# Licensing probabilistic Patents and Liability Rules: The Duopoly case

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2009

# Licensing probabilistic patents and liability rules

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September 30, 2009

**Abstract**

In this paper a dynamic game is used to compare licensing of a cost reduction innovations under lost profits (LP) and unjust enrichment (UE), both damage rules used by courts in the calculation of damages when a patent has been infringed.

The innovation, whose property right belongs to a firm (patent holder) has a positive probability to be declared invalid in a court. The market is composed by two homogeneous firms that compete in quantities (Cournot).

Licensing by using royalty rates is preferred compared with fixed fees, it is observable little licensing (just big innovations).

LP is better (almost all cases) than UE for the industry and society. However in the major of the cases consumers are better off under UE and in the major of cases LP benefits more to the patent holder.

## 1 Introduction

One of the most important mechanisms made for to compensate and to award innovation is the *Patent System*. In this system there is an authority (i.e. the EPO in Europe) that gives rights of property on pieces of knowledge to an agent, this rights are known as *patents*.

Once a patent is granted, the patent holder has the exclusivity right to exploit the commercial potential of a innovation through a monopoly, by licensing to others or under other kind of contracts ( i.e. cross licensing).

Economists have been interested in the incentives for licensing. By thinking in the case of a inventor that license to a firm, it does because is not able to exploit the commercial potential of the innovation. However, licensing between competitors produces ambiguous results, i.e in a Bertrand competition licensing splits the monopoly profit, however it is expected licensing in markets with high level of differentiation.

Other topic that has received great interest for economists is the contractual mechanism of licensing. Licensing contracts can be summarized in the groups : licensing by a royalty rate, an fixed fee and a combination of both. The common approach has been game theory. In this approach the patent holder and one or several players are involved in a game of three stages:

1. at the first stage of the game, the patent holder decides how much to ask for the licenses and how many licenses he will offer;

2. at the second stage potential licensees decide whether to get the license or to use the backstop technology [1];

3. finally in the last stage, firms compete in the market[2].

The early literature has assume that patents are indisputable property rights-*ironclad property rights*. Several authors have compared fixed fees against royalty rates under different conditions: insider/ outsider inventor and duopolistic/ monopolistic/ perfect competence. Sen and Tauman [2007] summarize and extend the early models. They consider a contract where royalties and fixed fees can be included together, the innovation is a cost reduction one (drastic or non-drastic[3]) and a outsider/incumbent inventor. They conclude that:

1. there is full diffusion of the innovation;

2. consumers are better off, firms are worse off and welfare is improved.

3. the optimal license contract includes a positive royalty rate for non-drastic innovations.

4. outsider innovator license by a fixed fee just if the market is a monopoly and the innovation is drastic.

[1]The best technology available without the use of the innovation

[2]see Kamien and Tauman [2002] and Sen and Tauman [2007] for a survey about licensing games under ironclad rights

[3]In the case of a drastic innovation, the industry becomes a monopoly unless licensing is allowed.

In an ideal world should be expectable that, just the ideas that increase the well being of the society should be patented. But, in theory to select which ideas are valuable or which are not valuable, creates some collateral effects. One of them is the fact that patents could be declared invalid in a court procedure.

Lemley and Shapiro [2005] and others have pointed out that half of all litigated patents are found to be invalid, some of them with considerable commercial importance. Nowadays economists have changed the concept of patent as the right to sue others for infringement.

When potential users decide to infringe a patent, the patent holder could enforce the property rights by using the legal system, in this arena the patent holder will try to prove *infringement* and the infringer(s) will try to invalid the patent.

If the patent holder is successful to prove infringement, the court could authorize the a compensation or *damage payments* and could order other actions in order to enforce the property rights. Commonly two liability rules are used for to calculate damage payments: *Lost Profits (LP)* and *Unjust Enrichment (UE)* [4].

When the inventor is incumbent in the market, damages could be calculated in different ways, the most common way to do it is using the LP rule or the UE rule, both rules are based in a profile scenario. This scenario is the *"no infringement"* scenario.

By comparing with the base scenario, LP compensate the share of profit lost by the patent holder caused by the infringement and UE transfers the competitor's profit excess to the patent holder.

The impacts of this damages have been studied in different contexts as vertical relationship and horizontal competition.

In the case of vertical relationship Schankerman and Scotchmer [2001] have analyzed how liability rules protect patents, they conclude that UE protect better the patent holder than LP in the case of research tools, however in the case of cost reduction innovations these results are reverse.

Anton and Yao [2007] explore the impacts of the LP rule on competence and innovation, assuming a linear demand scheme and a non-drastic innovation, they conclude that infringement is a dominant situation even under the use of different liability rules. In the other fold Choi [2009] compares different liability rules assuming a drastic innovation and a more general demand function, he concludes that LP benefits more to the patent holder.

The objective of this paper is to fill the gap left by the recent literature related with the licensing and damage rules, I try first to compare LP against UE, secondly

[4] see Heath et al. for a complete comparison of damage rules between countries.

I compare fixed fees against royalty rates and its relation with damage rules.

The starting point of my research is the contribution developed by Wang [1998], where he develops a duopoly model to study licensing under ironclad patents under a Cournot scenario. In this model royalty rate scheme is compared against fixed fee licensing for drastic and non-drastic innovations. Under this base model I added the development made by Anton and Yao and Choi (AYC) to include probabilistic patents in a take or leave it ex-post[5] licensing situation.

In a difference of AYC I use a simple linear demand with homogeneous firms and homogeneous costs, this specification allow me to study drastic and non-drastic innovations, also I compare the royalty rate scheme against the fixed fee scheme assuming probabilistic patents.

My results show that surprisingly licensing is not possible under UE and just big innovation are licensed under the LP rule, for another side it is showed that licensing using a royalty rate is better than a fixed fee scheme front the point of view of the patent holder. Finally, comparison analysis shows that LP protect better the patentee for big innovations and for small ones the patentee is better protected by UE.

The document is organized as follows. In the section 2 are established the assumptions and description of a licensing game. In the sections 3, 4 and 5 the game is solved. In section 6 a comparative analysis between LP and UE is executed. In section 7 the conclusions and important remarks of this work are analyzed. Proofs of the propositions are showed in the text and lengthy proofs are treated in an appendix.

# 2 The Game

The game is a non cooperative game that involves two players: patent holder (firm $1$) and a competitor (firm $2$), they produce the same good under fixed marginal costs $c$, and without loss of generality it is assume that $c = 0$.

Let $p = 1 - q_1 - q_2$ be the inverse linear demand function that both face, where is used the subindex $1$ for the patent holder and the subindex $2$ for the competitor.

The *firm* $1$ has a patented a cost reduction innovation that reduces the marginal cost in $\gamma$, where $0 < \gamma$.

Let $\pi_i^s(q_i, q_j) = (1+\gamma-q_1-q_2)q_i$ be the profit function associated with the use of the new cost reduction innovation by the firm $i$ and let $\pi_i^i(q_i, q_j) = (1-q_1-q_2)q_i$

[5] ex-post innovation or ex-ante trial.

be the profit function associated with the use of the old technology. Notice that the profit function for the patent holder is always $\pi_1^s$.

At the very beginning of the game the patent holder decides whether to license ($\mathcal{L}'$) or not ($\mathcal{N}'$), if decides licensing offers a fixed fee ($F$) or a royalty rate($r$), the offer is a take it or leave it.

In the second stage the competitor decides between three alternatives: 1) accept the offer of the patent holder when is offered ($\mathcal{L}$); 2) uses the backstop technology ($\mathcal{N}$ or $\mathcal{N}'$) and 3) Infringe the patent ($\mathcal{I}$ or $\mathcal{I}'$) (see Figure 1below).

In the last stage the firms decide the quantities offered in the market as solution of a Cournot game. Once the competitor infringes the patent the patent holder reacts by starting a process in a court, with the objective to enforce its property rights.

The result of the trial is unknown, but there is a common knowledge probability $\theta \in (0, 1)$ that the patent will be declared valid after the trial, this parameter also reflects the *strength of the patent*.

When the patent holder shows the existence of infringement, the court calculate damage payments, for calculations are considered LP and UE. The method going to be used by the court for calculate damages is common knowledge before trial.

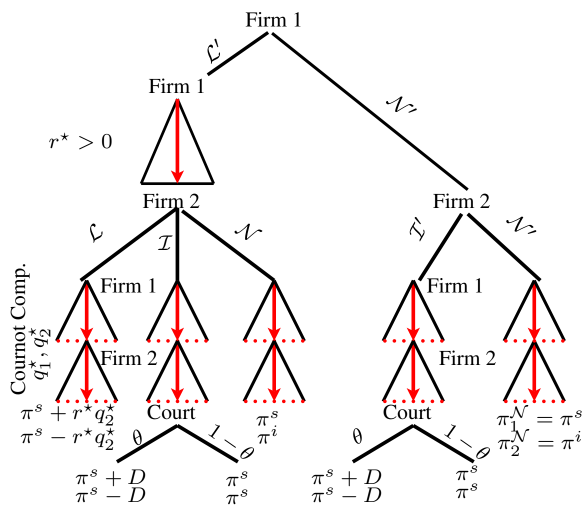


Figure 1: Game tree (the royalty rate case)

Payoffs are characterized through the actions of the competitor, by example if the patent holder plays $\mathcal{N}'$ and the competitor plays $\mathcal{N}$, the payoff obtained is the same that is obtained in the case when, the patent holder plays $\mathcal{L}'$ and asks a royalty rate $r$ and the competitor plays $\mathcal{N}'$, where in both situation players choose the same quantities.

Then by using this consideration the payoffs are:

1. $\pi_1^{\mathcal{N}} = \pi_1^s(q_1, q_2)$ and $\pi_2^{\mathcal{N}} = \pi_2^i(q_1, q_2)$, when competitor plays $\mathcal{N}$ the payoff for the players are.

2. $\pi_1^{\mathcal{L}} = \pi_1^s(q_1, q_2) + L(q_2)$ and $\pi_2^{\mathcal{L}} = \pi_2^s(q_1, q_2) - L(q_2)$, when competitor plays $\mathcal{L}$, notice that $L$ is the license's fee (a fixed fee or a royalty rate).

3. $\pi_1^{\mathcal{I}} = \pi_1^s(q_1, q_2) + \theta D(q_1, q_2)$ and $\pi_2^{\mathcal{I}} = \pi_2^s(q_1, q_2) - \theta D(q_1, q_2)$, when competitor plays $\mathcal{I}$. $D$ is the damage payment.

The solution criterion for the game described above is the *Sub-Game Perfect Nash Equilibrium (SPNE)*,that is going to be solved in the next three sections.

# 3 Competition Stage

Given a defined rule for the calculations of damages (LP or UE), a level of technology chosen by the incumbent firm ($\mathcal{N}$, $\mathcal{I}$, $\mathcal{L}$) and a licensing policy defined by the patent holder (to offer or not a license to the competitor using a fixed fee or a royalty rate), both firms compete by choosing quantities. This section is devoted to calculate the payoffs under different scenarios as a solution of the Cournot problem.

When the competitor decides to use the backstop technology ($\mathcal{N}$), the *Nash Equilibrium (NE)* is granted when

$$(q_1^{\mathcal{N}}, q_2^{\mathcal{N}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}, 0\right) & \text{if } 1 \leq \gamma \end{cases} \tag{1}$$

As was noted by Arrow [1962] big innovations could permit to the patent holder to reduce the price till levels below the competitive prices, meaning that just the patent holder can remain in the market, this kind of innovations are called *drastic*. In this particular setup an innovation is *non-drastic* if $0 \leq \gamma < 1$ and is defined *drastic* if $\gamma \geq 1$.

Payoffs are

$$
\begin{aligned}
\pi_1^{\mathcal{N}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \\
\pi_2^{\mathcal{N}} &= \begin{cases} \left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ 0 & \text{if } 1 \leq \gamma \end{cases}
\end{aligned}
\tag{2}
$$

A more complex situation emerges when the competitor infringes the patent ($\mathcal{I}$), once infringement is played the patent holder will try to enforce the property rights by suing the incumbent firm. When the patent holder is successful in the court (gains the trial), it is assumed here that the court will calculate a damage payment based in the LP or UE rule.

$$
\begin{aligned}
\pi_1^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_1 + \theta D(q_1, q_2) \\
\pi_2^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_2 - \theta D(q_1, q_2)
\end{aligned}
\tag{3}
$$

Then the payoffs are characterized by eq.(3), notice that the first term of the r.h.s. $(1 - q_i - q_j + \gamma)q_i$ is the profit gained by the sales and the second term are the damage payments $\theta D(q_1, q_2)$.

Basically UE and LP both need a comparison scenario of *"no infringement"*, here $\pi_1^{\mathcal{N}}$ is used as the comparison value when LP is the liability rule used by the court. Under LP damages are[6].

$$
D^{LP} = \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \tag{4}
$$

When the court uses UE as liability rule, the damage ($D^{UE}$) is the excess of profit of the competitor respect to $\pi_2^{\mathcal{N}}$, then

$$
D^{UE} = \max\left\{(1 - q_1 - q_2 + \gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \tag{5}
$$

The NE when damages are calculated by using the LP rule, and when the incumbent firm decides to infringe the patent deserves a special treatment. Given the structure of the damage rules quantities affect the level of damages, so then, the expected damage has effects on the equilibrium quantities.

[6]Interested readers could see Anton and Yao [2007] for a more detailed analysis for non drastic innovation and a more general linear demand.

**Lemma 1.** *The Cournot solution when competitor infringes and court uses LP rule for calculate damages is,*

$$(q_1^{\mathcal{I},\mathcal{LP}}, q_2^{\mathcal{I},\mathcal{LP}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \textit{if } \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right) & \textit{if } \gamma \geq \frac{\theta}{3-2\theta} \end{cases} \tag{6}$$

, it produces

$$\begin{aligned} \pi_1^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \\ \pi_2^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \end{aligned} \tag{7}$$

When $q_2^{\mathcal{I},\mathcal{LP}} = q_2^{\mathcal{N}}$ eq. (6), the patent holder gets the same profit that in the situation of no infringement. However, the competitor stays in a better situation because enjoys a lower cost and produce the same quantity that should be produced under no infringement. Anton and Yao [2007] calls this equilibrium *Passive Infringement*, because the damage payment does not reflect the effects of the infringement.

When $\gamma > \frac{\theta}{3-2\theta}$ damage payments calculated with the lost profit rule are positive in equilibrium, then in equilibrium a *Active Infringement* is present.

**Lemma 2.** *The Cournot solution when competitor infringes and court uses UE as liability rule is,*

$$(q_1^{\mathcal{I},\mathcal{UE}}, q_2^{\mathcal{I},\mathcal{UE}}) = \left((1-\theta)\frac{1+\gamma}{3-\theta}, \frac{1+\gamma}{3-\theta}\right) \tag{8}$$

Results in the lemmas 1 and 2 cannot be considered trivial, because the best replies that produces the NEs are non-smooth in both cases. Proofs of this lemmas are considered in the appendix[7].

[7] Anton and Yao [2007] have been proved the lemma 1 and have claim that the lemma 2 is true, in the appendix I offer the proof for the lemma 2 and an alternative proof for the lemma 1.

By using the lemma 2,

$$\begin{aligned}\pi_1^{\mathcal{I},\mathcal{UE}} &= \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0<\gamma<1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma\geq 1 \end{cases} \\ \pi_1^{\mathcal{I},\mathcal{UE}} &= \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0<\gamma<1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma\geq 1 \end{cases}\end{aligned} \tag{9}$$

When the competitor accepts the offer ($\mathcal{L}$) against a given fixed fee ($F$) or a given royalty rate ($r$), the following NEs are obtained: in the fixed fee case

$$\left(q_1^{\mathcal{L},\mathcal{F}}, q_2^{\mathcal{L},\mathcal{F}}\right) = \left(\frac{1+\gamma}{3}, \frac{1+\gamma}{3}\right) \tag{10}$$

; and for a given royalty rate ($r$)

$$\left(q_1^{\mathcal{L},\mathcal{R}}, q_2^{\mathcal{L},\mathcal{R}}\right) = \left(\frac{1+\gamma+r}{3}, \frac{1+\gamma-2r}{3}\right) \tag{11}$$

these results produce the following payoffs: for the fixed fee case

$$\begin{aligned}\pi_1^{\mathcal{L},\mathcal{F}} &= \left(\frac{1+\gamma}{3}\right)^2 + F \\ \pi_2^{\mathcal{L},\mathcal{F}} &= \left(\frac{1+\gamma}{3}\right)^2 - F\end{aligned} \tag{12}$$

; and

$$\begin{aligned}\pi_1^{\mathcal{L},\mathcal{R}} &= \left(\frac{1+\gamma+r}{3}\right)^2 + r\frac{1+\gamma-2r}{3} \\ \pi_2^{\mathcal{L},\mathcal{R}} &= \left(\frac{1+\gamma-2r}{3}\right)^2\end{aligned} \tag{13}$$

for the royalty rate case.

# 4 Competitor's Technology Stage

By assuming that the policy offered by the patent holder is known, it means the kind of contract offered (royalty rate or fixed fee). For to solve the game is necessary to analyze the behavior of the competitor respect to the technology choice, where the alternatives are:

1. not infringe the patent $\mathcal{N}$ (use the backstop technology);

2. infringe the patent $\mathcal{I}$ (use the new technology without a permission of the patent holder);

3. accept to pay for the use the new technology if a license is offered $\mathcal{L}$.

**Lemma 3.** *If the courts calculates damages using the* LP *rule or the* UE $\pi_2^{\mathcal{I}} \geq \pi_2^{\mathcal{N}}$.

Lemma 3 says that the competitor always prefer to infringe instead to use the backstop technology independently of the liability rule[8]. Then so, it is necessary to just compare the competitor's payoff under licensing $\pi_2^{\mathcal{L}}$ against the payoff under infringement $\pi_2^{\mathcal{I}}$.

Let $\underline{F}$ be a fixed fee $F \geq 0$ such that $\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}} = 0$, then

$$\underline{F} = \left(\frac{1+\gamma}{3}\right)^2 - \pi_2^{\mathcal{I}} \tag{14}$$

, notice that if $\underline{F}$ is negative there is no positive fixed fee that makes the licensing option as good as infringe for the competitor. In the appendix is proved that

**Lemma 4.** $\underline{F}^{\mathcal{LP}} \geq 0$ *but* $\underline{F}^{\mathcal{UE}} \geq 0$ *just if* $\gamma \geq \delta_1$, *where*

$$\delta_1 = \frac{12 - 5\theta + \theta^2 - 2\sqrt{(3-\theta)^2(3+\theta)}}{6 - 7\theta + \theta^2}$$

For the case of licensing under a royalty rate, let $\underline{r}$ be a royalty rate $r \geq 0$ that makes $\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}} = 0$, then if exists

$$\underline{r} = \frac{1+\gamma-3\sqrt{\pi_2^{\mathcal{I}}}}{2} \tag{15}$$

From eq. 12 and 13:

1. $\partial(\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}})/\partial F < 0$;

2. $\partial(\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}})/\partial r < 0$

3. $\pi^{\mathcal{L},\mathcal{F}} = \pi^{\mathcal{L},\mathcal{R}} > 0$ if $F = r = 0$

4. $\pi^{\mathcal{L},\mathcal{F}}$ and $\pi^{\mathcal{L},\mathcal{R}}$ are bounded.

[8]This result is also true under other liability rules as *lost royalties* and for more general specifications, see AYC.

then so, it is possible to create a one to one function between$\underline{r}$ and $\underline{F}$, in consequence

**Lemma 5.** $\underline{F} > 0$ *iff* $\underline{r} > 0$

by using the lemma 4 and 5, it is establish that

**Lemma 6.** *In the LP case always exist a positive fixed fee $\underline{F}$ (or royalty rate $\underline{r}$) such that $\pi_2^{\mathcal{L},\mathcal{F}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$ (or $\pi_2^{\mathcal{L},\mathcal{R}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$). However in the UE case the last statement is true just for $\gamma > \delta_1$.*

# 5 Licensing Stage

In a take it or leave it bargaining, the patent holder will ask for the fixed fee that makes the competitor indifferent between take the license or to infringe.

From eq. 12 it is observable that the patent holder will choose the greater $F$ that makes the competitor enjoys the same profit than under infringement, so then ,

$$F^\star = \underline{F}$$

In the case of the royalty rate from the eq. 13 it is easy to see that the patent holder gets the maximum level of fees when $r = \frac{1+\gamma}{2}$ and $\pi_2^{\mathcal{L},\mathcal{R}} = 0$, because $\underline{r} \leq \frac{1+\gamma}{2}$, so then, the patent holder will ask

$$r^\star = \underline{r}$$

as a royalty rate in exchange of a license, summarizing

**Lemma 7.** *When a licensing contract if offered the patent holder will ask for $F^\star = \underline{F}$ ($r^\star = \underline{r}$) when a fixed fee (royalty rate) is asked against the license.*

By using the definition of $\underline{r}$ (eq. 15) in the payoff function $\pi_1^{\mathcal{L},\mathcal{R}}$ from eq.13, the patent holder's payoff is

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}} \tag{16}$$

From eq. 14, $\pi_2^{\mathcal{I}} = \left(\frac{1+\gamma}{3}\right)^2 - F^\star$, by replacing $\pi_2^{\mathcal{I}}$ in the last equation produces

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left[\left(\frac{1+\gamma}{3}\right)^2 - F^\star\right] \tag{17}$$

and by using eq. 12, produces

$$\pi_1^{\mathcal{L},\mathcal{R}} - \pi_1^{\mathcal{L},\mathcal{F}} = F^\star/4 \geq 0$$

, summarizing.

**Proposition 1.** *The patent holder will prefer to license using a royalty rate scheme instead or a fixed fee scheme.*

When the patent holder does not offer a license, the competitor infringes the patent, so then the patent holder has to compare $\pi_1^{\mathcal{L}}$ against $\pi_1^{\mathcal{I}}$ in order to offer or not a license. Then by comparing this payoffs is observable that

**Proposition 2.** *The patent holder will never license under UE. However under LP a royalty rate's license is offered if $\gamma > \delta_2$, where*

$$\delta_2 = \frac{\theta(3-2\theta) + 3\sqrt{(3-\theta)^2(2-\theta)}}{18 - 15\theta + 4\theta^2}$$

.

In the last lemma $\delta_2$ is near to 1, meaning that just big innovations are licensed when courts used the LP rule. This result also coincides with the result by Choi [2009] when he concludes that the royalty rate under UE is lower than the one under LP in a general demand case.

Here, in fact when UE is use by the court, there is not royalty rate in equilibrium, because the royalty rate is not so big for to make the payoff under licensing enough big as the expected payoff under infringement, for the patent holder.

# 6 LP vs UE

In this section a comparison between LP and UE is made it using the results of previous sections.By summarizing,

**Lemma 8.** *When the LP rule is used there are three situations:*

1. *passive infringement $\gamma \leq \theta/(3-2\theta)$;*
2. *active Infringement $\gamma > \theta/(3-2\theta)$;*
3. *licensing by a royalty rate $\gamma > \delta_2$, where $0 \leq \theta/(3-2\theta) \leq \delta_2 \leq 1$.*

*However, when UE is used there is no licensing and the competitor infringes the patent in equilibrium.*

Then so, the equilibrium payoffs under both regimes are

$$\left(\pi_i^{\mathcal{LP}}, \pi_i^{\mathcal{UE}}\right) = \begin{cases} \left(\pi_i^{\mathcal{I},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & 0 \leq \gamma < \delta_2 \\ \left(\pi_i^{\mathcal{L},\mathcal{R},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & \gamma \geq \delta_2 \end{cases}$$

where $i = 1, 2$, in consequence by using eq. 16 and 7 it is easy to obtain the equilibrium payoffs under the LP regime

$$\pi_1^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma \leq \theta/(3-2\theta) \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \text{if } \delta_2 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \text{if } 1 \leq \gamma \end{cases} \tag{18}$$

$$\pi_2^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \theta/(3-2\theta) \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \theta/(3-2\theta) < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases}$$

Because in the case of UE infringement is always present, then from eq. 9 the payoffs are

$$\pi_1^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases}$$

$$\pi_2^{\mathcal{UE}} = \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{19}$$

and by comparing payoffs under LP against UE, is established that

**Proposition 3.** *The patent holder and the industry (competitor) are better off (is worse off) under LP (UE) for drastic and almost all the non-drastic innovations.*

As is Showed in the Figure 2 the patent holder is better off under LP (gray area) a mirror situation happens with the competitor that is better off under UE (black area), both situations are often observed except in a small area.

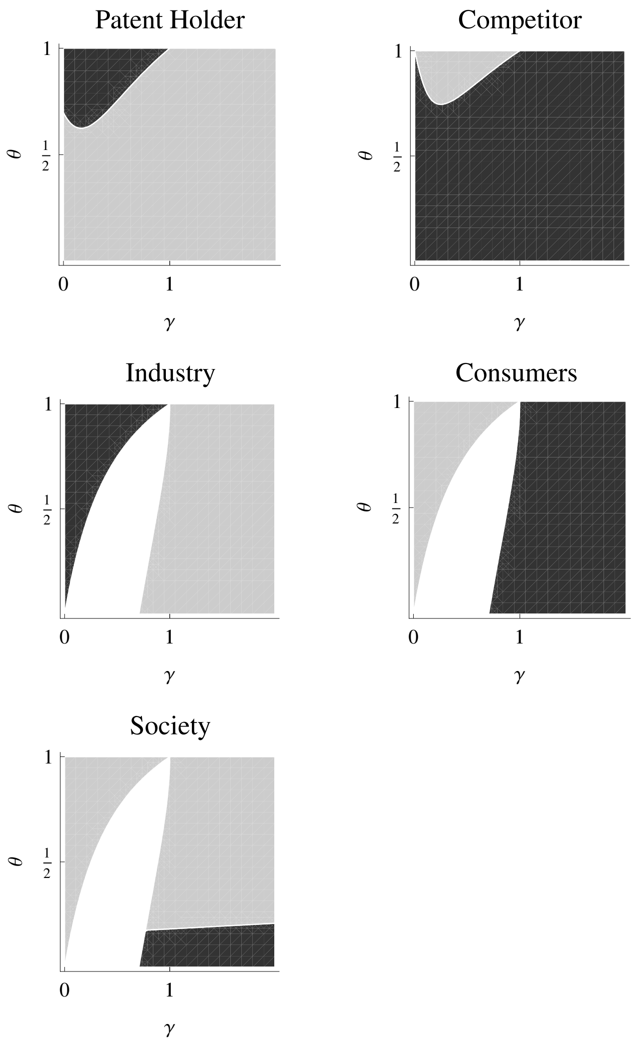


Figure 2: LP against UE: $LP \succ UE$ in gray; $LP \prec UE$ in black; and $LP \approx UE$ in white

Schankerman and Scotchmer [2005] said that under LP the competitor is worry about the looses of the patent holder, so then, its output is chosen endogenously for to compensate the damages of the patent holder, so LP turns in a collusive mechanism of profit transfer.

When the effect on the industry is compared, industry is better off under UE in the area on passive infringement (black area), the reason comes from the fact that under passive infringement there is not transfers from the competitor to the patent holder under LP, this reduces the possibility to reach a collusive profit for the industry.

However, when active infringement is present both rules produces the same industry profit, this result coincides with the one find it by Choi [2009] for a general quantity competition. Finally, when the innovation is drastic and LP is used there is licensing, this mechanism seems to be better that the UE damages infringement mechanism for to share the surplus of the innovation. In a consequence a inverse situation is going to be observed by the consumers whom are loosing surplus facing a higher price.(see lemma 4)

When a patent race is considered, the patent holder equilibrium's payoff is the reward of the winner and the competitor's payoff is the reward of the looser in the patent race, then as consequence of the proposition 3,

**Proposition 4.** *LP incentives more R&D than UE for drastic and almost all non-drastic innovations.*

In the case of the consumers, as consequence that the demand is linear, the consumer surplus is $(q_1 + q_2)^2/2 = Q^2/2$. Now, when the LP rule is used by using eq 6, 11 and lemma 8

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \theta/(3-2\theta) \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \frac{2(1+\gamma)-r}{3} & \text{if } \gamma \geq \delta_2 \end{cases}$$

, now by using eq 7

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \theta/(3-2\theta) \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2} & \text{if } \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2} & \text{if } \gamma \geq 1 \end{cases} \tag{20}$$

For UE from eq 8

$$Q^{\mathcal{UE}} = (2-\theta)\frac{1+\gamma}{3-\theta} \tag{21}$$

, so then, by direct comparison between $Q^{\mathcal{LP}}$ versus $Q^{\mathcal{UE}}$, it is establish that

**Proposition 5.** *The consumers are better off under UE for drastic innovations, but under non-drastic innovations LP is at least as good as UE.*

Let $SW = Q^2/2 + \sum \pi_i$ be the social welfare, then by using eq 17,18,19 and 20, comparisons shows that

**Proposition 6.** *Under non-drastic innovations LP is at least as good as UE, for the society, but for drastic innovations society is better off under the LP (UE) for strong (weak) patents.*

When innovations are drastic and LP is used, there is licensing against a royalty rate. However, when patents are weak (small $\theta$), patentees have less power of bargain, so then, patentees receive a small royalty rate, it produces a fall in the revenues of the patent holder and the industry (see Figure 2).

# 7 Conclusions

Throughout this article LP rule is compared against UE rule, as consequence of proposition 3, 5 is conclusive that LP should be preferred. At first because has a positive impact on $R\&D$. In second it is the positive net effect on the society, even being considered a collusive mechanism.

Damages has been prove to be important in to determine the licensing terms and critical for the existence of a licensing contract (fixed fee or royalty rate). The results show that there is no licensing under UE, nevertheless under LP just big innovations are licensing.

One implicit assumption of this model is the timing, this model lives and ends during the litigation time, by making so that injunctions have not been important, then even the results showed here predict no licensing, it could be possible when injunctions are considered[9].

Some questions remain unsolved, the first one is related to the objectives of damage rules in to deter infringement the question in this direction is: Which new

[9]see Farrell and Shapiro [2008] and Encaoua and Lefouili [2009] for the study of licensing when just injunctions are considered.

damage rule could deter infringement?, also an axiomatic point of view could be useful in such way could be important to know which ideal properties must be present in a damage rule.

One of the problems with non-drastic innovations and lost profits are the non-smooth of the payoff function, more work should be done trying to characterize some smooth approximation to this functions as was (i.e. Boone [2001]).

## Appendix

*Proof Lemma 1.* It is important to notice is that the best response function $\phi_1(q_2)$ is the same whether $D^{LP} > 0$ or $D^{LP} = 0$. The best response when $q_2 \in [0, a - c + \epsilon]$ is

$$\phi_1(q_2) = \frac{1 - q_2 + \gamma}{2}$$

The best response of the competitor deserves a special treatment. Let,

$$\begin{aligned} x(q_1, q_2) &= (1 - q_1 - q_2 + \gamma)q_2 - \theta \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \\ &= x_1(q_1, q_2) - \theta \max\{x_2(q_1, q_2), 0\} \end{aligned}$$

be the payoff of the competitor.

When $q_1 > 1 + \gamma$ the price becomes negative for any $q_2 \geq 0$, then in this case

$$\phi_2(q_1) = 0 \quad \text{if } q_1 > 1 + \gamma$$

.

If the innovation is drastic $\pi_1^{\mathcal{N}}$ is the monopoly profit in consequence $\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1 \geq 0$ for any $q_1, q_2 \geq 0$, then $D^{LP} > 0$ and, in consequence

$$\phi_2(q_1) = \frac{1 + \gamma - (1 + \theta)q_1}{2} \quad \text{if } \gamma/(1) \geq 1 \text{ and } q_1 \in [0, 1 + \gamma)$$

When the innovation is non drastic, for a given $q_1 \in [0, 1 + \gamma)$, $x(q_1, q_2)$ reach maximum at $\tilde{q_2}$, where $0 < \tilde{q_2} < \hat{q_2} = (1 + \gamma - q_1)/2$ and $\hat{q_2}$ is the maximum of $x_1(q_1, q_2)$. Then $\partial x_1(q_1, q_2)/\partial q_2 > 0$ for $q_2 \in [0, (1+\gamma-q_1)/2)$. And $\partial x_2(q_1, 0)/\partial q_2 = q_1$ for any $q_2$. Then the best response depends on the sign of $x_2(q_1, q_2)$, this sign could be positive, negative or zero. There are two values of $q_1$ that make $x_2(q_1, 0) = 0$,

$$\begin{aligned} q_1^{a,b} &= \frac{(1 + \gamma) \pm \sqrt{(1 + \gamma)^2 - 4\pi_1^{\mathcal{N}}}}{2} \\ &= \frac{(1 + \gamma)}{2} \pm \sqrt{\left(\frac{1 + \gamma}{2}\right)^2 - \left(\frac{1 + 2\gamma}{3}\right)^2} \end{aligned}$$

, where $a$ refers to the inferior value and $b$ to the superior one. For a given $q_1$ $x_2(q_1,0)$ reach minimum at $q_1^c = \frac{(1+\gamma)}{2}$, this results plus the fact that $\gamma/(1) < 1$ allow to see that

$$0 < q_1^a < q_1^{\mathcal{N}} < q_1^c < q_1^b < 1+\gamma$$

.

In consequence $x_2(q_1,0) > 0$ for $q_1 \in (0, q_1^a) \cup (q_1^b, 1+\gamma)$ and $x_2(0) \leq 0$ when $q_1 \in [q_1^a, q_1^b]$, then

$$\phi_2(q_1) = \frac{1+\gamma-(1+\theta)q_1}{2} \quad \text{if } q_1 \in (0,q_1^a) \cup (q_1^b, 1+\gamma) \text{ and } \gamma/(1) < 1$$

.

The next case appears when $q_1 \in [q_1^a, q_1^b]$, in consequence $x_2(q_1,0) \leq 0$, then by looking for some $q_2^a$ that makes $x_2(q_1, q_2^a) = 0$

$$\begin{aligned} q_2^a &= (1+\gamma) - q_1 - \pi_1^{\mathcal{N}}/q_1 \\ &\geq (1+\gamma) - q_1^b - \pi_1^{\mathcal{N}}/q_1^a = 0 \end{aligned}$$

, in consequence $0 \leq q_2^a < 1+\gamma$. Now by evaluating the derivative on the right of $x$ at $(q_1, q_2^a)$ (or directional derivative in the direction $(0,1)$),
$\partial^+ x/\partial q_2(q_1, q_2^a) = 1+\gamma-2q_2^a - q_1 - \theta q_1$, then

$$\phi_2(q_1) = q_2^a \quad \text{if } q_1 \in [q_1^a, q_1^b] \wedge \gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 \leq 0$$

or

$$\begin{aligned} \phi_2(q_1) = &\frac{1+\gamma-(1+\theta)q_1}{2} \\ &\text{if } q_1 \in [q_1^a, q_1^b] \wedge \gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 > 0 \end{aligned}$$

When is assumed that $x_2 > 0$ the Nash equilibrium is
$\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$
, now if $x_2 \leq 0$, $q_2^a(\phi_1) = q_2^{\mathcal{N}}$, so then $\phi_1(q_2^{\mathcal{N}}) = q_1^{\mathcal{N}}$ and the condition $\gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 > 0$ becomes in $\gamma/(1) < \theta/(3-2\theta)$, this condition implies that $\frac{1+\gamma}{3-\theta} \in [q_1^a, q_1^b]$, then if $\gamma/(1) < \theta/(3-2\theta)$ holds $x_2 < 0$ in equilibrium and the Nash equilibrium is $\left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right)$.

When $\gamma/(1) < \theta/(3-2\theta)$ does not hold $x_2 > 0$, the Nash equilibrium is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$
□

*Proof Lemma 2.* There is a symmetry respect to last proof, this time $\phi_2(q_1)$ is the same whether $D^{UE} > 0$ or $D^{UE} = 0$, when $q_1 \in [0, 1+\gamma]$ is

$$\phi_2(q_1) = \frac{1 - q_1 + \gamma}{2}$$

and 0 if $q_1 > 1 + \gamma$.

Let,

$$\begin{aligned} y(q_1, q_2) &= (1 - q_1 - q_2 + \gamma)q_1 + \theta \max\left\{(1 - q_1 - q_2 + \gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \\ &= y_1(q_1, q_2) - \theta \max\{y_2(q_1, q_2), 0\} \end{aligned}$$

be the payoff of the patent holder.

When $q_2 > 1 + \gamma$ the price becomes negative for any $q_1 \geq 0$, then in this case

$$\phi_1(q_2) = 0 \quad \text{if } q_2 > 1 + \gamma$$

.

If the innovation is drastic $\pi_2^{\mathcal{N}} = 0$ , then

$$\phi_1(q_2) = \frac{1 + \gamma - (1 + \theta)q_2}{2} \quad \text{if } \gamma/(1) \geq 1 \text{ and } q_2 \in [0, 1 + \gamma)$$

When the innovation is non drastic, for a given $q_2 \in [0, 1 + \gamma)$, $y(q_1, q_2)$ reach maximum at $\tilde{q_1}$, where $0 < \tilde{q_1} < \hat{q_1} = (1 + \gamma - q_2)/2$ and $\hat{q_1}$ is the maximum of $y_1(q_1, q_2)$. Then $\partial y_1(q_1, q_2)/\partial q_1 > 0$ for $q_1 \in [0, (1+\gamma-q_2)/2)$ and $\partial y_2(q_1, 0)/\partial q_1 = -q_2$ for any $q_1$. Then the best response depends on the sign of $y_2(q_1, q_2)$, this sign could be positive, negative or zero. There are two values of $q_2$ that make $y_2(0, q_2) = 0$,

$$\begin{aligned} q_2^{a,b} &= \frac{(1 + \gamma) \pm \sqrt{(1 + \gamma)^2 - 4\pi_2^{\mathcal{N}}}}{2} \\ &= \frac{(1 + \gamma)}{2} \pm \sqrt{\left(\frac{1 + \gamma}{2}\right)^2 - \left(\frac{1 - \gamma}{3}\right)^2} \end{aligned}$$

where $a$ refers to the inferior value and $b$ to the superior one. $y_2(q_1, 0)$ reach maximum at $q_2^c = \frac{(1+\gamma)}{2}$, in a consequence

$$0 < q_2^a < q_2^c < q_2^b < 1 + \gamma$$

.

Also $y_2(0, q_2) < 0$ for $q_1 \in (0, q_2^a) \cup (q_2^b, 1+\gamma)$ and $y_2(0, q_2) \geq 0$ when $q_2 \in [q_2^a, q_2^b]$, then

$$\phi_1(q_2) = \frac{1+\gamma-q_2}{2} \quad \text{if } q_2 \in [q_1^a, q_1^b]$$

.

if $\phi_1(q_2) = \frac{1+\gamma}{3}$ is played the best response of the other player is $\frac{1+\gamma}{3} \in [q_2^a, q_2^b]$, in a consequence it is not a NE.

Because $q_2^b > \hat{q_2}$, the best response belong to the interval $[0, q_2^a)$ when $y(0, q_2) > 0$. There is

$$\begin{aligned} q_1^a &= (1+\gamma) - q_2 - \pi_2^{\mathcal{N}}/q_2 \\ &\geq (1+\gamma) - q_2^b - \pi_1^{\mathcal{N}}/q_2^a = 0 \end{aligned}$$

that makes $y_2(q_1^a, q_2) = 0$, where $0 \leq q_2^a < 1+\gamma$.

then the derivative on the left (or in direction (-1,0)),
$\partial^- y/\partial q_1(q_1^a, q_2) = -(1+\gamma-2q_1^a - q_2 - \theta q_2)$, then

$$\phi_2(q_1) = q_1^a \quad \text{if } q_2 \in [0, q_1^a) \wedge \gamma/(1) < 1 \wedge -(1+\gamma-2q_1^a-q_2-\theta q_2) \leq 0$$

$$\begin{aligned} \phi_2(q_1) = \; & \frac{1+\gamma-(1+\theta)q_1}{2} \\ & \text{if } q_2 \in [0, q_1^a) \wedge \gamma/(1) < 1 \wedge -(1+\gamma-2q_1^a-q_2-\theta q_2) > 0 \end{aligned}$$

If is assumed that $y_2 > 0$ the Nash equilibrium is
$\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$
, now if $y_2 \leq 0$ in equilibrium, $q_2^a(\phi_1) = (1+5\gamma)/3$, so then $\phi_2((1+5\gamma)/3) = q_2^{\mathcal{N}}$ and the condition $-(\gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1) \leq 0$ becomes in $-(1-2\gamma)/3 + (1+\theta)(1-\gamma)/3 \leq 0$, but the first term is always positive then the unique Nash equilibrium is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ □

*Proof Lemma 3.* When $\gamma \leq \theta/(3-2\theta)$, $\pi_2^{\mathcal{I},\mathcal{LP}} = \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$. When $\theta/(3-2\theta) \leq \gamma < 1$,

$$G(\gamma,\theta) = \pi_2^{\mathcal{I},\mathcal{LP}} - \pi_2^{\mathcal{N}} = \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 - \left(\frac{1-\gamma}{3}\right)^2$$

, now notice that $G_{11} = \left(\frac{1}{3-\theta}\right)^2 - \frac{4\theta+1}{9}$, because at $\theta = 0$ $G_{11} = 0$ and because $dG_{11}/d\theta = 2(3-\theta)^{-3} - 4/9 < (2)^{-2} - 4/9 < 0$, $G_{11} < 0$ for $\theta \in (0,1)$, then $G$ is concave in $\gamma$ for $\theta \in (0,1)$.

$G(1,\theta) = \left(\frac{2}{3-\theta}\right)^2 - \theta$, moreover $G_2(1,\theta) = 8(3-\theta)^{-3} - 1 < 0$ for $\theta \in (0,1)$, $G(1,0) = \left(\frac{2}{3}\right)^2$ and $G(1,1) = 0$ then by continuity $G(1,\theta) > 0$ for $\theta \in (0,1)$.

$$G(\theta/(3-2\theta),\theta) = \left(\frac{1}{3-2\theta}\right)^2 - \theta\left(\frac{1}{3-2\theta}\right)^2 - \left(\frac{1-\theta}{3-2\theta}\right)^2 = \frac{\theta(1-\theta)}{(3-2\theta)^2} > 0$$

because $G$ is concave in $\gamma$ and $G(\theta/(3-2\theta),\theta), G(1,\theta) > 0$ $G > 0$ for $\gamma > \theta/(3-2\theta)$ and $\theta \in (0,1)$.

When $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{LP}} \geq \pi_2^{\mathcal{N}} = 0$.

For the UE case, if $\gamma < 1$

$\pi_2^{\mathcal{I},\mathcal{UE}} = (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$

and in the case $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{UE}} \geq \pi_2^{\mathcal{N}} = 0$

□

*Proof Lemma 4.* By using eq 2.14 this definition

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{22}$$

and after some algebra

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \frac{1}{9}\gamma(1+3\gamma) & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \frac{\theta\left(3-5\theta+\theta^2+\gamma^2\left(30-23\theta+4\theta^2\right)+\gamma\left(24-22\theta+4\theta^2\right)\right)}{9(3-\theta)^2} & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(57-50\theta+9\theta^2\right)}{36(3-\theta)} & \text{if } 1 \leq \gamma \end{cases}$$

it is straightforward to see that the first and third term are positive, in the case of the second term, this term is not always positive, but if $\frac{\theta}{3-2\theta} < \gamma < 1$, the term is also positive, then $\underline{F}^{\mathcal{LP}} \geq 0$.

Now in the case of UE,

$$\underline{F}^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{23}$$

and after some algebra

$$\underline{F}^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2+\gamma^2\left(6-7\theta+\theta^2\right)-2\gamma\left(12-5\theta+\theta^2\right)\right)}{9(-3+\theta)^2} & \text{if } 0 < \gamma < 1 \\ \frac{(1+\gamma)^2\theta(3+\theta)}{9(3-\theta)^2} & \text{if } 1 \leq \gamma \end{cases} \tag{24}$$

it is easy to see that the first term in this case is not always positive, however after find the roots of the polynomial it is easy to see that the expression is equal or greater than zero when $\gamma > \frac{12-5\theta+\theta^2-2\sqrt{27-9\theta-3\theta^2+\theta^3}}{6-7\theta+\theta^2}$. In the case of the second term is easy to see that is positive. Then $\underline{F}^{\mathcal{UE}} \geq 0$ if $\gamma > \frac{12-5\theta+\theta^2-2\sqrt{27-9\theta-3\theta^2+\theta^3}}{6-7\theta+\theta^2}$.

□

*Proof Proposition 2.* from 2.14 $\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}}$

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\frac{1+2\gamma}{3-\theta}\right)\left(\frac{1-\gamma}{3}\right) & 0 \leq \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \gamma \geq 1 \end{cases}$$

then after some algebra

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}} = \begin{cases} -\frac{(1-\gamma)(5\theta+\gamma(9+7\theta))}{36(3-\theta)} & 0 \leq \gamma < \frac{\theta}{3-2\theta} \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(-3+\theta)^2} & \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \frac{(1+\gamma)^2(1-\theta)^2\theta}{16(3-\theta)^2} & \gamma \geq 1 \end{cases}$$

it is easy to see that the first term is negative, the third one is positive and the second one could be positive or negative this case is not always positive, however after find the roots of the polynomial this expression is equal or greater than zero when $\gamma >$. In the case of the second term is easy to see that is positive. Then $\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}} \geq 0$ if $\gamma > \frac{\theta(3-2\theta)+3\sqrt{(3-\theta)^2(2-\theta)}}{18-15\theta+4\theta^2}$.

For the case of UE by preceding as in the LP case,

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4} * \left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta * \left(\frac{1-\gamma}{3}\right)^2\right) & 0 \leq \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4} * \left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2\right) & \gamma \geq 1 \end{cases}$$

then after some algebra and using the definition of $\pi_1^{\mathcal{I},\mathcal{UE}}$

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} - \pi_1^{\mathcal{I},\mathcal{UE}} = \begin{cases} -\frac{\theta\left(18-15\theta+\theta^2-2\gamma\theta(3+\theta)+\gamma^2\left(18-15\theta+\theta^2\right)\right)}{36(-3+\theta)^2} & 0 \le \gamma < 1 \\ \frac{(1+\gamma)^2(-1+\theta)\theta}{4(-3+\theta)^2} & \gamma \ge 1 \end{cases}$$

The second term is clearly negative, in the case of the roots of the polynomial are imaginary then the term is positive or negative, because at $\theta = \gamma = 1/2$ the value is $-187/7200$, then $\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} - \pi_1^{\mathcal{I},\mathcal{UE}} < 0$.

$\square$

*Proof Proposition 3.* After some algebra,

$$\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} = \begin{cases} \frac{\theta\left(3-5\theta+\theta^2\right)-2\gamma\left(-9+21\theta-8\theta^2+\theta^3\right)+\gamma^2\left(27-15\theta-2\theta^2+\theta^3\right)}{9(-3+\theta)^2} & 0 \le \gamma < \frac{\theta}{3-2*\theta} \\ \frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(-3+\theta)^2} & \frac{\theta}{3-2*\theta} \le \gamma < \delta_2 \\ \frac{\theta\left(2\gamma\theta(-9+2\theta)+3\left(3-5\theta+\theta^2\right)+\gamma^2\left(54-45\theta+8\theta^2\right)\right)}{12(-3+\theta)^2} & \delta_2 \le \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(21-26\theta+5\theta^2\right)}{16(-3+\theta)^2} & \gamma \ge 1 \end{cases}$$

It easy to see that the third and fourth cases are positive, the second case has two roots under $\theta/(2 - theta)$ and because at $\theta = \gamma = 1/2$ is positive the term is also positive in the region under study, the last case has both roots inside the study region, then after a straightforward analysis it is concluded that. $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ if $\theta < \frac{-9+21\theta-8\theta^2+\theta^3\pm3\sqrt{9-51\theta+85\theta^2-50\theta^3+12\theta^4-\theta^5}}{27-15\theta-2\theta^2+\theta^3}$.

In the case of the competitor, after some algebra

$$\pi_2^{\mathcal{LP}} - \pi_2^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2\right)+\gamma\left(9-30\theta+11\theta^2-2\theta^3\right)+\gamma^2\left(27-12\theta-4\theta^2+\theta^3\right)}{9(-3+\theta)^2} & 0 \le \gamma < \frac{\theta}{3-2*\theta} \\ -\frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(-3+\theta)^2} & \frac{\theta}{3-2*\theta} \le \gamma < 1 \\ -\frac{(1+\gamma)^2(-6+\theta)\theta^2}{9(-3+\theta)^2} & \gamma \ge 1 \end{cases}$$

in the last case is easy to see that the expression is positive, in the second case both roots are under $\theta/(3 - 2\theta)$ and the expression is negative at $\theta = \gamma = 1/2$, finally in the first case both roots are inside the region of interest then after some analysis is straightforward to see that $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ if $\theta > \frac{-9+30\theta-11\theta^2+2\theta^3-3\sqrt{9-132\theta+238\theta^2-116\theta^3+17\theta^4}}{2(27-12\theta-4\theta^2+\theta^3)}$.

In the case of the industry

$$\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} = \begin{cases} \frac{\theta(-3+2\theta)+\gamma^2\theta(-3+2\theta)+\gamma\left(9-12\theta+5\theta^2\right)}{9(-3+\theta)^2} & 0 \le \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \le \gamma < \delta_2 \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(-3+\theta)^2} & \delta_2 \le \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(189-138\theta+29\theta^2\right)}{144(-3+\theta)^2} & \gamma \ge 1 \end{cases}$$

In the first case both roots are outside the region of study and at $\theta = 1/2, \gamma = 1/10$ the expression is negative, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} < 0$ if $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, the third case follows by notice that one of the roots is $\delta_2$ (the other one is negative) and at $\theta = 1/10, \gamma = 9/10$ the expression is positive, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} > 0$ if $\frac{\theta}{3-2*\theta} \leq \gamma < \delta_2$, and the last case follows directly. $\square$

*Proof Proposition 4.* after some algebra

$$Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = \begin{cases} \frac{\theta-\gamma(3-2\theta)}{3(3-\theta)} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+2*\gamma}{3}\right)^2} & \delta_2 \leq \gamma < 1 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+\gamma}{2}\right)^2} & \gamma \geq 1 \end{cases}$$

in the case when $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, it is observable that $\partial(Q^{\mathcal{LP}} - Q^{\mathcal{UE}})/\partial\gamma < 0$ and at $\gamma = \frac{\theta}{3-2*\theta}$ $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = 0$, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $0 \leq \gamma < \frac{\theta}{3-2*\theta}$. In the third case, because both roots of the polynomial are below $\delta_2$ then the term is positive or negative, because at $\theta = 1/10, \gamma = 9/10$ the expression is negative, by noticing that if the third term is negative this implies that the fourth it is also negative, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $\gamma \geq \delta_2$. $\square$

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