Normal versus Noncentral Chi-square Asymptotics of Misspecified Models

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Abstract. The noncentral chi-square approximation of the distribution of the likelihood ratio (LR) test statistic is a critical part of the methodology in structural equations modeling (SEM). Recently, it was argued by some authors that in certain situations normal distributions may give a better approximation of the distribution of the LR test statistic. The main goal of this paper is to evaluate the validity of employing these distributions in practice. Monte Carlo simulation results indicate that the noncentral chi-square distribution describes behavior of the LR test statistic well under small, moderate and even severe misspecifications regardless of the sample size (as long as it is sufficiently large), while the normal distribution, with a bias correction, gives a slightly better approximation for extremely severe misspecifications. However, neither the noncentral chi-square distribution nor the theoretical normal distributions give a reasonable approximation of the LR test statistics under extremely severe misspecifications. Of course, extremely misspecified models are not of much practical interest.

Key Words. Model misspecification, covariance structure analysis, maximum likelihood, generalized least squares, discrepancy function, noncentral chi-square distribution, normal distribution, factor analysis.

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Introduction

It is well recognized that no model can represent real data exactly (e.g., see Browne & Cudeck, 1993). Therefore, even reasonably good models are often rejected for larger sample sizes by standard test statistics. This motivated investigations of the statistical properties of test statistics under alternative hypotheses. A classical result states that under a sequence of local alternatives, i.e., the so-called population drift, and certain regularity conditions, likelihood ratio (LR) test statistics asymptotically have a noncentral chi-square distribution. Thus, the noncentral chi-square distribution is widely used for model evaluation and power analysis of testing in structural equations modeling (SEM). In practice this means that rather than assuming an exact fit of the data to a considered model, one can estimate the population discrepancy with the model by employing an estimate of the corresponding noncentrality parameter. Usage of noncentral chi-square asymptotics has a long history in the statistics literature (e.g., see McManus, 1991, for a historical overview). In the analysis of covariance (moment) structures it goes back to Shapiro (1983) and J. H. Steiger et al. (1985).

One of the criticisms of this approach is that the assumption of the population drift, where the population covariance matrix is assumed to depend on the sample size, is unrealistic. Recently this issue was discussed in a number of publications with a suggestion that the normal distribution could sometimes be a better alternative for approximating the true distribution of the LR test statistics (e.g., Golden, 2003 ; Olsson, Foss, & Breivik, 2004 ; Yuan, Hayashi, & Bentler, 2007 ; Yuan, 2008).

In this paper, we empirically compare the noncentral chi-square distribution with the normal distribution in describing the behavior of the LR test statistics T_{ML} under a variety of sample sizes and model misspecifications. Our simulation results may be of some practical assistance to researchers facing model evaluation so that they can derive reasonable inferences.

This paper is organized as follows. Theoretical background regarding noncentral chi-square and normal approximations is given in the next section. Then the results of Monte Carlo experiments aimed at evaluation of the appropriateness of using the noncentral chi-square and normal distributions for LR test statistics are given. In particular, the Kolmogorov-Smirnov distance and quantile-quantile (QQ) plots are provided as measures of the distributions' fit. We also use the Thurstone data (Thurstone & Thurstone, 1941) from a classic study of mental ability for our illustration. Discussion section gives some remarks and suggestions for future directions of research.

Theoretical background

Let us start with a critical look at the noncentral chi-square distribution. Let $Y_1, ..., Y_k$ be a sequence of independent random variables having normal distributions with standard deviation 1 and respective means $\mu_1, ..., \mu_k$, i.e., $Y_i \sim N(\mu_i, 1)$, $i = 1, ..., k$. Then the random variable $V = Y_1^2 + ... + Y_k^2$ has noncentral chi-square distribution with k degrees of freedom and noncentrality parameter $\delta = \mu_1^2 + ... + \mu_k^2$, denoted $V \sim \chi_k^2(\delta)$. Note that the distribution of V depends only on the sum $\mu_1^2 + ... + \mu_k^2$, and not on the individual means μ_i. Therefore we can assume that $\mu_1 = \mu$ and $\mu_2 = ... = \mu_k = 0$. In that case $\delta = \mu^2$ and

$$V = (Z_1 + \mu)^2 + Z_2^2 + ... + Z_k^2 = \underbrace{Z_1^2 + Z_2^2 + ... + Z_k^2}_{W} + 2\mu Z_1 + \mu^2, \qquad (1)$$

where $Z_i \sim N(0, 1)$ are independent standard normal random variables.

The right hand side of (1) can be considered as the sum of two components, namely, the sum $W = Z_1^2 + ... + Z_k^2$ which has a (central) chi-square distribution with k degrees of freedom, and the term $2\mu Z_1 + \mu^2$ which has normal distribution $N(\mu^2, 4\mu^2)$. Moreover, variables Z_1^2 and Z_1 are uncorrelated, and hence these two terms are uncorrelated. Recall that the expected value of W is k and its variance is $2k$. For large values of the noncentrality parameter δ, the term $2\mu Z_1 + \mu^2$ becomes dominant and hence the corresponding noncentral chi-square distribution could be well approximated by the normal distribution $N(k+\delta, 2k+4\delta)$. It also could be noted that the random variable W is given by the sum of k independent identically distributed random variables, and hence by the Central Limit Theorem its distribution approaches normal with increase of the number of degrees of freedom k. In other words a noncentral chi-square distribution can be well approximated by the respective normal distribution if the number of degrees of freedom k is large even if the noncentrality parameter δ is small or even zero. That is, a noncentral chi-square distribution can be approximately normal if either the noncentrality parameter is large or the number of degrees of freedom is large or both.

Consider a covariance structure model $\mathbf{\Sigma} = \mathbf{\Sigma}(\boldsymbol{\theta})$ relating parameter vector $\boldsymbol{\theta} \in \mathbb{R}^q$ to $p \times p$ population covariance matrix. Let $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ be a random sample from the considered population, and $\boldsymbol{S} = (n-1)^{-1} \sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'$ be the corresponding sample covariance matrix. Recall that $\boldsymbol{S}$ is an unbiased estimate of the population covariance matrix $\mathbf{\Sigma}_0$. The popular test statistic for testing the model is $T_{ML} = n\hat{F}_{ML}$, where

$$\hat{F}_{ML} = \min_{\boldsymbol{\theta}} F_{ML}(\boldsymbol{S}, \mathbf{\Sigma}(\boldsymbol{\theta})) \tag{2}$$

and

$$F_{ML}(\boldsymbol{S}, \mathbf{\Sigma}) = \log|\mathbf{\Sigma}| + \operatorname{tr}(\boldsymbol{S}\mathbf{\Sigma}^{-1}) - \log|\boldsymbol{S}| - p. \tag{3}$$

We say that the *normality assumption* holds if the population, from which the random sample is drawn, has *normal* distribution, i.e., $\boldsymbol{X}_i \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}_0)$, $i = 1, ..., n$. In that case $\frac{n-1}{n}\boldsymbol{S}$ becomes the Maximum Likelihood estimator[1] of the population covariance matrix and T_{ML} becomes the corresponding likelihood ratio test statistic. This is why T_{ML} is referred to as the ML test statistic. Of course, this test statistic can be computed whether the population distribution is normal or not. We will discuss this point later.

The classical result, going back to Wilks (1938), is that if the model is correct, i.e.,

$\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\boldsymbol{\theta}_0)$ for some value $\boldsymbol{\theta}_0$ of the parameter vector, then under the normality assumption and mild regularity conditions the asymptotic distribution of the test statistic T_{ML} is central chi-square with $df = p(p+1)/2 - q$ degrees of freedom. Let us briefly outline arguments behind this theoretical result. Consider the function

$$f(\boldsymbol{Z}) = \min_{\boldsymbol{\theta}} F_{ML}(\boldsymbol{Z}, \boldsymbol{\Sigma}(\boldsymbol{\theta})) \tag{4}$$

of a $p \times p$ positive definite symmetric matrix variable $\boldsymbol{Z}$. Note that here $\boldsymbol{Z}$ is a general (matrix valued) variable while $\boldsymbol{S}$ denotes the sample covariance matrix, so that for $\boldsymbol{Z} = \boldsymbol{S}$ we have that $\hat{F}_{ML} = f(\boldsymbol{S})$.

In the subsequent analysis we use notation $\boldsymbol{s}, \boldsymbol{\sigma}, \boldsymbol{z}$, for the $p^2 \times 1$ dimensional vectors[2] obtained by stacking columns of the respective matrices $\boldsymbol{S}, \boldsymbol{\Sigma}, \boldsymbol{Z}$, i.e., $\boldsymbol{s} = \mathrm{vec}(\boldsymbol{S})$, etc. Observe that the ML discrepancy function F_{ML} has the following properties. For any positive definite symmetric matrices $\boldsymbol{Z}$ and $\boldsymbol{\Sigma}$, it holds that $F_{ML}(\boldsymbol{Z}, \boldsymbol{\Sigma}) \geq 0$ and $F_{ML}(\boldsymbol{Z}, \boldsymbol{\Sigma}) = 0$ if and only if $\boldsymbol{Z} = \boldsymbol{\Sigma}$. This implies that $f(\boldsymbol{z}) \geq 0$ for any $\boldsymbol{z}$, and $f(\boldsymbol{z}) = 0$ for $\boldsymbol{z} = \boldsymbol{\sigma}_0$. That is, if the model is correct, then the function $f(\boldsymbol{z})$ attains its minimum (equal zero) at $\boldsymbol{z} = \boldsymbol{\sigma}_0$, and hence vector $\partial f(\boldsymbol{\sigma}_0)/\partial \boldsymbol{z}$, of partial derivatives at $\boldsymbol{z} = \boldsymbol{\sigma}_0$, is zero. By using the *second order* Taylor expansion of $f(\boldsymbol{z})$ at the point $\boldsymbol{z} = \boldsymbol{\sigma}_0$, we can approximate

$$f(\boldsymbol{s}) \approx f(\boldsymbol{\sigma}_0) + (\boldsymbol{s} - \boldsymbol{\sigma}_0)'[\partial f(\boldsymbol{\sigma}_0)/\partial z] + (\boldsymbol{s} - \boldsymbol{\sigma}_0)'\boldsymbol{Q}(\boldsymbol{s} - \boldsymbol{\sigma}_0), \tag{5}$$

where $\boldsymbol{Q} = \frac{1}{2}\partial^2 f(\boldsymbol{\sigma}_0)/\partial \boldsymbol{z} \partial \boldsymbol{z}'$ is half the Hessian matrix of second order partial derivatives of $f(\boldsymbol{z})$ at $\boldsymbol{z} = \boldsymbol{\sigma}_0$. Since $T_{ML} = nf(\boldsymbol{s})$ and by the above the first two terms $f(\boldsymbol{\sigma}_0)$ and $(\boldsymbol{s} - \boldsymbol{\sigma}_0)'[\partial f(\boldsymbol{\sigma}_0)/\partial \boldsymbol{z}]$ in the above expansion vanish, it follows that

$$T_{ML} \approx [n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)]'\boldsymbol{Q}[n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)]. \tag{6}$$

Now by the Central Limit Theorem (CLT) we have that $n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)$ converges in distribution to a (multivariate) normal[3] with zero mean vector and a covariance matrix $\boldsymbol{\Gamma}$, given by

$$\boldsymbol{\Gamma} = \mathbb{E}\big\{\mathrm{vec}[(\boldsymbol{X}_i - \boldsymbol{\mu})(\boldsymbol{X}_i - \boldsymbol{\mu})']\mathrm{vec}'[(\boldsymbol{X}_i - \boldsymbol{\mu})(\boldsymbol{X}_i - \boldsymbol{\mu})']\big\} - \boldsymbol{\sigma}_0\boldsymbol{\sigma}_0'. \tag{7}$$

This implies that T_{ML} converges in distribution to the distribution of the quadratic form $\boldsymbol{Y}'\boldsymbol{Q}\boldsymbol{Y}$, where $\boldsymbol{Y}$ is a random vector having normal $N(\mathbf{0}, \boldsymbol{\Gamma})$ distribution. If the population has normal distribution, then the matrix $\boldsymbol{\Gamma}$ has a specific structure, which is a function of the covariance matrix $\boldsymbol{\Sigma}_0$ alone, i.e., does not involve calculation of forth order moments of the population distribution. We denote this matrix by $\boldsymbol{\Gamma}_N$ in order to emphasize that it is computed under the assumption of normality. The point is that under the normality assumption and standard regularity conditions, we have here that $\boldsymbol{Q}\boldsymbol{\Gamma}_N\boldsymbol{Q} = \boldsymbol{Q}$ and matrix $\boldsymbol{Q}$ has rank $p(p+1)/2 - q$. Then invoking some algebraic manipulations it is possible to show that the distribution of the quadratic form $\boldsymbol{Y}'\boldsymbol{Q}\boldsymbol{Y}$ is (central) chi-square with $df = p(p+1)/2 - q$ degrees of freedom (cf., Shapiro, 1983, Theorem 5.5).

It is worthwhile to point the following. In this derivation the only place where the assumption about normality of the population distribution was used is verification of the equation $\boldsymbol{Q}\boldsymbol{\Gamma}_N\boldsymbol{Q} = \boldsymbol{Q}$, which is based on a particular structure of the covariance matrix $\boldsymbol{\Gamma}_N$. In some cases this equation can be verified, and hence asymptotic chi-squaredness of the distribution of T_{ML} can be established, even without the normality assumption. This is a basis of the so-called asymptotic robustness theory of the ML discrepancy test statistic (cf., Browne & Shapiro, 1988).

Suppose now that the model is *misspecified*, i.e., the population covariance matrix $\boldsymbol{\Sigma}_0$ is different from $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ for any value of the parameter vector $\boldsymbol{\theta}$. We still have that $T_{ML} = nf(\boldsymbol{s})$ with function $f(\cdot)$ defined in (4) and, by the CLT, $n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)$ converges in distribution to (multivariate) normal $N(\mathbf{0}, \boldsymbol{\Gamma})$. However, now the term

$$F^*_{ML} = \min_{\boldsymbol{\theta}} F(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}(\boldsymbol{\theta})), \tag{8}$$

representing the discrepancy between the *population* value $\boldsymbol{\Sigma}_0$ of the covariance matrix and the model, is strictly positive. Consequently, the first term $f(\boldsymbol{\sigma}_0) = F^*_{ML}$ in the second order Taylor expansion, given in the right hand side of (5), does not vanish and is strictly positive. It follows that for large n the statistic T_{ML} can be approximated by nF^*_{ML} and will grow to infinity as $n \to \infty$. A more precise statement is that $n^{-1}T_{ML} = \hat{F}_{ML}$ converges with probability one (w.p.1) to F^*_{ML}. Also by employing the *first order* Taylor expansion at the point $\boldsymbol{z} = \boldsymbol{\sigma}_0$, i.e., by using first two terms in the right hand side of (5), we can write

$$n^{1/2}[f(\boldsymbol{s}) - f(\boldsymbol{\sigma}_0)] \approx [n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)]'[\partial f(\boldsymbol{\sigma}_0)/\partial \boldsymbol{z}]. \tag{9}$$

It is possible to show that

$$\frac{\partial f(\boldsymbol{\sigma}_0)}{\partial \boldsymbol{z}} = \frac{\partial F_{ML}(\boldsymbol{z}, \boldsymbol{\sigma}^*)}{\partial \boldsymbol{z}}\Big|_{\boldsymbol{z}=\boldsymbol{\sigma}_0}, \tag{10}$$

where $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\boldsymbol{\theta}^*)$ and $\boldsymbol{\theta}^*$ is the minimizer of the function $F_{ML}(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}(\boldsymbol{\theta}))$, provided that this minimizer is unique (equation (10) follows by the so-called Danskin Theorem). Recalling that $f(\boldsymbol{s}) = \hat{F}_{ML}$ and $f(\boldsymbol{\sigma}_0) = F^*_{ML}$ and that $n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)$ converges in distribution to $\boldsymbol{Y} \sim N(\boldsymbol{0}, \boldsymbol{\Gamma})$, we obtain that $n^{1/2}\big(\hat{F}_{ML} - F^*_{ML}\big)$ converges in distribution to $\boldsymbol{\gamma}'\boldsymbol{Y} \sim N(0, \boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma})$, where

$$\boldsymbol{\gamma} = \frac{\partial F_{ML}(\boldsymbol{z}, \boldsymbol{\sigma}^*)}{\partial \boldsymbol{z}}\Big|_{\boldsymbol{z}=\boldsymbol{\sigma}_0} = \operatorname{vec}\left[(\boldsymbol{\Sigma}^*)^{-1} - \boldsymbol{\Sigma}_0^{-1}\right]. \tag{11}$$

This implies the following result (see Shapiro, 1983, section 5 for technical details):

- Let $\boldsymbol{\theta}^*$ be the unique minimizer of $F_{ML}(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}(\boldsymbol{\theta}))$. Then $n^{1/2}\big(\hat{F}_{ML} - F^*_{ML}\big)$

converges in distribution to normal $N(0, \boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is given in (11) and $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\boldsymbol{\theta}^*)$. In other words we can approximate the distribution of $T_{ML} = n\hat{F}_{ML}$ by the normal distribution with mean nF^*_{ML} and variance $n\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma}$.

The (asymptotic) covariance matrix $\boldsymbol{\Gamma}$ depends on the population distribution. In particular, if the population distribution is normal, then (cf., Shapiro, 2009)

$$\boldsymbol{\gamma}'\boldsymbol{\Gamma}_N\boldsymbol{\gamma} = 2\operatorname{tr}\left[\left(\boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}_0^{-1}\right)\boldsymbol{\Sigma}_0\left(\boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}_0^{-1}\right)\boldsymbol{\Sigma}_0\right] = 2\operatorname{tr}\left[\left(\boldsymbol{\Sigma}^{*-1}\boldsymbol{\Sigma}_0 - \boldsymbol{I}_p\right)^2\right]. \tag{12}$$

If the population distribution is normal, and hence T_{ML} becomes the likelihood ratio test statistic, then the above result can be also derived from Vuong (1989). Note, however, that the above asymptotic normality of T_{ML} holds even without the normality assumption, although in that case the right hand side of (12) may be not a correct formula for the asymptotic variance $\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma}$. We will discuss this issue further later.

Theoretically this is a correct result. However, in any real application the question is: "how good is this normal approximation for a *finite* sample?" Let us point to the obvious deficiencies of the normal approximation. Any normal distribution is symmetric around its mean. On the other hand, as it was mentioned earlier, the test statistic T_{ML} is always nonnegative and its distribution is typically skewed especially when $\boldsymbol{\Sigma}_0$ is not "too far" from the model and hence the (population) discrepancy F^*_{ML} is close to zero. In the extreme case when the model is correct, we have that $F^*_{ML} = 0$ and $\boldsymbol{\gamma} = \mathbf{0}$, and hence the normal approximation, of $n^{1/2}\hat{F}_{ML}$, degenerates into the identically zero distribution. This should be not surprising since in that case T_{ML} converges (in distribution) to a finite limit and hence $n^{1/2}\hat{F}_{ML} = n^{-1/2}T_{ML}$ tends (in probability) to zero. Of course, our primary interest in situations when the fit is not "too bad", and this is exactly where the normal approximation may not work well. Another deficiency of the above construction of normal approximation is that it is based on the *first order* Taylor expansion and does not take into account the third (quadratic) term in the right hand side of (5). It is possible to make a bias correction based on this quadratic term (cf., Shapiro, 1983, and see below), but yet the skewness problem may still persist.

In order to resolve these problems we can use the following idea. Instead of a second order Taylor expansion at the population point (covariance matrix) $\boldsymbol{\Sigma}_0$, let us consider the respective expansion at the point $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}(\boldsymbol{\theta}^*)$ satisfying the model. (Recall that $\boldsymbol{\Sigma}^*$ is the closest to $\boldsymbol{\Sigma}_0$, in terms of the F_{ML} discrepancy function, covariance matrix satisfying the considered model.) That is,

$$f(\boldsymbol{s}) \approx f(\boldsymbol{\sigma}^*) + (\boldsymbol{s} - \boldsymbol{\sigma}^*)'[\partial f(\boldsymbol{\sigma}^*)/\partial \boldsymbol{z}] + (\boldsymbol{s} - \boldsymbol{\sigma}^*)'\boldsymbol{Q}^*(\boldsymbol{s} - \boldsymbol{\sigma}^*), \tag{13}$$

where $\boldsymbol{Q}^* = \frac{1}{2}\partial^2 f(\boldsymbol{\sigma}^*)/\partial \boldsymbol{z}\partial \boldsymbol{z}'$. The above approximation (13) could be reasonable if $\boldsymbol{\sigma}^*$ is close to $\boldsymbol{\sigma}_0$, i.e., if the discrepancy between $\boldsymbol{\Sigma}_0$ and the model is not too bad. Again we have that first two terms in the right hand side of (13) vanish and hence

$$T_{ML} = nf(\boldsymbol{s}) \approx [n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}^*)]'\boldsymbol{Q}^*[n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}^*)]. \tag{14}$$

Since $\boldsymbol{S}$ is an unbiased estimate of $\boldsymbol{\Sigma}_0$, i.e., $\mathbb{E}[\boldsymbol{S}] = \boldsymbol{\Sigma}_0$, we have that $\mathbb{E}[\boldsymbol{s} - \boldsymbol{\sigma}^*] = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*$. Therefore we can approximate the distribution of T_{ML} by the distribution of the quadratic form $\boldsymbol{Y}'\boldsymbol{Q}^*\boldsymbol{Y}$, where $\boldsymbol{Y} \sim (\boldsymbol{\mu}, \boldsymbol{\Gamma})$ with $\boldsymbol{\mu} = n^{1/2}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*)$. This suggests approximating the distribution of T_{ML} by a noncentral chi-square distribution with $df = p(p+1)/2 - q$ degrees of freedom and noncentrality parameter $\delta = n(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*)'\boldsymbol{Q}^*(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*)$. Again by (13) we have that

$$(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*)'\boldsymbol{Q}^*(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*) \approx f(\boldsymbol{\sigma}_0) = F^*_{ML}, \tag{15}$$

and hence we can use $\delta = nF^*_{ML}$ as the noncentrality parameter as well. Since $F^*_{ML} > 0$ we have here that the noncentrality parameter δ tends to infinity as $n \to \infty$. In order to reconcile this problem we may assume that the population value $\boldsymbol{\sigma}_{0,n}$ depends on the sample size n in such a way that $n^{1/2}(\boldsymbol{\sigma}_{0,n} - \boldsymbol{\sigma}^*)$ converges to a fixed limit. This assumption implies that $\boldsymbol{\sigma}_{0,n}$ converges to $\boldsymbol{\sigma}^*$ at a rate of $O(n^{-1/2})$, and referred to as a *sequence of local alternatives* or the *population drift.* Note also that the approximation (15) makes sense only if $\boldsymbol{\sigma}_0$ is close to $\boldsymbol{\sigma}^*$, i.e., if the misspecification is not "too serious", and can be poor otherwise (e.g., see Sugawara & MacCallum, 1993).

The concept of the population drift is just a mathematical fabrication allowing to make an exact mathematical statement. It could be pointed, however, that the assumption about existence of an abstract population from which we can sample indefinitely, and hence to arrive at a limiting distribution as the sample size tends to infinity, is also a mathematical abstraction. In practice the sample is always finite, and the real question is how good a considered approximation is for a given sample. This, of course, depends on a particular application. One could be also tempted to use the second order Taylor approximation of the discrepancy function at the *population* point $\mathbf{\Sigma}_0$. However, for misspecified models the corresponding quadratic form does not have a (noncentral) chi-square distribution, even under the normality assumption (cf., Shapiro, 1983, Theorem 5.4(c)). Consequently asymptotics based on such approximation could be difficult to use in practice.

The noncentrality parameter $\delta = nF^*_{ML}$ can be large for two somewhat different reasons. Namely, it can happen that F^*_{ML} is large, i.e., the fit is bad, or that the sample size n is large amplifying a reasonably small discrepancy F^*_{ML}, and of course it could be both. If the noncentrality parameter is large because of the large sample size, while F^*_{ML} is reasonably small, then the noncentral chi-square approximation can be still reasonable. As it was discussed at the beginning of this section, for large δ the distribution $\chi^2_k(\delta)$ by itself can be approximately normal.

Let us finally mention that by taking into account the last (quadratic) term in the right hand side of (5) we can make the following correction for the normal distribution approximation. The expected value of this quadratic term can be approximated by $n^{-1}\mathrm{tr}(\mathbf{\Gamma}\boldsymbol{Q})$. In order to apply bias correction based on that term one would need to estimate matrices $\mathbf{\Gamma}$ and $\boldsymbol{Q}$, which may be not easy and will involve an error in any such estimation. Alternatively the term $\mathrm{tr}(\mathbf{\Gamma}_N\boldsymbol{Q})$ can be approximated by the number of degrees of freedom $df = p(p+1)/2 - q$. The variance of this quadratic term can be approximated by $n^{-2}(2df + 4\delta)$. Therefore, assuming that the population distribution is normal, we can use the corrected normal distribution approximation of the distribution of

T_{ML} with mean $nF^*_{ML} + df = \delta + df$ and variance

$$2n\,\mathrm{tr}\left[\left(\boldsymbol{\Sigma}^{*-1}\boldsymbol{\Sigma}_0 - \boldsymbol{I}_p\right)^2\right] + 2df + 4\delta. \tag{16}$$

Similar analysis can be performed for the Generalized Least Squares (GLS) discrepancy function

$$F_{GLS}(\boldsymbol{S}, \boldsymbol{\Sigma}) = \tfrac{1}{2}\mathrm{tr}\left\{[(\boldsymbol{S} - \boldsymbol{\Sigma})\boldsymbol{S}^{-1}]^2\right\}. \tag{17}$$

In that respect it is worthwhile to point the following. The second order Taylor expansion of the GLS discrepancy function, at a point satisfying the model, coincides with the corresponding second order Taylor expansion of the ML discrepancy function. Therefore, if the model is correct, then the test statistics T_{ML} and T_{GLS} are asymptotically equivalent (cf., Browne, 1974). In that case the numerical values of T_{ML} and T_{GLS}, for a given sample covariance matrix $\boldsymbol{S}$, should be close to each other. On the other hand for misspecified models, as the population covariance matrix moves away from the model, the test statistics T_{ML} and T_{GLS} diverge and the corresponding estimates of the noncentrality parameter based on these statistics could be quite different from each other. As far as the asymptotic normality is concerned the following result, similar to the ML case, holds:

- Let $\boldsymbol{\theta}^*$ be the unique minimizer of $F_{GLS}(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}(\boldsymbol{\theta}))$ and $\boldsymbol{\gamma} = \left.\frac{\partial F_{GLS}(\boldsymbol{z}, \boldsymbol{\sigma}^*)}{\partial \boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{\sigma}_0}$,

where $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\boldsymbol{\theta}^*)$. Then $n^{1/2}\left(\hat{F}_{GLS} - F^*_{GLS}\right)$ converges in distribution to normal $N(0, \boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma})$.

In particular, if the population distribution is normal, then the asymptotic variance associated with the GLS test statistic is given by the following formula (cf., Shapiro, 2009)

$$\boldsymbol{\gamma}'\boldsymbol{\Gamma}_N\boldsymbol{\gamma} = 2\,\mathrm{tr}\left[\left(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}^*\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}^* - \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}^*\right)^2\right]. \tag{18}$$

Note that here $\boldsymbol{\Sigma}^*$ corresponds to the minimizer $\boldsymbol{\theta}^*$ of the GLS discrepancy function and vector $\boldsymbol{\gamma}$ is given by derivatives of the GLS discrepancy function, and formula (18) for the asymptotic variance is different from the corresponding formula (12) for the ML discrepancy function.

Non-normal Distributions

The asymptotic normality of $\hat{F}_{ML}$, i.e., convergence in distribution of $n^{1/2}\big(\hat{F}_{ML} - F^*_{ML}\big)$ to $N(0, \boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma})$, holds without the assumption that the population has normal distribution as well. The asymptotic variance $\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma}$ can be estimated directly from the data by using formulas (7) and (11). That is, components of the matrix $\boldsymbol{\Gamma}$ and vector $\boldsymbol{\gamma}$ can be estimated by replacing the respective forth and second order moments with their sample estimates. Note, however, that estimation of matrix $\boldsymbol{\Gamma}$ involves estimation of $p(p+1)(p+2)(p+3)/4$ distinct forth order moments which can result in a significant estimation error. Therefore it could be desirable to consider specific situations where estimation of forth order moments can be avoided. One such case, other than normal, is the case of elliptical distributions.

Suppose now that the population distribution is elliptical. The elliptical class of distributions incorporates a single additional kurtosis parameter, κ, and is convenient for investigating the sensitivity of normal theory methods to the kurtosis of the population distribution. Note that kurtosis parameter $\kappa = \frac{1}{3}\gamma$, where γ is the (marginal) kurtosis of the multivariate distribution (e.g., Muirhead & Waternaux, 1980). The basic asymptotic result that we need here is that the corresponding matrix $\boldsymbol{\Gamma}$ has the following structure (e.g., Muirhead & Waternaux, 1980)

$$\boldsymbol{\Gamma} = (1+\kappa)\boldsymbol{\Gamma}_N + \kappa\boldsymbol{\sigma}_0\boldsymbol{\sigma}_0'. \tag{19}$$

Here, as it was defined before, $\boldsymbol{\Gamma}_N$ is the asymptotic covariance matrix of $n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_0)$ obtained under the assumption that the population has normal distribution. Consequently

$$\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma} = (1+\kappa)\boldsymbol{\gamma}'\boldsymbol{\Gamma}_N\boldsymbol{\gamma} + \kappa(\boldsymbol{\gamma}'\boldsymbol{\sigma}_0)^2, \tag{20}$$

where $\boldsymbol{\gamma}'\boldsymbol{\Gamma}_N\boldsymbol{\gamma}$ is given by the right hand side of (12) and represents the asymptotic variance of $n^{1/2}(\hat{F}_{ML} - F^*_{ML})$ under the normality assumption. Also by (11) we have

$$\kappa(\boldsymbol{\gamma}'\boldsymbol{\sigma}_0)^2 = \kappa\left[\mathrm{tr}\left(\boldsymbol{\Sigma}^{*-1}\boldsymbol{\Sigma}_0 - \boldsymbol{I}_p\right)\right]^2. \tag{21}$$

Let us also note that assuming that the model is invariant under a constant scaling factor, we have here that under a sequence of local alternatives the test statistic $(1+\kappa)^{-1}T_{ML}$ asymptotically has a noncentral chi-square distribution with $df = p(p+1)/2 - q$ degrees of freedom and noncentrality parameter $(1+\kappa)^{-1}\delta$, where $\delta = nF^*_{ML}$ (cf., Shapiro & Browne, 1987). Therefore, similar to (16), we can use the corrected normal distribution approximation of the distribution of T_{ML} with mean $nF^*_{ML} + (1+\kappa)df$ and variance

$$(1+\kappa)\boldsymbol{\gamma}'\boldsymbol{\Gamma}_N\boldsymbol{\gamma} + \kappa\left[\mathrm{tr}\left(\boldsymbol{\Sigma}^{*-1}\boldsymbol{\Sigma}_0 - \boldsymbol{I}_p\right)\right]^2 + (1+\kappa)^2(2df + 4\delta). \tag{22}$$

Numerical Illustrations

In this section we discuss Monte Carlo experiments aimed at an empirical evaluation of the suitability of the noncentral chi-square and normal distributions for the LR test statistic. We consider factor analysis models $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Psi}$ under varying conditions of model misspecification and sample size. Our study also includes different number of variables and factors. Furthermore, we use both normal and non-normal (elliptically distributed) data to investigate the robustness of test statistics to non-normality of the population distribution.

Normally distributed data

Our experiments included six sample sizes $n = 50, 100, 200, 400, 800, 1000$ with various degrees of model misspecification ranging from small to severe.

The population covariance matrices employed in Monte Carlo simulations, were constructed as follows. First, a $p \times p$ covariance matrix $\boldsymbol{\Sigma}^* = \boldsymbol{\Lambda}^*\boldsymbol{\Lambda}^{*\prime} + \boldsymbol{\Psi}^*$, satisfying the

Factor Analysis model, was constructed with specific values of elements of matrix $\boldsymbol{\Lambda}^*$ and diagonal elements of matrix $\boldsymbol{\Psi}^*$, as shown in Table 1 for Model 1, and Table 2 for Model 2. Model 1 has seven variables and one factor. Model 2 has twelve variables and three factors. Next, misspecified covariance matrices were generated of the form $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}^* + t\boldsymbol{E}$, where $\boldsymbol{E}$ is a $p \times p$ symmetric matrix and $t > 0$ is a scaling factor controlling the level of misspecification. The matrix $\boldsymbol{E}$ was chosen in such a way that the corresponding matrix $\boldsymbol{\Sigma}_0$ is positive definite and $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}(\boldsymbol{\theta}^*)$, where $\boldsymbol{\theta}^*$ is the minimizer of the right hand side of (8). That is, for $\boldsymbol{S} = \boldsymbol{\Sigma}_0$ the estimated covariance matrix obtained by applying the maximum likelihood(ML) procedure is the specified matrix $\boldsymbol{\Sigma}^*$, and hence $F^*_{ML} = F_{ML}(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}^*)$.

In order to construct matrix $\boldsymbol{E}$, producing a largest possible range of the discrepancy values, we used procedures developed in Cudeck & Browne, 1992 and Chun & Shapiro, 2008. Given the population covariance matrix $\boldsymbol{\Sigma}_0$, we randomly generated $M = 50000$ sample covariance matrices, corresponding to the specified population covariance matrix $\boldsymbol{\Sigma}_0$ and the sample size n, from a Wishart distribution $W_p\left(\frac{1}{n-1}\boldsymbol{\Sigma}_0, n-1\right)$. We used the Matlab function 'wishrnd' to generate random matrices having Wishart distribution. For each covariance matrix, sample values T_i, $i = 1, \ldots, M$, for the LR test statistics were calculated by maximum likelihood estimation. Estimation of factor loading matrix $\boldsymbol{\Lambda}$ was done by Matlab function 'factoran'.

For Model 1, the maximum discrepancy F^*_{ML} (corresponding to the largest value of the scaling parameter t) was computed to be 1.360. By using different values of the scaling parameter t we generated population covariance matrices, of the form $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}^* + t\boldsymbol{E}$, with discrepancy values in the ranges of 0.025 to 1.360. Similarly, population covariance matrices for Model 2 were generated with discrepancy values from 0.01 to 0.5. Discrepancy misspecification and corresponding population values of RMSEA are shown in Table 3 and Table 4. The RMSEA stands for *Root Mean Square Error of*

Approximation, and its (population) value is defined as

$$\text{RMSEA} = \sqrt{\frac{F^*_{ML}}{df}}$$

(cf., J. Steiger & Lind, 1980 ; Browne & Cudeck, 1992). In the present case $df = 14$ for Model 1 and $df = 33$ for Model 2.

We compare the noncentral chi-square distribution with the normal distribution for describing the behavior of the ML test statistic $T_{ML} = n\hat{F}_{ML}$. In the text and tables below the *noncentral chi-square distribution* is referred to as *ncx*. For the comparison we specify normal distributions with four different mean and variance values. Namely, mean $\delta = nF^*_{ML}$ and variance $n\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma}$, with $\boldsymbol{\gamma}'\boldsymbol{\Gamma}\boldsymbol{\gamma}$ given in (12) (referred to as nm); corrected mean $nF^*_{ML} + df$ and variance given in (16) (referred to as $nm2$); mean and variance estimated directly from the simulated values $T_1, ..., T_M$ by computing their average and sample variance (referred to as $nm3$); and mean $nF^*_{ML} + df$ and variance $2df + 4\delta$ (referred to as $nm4$). That is, nm corresponds to the direct normal approximation, $nm2$ corresponds to the normal approximation with the bias correction, $nm3$ corresponds to the normal approximation with mean and variance estimated directly from the sample, and $nm4$ corresponds to the normal approximation of the respective noncentral chi-square distribution. We refer to nm, $nm2$ and $nm4$ as *theoretical* normal approximations since their parameters (mean and variance) can be estimated from the data. On the other hand, sample mean and variance used in $nm3$ can be computed only in a simulation study.

We used several discrepancy measures to compare the fit of each distribution. One is the Kolmogorov-Smirnov (KS) distance defined as

$$K = \sup_{t \in \mathbb{R}} \left| \hat{F}_M(t) - F(t) \right|, \tag{23}$$

where $\hat{F}_M(t) = \frac{\#\{T_i \leq t\}}{M}$ is the empirical cumulative distribution function (cdf) based on Monte Carlo sample $T_1, ..., T_M$ of M computed values of the test statistic, and $F(t)$ is the theoretical cdf of the respective approximations ncx, nm, $nm2$, $nm3$ and $nm4$ of the test statistic. We also consider the *average* Kolmogorov-Smirnov distance (AK), defined as

$$AK = \frac{1}{M}\sum_{i=1}^{M} K_i, \tag{24}$$

where

$$K_i = \max\left\{\left|\frac{i-1}{M} - F(T_{(i)})\right|, \left|\frac{i}{M} - F(T_{(i)})\right|\right\},$$

with $T_{(1)} \leq \ldots T_{(M)}$ being the respective order statistics. The computed values of the KS distances are denoted as $ncxK$, nmK, $nm2K$, $nm3K$ and $nm4K$, respectively, and the computed values of the AK distances are denoted as $ncxAK$, $nmAK$, $nm2AK$, $nm3AK$ and $nm4AK$, respectively. These measures were used in Yuan et al. (2007).

Table 5 contains Kolmogorov-Smirnov distances (K) for Model 1 with sample sizes $n = 400$ and $n = 1000$, and nine degrees of misspecification F^*_{ML} ranging from 0.025 to 1.360. Corresponding $\delta = nF^*_{ML}$ values are from 9.8 to 544 for $n = 400$, and from 24.50 to 1360 for $n = 1000$. From this table we can compare the performance of each distribution for different degrees of discrepancy F^*_{ML} for Model 1. We can see that, for small to severe misspecification F^*_{ML} (with respective RMSEA values ranging from 0.042 to 0.116), $ncxK$ is smaller than nmK and $nm2K$, but the status of those measures is reverse for extremely severe misspecifications (with RMSEA values greater than 0.151).

This shows that for small, moderate and even severe misspecifications, the noncentral distribution gives a better approximation. On the other hand, for extremely severe misspecifications the normal distribution with bias correction ($nm2$) gives a slightly better approximation. However, models with extremely severe misspecifications are rejected anyway, say by the RMSEA criterion, and are not of much practical interest. Moreover, these results indicate that neither noncentral chi-square or theoretical normal is a reasonable approximation for severely misspecified models. For all values of F^*_{ML}, we observe that $ncxK \leq nm4K$, and these values are getting close to each other as F^*_{ML}

increases implying that for large δ the noncentral chi-square distribution by itself can be approximated by a normal distribution, as it was discussed at the beginning of the section "Theoretical background". Note that the noncentrality parameter $\delta = nF^*_{ML}$ gets larger because the discrepancy F^*_{ML} gets bigger with fixed n here. It also could be noted that for large discrepancies the normal distribution with sample mean and variance (column $nm3K$) gives a good approximation. This, however, is of a little practical interest since these mean and variance could be computed only in simulation experiments.

Table 6 contains Average Kolmogorov-Smirnov distance(AK) for Model 1 with sample sizes $n = 400$ and $n = 1000$, and F^*_{ML} values ranging from 0.025 to 1.36. The patterns of changes in AK are very similar to those of K in Table 5, except that the respective values are smaller here. This is the result of the different calculation in (23) and (24). Thus, we could get a similar conclusion, namely, the noncentral chi-square and the normal distributions are becoming similar in describing T_{ML} as F^*_{ML} increases, but the noncentral chi-square is better than the normal distribution (nm) or normal with bias correction ($nm2$) for small, moderate, and severe misspecifications. Again, normal distribution with bias correction is a little better description for the distribution of T_{ML} under extremely severe misspecifications. Note that neither ncx nor $nm2$ is a reasonable approximation under extremely severe misspecifications.

The results in both Tables 5 and 6 do not tell us much about the effect of the sample size for a fixed discrepancy F^*_{ML}. Table 7 is designed to show the effect of sample size on AK for each distribution for Model 1. We present three values of F^*_{ML} for the comparison. The value of the noncentrality parameter $\delta = nF^*_{ML}$ varies from 4.52 to 1097.20. As we can see, $ncxAK$ is smaller than $nmAK$, $nm2AK$, and $nm4K$ for all sample sizes n except $n = 50$ when $F^*_{ML} = 0.090$, confirming our analysis. For $F^*_{ML} = 0.474$, normal approximation with bias correction ($nm2$) is slightly better than the noncentral chi-square for the sample size $n \geq 400$. The normal (nm) provides a better description on the behavior of T_{ML} when discrepancy is extremely large, that is $F^*_{ML} = 1.097$, but none of the distributions gives a reasonable description for T_{ML} under extremely severe misspecifications. Our simulation results also show that sample size effect was not as important as the degree of misspecification of the model.

Validity of confidence intervals for fit indices and methods of power estimation, that rely upon the test statistic T_{ML}, depend on the quality of employed theoretical approximations. In that respect, we generated 50000 sample test statistics for Model 1 and calculated the empirical quantile (denoted $Q - T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile (denoted $P - T_{ML}$) under four underlying distribution assumptions with two noncentrality parameter values, $\delta = 36.12$ (Table 8) and $\delta = 189.56$ (Table 9) for $n = 400$. Here $Q - ncx$ are the quantiles from $\chi^2_{df}(\delta)$ and $P - ncx$ is the percent of samples that is less than computed quantile $Q - ncx$. Other measures are defined for the four normal distributions in a similar way. Values in parentheses are the differences between empirical values and respective theoretical values from each distribution. For $\delta = 36.12$, measures from $\chi^2_{df}(\delta)$ are very similar to empirical values. On the other hand, theoretical values from the three normal distributions (nm, $nm2$, $nm4$) are quiet different from empirical ones. Moreover, we can observed that normal quantile values show skewness problem which was pointed out in section "Theoretical background". For the large value of $\delta = 189.56$, measures from $nm2$ are more similar than that from ncx, but none of them are close to empirical one. Also, skewness problem still exists.

Figures 1–2 and 3–4 provide the quantile-quantile (QQ) plots for T_{ML} against ncx and $nm2$ for $n = 400$ with $\delta = 36.1229$ and $\delta = 189.555$ from Model 1 . When $\delta = 36.1229$, $\chi^2_{df}(\delta)$ describes the behavior of T_{ML} pretty well (Figure 1), while normal distribution with bias correction ($nm2$) works poorly (Figure 2). These plots confirm the skewness problem again. When $\delta = 189.555$, Figure 3 and Figure 4 show very similar pattern since $\chi^2_{df}(\delta)$ and normal distribution gets similar in terms of performance of describing T_{ML}. We could not see a difference between them from the plots.

We present similar results for Model 2 in Table 10 and Table 11. As we can see, $ncxK$ is smaller than nmK and $nm2K$ for small, moderate, and severe misspecification. Similarly, $ncxAK$ is smaller than $nmAK$ and $nm2AK$ for most cases. That is, $\chi^2_{df}(\delta)$ is a better approximation for T_{ML} under small to severe misspecification. Normal with bias correction($nm2$) is slightly better for extremely severe misspecification, but none of distributions gives a reasonable approximation in that case.

Quantile-quantile (QQ) plots for T_{ML} against $\chi^2_{df}(\delta)$ and normal distributions for $n = 400$ with $\delta = 39.95$ and $\delta = 80.05$ from Model 2 are provided (Figure 5-6 and Figure 7-8). As we can see, $\chi^2_{df}(\delta)$ describes the behavior of T_{ML} pretty well (Figure 5, Figure 7) while normal distribution with bias correction ($nm2$) shows poor performance (Figure 6, Figure 8). Skewness problem of normal approximation is very clear.

Non-normally distributed data

We also use non-normally (elliptically) distributed data to empirically illustrate the robustness of LR test statistics as we explained in section "Non-normal distributions". In order to generate data with an elliptical distribution we proceed as follows. Let $\boldsymbol{X} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ be a random vector having (multivariate) normal distribution and W be a random variable independent of $\boldsymbol{X}$. Then the random vector $\boldsymbol{Y} = W\boldsymbol{X}$ has an elliptical distribution with zero mean vector, covariance matrix $\alpha\boldsymbol{\Sigma}$, where $\alpha = \mathbb{E}[W^2]$, and the kurtosis parameter $\kappa = \frac{\mathbb{E}[W^4]}{(\mathbb{E}[W^2])^2} - 1$ (see the Appendix).

We consider the same structure as in Model 1 discussed in section "Normally distributed data", but with elliptically distributed data. That is, we directly calculate sample covariance matrices from the generated elliptically distributed data instead of using Wishart distribution. See Table 1 for generated parameters and Table 3 for discrepancy misspecification values. We generated two sets of elliptical distributions with different kurtosis parameter κ. Model 3 involves elliptically distributed data with random variable W taking two values, 1.2 with probability 0.45 and 0.8 with probability 0.55. Model 4 involves W taking two values, 2 with probability 0.2 and 0.5 with probability 0.8. The kurtosis parameter of these elliptical distributions is $\kappa = 0.1584$ (Model 3) and $\kappa = 2.25$ (Model 4). Note that in both cases $\mathbb{E}[W^2] = 1$, so that the covariance matrices of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equal to each other.

Table 12 and Table 13 contain Kolmogorov-Smirnov distance (K) and Average Kolmogorov-Smirnov distance (AK) for Model 3 with sample sizes $n = 400$ and $n = 1000$, and nine degrees of misspecification, $F^*_{ML} = 0.025, \ldots, 1.360$. We can see that for small to severe misspecification F^*_{ML} (with RMSEA values ranging from 0.042 to 0.116), $ncxK$ is smaller than $nm2K$ and $ncxAK$ is smaller than $nm2AK$, but the status of those measures reverse for extremely severe misspecifications (with RMSEA values greater than 0.151). This implies that for small, moderate and severe misspecifications, $\chi^2_{df}(\delta)$ is a better approximation. On the other hand, for extremely severe misspecifications the normal distribution with bias correction ($nm2$) gives a slightly better approximation, but none of distributions gives reasonable description for T_{ML} under extremely misspecified model. These results are consistent with the corresponding results of section “Normally distributed data”.

Quantile comparisons are done to investigate the quality of each theoretical approximation with respect to the validity of confidence intervals or fit indices and methods of power estimation. We calculated the empirical quantile (denoted $Q - T_{ML}$ and $(1+\kappa)^{-1}T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile (denoted $P - T_{ML}$) with $M = 50000$ sample test statistics of Model 3 under four underlying distribution assumptions with two noncentrality parameter values, $\delta = 36.12$ (Table 14) and $\delta = 189.56$ (Table 15) for $n = 400$. Here $Q - ncx$ are the quantiles from $\chi^2_{df}((1+\kappa)^{-1}\delta)$ and $P-ncx$ is the percent of samples that is less than computed quantile $Q-ncx$. Other measures are defined for the normal distributions in a similar way. Values in parentheses are the differences between empirical values and respective theoretical values from each distribution. For both $\delta = 36.12$ and $\delta = 189.56$, measures from $\chi^2_{df}(\delta)$ are very similar to empirical values. On the other hand, theoretical values from the normal distributions (nm, $nm2$) are very different from empirical ones. Again, we can observe that normal quantile values show skewness problem which was pointed out in section "Theoretical background".

Figures 9-10 and 11-12 provide the quantile-quantile (QQ) plots for T_{ML} against $\chi^2_{df}(\delta)$ and normal distributions for $n = 400$ with $\delta = 36.1229$ and $\delta = 189.555$ from Model 3 . For both $\delta = 36.1229$ and $\delta = 189.555$, $\chi^2_{df}(\delta)$ describes the behavior of T_{ML} pretty well (Figure 9, Figure 11) while normal distribution with bias correction ($nm2$) works poorly (Figure 10, Figure 12). We can confirm strong skewness problem of normal approximation.

Similar results are shown for Model 4 (Table 16, Table 17, and Figure 13-16). It is interesting to see that $\chi^2_{df}(\delta)$ describes the behavior of T_{ML} better than normal distribution under small, moderate, severe, and even extremely severe misspecification for Model 4. Quantile-Quantile(QQ) plots confirm same conclusion, especially clear skewness of normal approximation.

Empirical data

We consider the Thurstone data (Thurstone & Thurstone, 1941). The data matrix is generated by 60 test scores from a classic study of mental ability. We use nine variable Thurstone problem which is discussed in detail by McDonald (1999). The nine variables are: "Sentences", "Vocabulary", "Sentence completion", "First Letters", "Four letter words", "Suffixes", "Letter series", "Pedigrees" and "Letter Grouping", which measure verbal ability, word fluency, and reasoning ability.

We apply one factor model (denoted $Thurstone - 1$) and three factor model (denoted $Thurstone - 3$) to these data with 213 observations. Estimated parameters and RMSEA values for each model are in Table 18 and Table 19. Note that one factor model indicates an extremely poor fit (with RMEA value 0.2036) while three factor model shows a good fit (with RMSEA value 0.0408). In order to evaluate statistical properties of the corresponding LR test statistics we employ the parametric bootstrap approach (see Efron & Tibshirami, 1993, section 6.5). That is, in the Monte Carlo sampling the (unknown) population covariance matrix is replaced by the sample covariance matrix. Consequently, we randomly generate 50000 sample covariance matrices from the respective Wishart distribution and calculate the LR test statistics T_{ML}. Quantile-quantile (QQ) plots for T_{ML} against noncentral chi-square and normal distribution for $Thurstone - 1$ and $Thurstone - 3$ models are provided (Figure 17–18 and Figure 19–20). Noncentral chi-square distribution describes the distribution of test statistics pretty well for both models while normal distribution with bias correction shows a poor performance especially for the three factor model. For both models the skewness problem of normal approximation is present and is especially bad for the three factor model (Figure 20).

Discussion

The noncentral chi-square distribution is widely used to describe the behavior of LR test statistics T_{ML} in structural equation modeling (SEM) for the computation of fit indices and evaluation of statistical power. Recently, it was suggested by several authors that T_{ML} could be better described by the normal than the noncentral chi-square distribution. In this paper, we discuss the underlying theory of both approximations, normal and noncentral chi-square, and present some numerical experiments aimed at empirical comparison of the performance of two distributions in describing the distribution of the test statistic T_{ML}.

Monte Carlo experiments are conducted for several factor analysis models. Furthermore, we use both normal and non-normal data to investigate the robustness of test statistics to nonnormality. For each model, we considered different sample sizes ranging from 50 to 1000, and varying conditions of model misspecification ranging from small to extremely severe. Several discrepancy measures based on the Kolmogorov-Smirnov distance were used to compare the noncentral chi-square distribution with normal distributions. Respective quantiles are compared in order to investigate the behavior of tails in each distribution as well. Empirical results indicate that the distribution of T_{ML} is described well by the noncentral chi-square distribution under small, moderate, and even severe misspecifications irrespective of the sample size. For the extremely misspecified model, the normal distribution with a bias correction is slightly better than the noncentral chi-square distribution.

It could be noted that normal distribution with *estimated* sample mean and variance gives a better approximation for *larger* discrepancy values (see columns nm3K and nm3AK in the tables). This, however, is of a little practical significance since the corresponding mean and variance could be computed only in simulation experiments and will be unavailable for a given data set.

In summary, the noncentral chi-square approximation of the ML test statistic is valid under reasonable misspecifications and models. The normal distribution with a bias correction may perform slightly better under extreme misspecifications. However, neither the noncentral chi-square distribution nor the theoretical normal distributions give reasonable approximations of LR test statistics under extremely severe misspecifications. Of course, extremely misspecified models are unacceptable anyway for a reasonable statistical inference. These findings may differ with variations in model complexity, model parameterization and underlying data structure.

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Appendix

Let $\boldsymbol{X} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ be a random vector having normal distribution and W be a random variable independent of $\boldsymbol{X}$. Then $\boldsymbol{Y} = W\boldsymbol{X}$ has elliptical distribution with $\mathbb{E}[\boldsymbol{Y}] = \mathbf{0}$ and characteristic function

$$\phi(\boldsymbol{t}) = \mathbb{E}\left[\exp(iW\boldsymbol{t}'\boldsymbol{X})\right] = \mathbb{E}\left[\mathbb{E}\left\{\exp(iW\boldsymbol{t}'\boldsymbol{X})\middle|W\right\}\right] = \mathbb{E}\left[\exp\left\{-\tfrac{1}{2}W^2\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}\right\}\right].$$

That is, $\phi(\boldsymbol{t}) = \psi(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$, where $\psi(z) = \mathbb{E}\left[\exp\left\{-\frac{1}{2}W^2 z\right\}\right]$. Then it follows that the covariance matrix of $\boldsymbol{Y}$ is $\alpha\boldsymbol{\Sigma}$, where $\alpha = -2\psi'(0) = \mathbb{E}[W^2]$. It also follows that the kurtosis parameter is

$$\kappa = \frac{\psi''(0) - \psi'(0)^2}{\psi'(0)^2} = \frac{\mathbb{E}[W^4]}{(\mathbb{E}[W^2])^2} - 1$$

(cf., Muirhead & Waternaux, 1980). For example, if W can take two values, a with probability p and b with probability $1-p$, then $\alpha = a^2p + b^2(1-p)$ and

$$1 + \kappa = \frac{a^4p + b^4(1-p)}{(a^2p + b^2(1-p))^2}.$$

Footnotes

[1]Of course, for large n the factor $\frac{n-1}{n}$ is close to one, and for asymptotic results this correction does not matter.

[2]Note that since matrices $\boldsymbol{S}, \boldsymbol{\Sigma}, \boldsymbol{Z}$ are symmetric, the corresponding $p^2 \times 1$ dimensional vectors have no more than $p(p+1)/2$ nonduplicated elements. We use here the respective $p^2 \times 1$, rather than $p(p+1)/2 \times 1$, dimensional vectors for the sake of an algebraic convenience. Note also the corresponding gradient vectors $\partial f(\boldsymbol{\sigma})/\partial \boldsymbol{z}$ have the same structure of duplicated components.

[3]For this to hold we only need to verify that the population distribution has finite forth order moments.

Table 1

Generated Parameters for Model 1

Λ^*	0.6916	Ψ^*	0.8727
	1.2404		0.6480
	0.7971		1.0672
	0.9011		1.0614
	0.5761		3.0594
	1.5620		1.8551
	0.8117		1.3567

Table 2

Generated Parameters for Model 2

Λ^*	0.9644	0	0	Ψ^*	0.0699
	0.9644	0	0		0.0699
	0.9644	0	0		0.0699
	0.9644	0	0		0.0699
	0.9644	0	0		0.0699
	0	0.7182	0		0.4842
	0	0.7182	0		0.4842
	0	0.7182	0		0.4842
	0	0.7182	0		0.4842
	0	0	0.5052		0.7448
	0	0	0.5052		0.7448
	0	0	0.5052		0.7448

Table 3

Degree of discrepancy misspecification for Model 1

F^*_{ML}	0.025	0.090	0.185	0.318	0.474	0.655	0.863	1.097	1.360
RMSEA	0.042	0.080	0.116	0.151	0.184	0.216	0.248	0.280	0.312

Table 4

Degree of discrepancy misspecification for Model 2

F^*_{ML}	0.010	0.050	0.100	0.200	0.300	0.400	0.500
RMSEA	0.017	0.039	0.055	0.078	0.095	0.110	0.123

Table 5

Kolmogorov-Smirnov distance(K) for Model 1, $df = 14$

n	F^*_{ML}	δ	RMSEA	$ncxK$	nmK	$nm2K$	$nm3K$	$nm4K$
400	0.025	9.80	0.042	0.009	0.686	0.063	0.045	0.042
	0.090	36.12	0.080	0.022	0.421	0.051	0.031	0.037
	0.190	75.88	0.116	0.034	0.313	0.041	0.024	0.042
	0.318	127.32	0.151	0.044	0.252	0.037	0.019	0.048
	0.474	189.56	0.184	0.053	0.211	0.038	0.016	0.057
	0.655	262.16	0.216	0.065	0.174	0.047	0.013	0.068
	0.863	345.16	0.248	0.090	0.117	0.080	0.008	0.094
	1.097	438.88	0.280	0.174	0.084	0.170	0.004	0.181
	1.360	544.00	0.312	0.360	0.266	0.327	0.006	0.365
1000	0.025	24.50	0.042	0.012	0.487	0.065	0.038	0.040
	0.090	90.30	0.080	0.025	0.283	0.049	0.024	0.033
	0.190	189.70	0.116	0.035	0.217	0.036	0.018	0.039
	0.318	318.30	0.151	0.044	0.188	0.030	0.015	0.044
	0.474	473.90	0.184	0.051	0.166	0.028	0.012	0.052
	0.655	655.40	0.216	0.061	0.143	0.031	0.010	0.061
	0.863	862.90	0.248	0.082	0.101	0.059	0.007	0.083
	1.097	1097.20	0.280	0.196	0.145	0.179	0.004	0.200
	1.360	1360.00	0.312	0.490	0.442	0.435	0.005	0.493

[a] Kolmogorov-Smironov distance (K) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

[b] ncx stands for $\chi^2_{df}(\delta)$, nm stands for $N\big(\delta, 2n\,\mathrm{tr}[(\Sigma^{*-1}\Sigma_0 - I_p)^2]\big)$, $nm2$ stands for $N\big(\delta + df, 2n\,\mathrm{tr}[(\Sigma^{*-1}\Sigma_0 - I_p)^2]\big)$, $nm3$ stands for Normal with sample mean and variance, and $nm4$ stands for $N(\delta + df, 2df + 4\delta)$.

Table 6

Average Kolmogorov-Smirnov distance(AK) for Model 1, $df = 14$

n	F^*_{ML}	δ	RMSEA	$ncxAK$	$nmAK$	$nm2AK$	$nm3AK$	$nm4AK$
400	0.025	9.80	0.042	0.005	0.427	0.035	0.024	0.023
	0.090	36.12	0.080	0.013	0.284	0.031	0.017	0.021
	0.190	75.88	0.116	0.019	0.203	0.025	0.013	0.023
	0.318	127.32	0.151	0.024	0.154	0.021	0.010	0.027
	0.474	189.56	0.184	0.028	0.120	0.021	0.008	0.031
	0.655	262.16	0.216	0.034	0.093	0.025	0.006	0.037
	0.863	345.16	0.248	0.050	0.060	0.046	0.004	0.054
	1.097	438.88	0.280	0.120	0.045	0.105	0.002	0.123
	1.360	544.00	0.312	0.246	0.181	0.211	0.002	0.248
1000	0.025	24.50	0.042	0.008	0.329	0.038	0.019	0.020
	0.090	90.30	0.080	0.015	0.190	0.030	0.012	0.019
	0.190	189.70	0.116	0.021	0.131	0.023	0.009	0.023
	0.318	318.30	0.151	0.025	0.101	0.018	0.007	0.027
	0.474	473.90	0.184	0.029	0.085	0.016	0.006	0.031
	0.655	655.40	0.216	0.033	0.072	0.018	0.005	0.035
	0.863	862.90	0.248	0.042	0.054	0.034	0.003	0.044
	1.097	1097.20	0.280	0.133	0.085	0.116	0.002	0.135
	1.360	1360.00	0.312	0.326	0.295	0.284	0.002	0.326

[a] Average Kolmogorov-Smironov distance (AK) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

[b] ncx stands for $\chi^2_{df}(\delta)$, nm stands for $N\big(\delta, 2n\,\mathrm{tr}[(\Sigma^{*-1}\Sigma_0 - I_p)^2]\big)$, $nm2$ stands for $N\big(\delta + df, 2n\,\mathrm{tr}[(\Sigma^{*-1}\Sigma_0 - I_p)^2] + 2[df + 2\delta]\big)$, $nm3$ stands for Normal with sample mean and variance, $nm4$ stands for $N(\delta + df, 2df + 4\delta)$.

Table 7

Average Kolmogorov-Smirnov distance(AK) for Model 1 with different sample sizes, df $= 14$

F^*_{ML}	RMSEA	n	δ	$ncxAK$	$nmAK$	$nm2AK$	$nm3AK$	$nm4AK$
0.090	0.080	50	4.52	0.036	0.483	0.021	0.026	0.020
		100	9.03	0.011	0.439	0.026	0.025	0.019
		200	18.06	0.010	0.369	0.030	0.022	0.020
		400	36.12	0.022	0.421	0.051	0.031	0.037
		800	72.24	0.014	0.213	0.030	0.013	0.018
		1000	90.30	0.025	0.283	0.049	0.024	0.033
0.474	0.184	50	23.70	0.019	0.326	0.028	0.018	0.028
		100	47.39	0.028	0.234	0.030	0.015	0.036
		200	94.78	0.028	0.170	0.024	0.011	0.032
		400	189.56	0.053	0.211	0.038	0.016	0.057
		800	379.12	0.029	0.092	0.017	0.007	0.030
		1000	473.90	0.051	0.166	0.028	0.012	0.052
1.097	0.280	50	54.86	0.097	0.148	0.091	0.008	0.104
		100	109.72	0.110	0.064	0.099	0.005	0.115
		200	219.44	0.111	0.031	0.099	0.003	0.115
		400	438.88	0.174	0.084	0.170	0.004	0.181
		800	877.76	0.128	0.074	0.111	0.002	0.130
		1000	1097.20	0.196	0.145	0.179	0.004	0.200

[a] Average Kolmogorov-Smironov distance (AK) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

Table 8

Quantile comparison for Model 1 ($n = 400$, $\delta = 36.12$)

$Q - T_{ML}$	22.0196	28.7702	32.7764	68.9375	75.4350	87.9215
$P - T_{ML}$	1%	5%	10%	90%	95%	99%
$Q - ncx$	23.8941	30.2580	34.0014	67.4783	73.2377	84.7275
$Q_{diff} - ncx$	(-1.8745)	(-1.4878)	(-1.2250)	(1.4592)	(2.1973)	(3.1940)
$P - ncx$	1.65%	6.63%	11.96%	88.39%	93.60%	98.45%
$P_{diff} - ncx$	(-0.65)	(-1.63)	(-1.96)	(1.61)	(1.40)	(0.55)
$Q - nm$	12.1095	19.1462	22.8974	49.3624	53.1137	60.1503
$Q_{diff} - nm$	(9.9101)	(9.6240)	(9.8790)	(19.5751)	(22.3213)	(27.7712)
$P - nm$	0.01%	0.44%	1.24%	50.95%	61.24%	77.19%
$P_{diff} - nm$	(0.99)	(4.56)	(8.76)	(39.05)	(33.76)	(21.81)
$Q - nm2$	11.2630	22.6489	28.7187	71.5412	77.6109	88.9969
$Q_{diff} - nm2$	(10.7566)	(6.1213)	(4.0577)	(-2.6037)	(-2.1759)	(-1.0754)
$P - nm2$	0.00%	1.16%	4.95%	92.39%	96.10%	99.17%
$P_{diff} - nm2$	(1.00)	(3.84)	(5.05)	(-2.39)	(-1.10)	(-0.17)
$Q - nm3$	17.0173	26.7196	31.8918	68.3821	73.5543	83.2566
$Q_{diff} - nm3$	(5.0023)	(2.0506)	(0.8846)	(0.5554)	(1.8807)	(4.6649)
$P - nm3$	0.18%	3.29%	8.77%	89.41%	93.80%	98.10%
$P_{diff} - nm3$	(0.82)	(1.71)	(1.23)	(0.59)	(1.20)	(0.90)
$Q - nm4$	19.5741	28.5253	33.2972	66.9627	71.7345	80.6857
$Q_{diff} - nm4$	(2.4455)	(0.2449)	(-0.5208)	(1.9748)	(3.7005)	(7.2358)
$P - nm4$	0.51%	4.78%	10.85%	87.80%	92.54%	97.32%
$P_{diff} - nm4$	(0.49)	(0.22)	(-0.85)	(2.20)	(2.46)	(1.68)

[a] Empirical quantile ($Q - T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile for $\delta = 36.12$ with $n = 400$.

[b] $Q - distribution$ are the quantiles from $\chi^2_{df}(\delta)$ and $P - ncx$ is the percent of samples that is less than computed quantile $Q - distribution$

[c] Values in parentheses (Q_{diff}, P_{diff}) are the differences between empirical values and respective theoretical values from each distribution.

Table 9

Quantile comparison for Model 1($n = 400$, $\delta = 189.56$)

$Q - T_{ML}$	130.7591	149.9324	161.0327	245.6207	259.2872	284.4579
$P - T_{ML}$	1%	5%	10%	90%	95%	99%
$Q - ncx$	142.7670	159.1783	168.3016	240.0793	251.3053	273.0689
$Q_{diff} - ncx$	(-12.0079)	(-9.2459)	(-7.2689)	(5.5414)	(7.9819)	(11.3890)
$P - ncx$	2.87%	9.01%	14.89%	87.00%	92.38%	97.84%
$P_{diff} - ncx$	(-1.87)	(-4.01)	(-4.89)	(3.00)	(2.62)	(1.16)
$Q - nm$	136.6503	152.1485	160.4106	218.6997	226.9618	242.4601
$Q_{diff} - nm$	(-5.8912)	(-2.2161)	(0.6221)	(26.9210)	(32.3254)	(41.9978)
$P - nm$	1.73%	5.83%	9.65%	69.89%	77.62%	88.30%
$P_{diff} - nm$	(-0.73)	(-0.83)	(0.35)	(20.11)	(17.38)	(10.70)
$Q - nm2$	119.5678	144.1716	157.2878	249.8226	262.9388	287.5425
$Q_{diff} - nm2$	(11.1913)	(5.7608)	(3.7449)	(-4.2019)	(-3.6516)	(-3.0846)
$P - nm2$	0.30%	3.24%	8.08%	91.82%	95.90%	99.20%
$P_{diff} - nm2$	(0.70)	(1.76)	(1.92)	(-1.82)	(-0.90)	(-0.20)
$Q - nm3$	125.3587	147.9460	159.9872	244.9381	256.9793	279.5666
$Q_{diff} - nm3$	(5.4004)	(1.9864)	(1.0455)	(0.6826)	(2.3079)	(4.8913)
$P - nm3$	0.56%	4.23%	9.40%	89.64%	94.35%	98.56%
$P_{diff} - nm3$	(0.44)	(0.77)	(0.60)	(0.36)	(0.65)	(0.44)
$Q - nm4$	138.3253	157.4341	167.6210	239.4894	249.6763	268.7851
$Q_{diff} - nm4$	(-7.5662)	(-7.5017)	(-6.5883)	(6.1313)	(9.6109)	(15.6728)
$P - nm4$	1.99%	8.17%	14.37%	86.69%	91.77%	97.12%
$P_{diff} - nm4$	(-0.99)	(-3.17)	(-4.37)	(3.31)	(3.23)	(1.88)

[a] Empirical quantile ($Q - T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile for $\delta = 189.56$ with $n = 400$.

[b] $Q - distribution$ are the quantiles from $\chi^2_{df}(\delta)$ and $P - ncx$ is the percent of samples that is less than computed quantile $Q - distribution$

[c] Values in parentheses (Q_{diff}, P_{diff}) are the differences between empirical values and respective theoretical values from each distribution.

Table 10

Kolmogorov-Smirnov distance(K) for Model 2, df = 33

n	F^*_{ML}	δ	RMSEA	$ncxK$	nmK	$nm2K$	$nm3K$	$nm4K$
400	0.010	4.00	0.017	0.012	0.997	0.038	0.034	0.027
	0.050	20.00	0.039	0.019	0.895	0.071	0.029	0.046
	0.100	40.00	0.055	0.051	0.747	0.105	0.026	0.075
	0.200	80.00	0.078	0.115	0.533	0.158	0.020	0.134
	0.300	120.00	0.095	0.168	0.387	0.198	0.017	0.184
	0.400	160.00	0.110	0.201	0.294	0.226	0.013	0.215
	0.500	200.00	0.123	0.231	0.231	0.252	0.011	0.243
1000	0.010	10.00	0.017	0.004	0.970	0.050	0.032	0.032
	0.050	50.00	0.039	0.011	0.717	0.074	0.023	0.033
	0.100	100.00	0.055	0.023	0.547	0.090	0.017	0.040
	0.200	200.00	0.078	0.075	0.355	0.134	0.014	0.088
	0.300	300.00	0.095	0.124	0.236	0.172	0.011	0.135
	0.400	400.00	0.110	0.160	0.173	0.202	0.009	0.169
	0.500	500.00	0.123	0.210	0.123	0.240	0.008	0.218

[a] Kolmogorov-Smironov distance (K) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

Table 11

Average Kolmogorov-Smirnov distance(AK) for Model 2, df = 33

n	F^*_{ML}	δ	RMSEA	$ncxAK$	$nmAK$	$nm2AK$	$nm3AK$	$nm4AK$
400	0.010	4.00	0.017	0.006	0.500	0.020	0.017	0.013
	0.050	20.00	0.039	0.012	0.490	0.044	0.014	0.025
	0.100	40.00	0.055	0.034	0.450	0.059	0.012	0.043
	0.200	80.00	0.078	0.079	0.351	0.087	0.010	0.086
	0.300	120.00	0.095	0.116	0.263	0.108	0.008	0.121
	0.400	160.00	0.110	0.139	0.198	0.123	0.006	0.144
	0.500	200.00	0.123	0.160	0.147	0.136	0.005	0.163
1000	0.010	10.00	RMSEA	0.002	0.499	0.030	0.016	0.016
	0.050	50.00	0.039	0.007	0.438	0.047	0.012	0.017
	0.100	100.00	0.055	0.016	0.359	0.055	0.009	0.023
	0.200	200.00	0.078	0.053	0.245	0.072	0.007	0.057
	0.300	300.00	0.095	0.088	0.164	0.090	0.006	0.091
	0.400	400.00	0.110	0.111	0.110	0.106	0.005	0.115
	0.500	500.00	0.123	0.146	0.063	0.127	0.004	0.149

[a] Average Kolmogorov-Smironov distance (AK) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

Table 12

Kolmogorov-Smirnov distance (K) for Model 3, df $= 14$, $\kappa = 0.1584$

n	F^*_{ML}	δ	RMSEA	$ncxK$	nmK	$nm2K$	$nm3K$
400	0.025	9.80	0.042	0.008	0.716	0.069	0.046
	0.090	36.12	0.080	0.025	0.443	0.061	0.033
	0.190	75.88	0.116	0.037	0.322	0.050	0.026
	0.318	127.32	0.151	0.048	0.256	0.047	0.023
	0.474	189.56	0.184	0.059	0.212	0.049	0.018
	0.655	262.16	0.216	0.071	0.172	0.059	0.014
	0.863	345.16	0.248	0.096	0.111	0.093	0.008
	1.097	438.88	0.280	0.174	0.064	0.173	0.005
	1.360	544.00	0.312	0.345	0.230	0.311	0.005
1000	0.025	24.50	0.042	0.011	0.525	0.076	0.038
	0.090	90.30	0.080	0.021	0.306	0.062	0.024
	0.190	189.70	0.116	0.032	0.230	0.050	0.019
	0.318	318.30	0.151	0.041	0.193	0.041	0.015
	0.474	473.90	0.184	0.049	0.167	0.038	0.012
	0.655	655.40	0.216	0.059	0.140	0.043	0.010
	0.863	862.90	0.248	0.079	0.092	0.070	0.006
	1.097	1097.20	0.280	0.193	0.099	0.219	0.004
	1.360	1360.00	0.312	0.456	0.362	0.401	0.004

[a] Kolmogorov-Smironov distance (K) for different sample size n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$.

[b] ncx stands for $\chi^2_{df}((1+\kappa)^{-1}\delta)$, nm stands for $N(\delta, \omega)$, where $\omega = n\left\{2(1+\kappa)\,\mathrm{tr}[(\Sigma^{*-1}\Sigma_0 - I_p)^2] + \kappa[\mathrm{tr}(\Sigma^{*-1}\Sigma_0 - I_p)]^2\right\}$, $nm2$ stands for $N\big(\delta + (1+\kappa)df, \omega + (1+\kappa)^2[2df + 4\delta]\big)$, and $nm3$ stands for Normal with sample mean and variance.

Table 13

Average Kolmogorov-Smirnov distance(AK) for Model 3, df $= 14$, $\kappa = 0.1584$

n	F^*_{ML}	δ	RMSEA	$ncxAK$	$nmAK$	$nm2AK$	$nm3AK$
400	0.025	9.80	0.042	0.005	0.440	0.039	0.025
	0.090	36.12	0.080	0.014	0.299	0.037	0.018
	0.190	75.88	0.116	0.020	0.213	0.032	0.013
	0.318	127.32	0.151	0.026	0.161	0.029	0.011
	0.474	189.56	0.184	0.031	0.124	0.028	0.009
	0.655	262.16	0.216	0.036	0.093	0.033	0.006
	0.863	345.16	0.248	0.053	0.056	0.051	0.004
	1.097	438.88	0.280	0.120	0.034	0.102	0.002
	1.360	544.00	0.312	0.237	0.159	0.194	0.002
1000	0.025	24.50	0.042	0.005	0.349	0.043	0.020
	0.090	90.30	0.080	0.013	0.207	0.037	0.012
	0.190	189.70	0.116	0.019	0.144	0.030	0.009
	0.318	318.30	0.151	0.024	0.109	0.026	0.007
	0.474	473.90	0.184	0.028	0.089	0.024	0.006
	0.655	655.40	0.216	0.032	0.072	0.025	0.004
	0.863	862.90	0.248	0.041	0.047	0.038	0.002
	1.097	1097.20	0.280	0.135	0.066	0.115	0.001
	1.360	1360.00	0.312	0.306	0.241	0.233	0.002

[a] Average Kolmogorov-Smironov distance (AK) for different sample size n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$, and with ncx etc as in Table 12.

Table 14

Quantile comparison for Model 3 ($n = 400$, $\delta = 36.12$, $\kappa = 0.1584$)

$Q - T_{ML}$	22.149	29.2531	33.364	72.7484	79.7355	93.7186
$(1+\kappa)^{-1}T_{ML}$	19.12034	25.25302	28.8018	62.80076	68.83244	80.90349
$P - T_{ML}$	1%	5%	10%	90%	95%	99%
$Q - nxc$	20.7295	26.5867	30.0555	61.5349	67.005	77.9491
$Q_{diff} - ncx$	(-1.60916)	(-1.33368)	(-1.2537)	(1.26586)	(1.827441)	(2.954388)
$P - nxc$	1.62	6.65	12.208	88.518	93.786	98.43
$P_{diff} - nxc$	(-0.62)	(-1.65)	(-2.208)	(1.482)	(1.214)	(0.57)
$Q - nm$	12.1095	19.1462	22.8974	49.3624	53.1137	60.1503
$Q_{diff} - nm$	(10.0395)	(10.1069)	(10.4666)	(23.386)	(26.6218)	(33.5683)
$P - nm$	0.012	0.448	1.23	45.752	55.43	71.456
$P_{diff} - nm$	(0.988)	(4.552)	(8.77)	(44.248)	(39.57)	(27.544)
$Q - nm2$	8.4067	21.279	28.1412	76.5539	83.416	96.2883
$Q_{diff} - nm2$	(13.7423)	(7.9741)	(5.2228)	(-3.8055)	(-3.6805)	(-2.5697)
$P - nm2$	0.002	0.78	4.034	93.052	96.616	99.25
$P_{diff} - nm2$	(0.998)	(4.22)	(5.966)	(-3.052)	(-1.616)	(-0.25)
$Q - nm3$	16.2584	26.8093	32.434	72.116	77.7406	88.2915
$Q_{diff} - nm3$	(5.8906)	(2.4438)	(0.93)	(0.6324)	(1.9949)	(5.4271)
$P - nm3$	0.13	3.116	8.666	89.376	93.852	98.044
$P_{diff} - nm3$	(0.87)	(1.884)	(1.334)	(0.624)	(1.148)	(0.956)

[a] Compare empirical quantile ($Q - T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile for $\delta = 36.12$ with $n = 400$, $\kappa = 0.1584$.

[b] $(1+\kappa)^{-1}T_{ML}$ is calculated for the comparison to noncentral chi-square distribution.

[c] $Q - distribution$ are the quantiles from $\chi^2_{df}(\delta)$ and $P - ncx$ is the percent of samples that is less than computed quantile $Q - distribution$

[d] Values in parentheses (Q_{diff}, P_{diff}) are the differences between empirical values and respective theoretical values from each distribution.

Table 15

Quantile comparison for Model 3 ($n = 400$, $\delta = 189.56$, $\kappa = 0.1584$)

$Q - T_{ML}$	128.2762	148.0063	159.1807	250.8239	265.2447	293.5672
$(1+\kappa)^{-1}T_{ML}$	110.7357	127.7679	137.4143	216.5262	228.9751	253.4247
$P - T_{ML}$	1%	5%	10%	90%	95%	99%
$Q - nxc$	121.2977	136.4081	144.8371	211.702	222.2301	242.6834
$Q_{diff} - ncx$	(-10.562)	(-8.64023)	(-7.42282)	(4.824157)	(6.744952)	(10.74132)
$P - nxc$	2.86	9.338	15.498	87.172	92.666	97.848
$P_{diff} - nxc$	(-1.86)	(-4.338)	(-5.498)	(2.828)	(2.334)	(1.152)
$Q - nm$	136.6503	152.1485	160.4106	218.6997	226.9618	242.4601
$Q_{diff} - nm$	(-8.3741)	(-4.1422)	(-1.2299)	(32.1242)	(38.2829)	(51.1071)
$P - nm$	2.1	6.534	10.652	66.722	74.496	85.546
$P_{diff} - nm$	(-1.1)	(-1.534)	(-0.652)	(23.278)	(20.504)	(13.454)
$Q - nm2$	109.88	137.9714	152.9468	258.5988	273.5742	301.6656
$Q_{diff} - nm2$	(18.3962)	(10.0349)	(6.2339)	(-7.7749)	(-8.3295)	(-8.0984)
$P - nm2$	0.154	2.332	6.878	93.046	96.704	99.382
$P_{diff} - nm2$	(0.846)	(2.668)	(3.122)	(-3.046)	(-1.704)	(-0.382)
$Q - nm3$	120.9626	145.3586	158.3641	250.1175	263.123	287.519
$Q_{diff} - nm3$	(7.3136)	(2.6477)	(0.8166)	(0.7064)	(2.1217)	(6.0482)
$P - nm3$	0.564	4.192	9.53	89.686	94.422	98.514
$P_{diff} - nm3$	(0.436)	(0.808)	(0.47)	(0.314)	(0.578)	(0.486)

[a] Compare empirical quantile ($Q - T_{ML}$) and percent of samples from the simulation that covered theoretical distribution quantile for $\delta = 189.56$ with $n = 400$, $\kappa = 0.1584$.

[b] $(1+\kappa)^{-1}T_{ML}$ is calculated for the comparison to noncentral chi-square distribution.

[c] $Q - distribution$ are the quantiles from $\chi^2_{df}(\delta)$ and $P - ncx$ is the percent of samples that is less than computed quantile $Q - distribution$

[d] Values in parentheses (Q_{diff}, P_{diff}) are the differences between empirical values and respective theoretical values from each distribution.

Table 16

Kolmogorov-Smirnov distance(K) for Model 4, $df = 14$, $\kappa = 2.25$

n	F^*_{ML}	δ	RMSEA	$ncxK$	nmK	$nm2K$	$nm3K$
400	0.025	9.80	0.042	0.012	0.896	0.106	0.048
	0.090	36.12	0.080	0.013	0.659	0.138	0.046
	0.190	75.88	0.116	0.030	0.484	0.138	0.041
	0.318	127.32	0.151	0.046	0.368	0.134	0.035
	0.474	189.56	0.184	0.060	0.284	0.134	0.029
	0.655	262.16	0.216	0.077	0.211	0.150	0.022
	0.863	345.16	0.248	0.106	0.135	0.181	0.017
	1.097	438.88	0.280	0.166	0.063	0.229	0.012
	1.360	544.00	0.312	0.267	0.099	0.290	0.011
1000	0.025	24.50	0.042	0.010	0.744	0.136	0.048
	0.090	90.30	0.080	0.019	0.463	0.149	0.038
	0.190	189.70	0.116	0.031	0.328	0.143	0.030
	0.318	318.30	0.151	0.042	0.250	0.137	0.026
	0.474	473.90	0.184	0.053	0.194	0.131	0.021
	0.655	655.40	0.216	0.066	0.143	0.139	0.017
	0.863	862.90	0.248	0.092	0.076	0.168	0.012
	1.097	1097.20	0.280	0.170	0.040	0.227	0.008
	1.360	1360.00	0.312	0.331	0.205	0.318	0.007

[a] Kolmogorov-Smironov distance (K) for different sample sizes n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$, and ncx etc as in Table 12.

Table 17

Average Kolmogorov-Smirnov distance(AK) for Model 4, $df = 14$, $\kappa = 2.25$

n	F^*_{ML}	δ	RMSEA	$ncxAK$	$nmAK$	$nm2AK$	$nm3AK$
400	0.025	9.80	0.042	0.007	0.490	0.056	0.027
	0.090	36.12	0.080	0.008	0.416	0.078	0.025
	0.190	75.88	0.116	0.017	0.326	0.082	0.021
	0.318	127.32	0.151	0.024	0.254	0.083	0.018
	0.474	189.56	0.184	0.031	0.198	0.085	0.015
	0.655	262.16	0.216	0.042	0.150	0.090	0.012
	0.863	345.16	0.248	0.067	0.099	0.100	0.008
	1.097	438.88	0.280	0.115	0.036	0.119	0.006
	1.360	544.00	0.312	0.187	0.053	0.147	0.005
1000	0.025	24.50	0.042	0.004	0.449	0.075	0.026
	0.090	90.30	0.080	0.012	0.314	0.085	0.020
	0.190	189.70	0.116	0.019	0.225	0.084	0.016
	0.318	318.30	0.151	0.024	0.170	0.083	0.013
	0.474	473.90	0.184	0.029	0.130	0.083	0.011
	0.655	655.40	0.216	0.034	0.096	0.085	0.008
	0.863	862.90	0.248	0.052	0.054	0.094	0.006
	1.097	1097.20	0.280	0.116	0.021	0.117	0.004
	1.360	1360.00	0.312	0.228	0.127	0.162	0.003

[a] Average Kolmogorov-Smironov distance (AK) for different sample size n with discrepancy F^*_{ML} and noncentral parameter $\delta = nF^*_{ML}$, and ncx etc as in Table 12.

Table 18

Model $Thurstone-1$ *with* $n = 213$, $df = 27$, $\delta = 234.6408$ *and RMSEA=0.2036*

Λ^*	0.8828	Ψ^*	0.2207
	0.8957		0.1978
	0.848		0.281
	0.5899		0.652
	0.5701		0.675
	0.5652		0.6806
	0.5429		0.7053
	0.6324		0.6001
	0.4869		0.7629

Table 19

Model $Thurstone-3$ with $n = 213$, $df = 9$, $\delta = 2.9181$ and RMSEA=0.0408

Λ^*	0.8674	-0.2686	0.0208	Ψ^*	0.1749
	0.8808	-0.237	-0.0572		0.1647
	0.8258	-0.2223	-0.0311		0.2677
	0.657	0.4448	-0.3202		0.268
	0.6297	0.4288	-0.2187		0.3718
	0.5965	0.2371	-0.2897		0.504
	0.6027	0.32	0.5026		0.2817
	0.6456	0.0526	0.2909		0.4959
	0.5402	0.3806	0.3008		0.4728

Figure Captions

Figure 1. *QQ plot of* T_{ML} *against ncx with* $\delta = 36.12$ *for Model 1*

Figure 2. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 36.12$ *for Model 1*

Figure 3. *QQ plot of* T_{ML} *against ncx with* $\delta = 189.56$ *for Model 1*

Figure 4. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 189.56$ *for Model 1*

Figure 5. *QQ plot of* T_{ML} *against ncx with* $\delta = 39.95$ *for Model 2*

Figure 6. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 39.95$ *for Model 2*

Figure 7. *QQ plot of* T_{ML} *against ncx with* $\delta = 80.05$ *for Model 2*

Figure 8. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 80.05$ *for Model 2*

Figure 9. *QQ plot of* T_{ML} *against ncx with* $(1+\kappa)^{-1}nF^*_{ML} = 31.19$ *for Model 3*

Figure 10. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 36.12$ *for Model 3*

Figure 11. *QQ plot of* T_{ML} *against ncx with* $(1+\kappa)^{-1}nF^*_{ML} = 163.64$ *for Model 3*

Figure 12. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 189.56$ *for Model 3*

Figure 13. *QQ plot of* T_{ML} *against ncx with* $(1+\kappa)^{-1}nF^*_{ML} = 11.12$ *for Model 4*

Figure 14. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 36.12$ *for Model 4*

Figure 15. *QQ plot of* T_{ML} *against ncx with* $(1+\kappa)^{-1}nF^*_{ML} = 58.325$ *for Model 4*

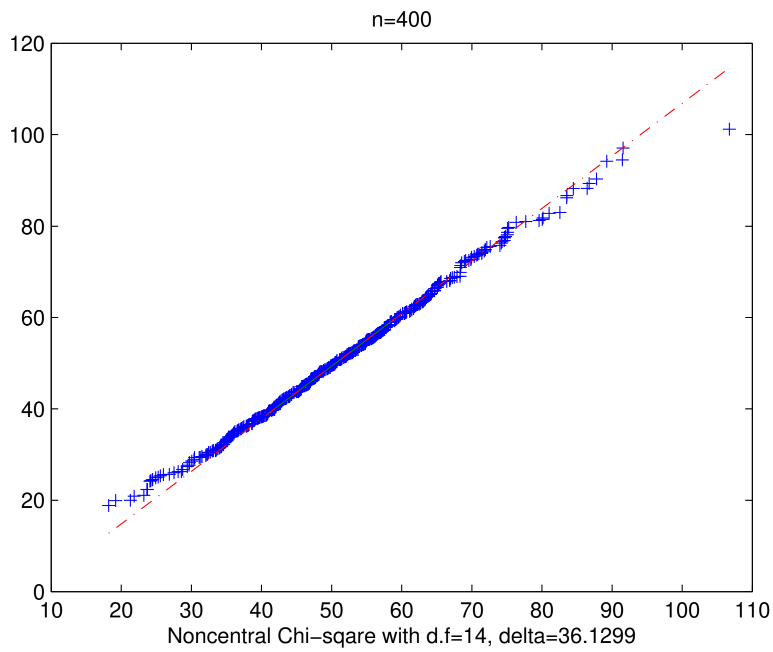
Figure 16. *QQ plot of* T_{ML} *against* $nm2$ *with* $\delta = 189.56$ *for Model 4*

Figure 17. *QQ plot of* T_{ML} *against ncx for Model* $Thurstone - 1$

Figure 18. $QQ\ plot\ of\ T_{ML}\ against\ nm2\ for\ Model\ Thurstone - 1$

Figure 19. $QQ\ plot\ of\ T_{ML}\ against\ ncx\ for\ Model\ Thurstone - 3$

Figure 20. $QQ\ plot\ of\ T_{ML}\ against\ nm2\ for\ Model\ Thurstone - 3$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100
80
60
40
20
0
10
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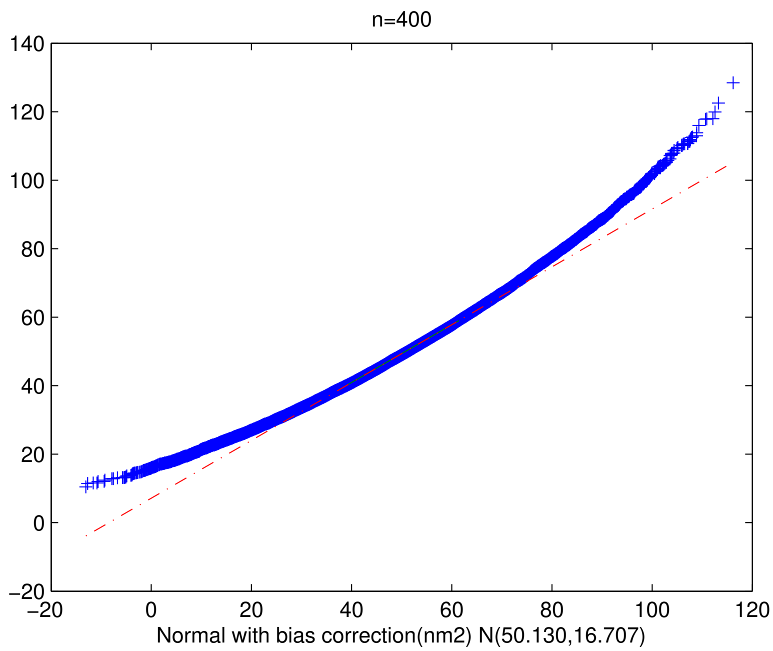
30
40
50
60
70
80
90
100
110
Noncentral Chi−sqare with d.f=14, delta=36.1299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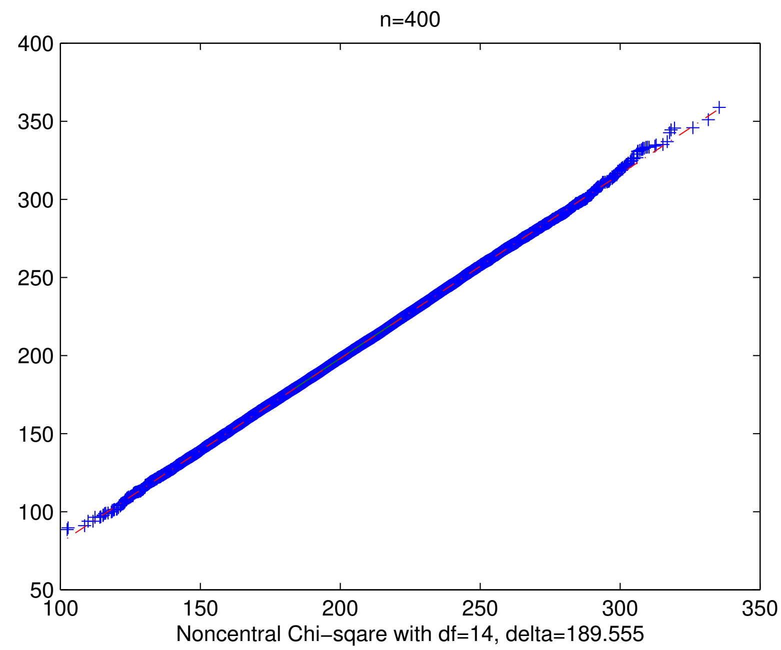
140
120
100
80
60
40
20
0
−20
−20
0
20
40
60
80
100
120
Normal with bias correction(nm2) N(50.130,16.707)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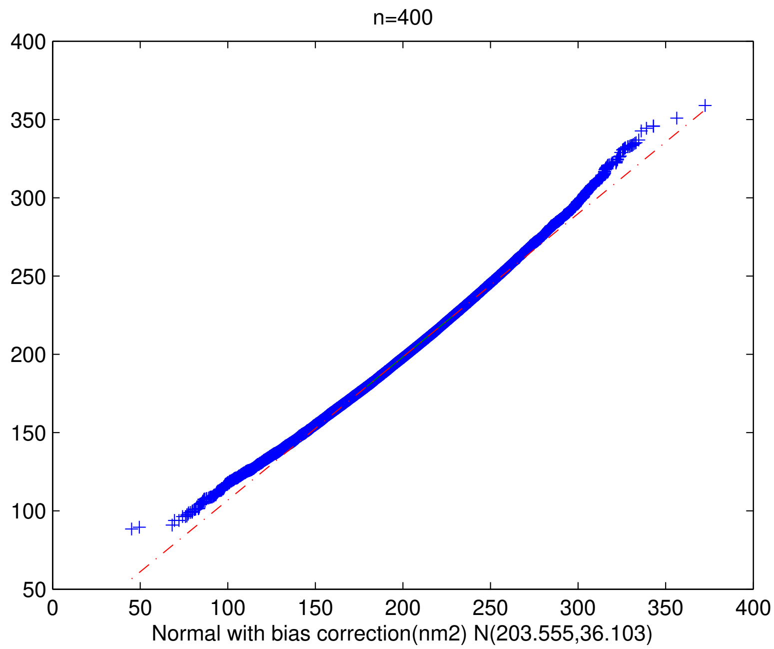
100
150
200
250
300
350
Noncentral Chi-sqare with df=14, delta=189.555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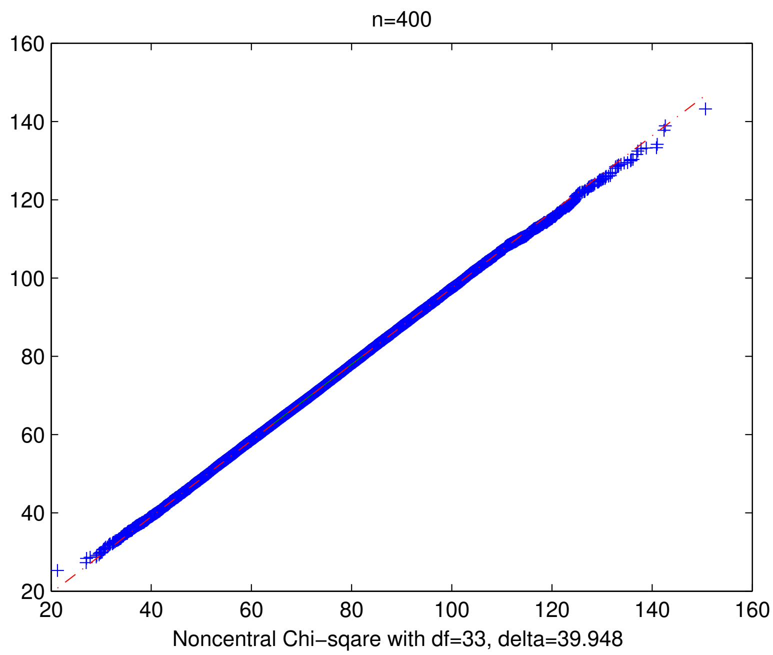
350
300
250
200
150
100
50
0
50
100
150
200
250
300
350
400
Normal with bias correction(nm2) N(203.555,36.103)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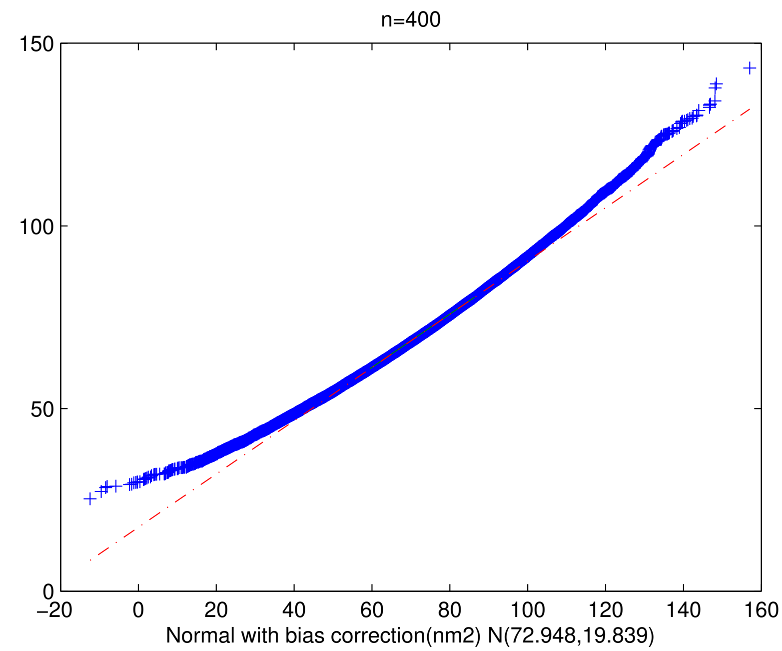
140
120
100
80
60
40
20
20
40
60
80
100
120
140
160
Noncentral Chi−sqare with df=33, delta=39.948


n=400
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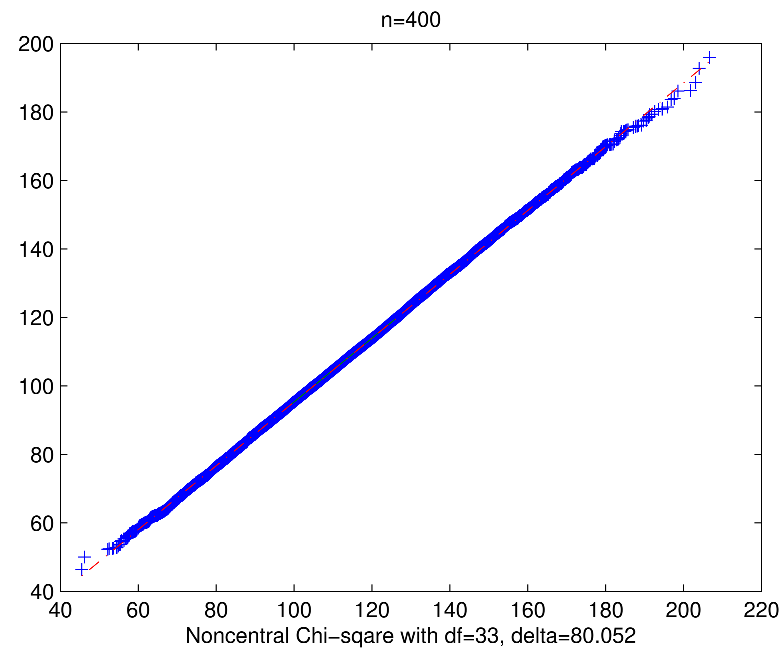
100
50
0
−20
0
20
40
60
80
100
120
140
160
Normal with bias correction(nm2) N(72.948,19.839)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160
140
120
100
80
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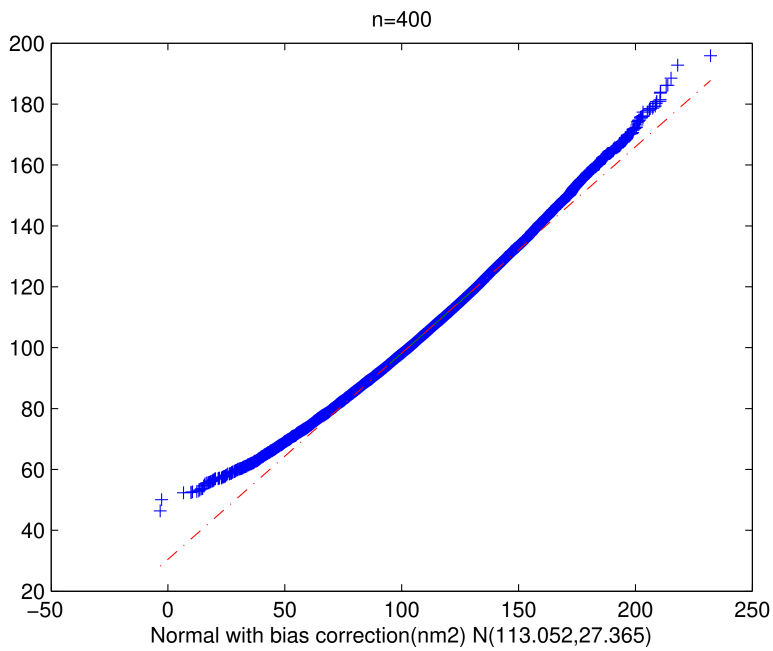
40
40
60
80
100
120
140
160
180
200
220
Noncentral Chi-sqare with df=33, delta=80.052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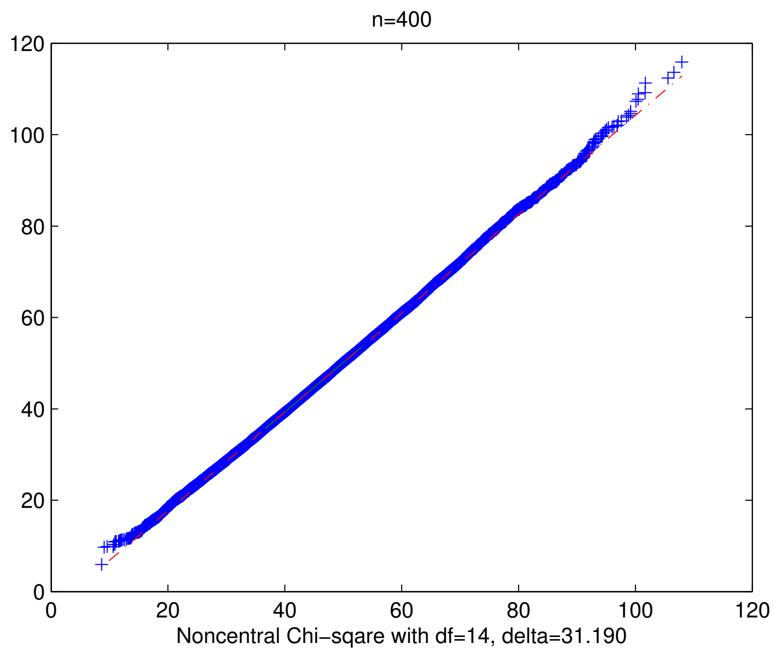
200
180
160
140
120
100
80
60
40
20
−50
0
50
100
150
200
250
Normal with bias correction(nm2) N(113.052,27.365)


n=400
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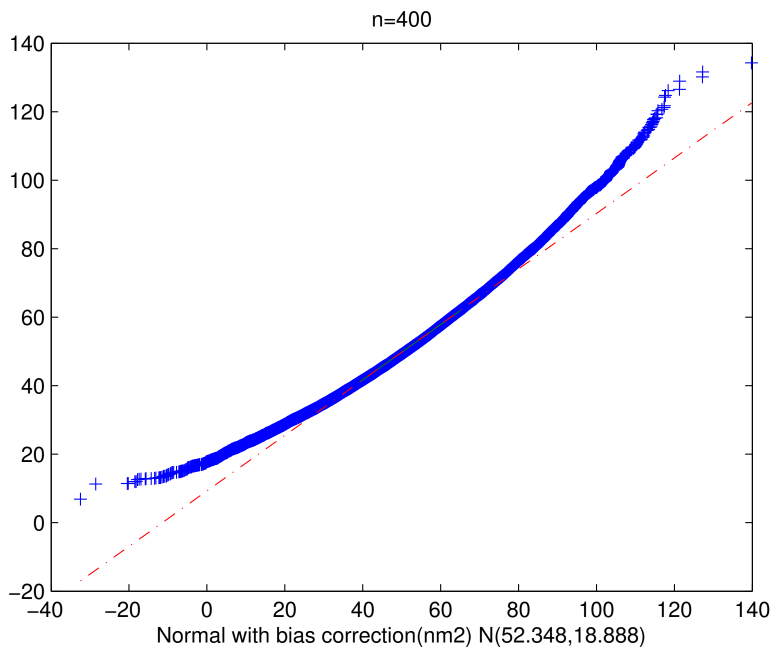
80
60
40
20
0
0
20
40
60
80
100
120
Noncentral Chi−sqare with df=14, delta=31.190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120
100
80
60
40
20
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−20
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−20
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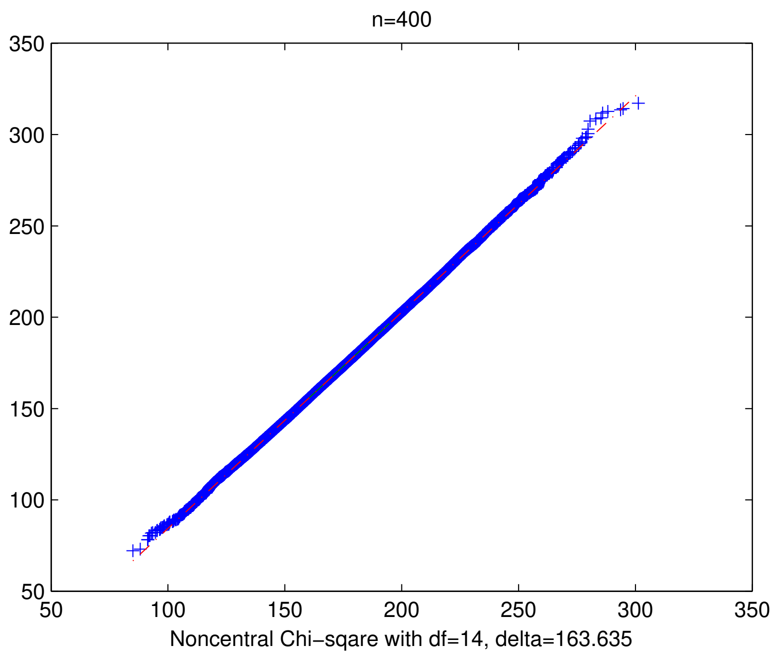
40
60
80
100
120
140
Normal with bias correction(nm2) N(52.348,18.888)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100
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200
250
300
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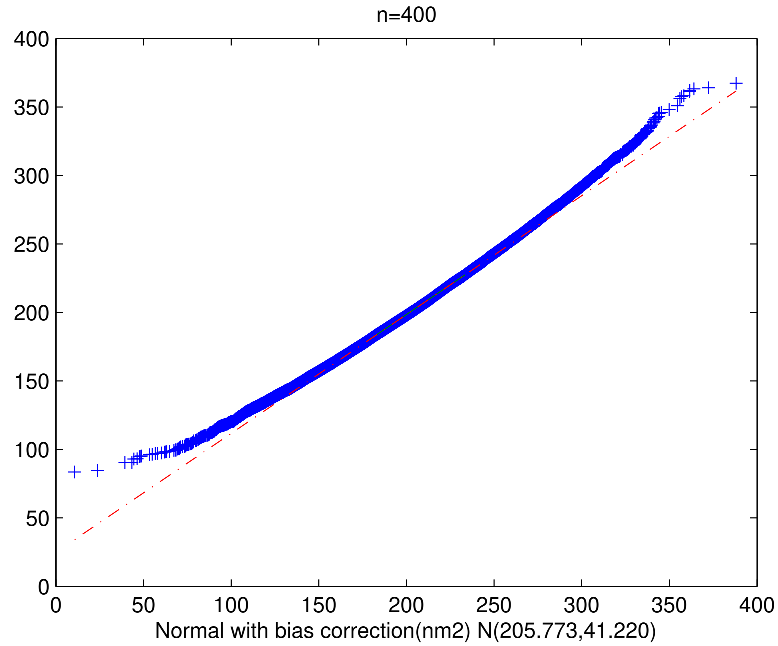
Noncentral Chi−sqare with df=14, delta=163.635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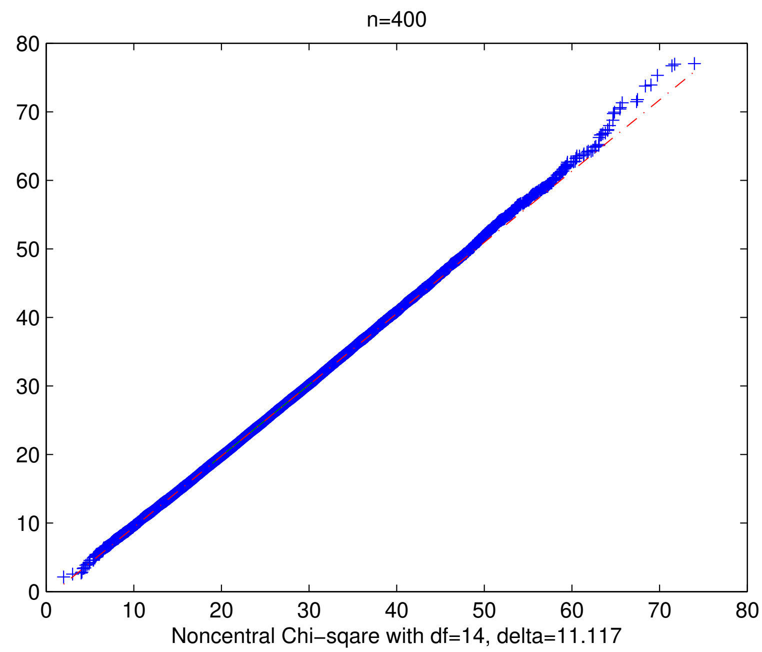
350
300
250
200
150
100
50
0
0
50
100
150
200
250
300
350
400
Normal with bias correction(nm2) N(205.773,41.220)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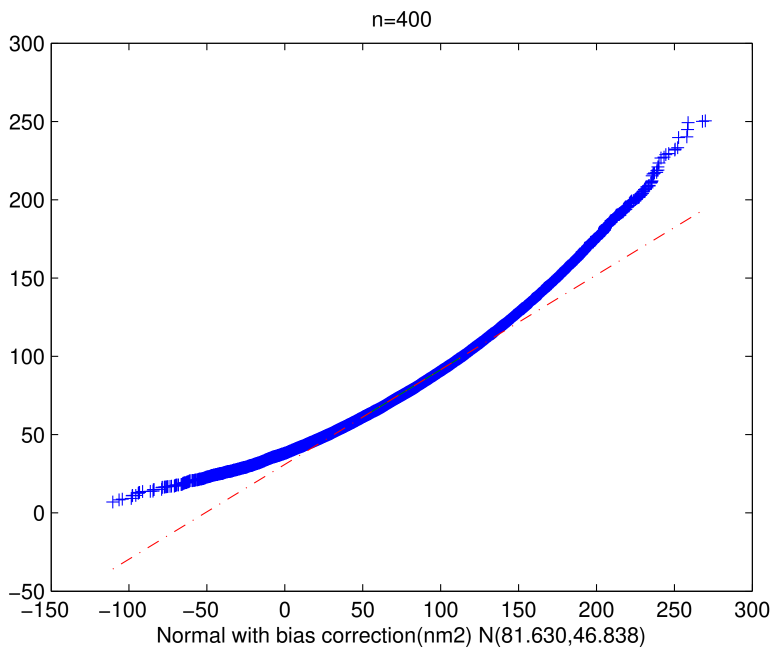
20
30
40
50
60
70
80
Noncentral Chi−sqare with df=14, delta=11.117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250
200
150
100
50
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−50
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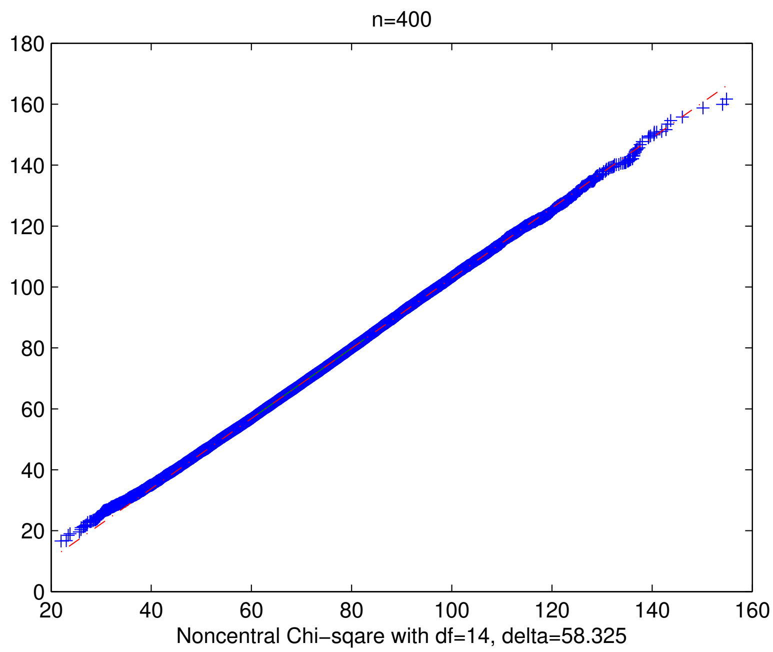
−50
0
50
100
150
200
250
300
Normal with bias correction(nm2) N(81.630,46.838)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160
140
120
100
80
60
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20
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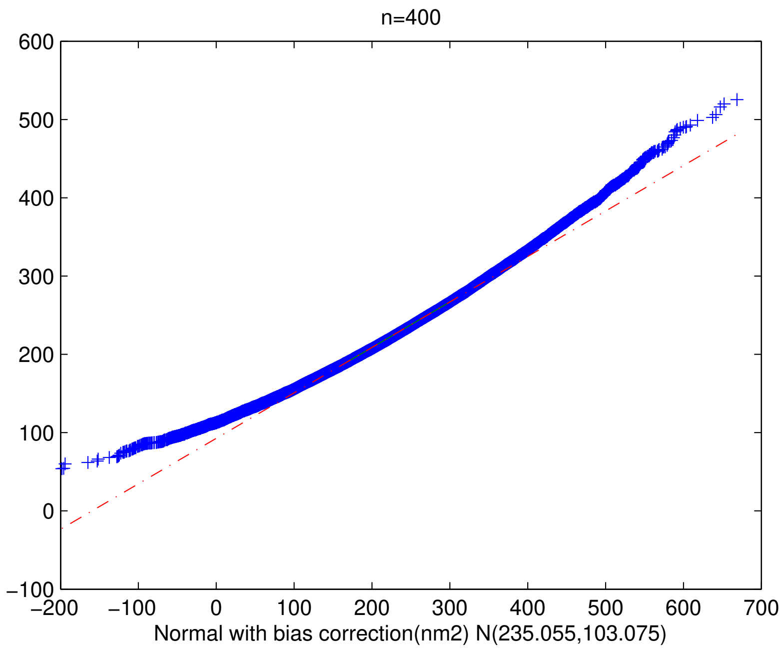
20
40
60
80
100
120
140
160
Noncentral Chi−sqare with df=14, delta=58.325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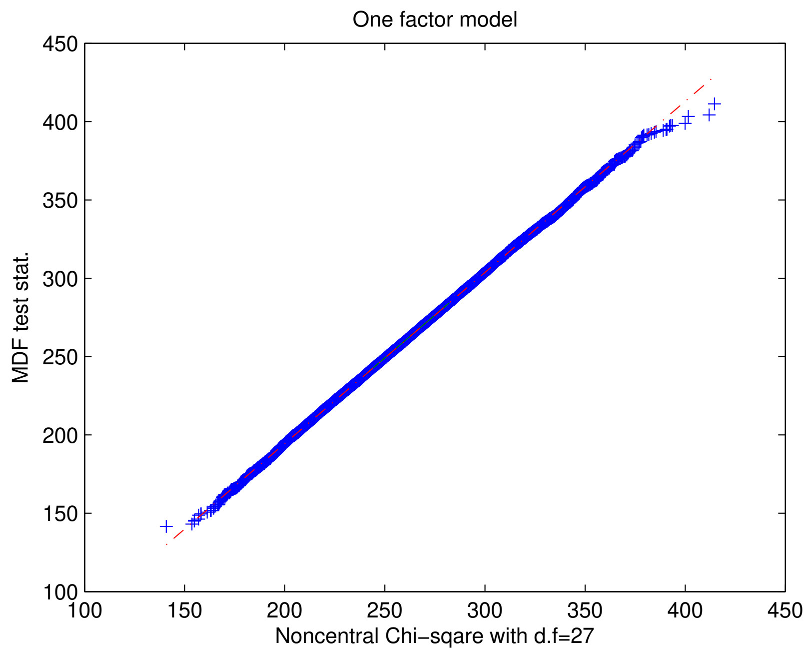
600
500
400
300
200
100
0
−100
−200
−100
0
100
200
300
400
500
600
700
Normal with bias correction(nm2) N(235.055,103.075)


One factor model
MDF test stat.
Noncentral Chi−sqare with d.f=27
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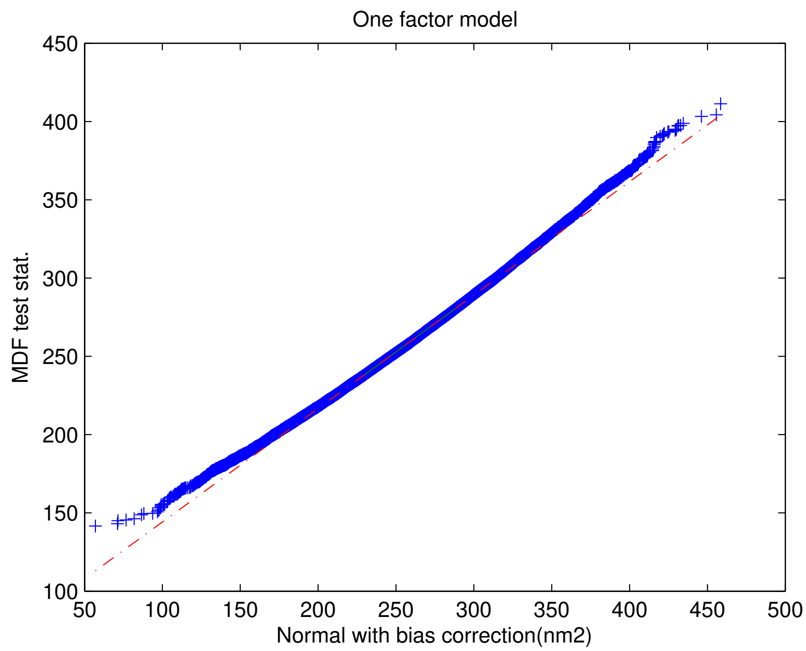
150
200
250
300
350
400
450


One factor model
MDF test stat.
Normal with bias correction(nm2)
450
400
350
300
250
200
150
100
50
100
150
200
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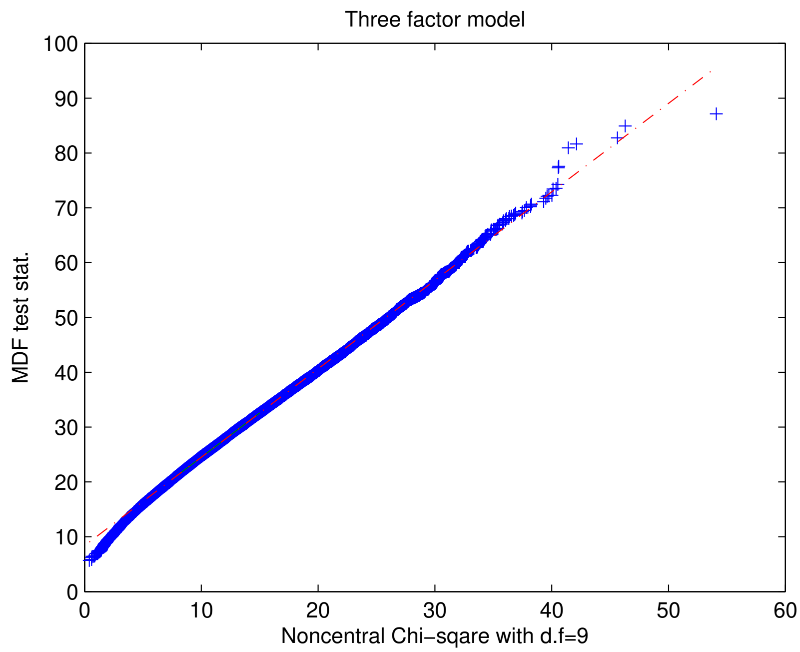
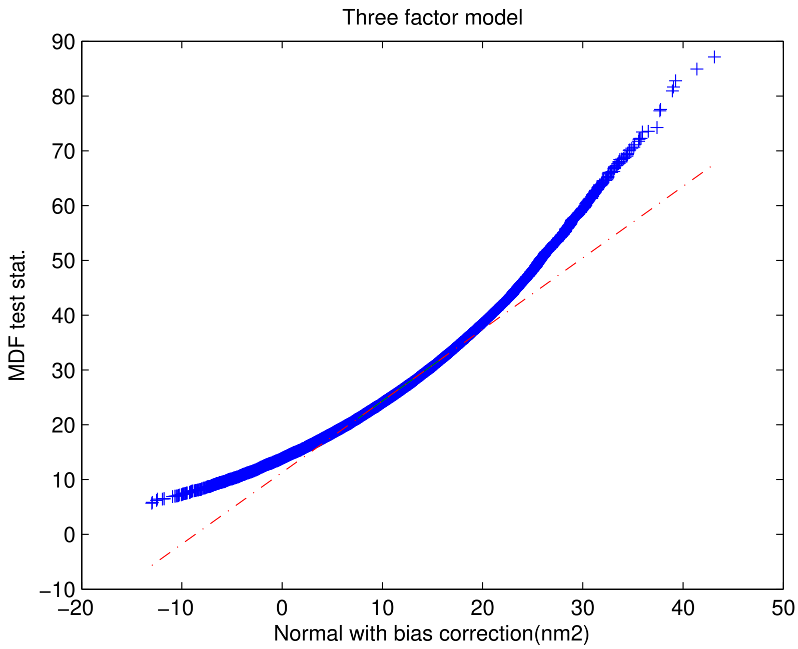
300
350
400
450
500



Three factor model
MDF test stat.
Noncentral Chi−sqare with d.f=9
0
10
20
30
40
50
60
70
80
90
100


Three factor model
MDF test stat.
Normal with bias correction(nm2)
90
80
70
60
50
40
30
20
10
0
−10
−20
−10
0
10
20
30
40
50