Improving Modeling of Extreme Events using Generalized Extreme Value Distribution or Generalized Pareto Distribution with Mixing Unconditional Disturbances

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Working Paper

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Abstract

In this paper an alternative non-parametric historical simulation approach, the Mixing Unconditional Disturbances model with constant volatility, where price paths are generated by reshuffling disturbances for S&P 500 Index returns over the period 1950 - 1998, is used to estimate a Generalized Extreme Value Distribution and a Generalized Pareto Distribution. An ordinary back-testing for period 1999 - 2008 was made to verify this technique, providing higher accuracy returns level under upper bound of the confidence interval for the Block Maxima and the Peak-Over Threshold approaches with Mixing Unconditional Disturbances. This method can be an effective tool to create value for stress-testing valuation.

Introduction

Nassim Taleb in his famous book, *The Black Swan: The Impact of the Highly Improbable* (2007), defines *black swan* as rare event of high-impact and hard to predict.

Extreme events ("*black swan*") analysis has progressed over the last decades with technology's advance, nowadays Block Maxima and Peak-Over Threshold methodologies are frequently applied in areas as weather and climate, engineering, finance and insurance (see e.g. Reiss and Thomas, 2007, and Castillo et al., 2005).

Nassim Tabel pointed how models consider the "known unknowns" but ignore the "unknown unknowns", in this sense, simulations approach must evolve to generate "unknowns" data to be incorporated in analysis, because of, as Emil Gumbel quoted, "*It's impossible that the improbable never arrives*".

In this sense, one of the greatest challenges to risk manager is to implement risk management models which allow for rare but damaging events, and permit the measurement of their consequences (McNeil, 1999). Thus, extreme values theory offers an important set of techniques for quantifying the boundaries between expected, unexpected and stress loss classes (Embrechts et at. 1999).

We propose a simulation approach, the Mixing Unconditional Disturbances model, to generate alternative variable paths prior to fit a Generalized Extreme Value Distribution or a Generalized Pareto Distribution to calculate a returns level of time series, providing a tool to achieve stressed values that can be use to complement managing risks.

Extreme value theory can be a useful tool to understand and protect against "catastrophic" risks, nevertheless mathematical aspects of risk can't be summarized into one number (Rootzen and Kluppelberg, 1999).

The paper is organized as follows: Section 1 describes a brief review of Extreme Value Theory, particularly illustrating the Block Maxima and the Peak-Over Threshold approaches with an application for S&P 500 Index daily returns. Section 2 develops the Mixing Unconditional Disturbances model and shows an example of a price path simulation for the Index. In Section 3 the Block Maxima and the Peak-Over Threshold methods are estimated with and without the Mixing Unconditional Disturbances model, ten years return level for S&P 500 Index ten days loss are back-tested to compare accuracy of estimations. Section 5 contains some concluding remarks and suggestions for future research.

1. Extreme Values

Extreme values reflect rare and high impact events that have an unlikely occurrence.

Extreme events occur when a risk takes values from the tail of its distribution (McNeil, 1999).

Modeling of extreme cases, generally treated as "outliers" in classic statistical methods, are approached using Block Maxima (BM), where the maxima (or minima) a variable takes in successive periods are selected, or using Peak-Over Threshold (POT), where the observations that exceed a given threshold are considered.

Block Maxima follows a General Extreme Value Distribution (GEV) while Peak-Over Threshold follows a General Pareto Distribution (GPD), both distributions can be seen in Figure 1.

Coles (2001) is recommended for people interested in an extreme value theory's mathematical treatment.

Selection of the size of the block or the threshold is subject to a trade-off between variance and bias. A raise in the number of observations, choosing lower block size or low threshold, reduces variance but increase bias.

If blocks are sufficiently large, BM approach can avoid dealing with data clustering issue. In addition, the estimation is simplified as the selection of a threshold is not needed. However, POT approach seem to become the method of choice in recent application because of this method uses data more efficiently (Gilli and Kellezi, 2006).

McNeil (1999) also commented that POT models are generally considered to be the most useful for practical applications, due to their more efficient use of the (often limited) data on extreme values.

In practice parameters of GEV and GPD are usually estimated with the maximum likelihood method. Other techniques used to approximate parameters are: the method of moments and the method of probability weighted moments (Habiboellah, 2005).

One extreme value theory restriction can emerge in multivariate case (e.g. assets' portfolio) where the joint distribution of the marginal extreme distributions is not necessarily an extreme distribution (Bensalah, 2000).

An interesting discussion of extreme value theory's limitations can be seen in Embrechts (2000) and Diebold et. al. (1998).

1.1. Block Maxima (BM)

Let $X = (X_1, \ldots, X_n)$ be independent identically distributed random variables with a unknown distribution function F.

The sample maximum, M_n, with *n* the size of the block is defined $M_n = \max(X_1, \ldots, X_n)$.

Under the Fisher - Tippett Theorem the sequence of normalized maxima converges in distribution:

$$H(x) = \begin{cases} \exp\left(-1\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right) & \text{for } \xi \neq 0 \\ \exp\left(-\exp^{-(x-\mu/\sigma)}\right) & \text{for } \xi = 0 \end{cases}$$

where μ is the location parameter, σ is the scale parameter and ξ is the shape parameter. For ξ > 0 and ξ < 0 the GEV distributions corresponds to Frechet Distribution and Weibull Distribution, respectively. The case ξ = 0 corresponds to Gumbel Distribution.

The *return level* R_k^n is the level expected, on average, to be exceeded in one out of *k* periods of length *n*. The return period is the amount of time expected to wait for particular return level to be exceed; return period is the inverse of the probability of an event (e.g. a called "100 years event" has a 1% probability of exceed the record level in a given year).

In this sense, return level is simply the calculation of quantiles from the Generalized Extreme Events Distribution, specifically:

$$\Pr(M_n \geq R_k^n) = 1/k$$

$$R_k^n \approx H^{-1}_{\xi,\mu,\sigma}(1 - 1/k) \approx \tilde{\mu} - \frac{\tilde{\sigma}}{\tilde{\xi}}\left(1 - \left(-\ln\left(1 - 1/k\right)\right)^{-\xi}\right) \quad \text{for } \tilde{\xi} \neq 0$$

$$\tilde{\mu} - \tilde{\sigma}\ \ln\left(-\ln\left(1 - 1/k\right)\right) \quad \text{for } \tilde{\xi} = 0$$

To apply BM method, first sample is divided in blocks, second the maximum value in each block is collected, and then GEV is fitted.

To illustrate this method we use daily returns of SP&500 Index from January 4, 1950 to December 31, 1998, downloaded from Yahoo Finance, for a total 12,329 observations. The data sample was divided in 49 been non-overlapping sub-samples of each calendar years. The minimum daily returns in each of the blocks constitute the data points used to estimate GEV.

Figure 2 and Figure 3 shows SP&500 daily returns of the period and the maxima daily loss of each year, respectively.

Using the extRemes Toolkit developed by Eric Gilleland, within statistical software R, we estimated GEV.

Figure 4 presents the estimated Frechet Distribution where GEV has location parameter (μ): 2.17863, scale parameter (σ): 0.87080 and shape parameter (ξ): 0.52698, and Figure 5 plots return level of S&P 500 daily returns.

The estimated 100 years return level (R_{100}) is 19.19%, with 95% confidence interval of (9.875%, 44.842%), meaning, on average, that the maximum daily loss during a period of one year exceeds that level in one out of hundred years, in other words a Black Monday[+] happens just once in a century.

[+] Black Monday refers to Monday, October 19, 1987, the largest one-day percentage decline in stock market's history. The S&P 500 Index fell 22.9% that day.

1.2. Peak-Over Threshold (POT)

Let $X = (X_1, \ldots, X_n)$ be independent identically distributed random variables with a unknown distribution function F.

Then for a large enough threshold μ, the conditional distribution of $Y = (X_i - \mu \mid X_i > \mu)$, under the Balkema and de Haan - Pickands Theorem, is approximately:

$$H(y) = \begin{cases} 1 - \left(1 + y\frac{\xi}{\sigma}\right)^{-1/\xi} & \text{for } \xi \neq 0 \\ 1 - \exp^{-(y/\sigma)} & \text{for } \xi = 0 \end{cases}$$

For a GPD, the *k*-year return level is defined:

$$R_k \approx \begin{cases} \mu + \frac{\tilde{\sigma}}{\tilde{\xi}}\left(\left(k * n_y * \Pr(X > \mu)\right)^{\xi} - 1\right) & \text{for } \tilde{\xi} \neq 0 \\ \mu + \tilde{\sigma}\ \ln\left(k * n_y * \Pr(X > \mu)\right) & \text{for } \tilde{\xi} = 0 \end{cases}$$

where n_y is the number of observations per year and $\Pr(X > \mu)$ is equal to number of exceedances of threshold divided by total number of observations.

To apply POT method, first a threshold μ must be selected, and second GPD is estimated.

To illustrate this method we also used same SP&500 Index data sample and utilized extRemes to conduct a visual inspection of mean excess plot and graph of the estimated parameter as functions of μ to define the threshold value, and then we recorded GPD.

Figure 6 shows Mean Residual Life plot, looking for lowest threshold where plot is nearly linear. Figure 7 presents Fitting Data to a GPD Over a Range of Thresholds to check the stability in the parameter. A threshold of 2.5% daily lose was selected, for a 74 exceedances of threshold.

Figure 8 presents GPD with parameters (σ): 0.61790 and (ξ): 0.53758. Figure 9 plots return level of S&P 500 daily returns with (n_y): 252 days

The estimated 100 years return level (R_{100}) is 18.42% with 95% confidence interval of (9.984%, 27.642%).

2. Mixing Unconditional Disturbances (MUD)

Tompkin and D'Ecclesia (2006) introduce the MUD model where simulations of path are obtained re-writing history, under this approach parameter estimation and distributional assumption are not required and the statistical characteristics of the original path are conserved.

Given the historical (e.g. daily) series for a variable X_t (e.g. returns), for t = 0,...,T, the unconditional mean μ, and standard deviation σ, are estimated (e.g. using Excel functions AVERAGE and STDEV).

Normalizing the sequence of the variable yields:

$$Z_t = X_t - \mu / \sigma \qquad (1)$$

where Z_t is the series of standardized "disturbances" from 1 to T. By design, resulting disturbances have a mean of 0 and standard deviation of 1.

The simulated variable $\tilde{X}_t$ at each time t>0 are obtained using the standardized disturbances, to generate new path we "freeze" the Z_t and use formulation:

$$\tilde{X}_t = Z_t{*}\sigma + \mu \qquad (2)$$

Given the unconditional disturbances, Z_t, *is estimated using equation* (1) *and same* unconditional mean and standard deviation of the original series, we randomly (e.g. using Excel functions RAND) mix the Z_t, by drawing without replacement until all the disturbances are selected. With these new sequences of disturbances, new variable paths can be estimated using equations (2), for constant volatility assumptions.

We will use daily returns of the SP&500 Index from 1950 to 1998 to demonstrate this approach, too. Figure 10 shows difference between the original price path and the simulated one.

3. S&P 500 Index Stress Valuation

The intent of stress testing is to identify and measure exposure to risk in those economic environments that can be characterized as unlikely, but plausible. Such environments differ from "normal" markets not only in the increased magnitude of movements of variables but also in the changed relations among them (Schachter, 1998).

Stress test can provide useful information about a firm's risk exposure that Value at Risk (VaR) methods can easily miss, particularly if VaR models focus on "normal" market risk rather than the risk associated with rare or extreme events (Aragones et al. 2000).

3.1. Estimating BM with and without MUD

3.1.1. Estimating BM without MUD

Block Maxima approach is applied to two weeks* returns of SP&500 Index from January 4, 1950 to December 31, 1998. Data sample was divided once more in 49 been sub-samples of each calendar years. Minimum two weeks returns in each year compose the information used to estimate GEV using extRemes.

Figure 11 shows maxima two weeks loss of each year for SP&500 Index, the worse lost of 33.79% was registered for period October 6, 1987 to October 19, 1887, it included the Black Monday.

In addition, return level of S&P 500 Index two weeks returns graph (μ= 5.47859; σ= 2.29282; ξ= 0.20364) can be seen in Figure 12.

The estimated 10 years return level (R_{10}) is 12.02% with 95% confidence interval of (10.052%, 15.812%), meaning, on average, only once in ten years the two weeks loss will exceed that level.

3.1.2. Estimating BM with MUD

We made 1,000 price paths simulations of SP&500 Index for each year. Figure 13 illustrates 200 simulations for 1987.

Within the one thousand generated price paths for every year, from 1950 to 1998, we looked for the maxima two weeks loss of SP&500 Index in each year, and then we computed GEV with extRemes.

Figure 14 shows maxima ten days loss of each year for S&P 500 Index simulations against historical maxima two week loss.

Additionally, return level of S&P 500 Index two weeks returns of simulations (μ= 9.08622; σ= 3.15378; ξ= 0.15401) can be observed in Figure 15.

The estimated 10 years return level (R_{10}), with MUD model, is 17.57% with 95% confidence interval of (15.108%, 22.188%), meaning, on average, only once in ten years the two weeks negative return will exceed that level.

* We define a 10 days holding period returns considering that Basel II stipulates a ten-day horizon to estimate value at risk for banks.

3.2. Estimating POT with and without MUD

3.2.1. Estimating POT without MUD

Peak-Over Threshold method is applied to ten days returns of SP&500 Index 1950 – 1998, too.

Figure 16 and Figure 17 shows, in that order, Mean Excess graph and Fitting Data to a GPD Over a Range of Thresholds plot. A threshold of 4% two weeks negative returns was selected, for 82 exceedances of threshold of 1,184 two weeks returns observations.

Figure 18 presents return level of S&P 500 Index two weeks returns plot (σ= 2.32365; ξ= 0.15786; and n_y = 25).

The registered 10 years return level (R_{10}) is 12.37% with 95% confidence interval of (10.471%, 15.965%). Figure 19 presents return level plot.

3.2.2. Estimating POT with MUD

We employed, as time series to estimate the GDP, the two weeks returns of the price path simulated that generated the maxima two weeks loss of the SP&500 Index in each year.

Figure 19 shows ten days returns of S&P 500 Index simulations against historical two weeks returns for the period 1950 – 1998.

Figure 20 presents return level of S&P 500 Index ten days returns graph (μ=4.00; σ= 4.87936; ξ=0.03943 ; and n_y = 25).

The estimated 10 years return level (R_{10}) is 18.66% with 95% confidence interval of (15.731%, 24.100%).

3.3. Back-Testing

We estimated two weeks maxima loss of each year for the SP&500 Index for the period January 4, 1999 to December 31, 2008.

Table 1. S&P 500 Index Maxima Two Weeks Loss per Year, 1999 – 2008.

1999	2000	2001	2002	2003	2004	2005	2006	2007	2008
6.56%	9.96%	16.02%	17.77%	9.03%	4.62%	4.23%	5.31%	6.83%	29.95%

The worse loss of the period 1999 - 2008, 29.95%, was registered from September 29, 2008 to October 10, 2008, during a high volatility period of the Index, mainly due to the financial sector crisis.

Ten days returns from 1999 to 2008, 240 observations, are comparing with 10 years return level and his 99.99999% confidence intervals upper bound projected, using the BM and POT approaches with and without MUD model.

Table 2 summarized, for each method, returns levels, upper bound confidence interval estimated, and the number of violations for those negative returns higher than the estimated losses.

Table 2. S&P 500 Index Two Weeks Returns Back-testing, 1999 – 2008.

Method	Return Level	Violations	99.99% C.I.	Violations
Block Maxima	-12.02%	4 (1.67%)	-22.32%	1 (0.42%)
Peak-Over Threshold	-12.37%	4 (1.67%)	-22.25%	1 (0.42%)
Block Maxima w/MOU	-17.57%	2 (0.83%)	-30.26%	0 (0.00%)
Peak-Over Threshold w/MOU	-18.66%	1 (0.42%)	-36.30%	0 (0.00%)

4. Conclusions and Extensions

In this paper, we propose a new approach that combines variable simulations using the Mixing Unconditional Disturbances model with a Block Maxima or Peak-Over Threshold methods, this approach is apply for S&P 500 Index returns over the period 1950 to 1998, and results are verified with an ordinary back-test for period 1999 to 2008. The new technique provides higher accuracy returns level.

This method can be an effective tool for stress-testing valuation, because the perturbations of the unconditional historical return process generate more information to analyze extreme values.

Future lines of research could apply the technique in other areas and comparing it with other methods used to model tail risk measures, including the analyses of extreme events through MUD-model with stochastic volatility, and extreme value analysis under the presence of dependence.

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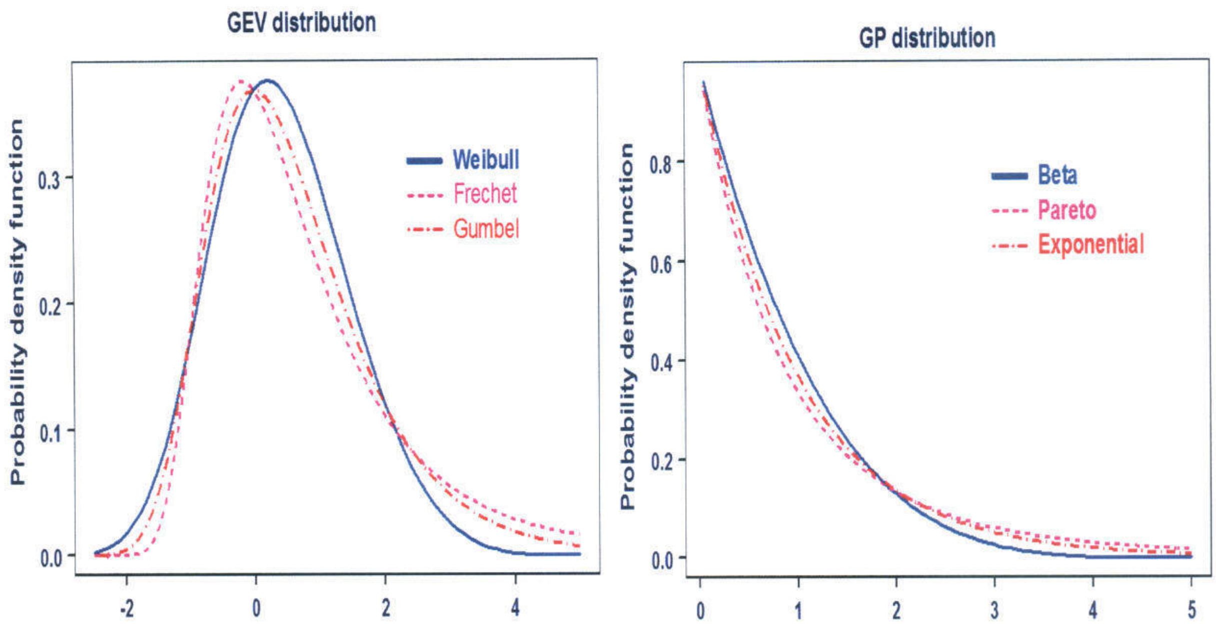


Figure 1. GEV and GPD Distributions. Source: www.isse.ucar.edu/extremevalues/back.html

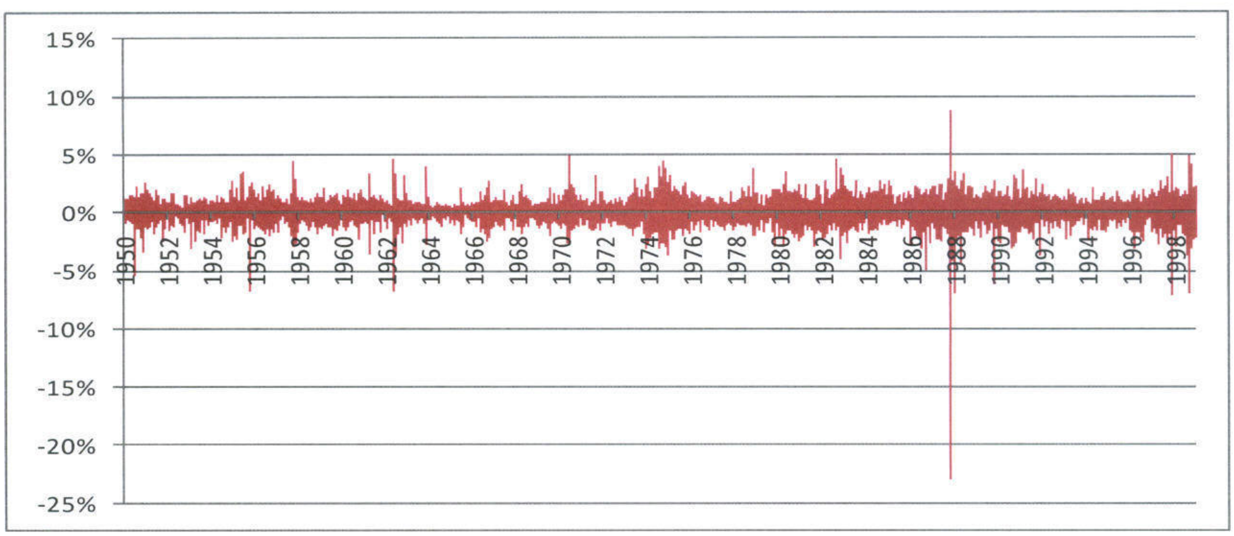


Figure 2. S&P 500 Index Daily Returns, 1950 – 1998.

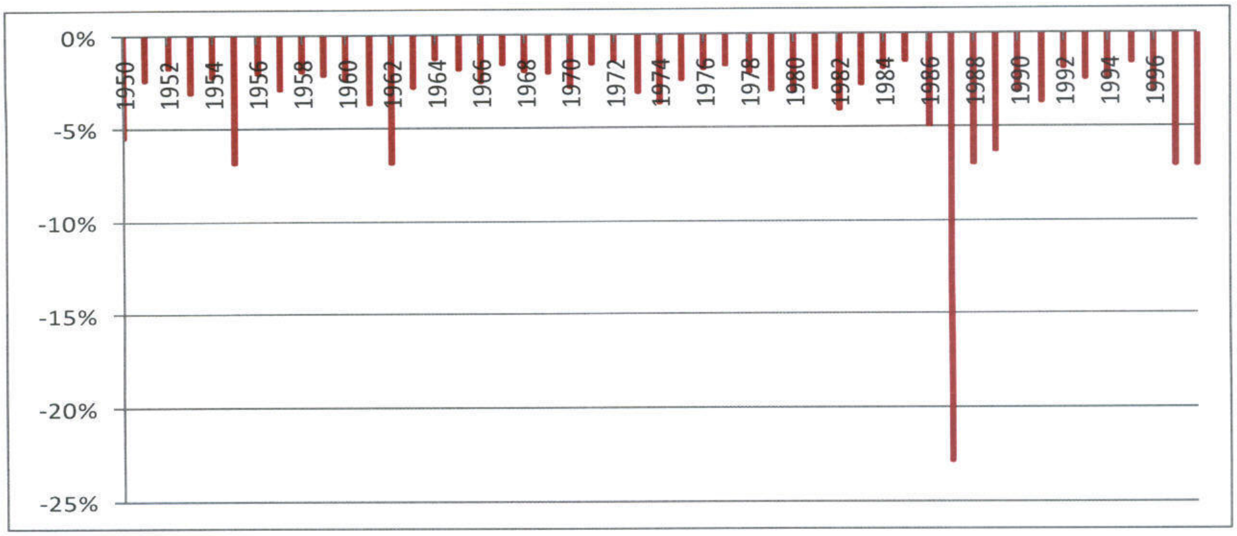


Figure 3. S&P 500 Index Maxima Daily Loss per Year, 1950 – 1998.

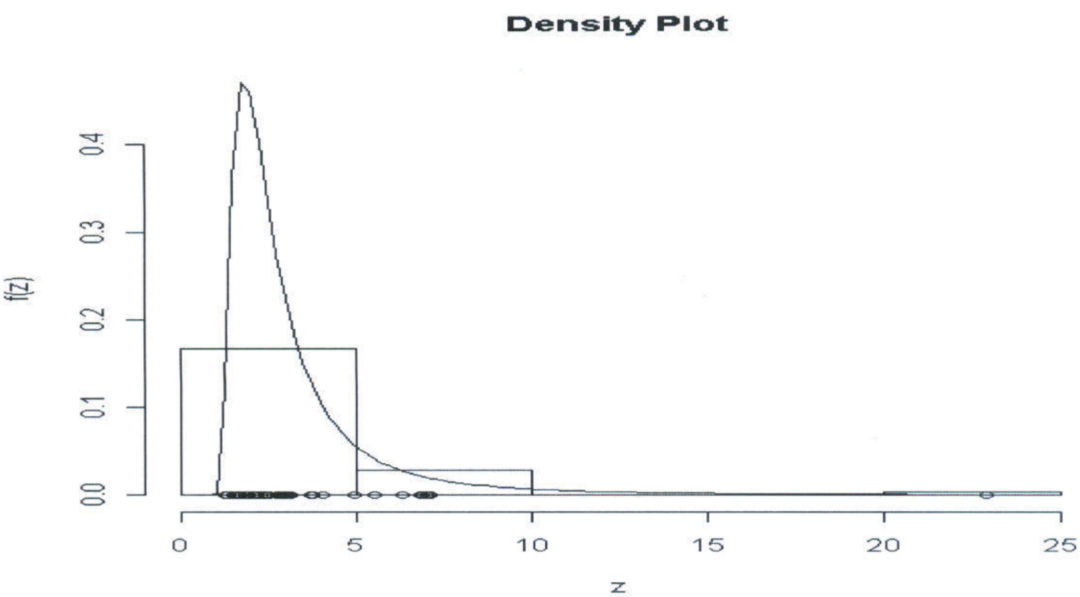


Figure 4. S&P 500 Index Daily Returns GEV Distribution, 1950 – 1998.

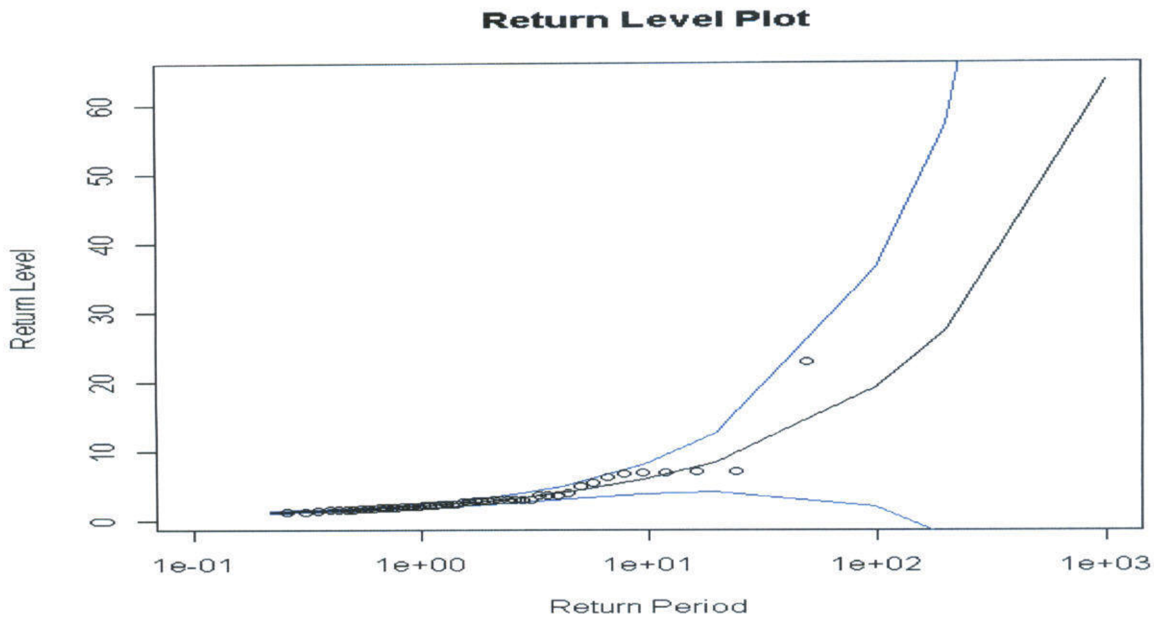


Figure 5. BM Return Level of S&P 500 Index Daily Returns, 1950 – 1998.

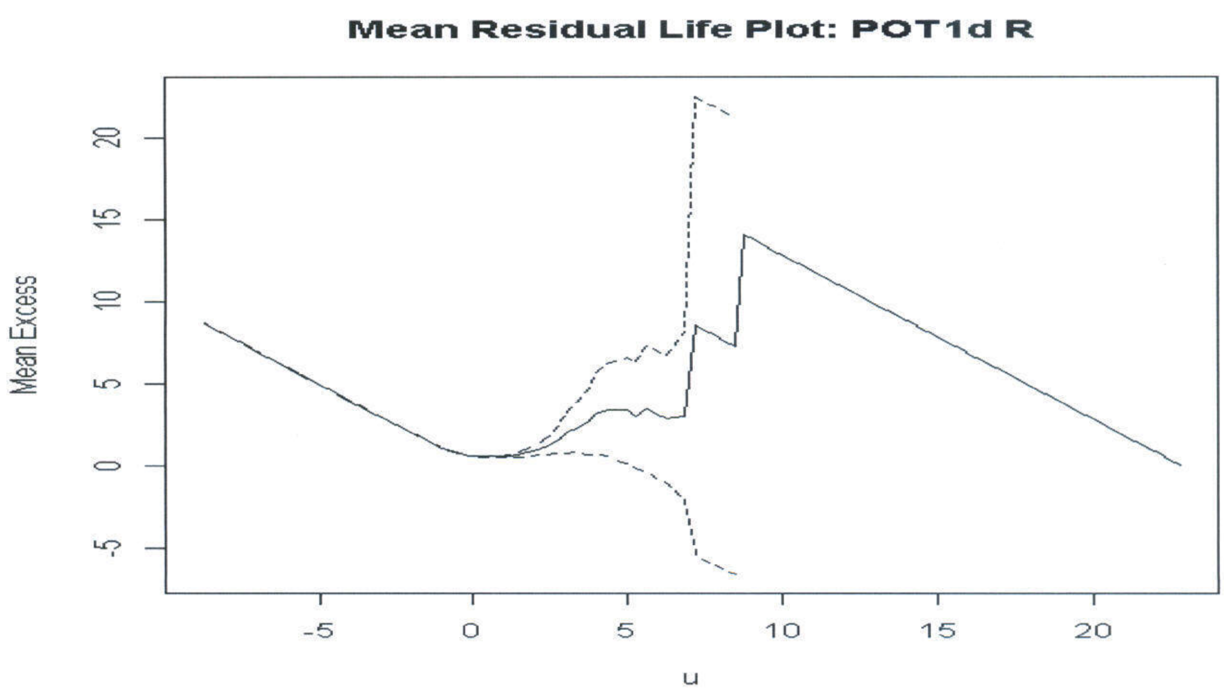


Figure 6. Mean Residual Life Plot, S&P 500 Index Daily Returns, 1950 – 1998.

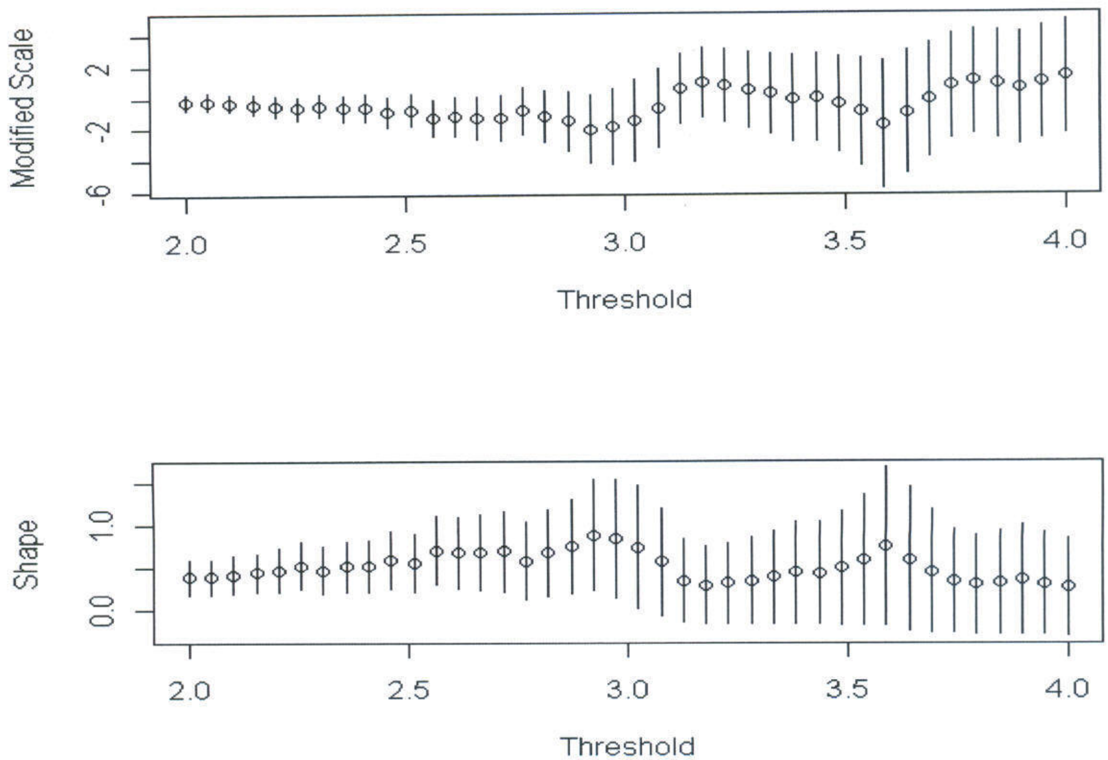


Figure 7. Fitting GDP for Thresholds Ranges, S&P 500 Index Daily Returns, 1950 – 1998.

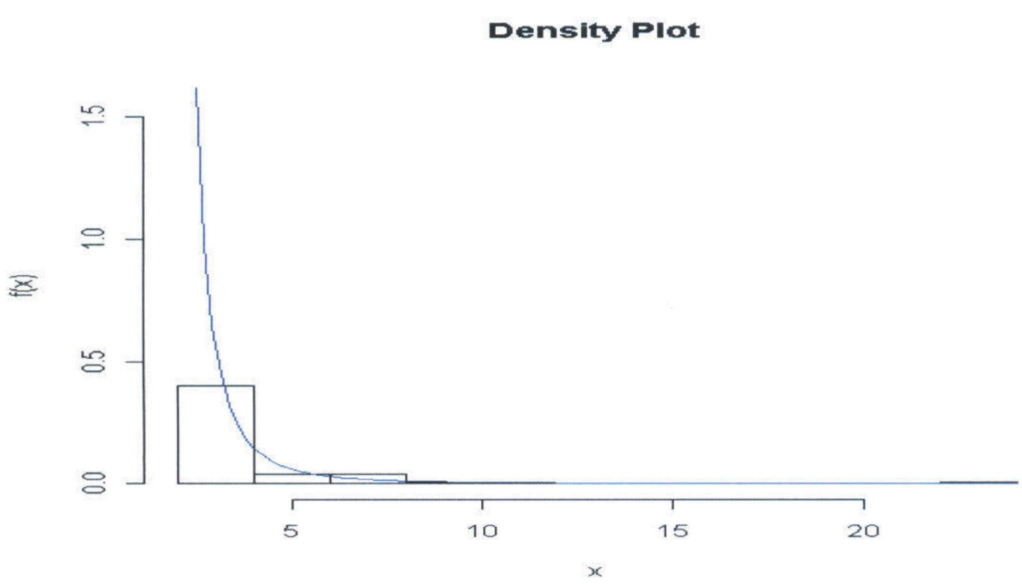


Figure 9. S&P 500 Index Daily Returns GPD, 1950 – 1998.

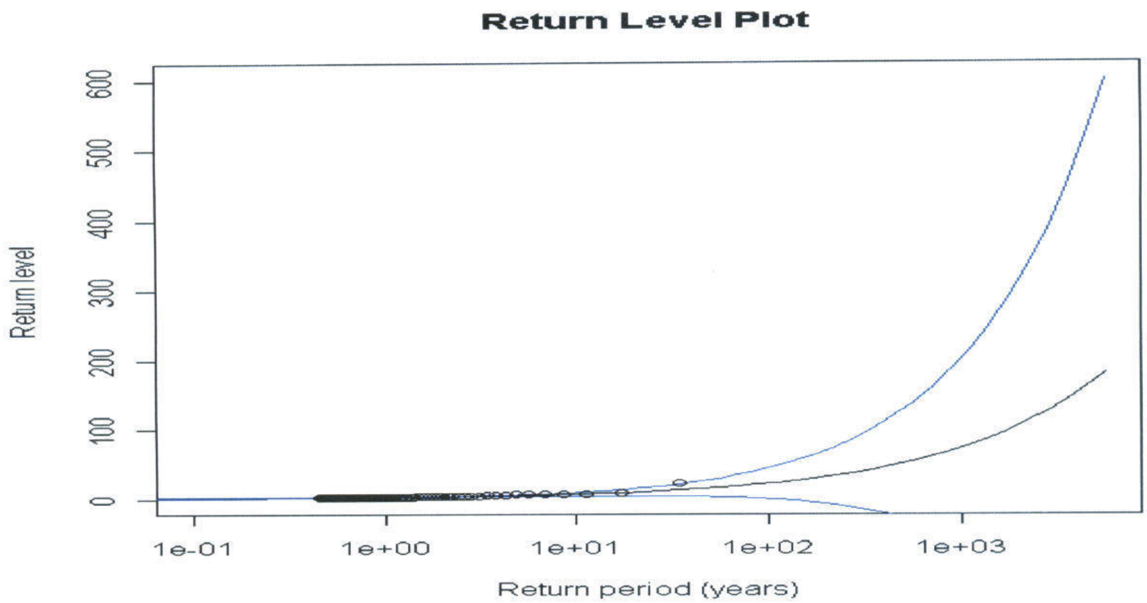
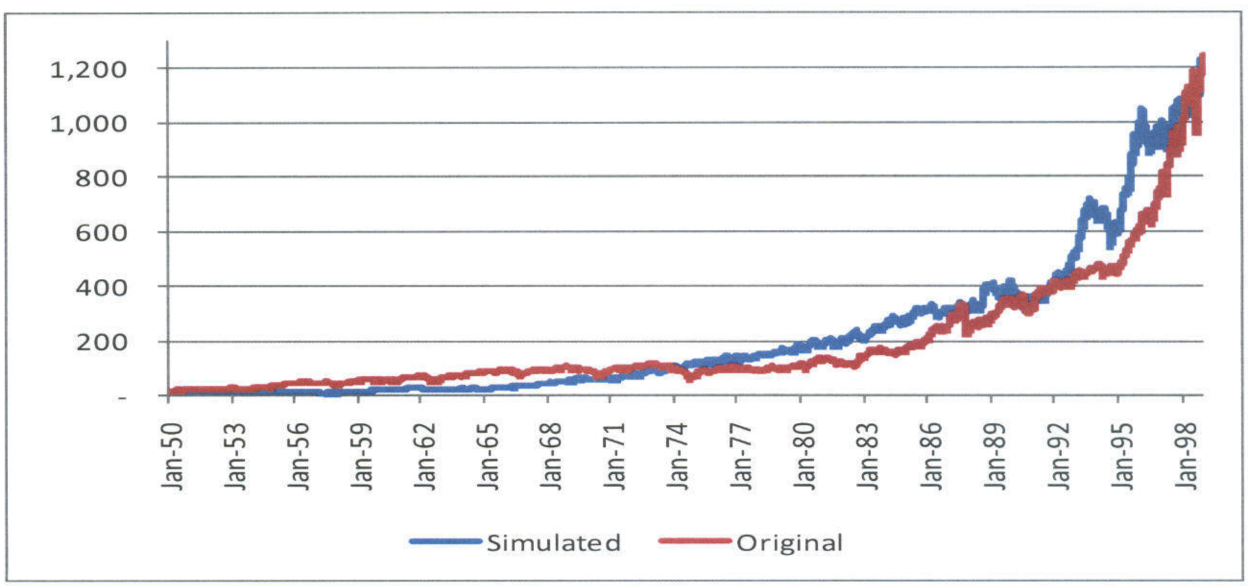


Figure 9. POT Return Level of S&P 500 Index Daily Returns, 1950 – 1998.



S&P 500 Daily Returns	Original	Standardized	Simulated
μ	0.03%	0.00	0.03%
σ	0.85%	1.00	0.85%

Figure 10. Price paths S&P 500 Index, 1950 – 1998.

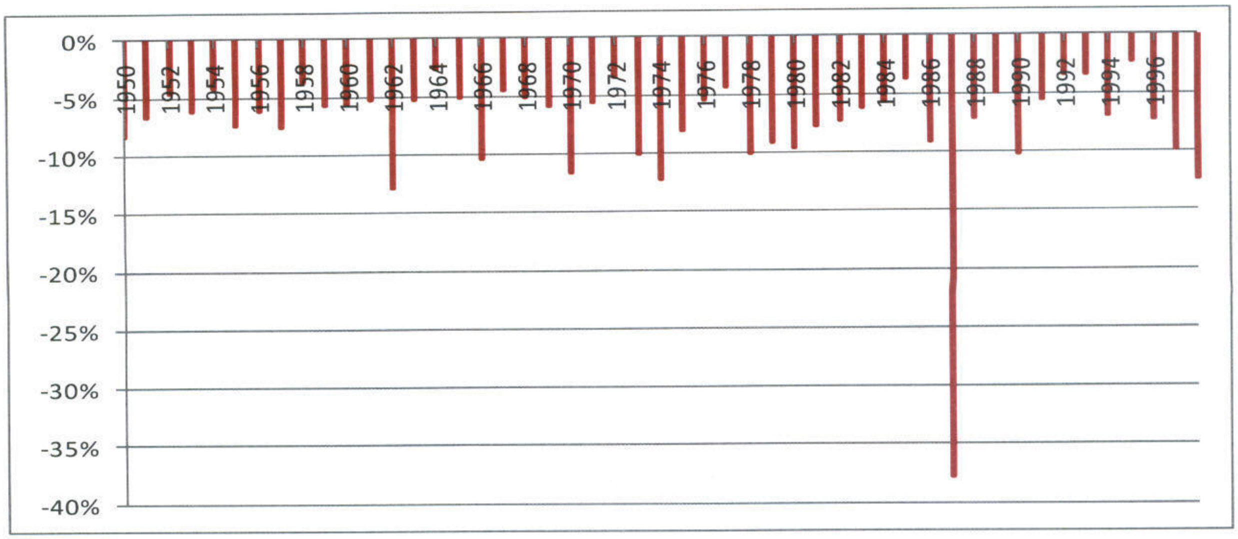


Figure 11. S&P 500 Index Maxima Two Weeks Loss per Year, 1950 – 1998.

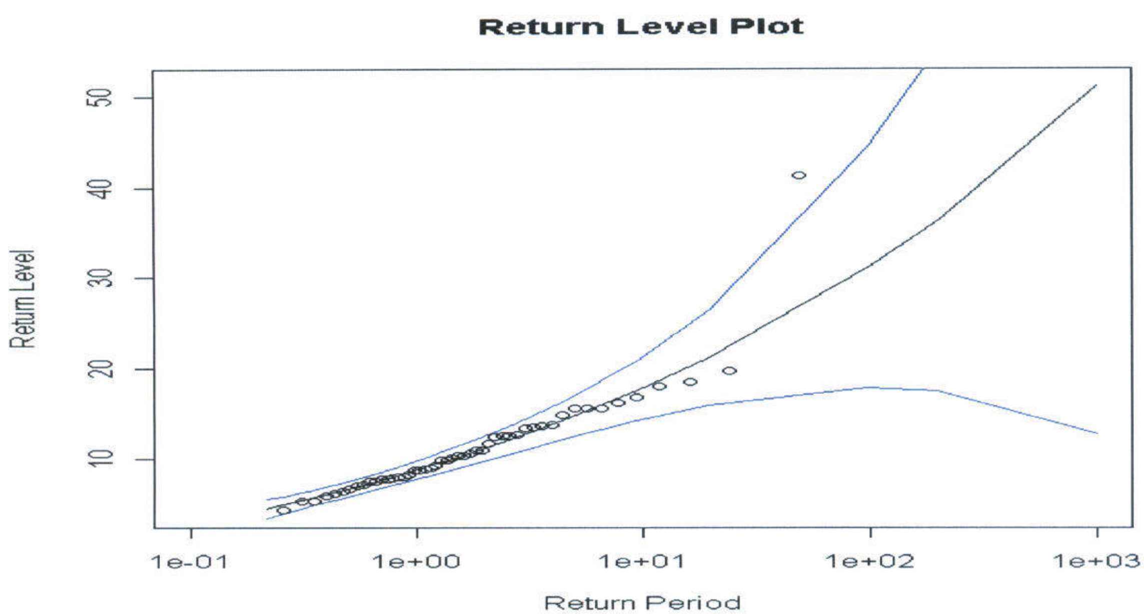


Figure 12. BM Return Level of S&P 500 Index Two Weeks Returns, 1950 – 1998.

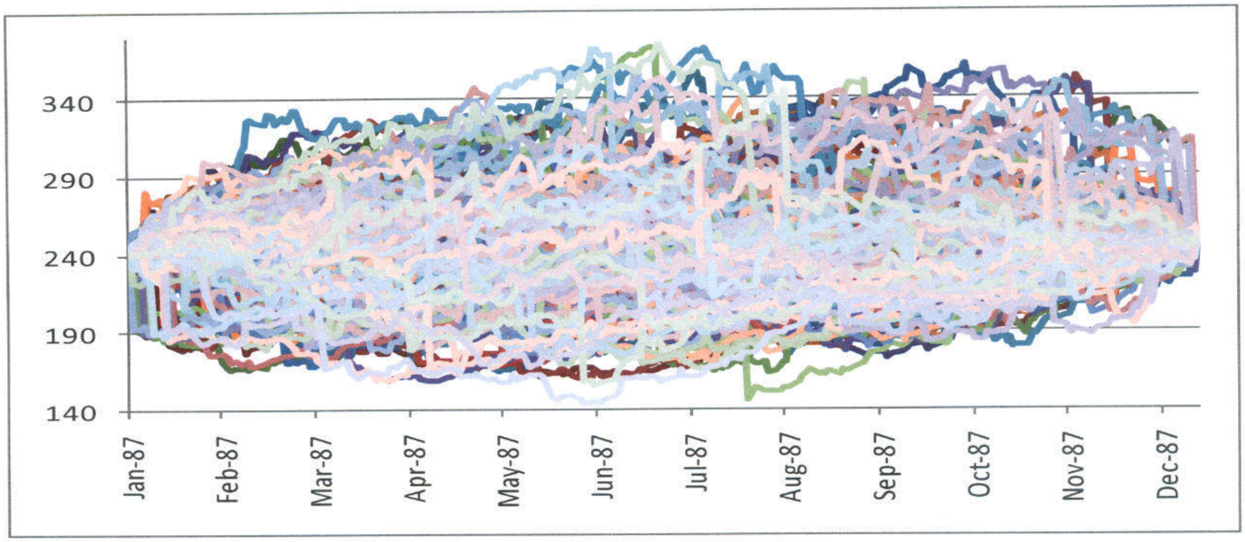


Figure 13. Price paths S&P 500 Index, 1987.

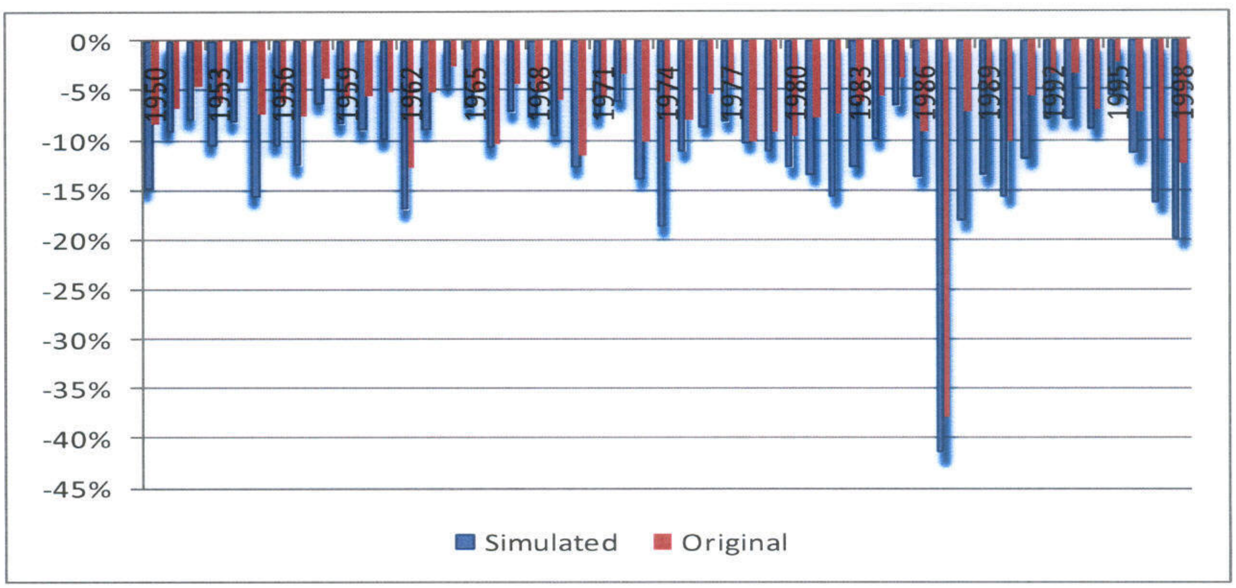


Figure 14. S&P 500 Index Maxima Two Weeks Returns Simulated, 1950 – 1998.

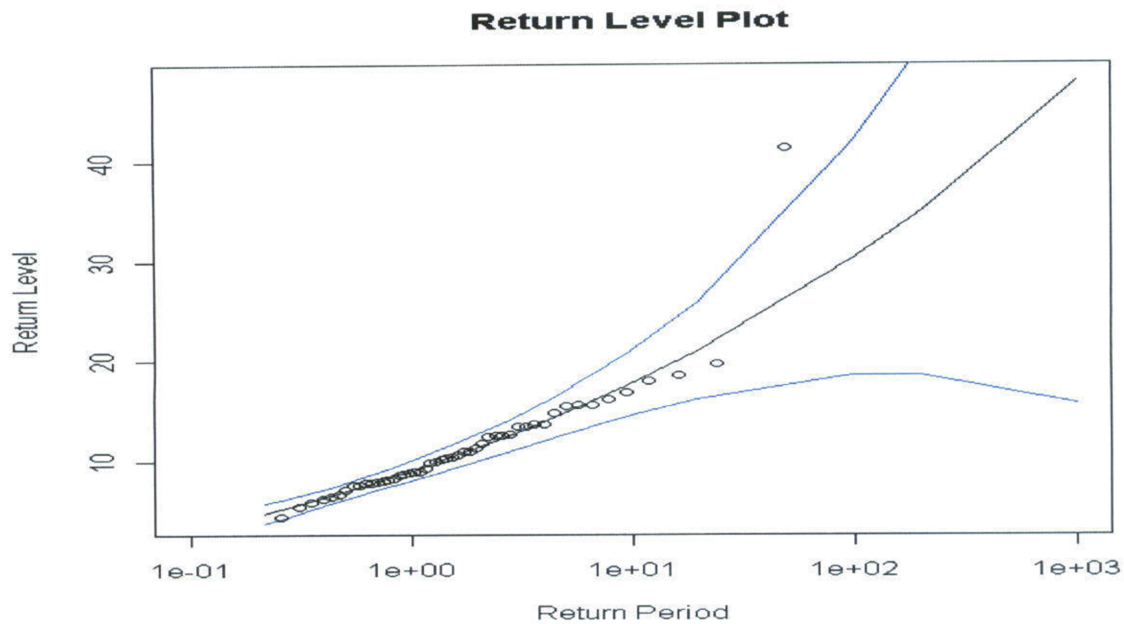


Figure 15. BM Return Level of S&P 500 Index Two Weeks Returns Simulated, 1950 – 1998.

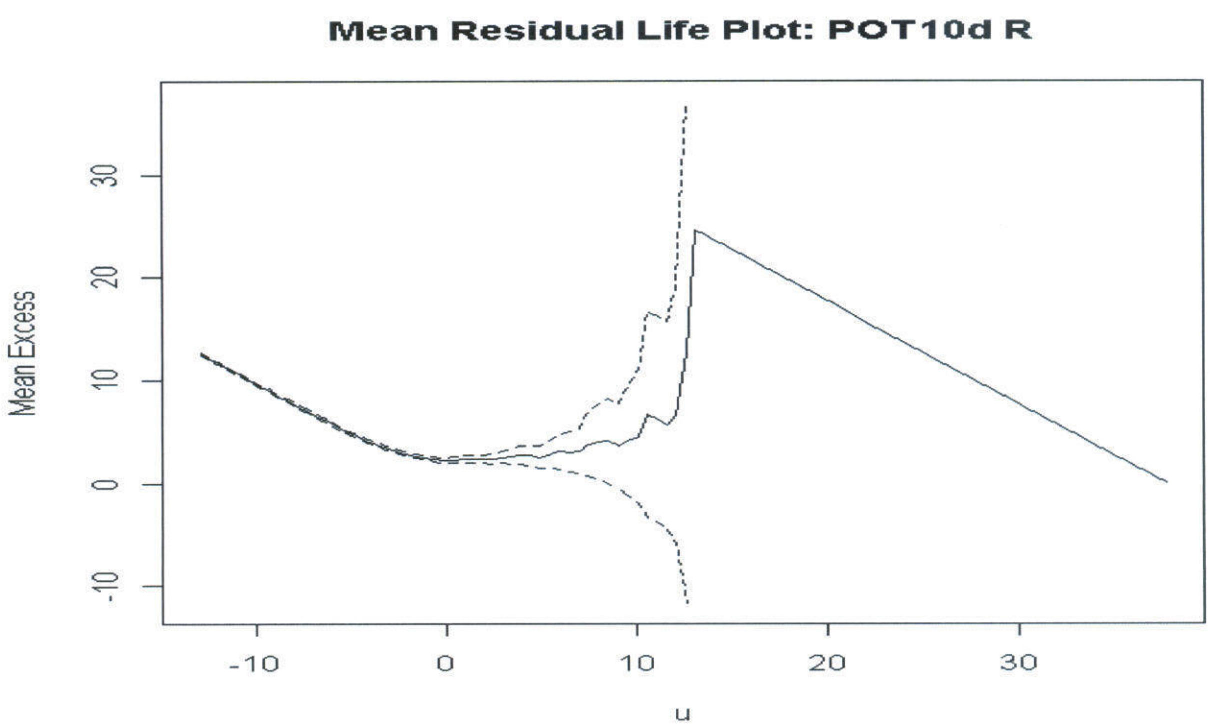


Figure 16. Mean Residual Life Plot, S&P 500 Index 10 Days Returns, 1950 – 1998.

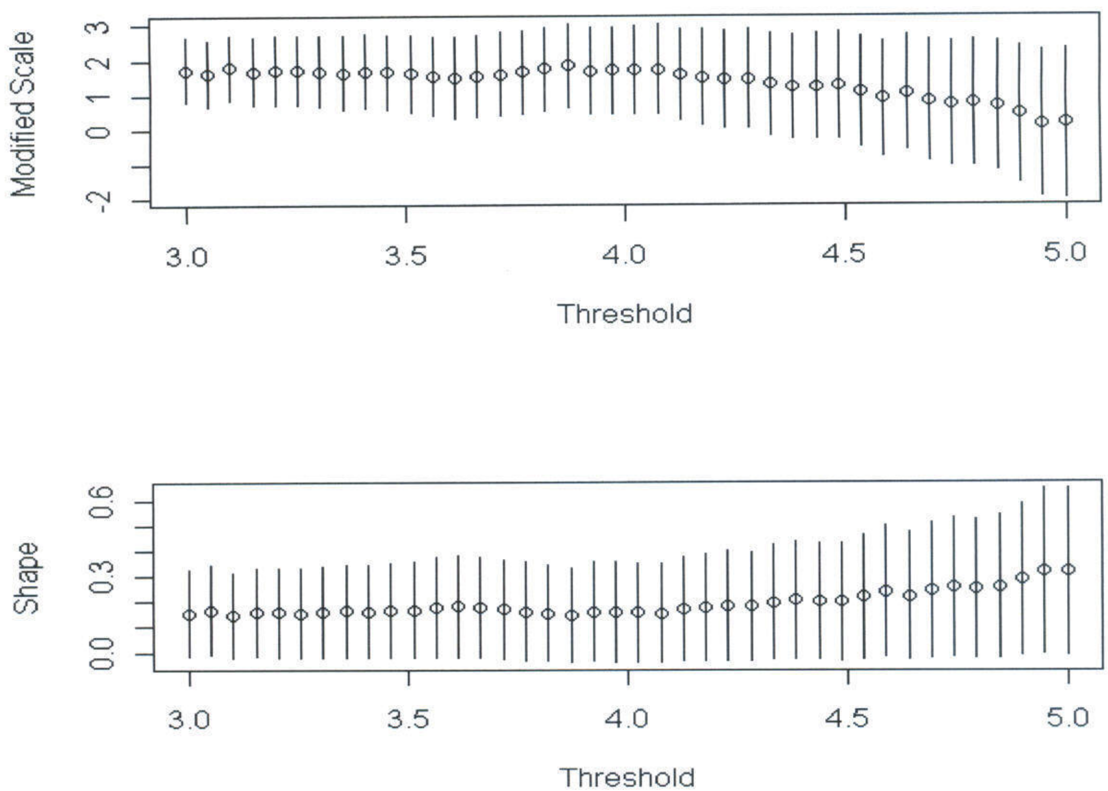


Figure 17. Fitting GDP for Thresholds Ranges, S&P 500 Index 10 Days Returns, 1950 – 1998.

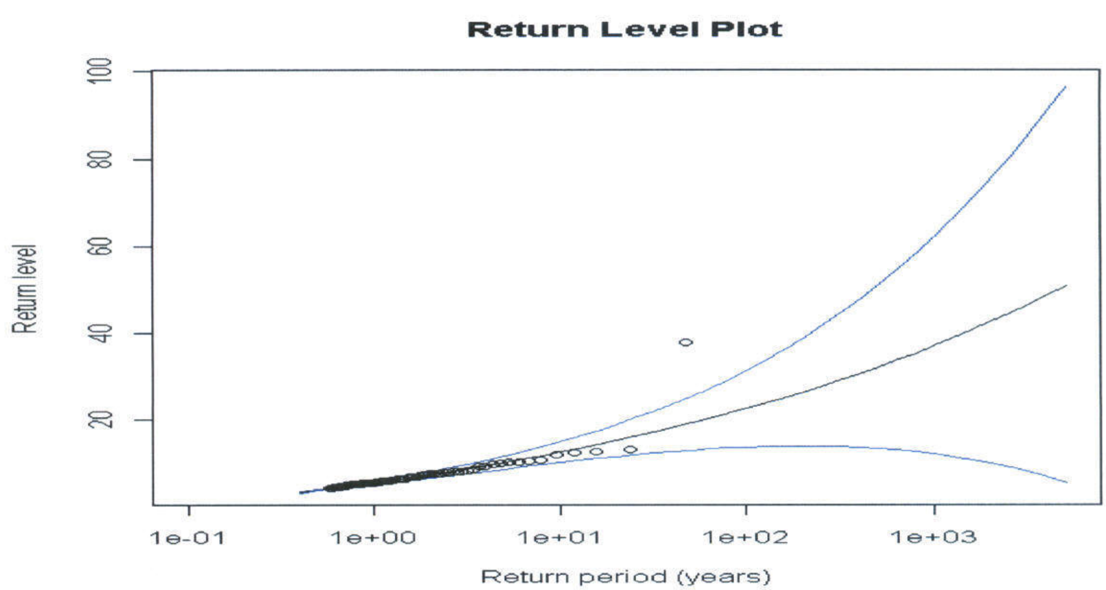


Figure 18. POT Return Level of S&P 500 Index 10 Days Returns, 1950 – 1998.

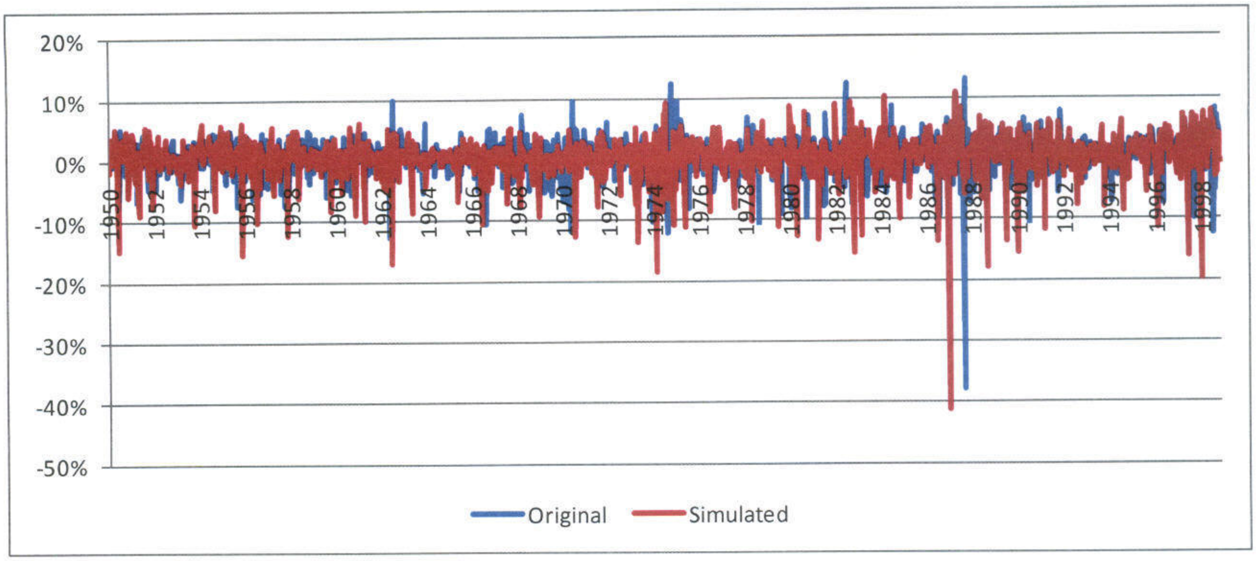


Figure 19. S&P 500 Index Two Weeks Returns Simulated, 1950 – 1998.

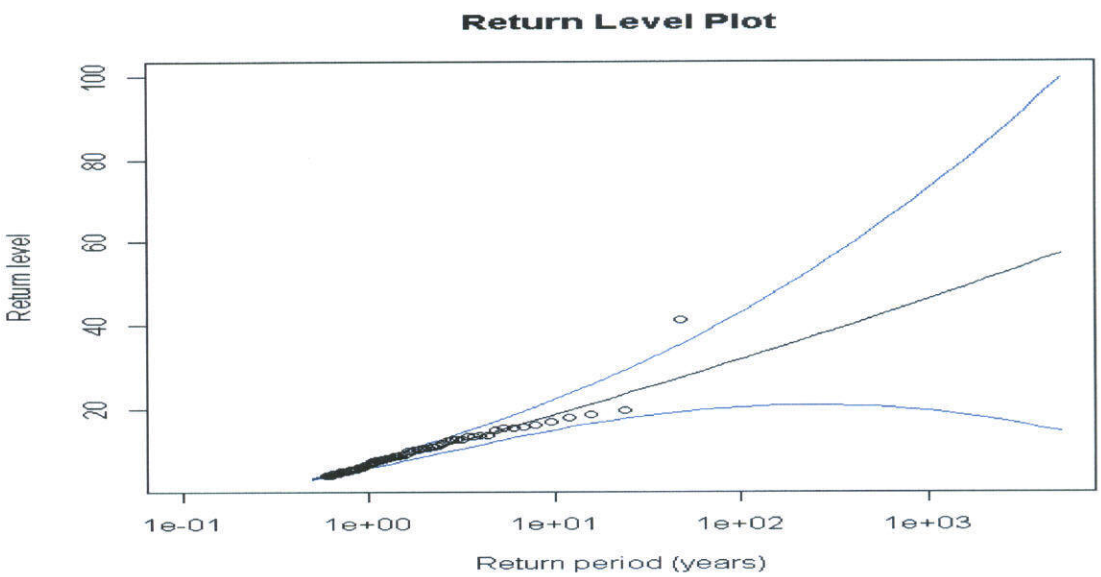


Figure 20. POT Return Level of S&P 500 Index 10 Days Returns Simulated, 1950 – 1998.