$ the result of matchings $\mathbf{M}_{k_1}, \mathbf{M}_{k_2}, \ldots, \mathbf{M}_{k_l}$ is the same, because it does not matter in which order to change the digits. Therefore, for any set of indexes $k_1, k_2, \ldots, k_l$ and for any permutation τ, the following commutativity equation holds:

$$\mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l} \equiv \mathbf{M}_{k_{\tau(1)}} \circ \mathbf{M}_{k_{\tau(2)}} \circ \ldots \circ \mathbf{M}_{k_{\tau(l)}}.$$

This commutativity of the matchings is illustrated in Fig. 6. For example, for Agent 1 holds $\mathbf{M}_0 \circ \mathbf{M}_1(a_1) = a_4 = \mathbf{M}_1 \circ \mathbf{M}_0(a_1)$.

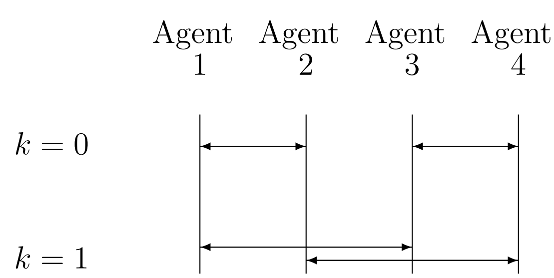

FIGURE 6. Commutativity of the matchings.

Though the matchings we constructed are commutative, by no means do we want to say that commutativity is required for any of the properties we considered (e.g., strong anonymity or independence).

σ-Algebra

Following the definition, σ-algebras $\mathcal{A}_t$ and $\mathcal{A}'_t$ are generated by the maximal histories:

$$\mathcal{A}_t = \sigma\left(\{\xi_t\}, \{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \le t_1 < t_2 < \ldots < t_l < t, l \ge 0}\right);$$

$$\mathcal{A}'_t = \sigma\left(\mathcal{A}_{t-1}, \mathbf{M}_{t-1}(\mathcal{A}_{t-1})\right).$$

Note that $\mathcal{A}_t \subseteq \mathcal{A}'_{t+1} \subseteq \mathcal{A}_{t+1}$. Define σ-algebra $\mathcal{A}$ as the minimal σ-algebras in which all shocks and matchings are measurable:

$$\mathcal{A} = \sigma\left(\{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \leq t_1 < \ldots < t_l, l \geq 0}\right).$$

By defining σ-algebra $\mathcal{A}$ in this way, we made all the shocks and matchings measurable. Thus, for matchings $\{\mathbf{M}_t\}_{t \in \mathbb{T}}$ property C3 (Measurability) is satisfied, and shocks $\{\xi_t\}_{t \in \mathbb{T}}$ are random variables for σ-algebra $\mathcal{A}$.

Probability

In order to define a probability measure on $(A, \mathcal{A})$, we use probability measure ν on $(\Theta, \mathcal{Q})$. Namely, for any $B \in \mathcal{A}$ define

$$\mu(B) = \lim_{j \to \infty} \frac{1}{j+1} \sum_{i=0}^{j} \nu(S^{-1}(B \cap A_i)).$$

To show that the probability is correctly defined for any $B \in \mathcal{A}$, notice that $\mathcal{A}$ is generated by shocks and matchings. Thus, it is enough to show that the definition is correct for any B from a σ-algebra generated by a finite number of shocks and matchings. The latter is equivalent to showing that the definition is correct for any

$$B = \bigcap_{(k_0, \ldots, k_l) \in W} \{a \in A : \xi_{k_0} \circ \mathbf{M}_{k_1} \circ \ldots \circ \mathbf{M}_{k_l}(a) \in B_{k_0 \ldots k_l}\},$$

where $W = \{(k_0, \ldots, k_l)\}_{k_0, \ldots, k_l, l \geq 0}$ is a finite collection of different ordered index sets, and $\{B_{k_0 \ldots k_l}\}_{(k_0, \ldots, k_l) \in W}$ is a collection of Borel sets indexed by W. From Eq. (4) and Eq. (3), for any $a \in A$ holds

$$\xi_{k_0} \circ \mathbf{M}_{k_1} \circ \ldots \circ \mathbf{M}_{k_l}(a) = \xi_{k_0}^{G_{k_1 k_2 \ldots k_l}(s(a))}(S^{-1}(a)),$$

and

$$\begin{aligned} B &= \bigcap_{(k_0, \ldots, k_l) \in W} \left\{a \in A : \xi_{k_0}^{G_{k_1 k_2 \ldots k_l}(s(a))}(S^{-1}(a)) \in B_{k_0 \ldots k_l}\right\} \\ &= \bigcap_{(k_0, \ldots, k_l) \in W} \left\{\bigcup_{i \in \mathbb{N}} \left\{a \in A_i : \xi_{k_0}^{G_{k_1 k_2 \ldots k_l}(i)}(S^{-1}(a)) \in B_{k_0 \ldots k_l}\right\}\right\} \\ &= \bigcup_{i \in \mathbb{N}} \left\{\bigcap_{(k_0, \ldots, k_l) \in W} \left\{a \in A_i : \xi_{k_0}^{G_{k_1 k_2 \ldots k_l}(i)}(S^{-1}(a)) \in B_{k_0 \ldots k_l}\right\}\right\}. \end{aligned}$$

Hence, values

$$\begin{aligned}\nu(S^{-1}(B\cap A_i)) &= \nu\left(S^{-1}\left(\bigcap_{(k_0,\ldots,k_l)\in W}\left\{a\in A_i : \xi_{k_0}^{G_{k_1k_2\ldots k_l}(i)}(S^{-1}(a))\in B_{k_0\ldots k_l}\right\}\right)\right)\\ &= \nu\left(\bigcap_{(k_0,\ldots,k_l)\in W}\left\{\theta\in\Theta : \xi_{k_0}^{G_{k_1k_2\ldots k_l}(i)}(\theta)\in B_{k_0\ldots k_l}\right\}\right)\end{aligned}$$

do not depend on i because of the independence and identical distribution of random variables $\left\{\xi_{k_0}^{G_{k_1k_2\ldots k_l}(i)}(\cdot)\right\}_{(k_0,\ldots,k_l)\in W}$. Therefore, for an arbitrary $B\in\mathcal{A}$ measure $\mu(\cdot)$ is defined correctly. We have also proved that for any $i\in\mathbb{N}$ and $B\in\mathcal{A}$ holds

$$\mu(B)=\nu(S^{-1}(B\cap A_i)).$$

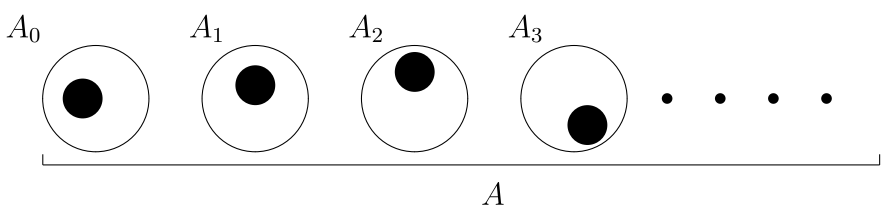

Figure 7. A typical measurable set.

Fig. 7 illustrates a typical measurable set: if $B\in\mathcal{A}$ contains some part of A_{i_0}, then it contains equivalent parts of all other A_i's, in accordance with measure $\nu\circ S_i^{-1}$.

Shocks $\xi_t(\cdot)$ have desired distribution function $F(X)$, because for any measurable C

$$\mu(\xi_t\in C)=\nu(S^{-1}(A_i\cap\{a:\xi_t\in C\}))=\nu(\xi_t^i\in C).$$

By definition, matching $\mathbf{M}_k$ naturally maps A_i onto A_j, where j's binary expansion differs from the i's binary expansion in the k-th digit. Therefore, for any $B\in\mathcal{A}$ sets $\mathbf{M}_k(B)$ and B have the same measure, i.e., property C4 (Measure Preserving) is satisfied.

Independence

Now we can show that matchings $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ and shocks $\{\xi_t\}_{t\in\mathbb{T}}$ are independent, i.e., properties C5 (Shock Independence) and C6 (Matching Independence) hold.

For property A5 (Shock Independence), the history of an agent before the shock at time t is a function of the previous shocks and matchings. Therefore, for any $i \in \mathbb{N}$

$$\begin{aligned}\mathcal{A}'_t \cap A_i &\subset S_i\left(\sigma\left(\left\{\xi^j_{t'}\right\}_{t'<t, j\in\mathbb{N}}\right)\right);\\ \sigma(\xi_t) \cap A_i &= S_i\left(\sigma\left(\xi^i_t\right)\right).\end{aligned}$$

Since random variables $\{\xi^i_t\}$ are independent, we have

$$\begin{aligned}\sigma\left(\left\{\xi^j_t\right\}_{t<t, j\in\mathbb{N}}\right) &\perp \sigma(\xi^i_t);\\ \mathcal{A}'_t \cap A_i &\perp \sigma(\xi_t) \cap A_i;\\ \mathcal{A}'_t &\perp \xi_t,\end{aligned}$$

i.e., shocks $\{\xi_t\}_{t\in\mathbb{T}}$ are independent from the history.

To prove property A6 (Matching Independence), denote W_i—the space indexes of the agents whose shocks are known to the agents from set A_i before the matching at time t, and W'_i—the space indexes of the agents whose shocks are known to the partners of agents from set A_i before the matching at time t:

$$\begin{aligned}W_i &= \bigcup_{a\in A_i, l\geq 0, t_1<\ldots<t_l<t} \{s(\mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(a))\};\\ W_i &= \bigcup_{a\in A_i, l\geq 0, t_1<\ldots<t_l<t} \{s(\mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l} \circ \mathbf{M}_t(a))\}.\end{aligned}$$

Due to strong anonymity, sets W_i and W'_i do not intersect. From the definition of sets W_i and W'_i follows

$$\begin{aligned}\mathcal{A}_t \cap A_i &\subset S_i\left(\sigma\left(\left\{\xi^j_{t'}\right\}_{t'\leq t, j\in W_i}\right)\right);\\ (\mathcal{A}_t \circ \mathbf{M}_t) \cap A_i &\subset S_i\left(\sigma\left(\left\{\xi^j_{t'}\right\}_{t'\leq t, j\in W'_i}\right)\right).\end{aligned}$$

Since random variables $\{\xi^j_t\}_{t\in\mathbb{T}, j\in\mathbb{N}}$ are independent and sets W and W' do not intersect, we have

$$\begin{aligned}\sigma\left(\left\{\xi^j_{t'}\right\}_{t'\leq t, j\in W_i}\right) &\perp \sigma\left(\left\{\xi^j_{t'}\right\}_{t'\leq t, j\in W'_i}\right);\\ (\mathcal{A}_t \cap A_i) &\perp (\mathcal{A}_t \circ \mathbf{M}_t \cap A_i);\\ \mathcal{A}_t &\perp (\mathcal{A}_t \circ \mathbf{M}_t);\\ \mathbf{M}_t &\perp \mathcal{A}_t,\end{aligned}$$

i.e., matchings $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ are independent from the history.■

5. Discussion

In this paper, we have formulated a discrete time model of a continuum of agents with a consistent set of assumptions C1-C6 about shocks and matchings and proved its existence. The shocks are independent from the agents' histories (idiosyncratic shocks). The histories of the meeting agents are also independent (independent matchings). The matchings are strongly anonymous, and this means that the meeting agents cannot meet again in the future, directly of through the partners. An important result of the paper is that the properties hold for all history-generated agent subsets. This result was not obtained in any previous paper.

The model suggested in this paper is flexible and allows us to explore a wide range of extensions. Some of these extensions are as follows:

(1) Dependent shocks. In addition to independent matchings, the agents can experience shocks that depend on the histories.

(2) Dependent partners. The agents might not meet randomly: instead, the agents with high shocks might have a higher chance to meet other agents with high shocks.

(3) Finitely lived agents. Some agents die and some agents are born during every period of time.

(4) Nonanonymous matchings. The agents have a non-zero chance of meeting the current partner in the future.

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