Analysis of Current Penalty Schemes for Violations of Antitrust Laws

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Analysis of Current Penalty Schemes for Violations of Antitrust Laws[1]

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Abstract. The main feature of the penalty schemes described in current sentencing guidelines is that the fine is based on the accumulated gains from cartel activities or price-fixing activities for the firm. The regulations suggest modeling the penalty as an increasing function of the accumulated illegal gains from price fixing to the firm, so that the history of the violation is taken into account. We incorporate these features of the penalty scheme into an optimal control model of a profit-maximizing firm under antitrust enforcement. To determine the effect of taking into account the history of the violation, we compare the outcome of this model with a model where the penalty is fixed. The analysis of the latter model implies that complete deterrence can be achieved only at the cost of shutting down the firm. The proportional scheme improves upon the fixed penalty, since it can ensure complete deterrence in the long run, even when penalties are moderate. Phase-diagram analysis shows that, the higher the probability and severity of punishment, the sooner cartel formation is blocked. Further, a sensitivity analysis is provided to show which strategies are most successful in reducing the degree of price fixing. It turns out that, when the penalties are already high, the antitrust policy aiming at a further increase in the severity of punishment is less efficient than the policy that increases the probability of punishment.

Key Words. Optimal control, dynamic analysis, antitrust policy, antitrust laws.

1. Introduction

This paper analyzes the optimal policies for the deterrence of violations of antitrust law. We study the effects of penalty schemes, determined according to

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the current US and EU antitrust laws, on the behavior of the firm. We investigate the intertemporal aspects of this problem using a dynamic optimal control model of utility maximization by the firm under antitrust enforcement.

This paper addresses the problem of whether the fine, determined on the basis of accumulated turnover of the firm participating in a cartel, can provide a complete deterrence outcome. We assume that the imposed fine takes into account the history of the violation. This means that, when the violation of antitrust law is discovered, the regulator is able to observe all the accumulated rent from cartel formation. Consequently, it will impose a fine that takes into account this information. We compare also the deterrence power of this system with the fixed penalty scheme.

The 2002 OECD report provides a description of the available sanctions for cartels according to the laws of member countries (Ref. 1). Those laws allow for considerable fines against enterprises found to have participated in price-fixing agreements. In some cases, however, the maximal fines determined by these laws may not be sufficiently large to accommodate multiples of the gain to the cartel, as suggested by the expected utility theory. In most of the countries, the maximal fines are expressed either in absolute terms or as a percentage (10%) of the overall annual turnover of the firm (Ref. 2). However, according to expert estimations, the best policy is to impose penalties which are a multiple of the illegal gains from price-fixing agreements to the firms. Of course, this would be difficult to estimate in real life, so it is still common practice to use the percentage of turnover as a proxy of the gains from price-fixing activities.

Several countries, namely the US, Germany, and New Zealand, have already accommodated this more advanced system, where the fine is stated in terms of unlawful gains (Ref. 3). In general, the determination of the final amount of the fine, to be paid by the firm in each particular case, is based on the degree of offense, which is proportional either to the amount of accumulated illegal gains from the cartel or to its proxy, turnover involved throughout the entire duration of the infringement. At the same time, there exists an upper bound for the penalties for violations of antitrust law. The fine is constrained from above by the maximum of a certain monetary amount, a multiple of the illegal gains from the cartel, or if the illegal gain is not known, 10% of the total annual turnover of the enterprise. The idea of the current paper is to incorporate these features of the current penalty systems into a dynamic model of intertemporal utility maximization by a firm, which is subject to antitrust enforcement.

Similar to Font et al. (Ref. 4) or Leung (Ref. 5), the set up of the problem leads to an optimal control model. Compared to Ref. 4 or Ref. 6, the main difference is that the gain from the cartel accumulated by the firm over the period of infringement takes the role of a state variable, whereas the idea of Ref. 4 was to take the offender's criminal record as a state variable of the dynamic game. Thus, an increase in the state variable is positively related to the degree of price fixing by the firm and increases the fine which the firm can expect in the case of being convicted.

Furthermore, this framework allows us to analyze the consequences of two major modifications of the penalty system for violations of competition law, which have been suggested recently by the OECD and the US Department of Justice (DOJ). The modification suggested by the OECD was concerned with the increase of the multiplier for the base fine, while DOJ (Ref. 7) suggests to increase the upper bound for the fine up to \$100 million. By solving the optimal control problem of the firm under antitrust enforcement, we will investigate the implications of the different penalty schedules.

The main results are that, for the benchmark case (i.e., when the penalty is fixed), the outcome with complete deterrence of cartel formation is possible but only at the cost of shutting down the firm. In other words, the fixed penalty, which can ensure complete deterrence, is too high, because it leads to immediate bankruptcy. However, the result can be improved by relating the penalty to the illegal gains from price fixing. The proportional scheme appears to be more appropriate than the fixed penalty, since it can ensure complete deterrence in the long run even in the case where the penalties are moderate. We study also the impact of the main parameters of the penalty scheme (probability and severity of punishment) on the efficiency of deterrence and analyze the optimal tradeoff between changes in the scale parameter of the proportional penalty scheme and probability of law enforcement. It turns out that, the higher the probability and severity of punishment, the earlier the cartel formation is blocked. The sensitivity analysis shows that, when the penalties are already high, the antitrust policy aiming at a further increase in the severity of punishment is less efficient than the policy that increases the probability of punishment.

The paper is organized as follows. In Section 2, we describe the general setup of an optimal control model of the firm under antitrust enforcement. In Section 3, we consider the case where the upper bound for the penalty is an exogenously given fixed monetary amount. Moreover, we will derive an analytical expression for this upper bound, which allows achieving the result of complete deterrence of price fixing. In Section 4, we investigate the implications of the penalty being proportional to the accumulated gains from price fixing. We conduct also sensitivity analysis of the equilibrium values of the variables of the model with respect to the parameters of the penalty scheme.

2. Optimal Control Model: General Setup

We introduce the basic ingredients of the intertemporal optimization problem of a profit maximizing firm, which participates in an illegal cartel. The key variable is the accumulated gain from prior criminal offenses (in the case of a cartel, these offenses are price-fixing activities).

2.1. Dynamics of the Accumulated Rent from Price Fixing. The accumulated rent from price fixing $w(t)$ is the state variable of the model, which increases depending on the degree of offense (price fixing). Using a continuous-time scale, the dynamics of the accumulated rent from price fixing equals[4]

$$\dot{w}(t) = \pi^m q(t)[2 - q(t)], \quad \text{with } w(0) = w_0 \geq 0; \tag{1}$$

here $\dot{w}(t)$ stands for the change in the value of the state variable, $q(t)$ denotes the degree of price fixing by the firm at instant t, and w_0 is the initial wealth of the firm before the start of the planning horizon. The expression (1) rests on the assumption of the demand function being linear. A complete derivation of expression (1) is given in the Appendix (Section 5), where $\dot{w}(t)$ is associated with instantaneous producer surplus for the firm caused by fixing price levels above the competitive. The main idea behind this formulation is that cartel formation leads to higher prices. The normal price is c (competitive equilibrium) leading to zero profits. Then, q denotes the degree of violation, i.e. when the cartel fixes a higher price than normal. From the definition of q in the Appendix, it is clear that, in the case of such a violation (i.e. when price is higher than competitive level), q is positive. Based on a simple linear demand function[5], profit or producer surplus can be expressed as a concave function of q. Now, the state variable $w(t)$ adds up the profits over time and as such $w(t)$ is the total gain from crime (too high prices) from time 0 up to time t.

There are strong legal and economic reasons for the introduction of a state variable in the form of accumulated rent from price fixing. It is related to the fact that, in the US and EU guidelines for imposition of fines for antitrust violations, the penalty imposed in many cases is based mainly on the turnover involved in the infringement throughout the entire duration of the infringement. Clearly, the accumulated turnover serves as a proxy for accumulated gains from cartel or price-fixing activities for the firm.

2.2. Profit Function. The instantaneous illegal gains from price fixing for the firm equal $\pi^m q(t)[2 - q(t)]$; this function has been derived from the microeconomic model underlying the problem of price fixing[6]. Obviously, this function implies that the marginal profit for the firm is always positive and strictly declining in the interval $q(t) \in [0, 1]$. Moreover, for each positive level of offense, the profit is also positive.

The instantaneous profit at time t will also be influenced by the accumulated rent from price fixing. This variable measures also the experience the firm has in

[4] To simplify the analysis for the rest of this section, we assume $w_0 = 0$. However, relaxing this assumption does not change the results stated in the propositions of the paper.

[5] See Appendix, Section 5.

[6] For complete derivation of this expression, see the Appendix (Section 5).

forming a cartel. The more the experience, the more efficiently the firm colludes; consequently, the higher the instantaneous profits from price fixing. This influence is reflected in the term $\gamma w(t)$, which enters additively the objective function of the firm; see the expression (4) below[7].

2.3. Law Enforcement Policy. The goal of the current section is to incorporate the features of the penalty system for antitrust law violations, described above, into the optimal control model of intertemporal utility maximization by the firm in the presence of a benevolent antitrust authority, whose aim is to minimize the loss of consumer surplus, i.e. to block any degree of price fixing. So, in order to capture the specifics of the sentencing guidelines and current antitrust practice, we model the penalty for violations of antitrust law as a linearly increasing function of the accumulated rent from price-fixing for the firm. Therefore, it can be written as

$$s(w(t)) = \alpha w(t). \tag{2}$$

This setup will allow also to study the effects of the changes of the multiplier for the base fine (refinement suggested by OECD) on the deterrence power of the penalty scheme.

According to Becker (Ref. 8), the cost of different punishment to an offender can be made comparable by converting them into their monetary equivalent or worth. And this is satisfied in our model, since we measure the accumulated rent from price fixing for the firm in monetary units. Moreover, our specification of the penalty function satisfies three main conditions specified in Ref. 4, namely: it is strictly increasing in the level of offense [since $w(t)$ is strictly increasing in $q(t)$], firms which do not collude at all should not be punished [$s(w_0) = 0$], and any detected positive level of offense should lead to a positive amount of punishment [$s(w(t)) > 0$, for any $w(t) > w_0$, which is equivalent to $q(t) > 0$ for some $t \in [0, T]$]. This implies that, if the firm has been checked, violated the law in the current period, and participated in the cartel in some of the previous periods, the fine will be imposed on the basis of the whole accumulated gains from price fixing $w(t)$ and thus not only on the basis of the current degree of offense $q(t)$.

[7] It may be more realistic to express this term as a nonlinear function of w. In particular, a concave formulation may be very tractable, since there might be decreasing marginal returns from experience. However, it will not change the results of the paper in a qualitative sense. The solution of the model in case the experience gain is modeled as $\gamma\sqrt{w}$ gives the outcome with complete deterrence similar to Proposition 2 and results of sensitivity analysis for the model with proportional penalty still hold. The analysis of the model, where penalty is fixed, with $\gamma\sqrt{w}$ term gives the same qualitative result but the model can only be solved numerically. A complete proof of this statement is available from the author upon request.

Further, we will compare the efficiency and deterrence power of the penalty systems for a model in which the penalty is given by the expression (2) and a model in which the penalty is fixed [$s(t) = S^{\max}$], where $S^{\max}$ is the fixed upper bound for the penalty introduced in the sentencing guidelines, which is not related to the level of offense.

2.4. Costs of Being Punished. The cost of being punished at time t equals the expected value of the fine that has to be paid. This will be defined as a multiple of the probability of being checked by the antitrust authority p (level of law enforcement) times the degree of offense at time t, $q(t)$, times the level of punishment, which depends on time as well,

$$\text{expected penalty} = s(t)q(t)p, \quad \text{with } p \in [0, 1]. \tag{3}$$

So, the expected penalty is determined by the expression (3), where $pq(t)$ is the probability of being punished at time t and $s(t)$ is the fine, which may either be fixed or can be expressed as a function of the accumulated gains from price fixing.

We should stress here that the firm can only be caught at time t if $q(t) > 0$, i.e. the offense is committed exactly at this time. Of course, this need not be the case for criminal acts in general: you can convict a thief, if the police has found the stolen things without having caught the burglar in action.[8] However, it does apply to antitrust law practices. According to Refs. 1 and 3, investigation concerning past behavior starts only at the moment it is observed that the current price exceeds the competitive price, thus when $q(t) > 0$. After this is proved (usually on the basis of empirical analysis of price markups), the antitrust authority will start a more detailed investigation and get access to accounting books and documents that can prove the existence of a cartel agreement. Only after that the gains from price fixing $w(t)$ become perfectly observable, so that the court or competition authority can take them into account while determining the amount of fine to be paid.[9]

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[8] We thank an anonymous referee who pointed out this difference.

[9] Here, it is also important to realize that the probability of being caught at instant t is $pq(t)$, so that the firm can only be caught at time t_1 if it does price fixing on that date, so if $q(t_1) > 0$. Later in time, say at time $t_2 > t_1$, the firm cannot be punished because of the offence at time t_1. At t_2, it can only be caught and punished if $q(t_2) > 0$. At the moment the firm is caught, it has to pay a fine $s(t)$. In one scenario, this fine is an increasing function of $w(t)$. So, this means that, if the firm did a lot of price fixing in the past, implying that $w(t)$ is large, the fine will be larger. In this sense, repeated offenders are more heavily punished, and this is what happens quite frequently in modern democratic societies. So, if the firm is caught at time t_2, it is convicted for the crime on t_2; the level of the fine depends on what the firm did in the past, thus also what it did at time $t_1 < t_2$ as well. In other words, the higher the degree of price fixing at t_1, the larger the fine will be at t_2. This is independent of how many times

2.5. Optimization Problem. The firm making the decision about the degree of price fixing faces the following intertemporal decision problem:

$$\max J(q(t)) := \int_0^\infty e^{-rt}\{\pi^m q(t)[2-q(t)] + \gamma w(t) - s(t)pq(t)\}dt, \tag{4a}$$

$$\text{s.t. } \dot{w}(t) = \pi^m q(t)[2-q(t)] \quad \text{and} \quad q(t) \in [0, 1]. \tag{4b}$$

The parameter r is the discount rate. The objective functional $J(q(t))$ is the discounted profit stream gained from engaging in price-fixing activities. The term $\pi^m q(t)[2-q(t)]$ reflects the instantaneous rent from collusion; the term $-s(t)p(t)q(t)$ reflects the possible punishment for the firm, if it is caught. Note that, the higher the degree of collusion, the higher $q(t)$, the higher the expected punishment. $\gamma w(t)$ reflects the experience of the firm in cartel formation that increases future instantaneous gains from cartel formation.

Having made the assumptions of Section 2, we define the current value Hamiltonian

$$\begin{aligned} H^c(q, w, \mu) = {} & \pi^m q(t)[2-q(t)] + \gamma w(t) - s(t)pq(t) \\ & + \mu(t)\{\pi^m q(t)[2-q(t)]\}, \end{aligned} \tag{5}$$

where $\mu(t)$ is the current value adjoint variable representing the shadow price of the offense. The Hamiltonian is well-defined and differentiable for all nonnegative values of the state variable $w(t)$ and all values of the control variable $q(t)$ in its domain [0, 1].

3. Penalty Represented by a Fixed Monetary Amount

In this section, we model the situation where the penalty for violations of antitrust law is represented by a fixed monetary amount. In this case, we assume that the fine does not depend on the accumulated gains from price fixing and is constant over time. This might be a good framework to study the efficiency of antitrust enforcement in an environment where there exists an upper bound for penalties and offenses are so grave that the punishment always reaches its upper bound, which is true for highly cartelized markets. The analysis of this model is quite essential, since the imposition of the upper bound for penalties for violations of antitrust law is still a current practice in most countries. Only Norway and Denmark do not have this limitation. This model also allows to take into account the DOJ new policy that suggests to increase the upper bound for

the firm was caught in the past: the fine the firm paid before will not be subtracted from w. Since w is nondecreasing over time, it is taken into account implicitly that repeated offenders will be more heavily punished.

the fine for violations of antitrust law up to $100 million. We modify the model of Section 2 in such a way that the fine is given by some fixed monetary amount $S^{\max}$, which denotes the maximal penalty. In other words, the antitrust authority commits to a policy of the following form: the rate of law enforcement is constant $p(t) = p \in (0, 1]$, for all t; when the firm is inspected, the penalty is

$$s(t) = S^{\max}, \quad \text{if } q(t) > 0,$$
$$s(t) = 0, \quad \text{if } q(t) = 0.$$

In this section, we show that, if the fixed penalty or upper bound for the fine imposed by the law is not high enough, complete deterrence is never possible. Moreover, we derive an analytical expression for the upper bound, which allows achieving the result of complete deterrence of price fixing. The main difference with the proportional penalty model is that the penalty does not depend on the accumulated illegal gains. For simplicity, we assume that there is no discounting[10] ($r = 0$), the planning horizon is finite ($T < \infty$), the salvage values for both players are equal to zero, so that the transversality conditions are $\lambda(T) = 0$, $\mu(T) = 0$ for both players.

We derive the dynamic system for the optimal control $q(t)$ from the following necessary optimality conditions:

$$q(t) = \arg\max_q H^c(q, w, \mu), \tag{6}$$

$$\dot{\mu}(t) = -\partial H(q, w, \mu)/\partial w. \tag{7}$$

The expression (7) gives

$$\dot{\mu}(t) = -\gamma.$$

Solving this simple differential equation in the case of finite planning horizon, we get

$$\mu(t) = \gamma(T - t).$$

Consequently, we get

$$\mu(t) \geq 0, \quad \text{for all } t \in [0, T].$$

This allows us to conclude that the Hamiltonian (5) is strictly concave with respect to q. Therefore, the condition (6) is equivalent to $H_q^c = 0$. It leads to

$$q^*(t) = 1 - pS^{\max}/\{2\pi^m[1 + \gamma(T - t)]\} = C. \tag{8}$$

[10] To make the analysis more transparent and analytically solvable, we assume here that $r = 0$. However, imposing that $r > 0$ does not change the qualitative predictions of the model. Only the dynamics of the costate variable of the firm changes. The equation for $\mu(t)$ becomes $\mu(t) = (\gamma/r)[1 - \exp(t - T)]$. A complete proof of this statement is available from author upon request.

Table 1. Impact of penalty on the accumulated gains from collusion and degree of price fixing.

Penalty	Accumulated gains from collusion	Degree of price fixing
$S^{\max} = 2$	$w(t) = 0.5/(-12 + t) + 2t + 25/24$, $\to w(T) \approx 20.792$	$q^*(t) = 1 - 1/(24 - 2t)$ $\to q(T) \approx \frac{3}{4}$
$S^{\max} = 10$	$w(t) = 12.5/(-12 + t) + 2t + 49/24$, $\to w(T) \approx 15.792$	$q^*(t) = 1 - 5/(24 - 2t)$ $\to q(T) = 0$
$S^{\max} = 20$	$w(t) = 50/(-12 + t) + 2t + 31/6$, $\to w(T) \approx 0.166$	$q^*(t) = 1 - 10/(24 - 2t)$ $\to q(T) = 0$

However, the control region of the offense rate q is limited to $[0, 1]$ by construction. This implies that the expression for the optimal degree of price fixing by the firm is given by

$$q^*(t) = 0, \quad \text{if } C \leq 0,$$
$$q^*(t) = C, \quad \text{if } 0 < C \leq 1.$$

Following the expression (8), we can represent the optimal degree of price fixing by the firm q as a decreasing function of both the penalty for violation and time. The first part of this statement is quite intuitive, since a higher expected penalty will obviously increase the incentives for the profit maximizing firms to avoid participation in price fixing agreements and thus reduce the degree of offense q. The negative relationship between the degree of price fixing and time is related to the fact that higher gains from price fixing in the beginning imply that, for a longer time period, the firm can take advantage of it, in the sense that, due to increased experience, profits from price fixing will be higher. So, incentives to commit crime decrease over time and hence the degree of offense falls.

3.1. State-Control Dynamics. After we substitute (8) into (1), the differential equation describing the dynamics of the state variable will be as follows:

$$\dot{w}(t) = \pi^m \left[1 - \left\{ \frac{S^{\max} p}{2\pi^m [1 + \gamma(T - t)]} \right\}^2 \right]. \tag{9}$$

The results of the solution of this differential equation for different values of $S^{\max}$, the other parameters being

$$p = 1/2, \pi^m = 2, \gamma = 1/2, T = 10, w(0) = 1,$$

are summarized in Table 1.

Consequently, when all the parameters of the model are fixed, $w(t)$ is increasing over time and the degree of offense is a decreasing function of time. Unfortunately, we must conclude that, for example, when the fixed penalty equals

2, which is the instantaneous monopoly profit for the firm for these parameter values, it does not allow to achieve complete deterrence even in the last period. On the contrary, the last period degree of price fixing is quite high (75% out of 100%).

We can conclude that the policies with fixed penalty appear to be highly inefficient, since to achieve

$$q^*(t) = 0, \quad \text{for all } t \in [0, T],$$

we should have

$$1 - s(t)p/2\pi^m[1 + \gamma(T - t)] \leq 0,$$

which implies

$$s(t) \geq 2\pi^m[1 + \gamma(T - t)]/p.$$

In the example, with parameter values

$$T = 10,\ \pi^m = 2,\ \gamma = 1/2,\ p = 1/2,$$

we get

$$s(0) \geq 48 = 24\pi^m, \quad s(T) = s(10) \geq 8 = 4\pi^m.$$

This enormous penalty will drive the firm to bankruptcy immediately. Moreover, this result is counterintuitive and unfair, since the firm colluding for one period will obtain less extra gain than a firm colluding for ten periods; consequently, t should be punished less.

The main result of the analysis of the model with fixed penalty is represented in the following proposition.

Proposition 3.1. In the optimal control model, where $p(t) = p > 0$, for all $t \in [0, T]$, the no collusion outcome (i.e. complete deterrence of price fixing) occurs when $S^{\max}(t) \geq \{2\pi^m[1 + \gamma(T - t)]\}/p$ for all $t \in [0, T]$, thus when $S^{\max}(0) \geq 2\pi^m(1 + \gamma T)/p$.

The implication of this result is that the penalty for antitrust violation, which potentially can provide complete deterrence, should be imposed by the antitrust authority (not by the court), i.e. by the authority which has complete information about the probability of law enforcement.

The fine should be inversely related to the probability of investigation (similar to Ref. 8). Moreover, the penalty should be based mainly on the instantaneous monopoly profits in the industry. Of course, this value is different for each industry, so the specifics of the industry also should be taken into account when the optimal fine for antitrust violations is determined. The length of the planning horizon should also be taken into account.

However, in real life, the implementation of this scheme is problematic, since the court (not the antitrust authority) imposes the penalty; consequently, the parameter p cannot be verified.

Unfortunately, the fixed penalty system does not always work. For

$$S^{\text{fixed}} < 2\pi^m[1+\gamma(T-t)]/p, \quad \text{for some } t,$$

the result with no price-fixing outcome during the whole planning period is not possible. However, the new DOJ policy may be quite successful, since \$100 million seems to be higher than $2\pi^m(1+\gamma T)/p$ for reasonable parameter values, such as

$$p = 1/5, \quad \pi^m = \$1\text{million}, \quad \gamma = 1/5, \quad T = 10.$$

Moreover, this result resembles the result, of Emons (Ref. 9), where the subgame perfect punishment for repeated offenders in a repeated games setting was investigated. The final conclusion of that paper is that if the regulator's aim is to block violation at the lowest possible cost, the penalty should be a decreasing function of time. Moreover, Emons concludes that the first period penalty (penalty for the first detected violation) should be the highest and should extract the entire wealth of the offender. So, another drawback of this system is that it does not explain escalating sanctions based on offense history which are embedded in many penal codes and sentencing guidelines.

Another problem with this result is that the fixed penalty, which can ensure complete deterrence, is too high. It is clearly unbearable for the firm and leads to immediate bankruptcy. Already for the first violation, we have to punish twenty times more than the maximal per-period monopoly profit. To resolve this impossibility result, we look at the other scheme that relates the penalty to the illegal gains from price fixing. In particular, in the next section, we introduce the penalty as a linearly increasing function of the accumulated gains from price fixing for the firm given by the expression (2) above. The proportional scheme is preferred to the fixed penalty, since it can ensure complete deterrence in the long run, even in the case where the penalties are moderate.

4. Analysis of the Model with Penalty Schedule $s(t) = \alpha w(t)$

This setup reflects another important feature of the penalty systems for violations of antitrust law suggested by current sentencing guidelines. Namely, that the fine is proportional to the illegal gains from cartel formation. This more advanced system has been implemented already in the US, Germany, New Zealand, and some other countries.

4.1. Utility Maximization. As before, we derive the optimal control $q(t)$ from the following necessary optimality conditions:

$$q(t) = \arg\max_q H^c, \tag{10}$$

$$\dot{\mu}(t) - r\mu(t) = -\gamma + \alpha p q(t). \tag{11}$$

Since the control region of the offense rate q is limited by [0, 1], the maximization condition (10) is equivalent to

$$q^*(t) = \begin{cases} 0, & \text{if } C < 0 \\ C, & \text{if } 0 \le C \le 1, \\ 1, & \text{if } C > 1 \end{cases} \quad \text{where } C = 1 - \alpha w(t)p/\{2\pi^m[1+\mu(t)]\}. \tag{12}$$

We conclude that the optimal degree of price-fixing by the firm is a decreasing function of both the penalty for violation and the probability of law enforcement. This is also quite intuitive from an economic point of view. First, the profit maximizing firm will reduce its optimal degree of price fixing in response to the increase in the rate of law enforcement, since it makes conviction more likely; Second, increase in accumulated rent from collusion also rises the expected penalty; this gives an additional incentive for the firm to reduce the degree of price fixing. This allows the system to gradually converge to the socially desirable outcome with no price fixing.

4.2. State-Costate Dynamics. Substituting (12) into (1) and (11) gives the following system of differential equations:

$$\dot{w}(t) = \pi^m \left[1 - \left\{ \frac{\alpha w p}{[2\pi^m(1+\mu)]} \right\}^2 \right] = 0, \tag{13a}$$

$$\dot{\mu}(t) = -\gamma + \alpha p \left\{ 1 - \frac{\alpha w p}{[2\pi^m(1+\mu)]} \right\} + r\mu = 0. \tag{13b}$$

A stationary point can be obtained by intersecting the locuses $\dot{w} = 0$ and $\dot{\mu} = 0$, which are given respectively by

$$w(\mu) = 2\pi^m(\mu + 1)/p\alpha,$$

$$w(\mu) = 2\pi^m(-\gamma\mu - \gamma + p\alpha\mu + p\alpha + \mu^2 r + r\mu)/\alpha^2 p^2.$$

The steady state of the system (13), being located in the positive orthant, is given by

$$\mu^* = \gamma/r, \quad w^* = 2\pi^m(1 + \gamma/r)\,\alpha p.$$

This implies that $q^* = 0$.

The necessary conditions for the existence of stationary points in the positive orthant are $\gamma < r$ and $p > 0$.[11] This means that, when the extra benefits for the firm from cartel formation do not increase much with the experience of the firm in cartel formation ($\gamma < r$), the outcome with no collusion is more likely to be sustained in the long run, since it is less attractive for the firm to participate in the cartel agreements. So, a unique stationary point in the positive orthant always exists, except when $p = 0$ (i.e. the probability to be caught is zero) or when $\gamma > r$ (i.e. the extra benefits for the firm from cartel formation increase very fast when the experience of the firm in cartel formation increases). The optimal control problem does not have a stable solution in cases $p = 0$ or $\gamma > r$.

Example 4.1. Next, the solution procedure and construction of the phase portrait is illustrated via an example. We construct the phase portrait when the parameters are

$$\gamma = 0.5, \quad \pi^m = 1, \quad \alpha = 2, \quad p = 0.2, \quad r = 0.2.$$

The $\dot{w} = 0$ isocline is given by

$$\mu = -1 + (1/5)w.$$

Similarly, the $\dot{\mu} = 0$ isocline is given by

$$\mu = -1/4 + (1/20)\sqrt{(225 + 160w)}.$$

The stationary point then satisfies

$$w^* = 35/2 \ \text{ and } \ \mu^* = 2.5.$$

Studying the stability of the above steady-state equilibrium, we obtain the following expressions for the values of trace and determinant of the Jacobian matrix of the system (13):

$$\text{trace } J = 1/5 > 0, \quad \det J = -4/175 < 0.$$

This allows us to conclude that the point

$$w^* = 35/2, \ \mu^* = 2.5, \ q^* = 0$$

is a saddle point.

[11] The detailed proof of this statement is available from the authors upon request.

4.3. Stability Analysis. Starting with the system dynamics (13) in the state-costate space, we can calculate the Jacobian matrix

$$J = \begin{pmatrix} -[\alpha p/(1+\mu)]^2 2w/4\pi^m & 2(\alpha p w)^2/4\pi^m(1+\mu)^3 \\ -(\alpha p)^2/2\pi^m(1+\mu) & (\alpha p)^2 w/2\pi^m(1+\mu)^2 + r \end{pmatrix}.$$

Obviously, the determinant has to be evaluated in the steady state (μ^*, w^*, q^*). It turns out that trace $J > 0$ and det $J < 0$, so that the steady state is a saddle point.

In general, with arbitrary values of the parameters and arbitrary equilibrium values, the matrix J has two real eigenvalues of opposite sign and the steady state has the local saddle-point property. This means that there exists a manifold containing the equilibrium point such that, if the system starts at the initial time on this manifold and at the neighborhood of the equilibrium point, it will approach the equilibrium point at $t \to \infty$.

This proves the following proposition.

Proposition 4.1. The outcome with complete deterrence is sustainable in the long run, given that the parameter p is strictly greater than zero. The steady state with $\mu^* = \gamma/r$, $w^* = 2\pi^m(1 + \gamma/r)/\alpha r$, and $q^* = 0$ is a saddle point.

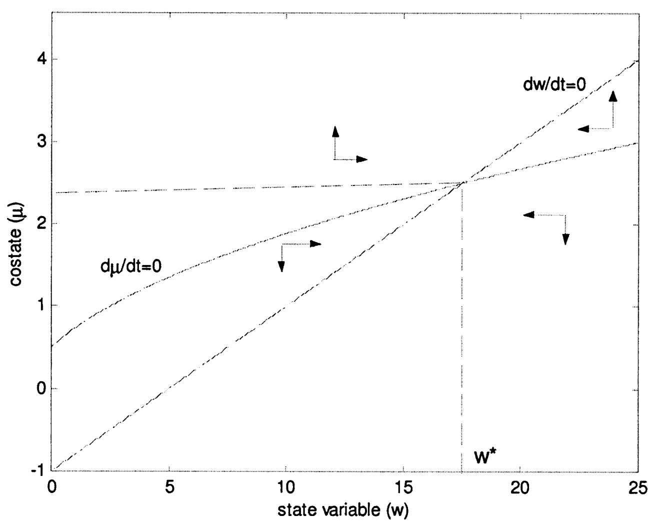

Fig. 1. Phase portrait in the (w,μ)-space for the optimal control model for the set of parameter values $\gamma = 0.5, \pi = 1, \alpha = 2, p = 0.2, r = 0.2$, where the penalty schedule is given by $s(t) = \alpha w(t)$.

The proposition implies that, in the long run, the full compliance behavior arises in the sense that the outcome with $q^* = 0$ is the saddle point equilibrium of the model. This means that one can choose always the initial value for the adjoint variable such that the equilibrium trajectory starts on the stable manifold and converges to the steady state. Economically speaking, the firm which maximizes profits over time under a proportional penalty scheme will gradually reduce the degree of violation to zero. However, there is one exception: for $p = 0$ the degree of offense is maximal. The parameter α influences only the speed of convergence to the steady-state value, not the steady-state value of the control variable. Clearly, a higher α increases the incentives for the firm to stop the violation earlier. Basically, deciding on the time of stopping the violation, the firm compares the expected punishment and expected benefits from crime. Consequently, since in the setup with proportional penalty the expected punishment rises also when the benefits from price fixing rise, in the long run the system will end up in the equilibrium with full compliance.

4.4. Sensitivity Analysis. Here, we investigate in which direction the saddle-point equilibrium moves if the set of parameter values changes. Analyzing the properties of the proportional penalty scheme [$s(t) = \alpha w(t)$], the main parameters of interest are the scale parameter of the penalty schedule α and the parameter which determines the certainty of punishment p. They appear to be also quite important parameters for the firm, whose objective is to maximize the expected rent from price-fixing in the presence of antitrust enforcement. Clearly, the firm will condition its behavior on the parameters of the penalty scheme, chosen by the regulator [see the expression (4)]. Moreover, the result obtained below will provide hints on how to choose the optimal enforcement policy to minimize the steady-state degree of price-fixing by the firms.

As a result of the necessary optimality conditions, in the steady-state equilibrium it holds that

$$
\begin{aligned}
&w(t) = f(q, w, \mu, \alpha) = \pi^m q(2 - q) = 0,\\
&\mu(t) = r\mu(t) - H_w(q, w, \mu, \alpha) = r\mu - \gamma + \alpha pq = 0,\\
&H_q(q, w, \mu, \alpha) = (2\pi^m - 2\pi^m q)(1 + \mu) - \alpha wp = 0.
\end{aligned}
$$

Computing the total derivatives of the above equations with respect to α and appling Cramer's rule[12], we obtain that

$$\partial\mu/\partial\alpha = -qp/r < 0.$$

[12] More detailed derivations of these results are available from the authors upon request.

In a similar way, we study the behavior of the costate variable with respect to a change in the probability of law enforcement,

$$\partial\mu/\partial p = -\alpha q/r < 0.$$

This means that the equilibrium steady-state value of the shadow price decreases when the slope of the penalty function α increases or the rate of law enforcement increases. The reason is that, with higher α or p, a higher accumulated wealth increases the expected punishment much faster than in the case when α or p are low.

In the same way, we can derive the sign of $\partial w/\partial\alpha$ and $\partial w/\partial p$. Again, application of Cramer's rule implies that

$$\partial w/\partial\alpha = -w/\alpha - 2\pi^m(1-q)q/r\alpha < 0.$$

Similar calculations for the parameter p give that

$$\partial w/\partial\alpha = -w/\alpha - 2\pi^m(1-q)q/r\alpha < 0.$$

This means that either an increase in the scale parameter of the penalty scheme or an increase in the certainty of punishment would cause a reduction of the equilibrium accumulated rent from collusion, so that the firms will try to reduce their gains in order to be punished less.

Finally, we take a look at the change of the offense level caused by a change in the slope of the punishment function or a change in the rate of law enforcement. This means that we are now interested in the signs of $\partial q/\partial\alpha$ and $\partial q/\partial p$. Computing the determinants, we find that

$$\partial q/\partial\alpha = \partial q/\partial p = 0.$$

So, we can conclude that the effect of either change in certainty or in severity of the penalty on the equilibrium value of the degree of offense is absent. It follows logically from the model, since $q^* = 0$ is a steady-state solution of the model and its absolute value and existence does not depend on the size of the parameters α and p.

The change in α or in p influences only the t^{**} value in Figure 2[13]. The numerical analysis of the behavior of the state and control variables of the model

[13] A no price-fixing outcome $[q(t) = 0]$ can be sustained, but it occurs only at the end of the planning period. To be more precise, the dynamics of the optimal behavior of the firm is such that, given the parameters of the penalty system (p and α), the firm gradually reduces the degree of offense to zero, which happens at time t^{**}. After that, no more collusion will take place. Consequently, the accumulated gains from price-fixing will gradually increase and after $t = t^{**}$ will stay at the level $w(t^{**})$. The parameters of the penalty system (p and α) have an impact on the optimal behavior of the firm and consequently on the deterrence power of the penalty system, which is measured by the timing of optimal deterrence or in other words by the value of t^{**}. The higher α and p, the closer the t^{**} to the origin, consequently, the earlier the cartel formation is blocked.

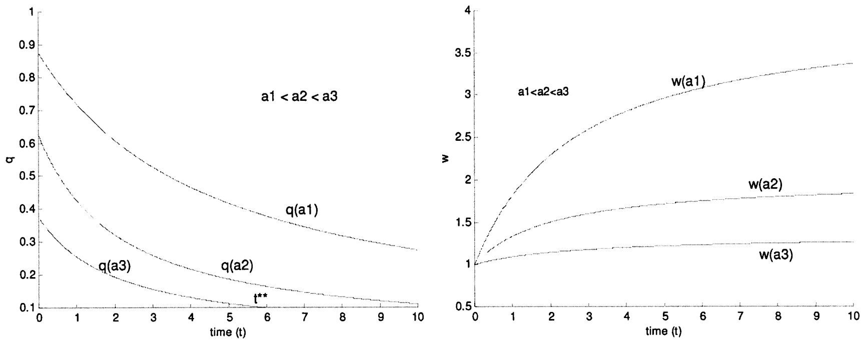

Fig. 2. Numerical analysis of the behavior of the state and control variables of the model with respect to the scale parameter of the penalty scheme α when the parameter values are $\gamma = 0.5$, $\pi = 1$, $p = 0.2$, $r = 0.2$.

with respect to the main parameters of the penalty scheme (α and p) shows that a higher α or p leads to earlier deterrence; i.e. t^{**} moves closer to the origin (see Figure 2). Consequently, the degree of price fixing is lower at each instant of time and the total accumulated gains from price fixing by the colluding firm are lower. Moreover, this policy allows to reduce the costs for society as well, since we can block violation earlier and hence reduce the control efforts earlier.

Looking at the partial derivatives of the state variable of the model with respect to the main parameters of the penalty scheme we obtain the following proposition.

Proposition 4.2.

(a) Under the policies that provide underdeterrence $\{$i.e. when α is low, i.e. $\alpha = p \in [0, 1]\}$, the effects of the detection probability and severity of punishment on the deterrence power of the penalty scheme in the steady state are equal.

(b) When α is high, i.e. under the policies that can potentially provide more efficient deterrence, the effect of the increase in the probability of punishment on the deterrence power of the penalty scheme in steady state is much stronger.

Proof. Consider the partial derivatives of the state variable of the model with respect to the main parameters of the penalty scheme. Following the above analysis, based on Cramer's rule, we derive

$$\partial w/\partial \alpha = -w/\alpha - 2\pi^m(1-q)q/r\alpha,$$

$$\partial w/\partial p = -w/p - 2\pi^m(1-q)q/rp.$$

Now, we can show that, when α is potentially higher than p (thus, for instance, when $\alpha > 1$), the decrease in w, in absolute terms, when α increases, is much less than the decrease in w, in absolute terms, when p increases. Assume $\alpha > 1$; then from the expression for $\partial w/\partial\alpha$, we obtain

$$|\partial w/\partial\alpha| < [wr + 2\pi^m(1-q)q]/r.$$

Similarly, keeping in mind that $p \in [0, 1]$ by construction, from the expression for $\partial w/\partial p$, we obtain that $|\partial w/\partial p| > [wr + 2\pi^m(1-q)q]/r$. □

The general conclusion of this subsection is that, when $w_0 = 0$, only partial deterrence is feasible. But nevertheless, $q(t) = 0$ for some $t \in [t^{**}, T]$ can be achieved in the model if $p(t) > 0$ for all $t \in [0, T]$ and the equilibrium with $q^* = 0$ can be sustained as the long run saddle point steady-state equilibrium of the model with penalty system given by $s(t) = \alpha w(t)$ and $p > 0$ under certain additional conditions on the parameters of the model.

Moreover, studying the sensitivity of the steady state values of the main variables of the model with respect to the parameters of the penalty scheme, we found an interesting result, which gives new insights into the problem of the optimal tradeoff between the probability and severity of punishment. This problem has been studied quite extensively in a static setting by Polinsky and Shavell (Ref. 10) and later by Garoupa (Refs. 11–12). The result, stated in Proposition 4.2, shows that, when the penalty is high, a further increase in the severity of punishment is less efficient than an increase in probability of punishment.

5. Appendix: Static Microeconomic Model of Price Fixing

Let us consider an industry with N symmetric firms engaged in a price fixing agreement. Assume that they can agree and increase prices from $p^c = c$ to $p > c$ each, where c is the marginal cost in the industry. Since the firms are symmetric, each of them has equal weight in the coalition; consequently, the total cartel profits will be divided equally among them.[14] Hence, the whole market for the product (in which the price-fixing agreement has been achieved) will be divided equally among N firms, so each firm operates in a specific market in which the inverse demand function equals

$$p(Q) = 1 - Q.$$

They are identical in all the submarkets. Under these assumptions, we can simplify the setting by considering not the whole cartel (group of violators), but only one

[14] We assume also that there is no strategic interaction between the firms in the coalition in the sense that we abstract from the possibility of self-reporting or any other noncooperative behavior of the firms toward each other.

firm, and apply similar sanctions to all the members of cartel.[15] Further, we denote by p^m the monopoly price in the industry under consideration and by $p = 1 - Q$ is the inverse demand for a particular firm. In order to be able to represent the consumer surplus and extra profits from price fixing for the firm π in terms of the degree of collusion, we specify the variable q as follows. Let $q = (p - c)/(p^m - c)$, where p^m is the monopoly price and p is the price level agreed by the firms. Then, we can conclude that $q \in [0, 1]$ and that the instantaneous extra profits from price fixing for this particular firm will be determined according to the following formula:

$$\pi = q\left[(1 - c)/(p^m - c) - q\right](p^m - c)^2.$$

Let

$$(p^m - c)^2 = A.$$

With linear demand

$$p = 1 - Q,$$

we observe that

$$p^m = (1 + c)/2,$$

so that

$$(1 - c)/(p^m - c) = 2;$$

consequently, it holds that

$$A = (1 - c)^2/4 = \pi^m$$

(monopoly profit in this particular market).

The instantaneous producer surplus, consumer surplus, and net loss in consumer surplus are represented in Figure 3.

So, the instantaneous producer surplus will be determined as

$$PS(q) = \pi(q) = \pi^m q(2 - q).$$

The net loss of consumer surplus will be the area of the right triangle, i.e.

$$\text{net loss of } CS = (1/2)\pi^m q^2.$$

The consumer surplus will be determined by the area of the triangle ABC,

$$CS(q) = (1/2)\pi^m(2 - q)^2.$$

[15] Of cause, in these settings the incentives of the firms to betray the cartel cannot be taken into account and the possibility to influence the internal stability of the cartel is not feasible. But this is the topic for another paper.

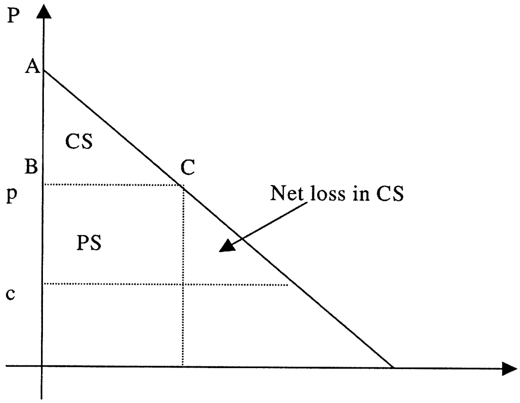

Fig. 3. Representation of producer and consumer surpluses in the price-quantity diagram.

Note that we can represent the consumer and producer surpluses as a continuous differentiable functions of the degree of price-fixing, i.e.

$$PS'(q) > 0, \text{ net loss of } CS'(q) > 0, \quad \text{and } CS'(q) < 0,$$

while

$$PS''(q) < 0, \text{ net loss of } CS''(q) > 0, \quad \text{and } CS''(q) > 0 \text{ for all } q \in [0, 1].$$

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